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CASCADE OF SINKS

BY
CLARK ROBINSON[1]

ABSTRACT. In this paper it is proved that if a one-parameter family $\{F_t\}$ of C^1 dissipative maps in dimension two creates a new homoclinic intersection for a fixed point P_t when the parameter $t = t_0$, then there is a cascade of quasi-sinks, i.e., there are parameter values t_n converging to t_0 such that, for $t = t_n$, F_t has a quasi-sink A_n with each point q in A_n having period n. A quasi-sink A_n for a map F is a closed set such that each point q in A_n is a periodic point and A_n is a quasi-attracting set (à la Conley), i.e., A_n is the intersection of attracting sets A_n^j, $A_n = \bigcap_j A_n^j$, where each A_n^j has a neighborhood U_n^j such that $\bigcap\{F^k(U_n^j)\colon k \geqslant 0\} = A_n^j$. Thus, the quasi-sinks A_n are almost attracting sets made up entirely of points of period n. Gavrilov and Silnikov, and later Newhouse, proved this result when the new homoclinic intersection is created nondegenerately. In this case the sets A_n are single, isolated (differential) sinks. In an earlier paper we proved the degenerate case when the homoclinic intersections are of finite order tangency (or the family is real analytic), again getting a cascade of sinks, not just quasi-sinks.

1. Definitions and statement of results. In order to state the result, several definitions are needed. For a more thorough introduction with compatible perspective see **[11]**. The first few definitions are given for C^1 maps (or families of maps) from a two-dimensional manifold M to itself, $F\colon M \to M$. A point p is called a *periodic saddle point* if $F^k(p) = p$ for some $k > 0$ and the eigenvalues λ_u and λ_s of the derivative $DF^k(p)$ are both real, $|\lambda_u| > 1$ and $|\lambda_s| < 1$. A saddle point is called *twisted* if the eigenvalue $\lambda_u < -1$. (Alligood and Yorke call such points *Mobius orbits*.) A saddle point p of period k is called *dissipative* if $|\det DF^k(p)| < 1$, so $|\lambda_u \lambda_s| < 1$. The *stable and unstable manifolds* of a periodic saddle point p are

$$W^s(p, F) = \{q\colon \text{distance}(F^j(p), F^j(q)) \to 0 \text{ as } j \to \infty\}$$

and

$$W^u(p, F) = \{q\colon \text{distance}(F^{-j}(p), F^{-j}(q)) \to 0 \text{ as } j \to \infty\},$$

respectively. Note, these are the stable manifolds of the point p and not the whole orbit of p. In the definition of $W^u(p, F)$, if F is not invertible then $F^{-j}(p)$ is a point on the periodic orbit through p and $F^{-j}(q)$ means there are choices q_{-j} for $F^{-j}(q)$ with $F(q_{-j-1}) = q_{-j}$ and distance$(F^{-j}(p), q_{-j}) \to 0$ as $j \to \infty$. If p is a periodic saddle point, then the stable manifold theorem for noninvertible maps

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means there are local C^1 curves through p, $W^s_{loc}(p, F)$ and $W^u_{loc}(p, F)$. This theorem is given in many places, but it is contained in [**8**, Theorem 5.1] for maps, as opposed to differential equations. See [**3**] for other references.

A point p is called a *periodic sink* if $F^k(p) = p$ for some $k > 0$ and $|\lambda_j| < 1$ for both eigenvalues λ_1 and λ_2 of $DF^k(p)$. A standard result shows that a sink is an attractor (asymptotically stable), i.e., there is a neighborhood U of the orbit of p, $\mathcal{O}(p) = \{F^j(p)\colon j \in Z\}$, such that $\bigcap\{F^i(U)\colon i \geqslant 0\} = \mathcal{O}(p)$. With the weak hypotheses in the Theorem, the periodic points which are created are not necessarily sinks, but are sets of periodic points which form a quasi-attracting set. A closed set A is called an *attracting set* if it is invariant and there is a neighborhood U of A such that $\bigcap\{F^i(U)\colon i \geqslant 0\} = A$. A closed set A is called a *quasi-attracting set* if A is the intersection of attracting sets A^j. A quasi-attracting set is not necessarily attracting, as an example in Remark 3 shows. Also see [**4** or **9**]. A closed set A is called a *quasi-sink of period n* if A is a connected quasi-attracting set and each point q in A is a periodic point of period n. In fact the quasi-sinks which are shown to exist below are either a single point or a closed curve segment.

The last few definitions and the Theorem concern a one-parameter family $\{F_t\}$ of C^1 maps from a two-dimensional manifold M to itself, $F_t\colon M \to M$, which depends continuously on the parameter t in the C^1 topology. A family $\{F_t\}$ is said to *create* (resp. *destroy*) *homoclinic intersections at q when $t = t_0$ for a periodic saddle* point P_t if there are ε, $k > 0$ and continuously varying subarcs $\gamma^s_t \subset W^s_{loc}(P_t, F_t)$ and $\gamma^u_t \subset W^u_{loc}(P_t, F_t)$ for $t_0 - \varepsilon \leqslant t \leqslant t_0 + \varepsilon$ such that

(i) $\gamma^s_t \cap F^k_t(\gamma^u_t) = \varnothing$ for $t_0 - \varepsilon \leqslant t < t_0$ (resp. $t_0 < t \leqslant t_0 + \varepsilon$);

(ii) for $t_0 < t \leqslant t_0 + \varepsilon$ (resp. $t_0 - \varepsilon \leqslant t < t_0$) $F^k_t(\gamma^u_t)$ crosses locally from one side of γ^s_t to the other and returns to the original side;

(iii) $q \in \gamma^s_t \cap \gamma^u_t$ for $t = t_0$.

Multiple crossings are allowed in (ii), and the crossings may be C^∞ tangent or whatever. A more precise way to state condition (ii) is that there are local coordinates $\{(x, y)\}$ near q (which depend continuously on t) and parametrizations $\{(x^\sigma_t(\tau), y^\sigma_t(\tau))\colon |\tau| \leqslant \delta\}$ for $\sigma = s$ and u of γ^s_t and $F^k_t(\gamma^u_t)$, respectively, such that (a) for $t = t_0$, $(x^s_t(0), y^s_t(0)) = (0, 0) = (x^u_t(0), y^u_t(0))$ corresponds to q, (b) $y^s_t(\tau) \equiv 0$, and (c) for $t_0 < t \leqslant t_0 + \varepsilon$, $y^u_t(-\delta)$ and $y^u_t(\delta)$ are of one sign and $y^u_t(\tau')$ is of another sign for some $|\tau'| < \delta$, where τ' may depend on t. The family *nondegenerately* creates the homoclinic intersection if the family is C^2 and, in addition to (i) and (ii), satisfies

(iii)$'$ for $t = t_0$, γ^s_t and γ^u_t have an intersection of order two at q (tangency of order one), i.e., for $t = t_0$,

$$y^u_t(0) = 0 = \frac{d}{d\tau} y^u_t(0) \quad \text{but} \quad \frac{d^2}{d\tau^2} y^u_t(0) \neq 0;$$

(iv) if y^*_t is the extreme value of $y^u_t(\tau)$ for $-\delta \leqslant \tau \leqslant \delta$ then $dy^*/dt \neq 0$ at $t = t_0$.

Note, when the family nondegenerately creates homoclinic intersections, then for $t > t_0$ there are two new transverse intersections and a Smale horseshoe.

A family $\{F_t\}$ has a *cascade of quasi-sinks* (resp. cascade of sinks) if there is a sequence of parameter values t_n converging to t_0 such that, for $t = t_n$, F_t has a quasi-sink p_n (resp. a sink) of period b_n and b_n goes to infinity as n goes to infinity. In the Theorem, where the cascade is caused by the creation of a homoclinic intersection for a periodic saddle point P_t of period a, then $b_n = na + k$, where $F_t^k(W^{\mathrm{u}}_{\mathrm{loc}}(P_t, F_t))$ intersects $W^{\mathrm{s}}_{\mathrm{loc}}(P_t, F_t)$. In particular if P_t is a fixed point then $b_n = n$ and all sufficiently large periods are attained. The Theorem can now be stated.

THEOREM. *Let $\{F_t\}$ be a one-parameter family of C^1 maps of a two-dimensional manifold M which creates a homoclinic intersection at q when $t = t_0$ for a dissipative periodic saddle point P_t of period a. Then $\{F_t\}$ has a cascade of quasi-sinks A_n of period b_n with the following properties.*

(1) *If P_t is a family of fixed points then the periods $b_n = n$ for n greater than some N. In general, $b_n = an + k$, where a is the period of P_t and $F_t^k(W^{\mathrm{u}}_{\mathrm{loc}}(P_t, F_t))$ intersects $W^{\mathrm{s}}_{\mathrm{loc}}(P_t, F_t)$.*

(2) *If F_t^a preserves the orientation of $W^{\mathrm{s}}(P_t, F_t)$ and $W^{\mathrm{u}}(P_t, F_t)$ then the parameter values t_n converge to t_0 from one side. In some cases $t_n > t_0$ and in other cases $t_n < t_0$. If F_t^a reverses either or both of the orientations, then the t_n for odd n converge to t_0 from one side and those for even n converge from the other side.*

(3) *As n goes to infinity, the quasi-sinks A_n converge to an invariant subset of the orbit F_{t_0} of $\gamma^{\mathrm{u}}_{t_0} \cap \gamma^{\mathrm{s}}_{t_0}$, $\mathcal{O}(\gamma^{\mathrm{u}}_{t_0} \cap \gamma^{\mathrm{s}}_{t_0})$.*

The proof is contained in §2–4. §2 considers the case where, for each t, P_t is a fixed point and F_t preserves the orientations of $W^{\mathrm{s}}(P_t, F_t)$ and $W^{\mathrm{u}}(P_t, F_t)$. §3 indicates the necessary changes when F_t reverses one of the orientations. Finally, §4 indicates the changes when P_t is a periodic point.

REMARK 1. As mentioned in the abstract, it has been previously proved that $\{F_t\}$ has a cascade of sinks (not just quasi-sinks) whenever the family nondegenerately creates a homoclinic intersection, or when it degenerately creates a homoclinic intersection of finite order. The nondegenerate case was proved by Gavrilov and Silnikov [**6**] and was also contained in the work of Newhouse [**10** or **11**]. The finite-order case was proved in our earlier paper [**12**] and always applies when the maps are real analytic. The present proof uses a lemma from this latter paper, which states that whenever a family creates a twisted periodic saddle point then somewhere before there is a periodic sink. However, in this paper an index argument is used rather than the analytic estimate, which shows a twisted periodic saddle is created whenever a finite-order homoclinic intersection is created. The present theorem is also related to, but different than, the recent result of Alligood and Yorke [**1**]. They assume a horseshoe is created as the parameter varies, which is true if the family nondegenerately creates homoclinic intersections, but is probably not true if it occurs degenerately. Their conclusion is much stronger than the above Theorem: rather than just one cascade of sinks, they show, via period doubling, there are many cascades of sinks. Thus, their assumptions and conclusions are both stronger. The paper by Aronson et al. [**2**] contains a numerical study where a homoclinic intersection is created for a noninvertible map. Thus, the Theorem proves the sinks they observe are actually there. For a map with singularities as simple as they study, there are parameter values with infinitely many sinks as in [**10** or **12**]. It is not clear what general assumptions insure infinitely many sinks for a noninvertible map.

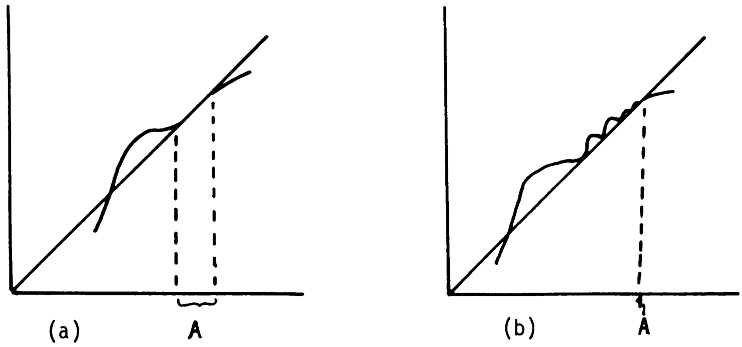

FIGURE 1

REMARK 2. The quasi-sinks which are not sinks have a lot of structure, and their basins of attraction are much like the basins of a sink. Let A_n be such a quasi-sink of period n. The points of A_n are center-stable periodic points with eigenvalue $\lambda_1 = 1$ and $|\lambda_2| < 1$. The Center Manifold Theorem implies that A_n lies on a C^1 curve segment γ_n, which is strongly contracting normally to γ_n. Moreover, the proof shows that $A_n = \bigcap_j A_n^j$, where A_n^j is a sequence of decreasing curve segments in γ_n with endpoints of period n and weakly attracting in γ_n. (The contraction is higher order because $\lambda_1 = 1$.) By minimality, A_n is either a point or a curve segment made up entirely of points of period n. If A_n is a curve segment then, since it is normally attracting, its stable manifold (or basin of attraction) will contain an open subset, so it is much like the basin of a sink. If A_n is a point then the points asymptotic to A_n might not contain an open set. In any case the stable manifolds of A_n^j contain an open set. With a given level of precision of iteration (on a computer for example), the set A_n would not be distinguished from A_n^j with j large, so the basin of attraction of A_n^j would be the effective basin of attraction of A_n. Thus, the quasi-sink would be visible.

REMARK 3. To give some understanding of how quasi-sinks occur, as opposed to sinks, consider a family of maps of the interval $f_t: I \to I$. Assume, for $t < 0$, that f_t has no fixed points and, for $t = 0$, that f_t has a degenerate fixed point. For all $t > 0$ it is possible that f_t has two intervals of fixed points, one of which is attracting (see Figure 1(a)). It is also possible that, for $t > 0$, f_t has a single point as a quasi-sink, but it is only quasi-attracting and not attracting, e.g., the quasi-sink could be the limit of a countable sequence of fixed points (saddle nodes). See Figure 1(b). For a discussion of such quasi-attracting sets see the papers by Conley [**4**] or Hurley [**9**].

2. Proof for orientation-preserving fixed saddle point. In this section it is assumed that the homoclinic intersection is created at q_0 for a fixed saddle point P_t. Also it is assumed that F_t preserves the orientation of $W^u(P_t, F_t)$ and $W^s(P_t, F_t)$. For simplicity of argument, F_t is assumed to be a C^2 local diffeomorphism, so there exist C^1 coordinates (x, y) on a neighborhood U of P_t in which F_t is linear, $F_t(x, y) = (\mu x, \lambda y)$ with $0 < \mu < 1$ and $1 < \lambda$. (If F_t cannot be linearized, e.g., if F_t is not invertible at P_t, then it is necessary to use C^1 coordinates where F_t is almost linear

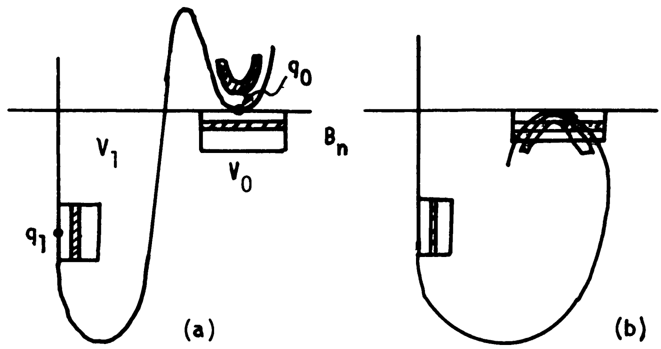

FIGURE 2

and the local stable and unstable manifolds are the coordinate axes. See [**12**, Appendix 5.4] for use of such coordinates.) The point P_t is dissipative, so $\lambda\mu < 1$ or $\mu < \lambda^{-1}$. By looking along the orbit of q_0, $q_0 = (x_0, 0)$ can be assumed to lie on the local stable manifold of P_t in U. Also, by taking k large enough there is a $q_1 = (0, y_1)$ on the local unstable manifold of P_t in U with $F_{t_0}^k(q_1) = q_0$.

Next, boxes B_n are chosen near q_0 lying in the quadrant abutting q_0 and q_1. Depending on the way the homoclinic intersection is formed, the boxes $F_t^n(B_n)$ are pulled off B_n for either $t > t_0$ or $t < t_0$. See Figures 2(a) and 2(b) for examples of the respective cases. It is proved below that as $F_t^n(B_n)$ is pulled off B_n, there has to be some value of t in which there is a quasi-sink. Take $\delta_1^{\mathrm{u}}, \delta_1^{\mathrm{s}}, \delta_0^{\mathrm{u}}, \delta_0^{\mathrm{s}} > 0$ and form

$$V_0 \equiv \{(x, y): |x - x_0| \leqslant \delta_0^{\mathrm{s}}, 0 \leqslant (\operatorname{sign} y_1)y \leqslant \delta_0^{\mathrm{u}}\},$$
$$V_1 \equiv \{(x, y): |y - y_1| \leqslant \delta_1^{\mathrm{u}}, 0 \leqslant (\operatorname{sign} x_0)x \leqslant \delta_1^{\mathrm{s}}\}.$$

The value δ_0^{s} has to be large enough so that $\gamma_{t_0}^{\mathrm{s}} \cap \gamma_{t_0}^{\mathrm{u}}$ is contained in the boundary of V_0. In fact, δ_0^{s} and γ_t^{s} are chosen such that $\gamma_t^{\mathrm{s}} \subset$ boundary V_0. Similarly, δ_1^{u} and γ_t^{u} are chosen so that $F_t^{-k}(\gamma_t^{\mathrm{u}}) \subset$ boundary V_1. For N large enough, $n \geqslant N$, and $m = n - k$, the box B_n is defined by

$$\begin{aligned} B_n &\equiv \operatorname{component}(V_0 \cap F_t^{-m}(V_1)) \\ &= \{(x, y): |x - x_0| \leqslant \delta_1^{\mathrm{s}}, |y - \lambda^{-m}y_1| = \lambda^{-m}\delta_1^{\mathrm{u}}\}. \end{aligned}$$

Thus, B_n is a horizontal strip near q_0. The component is chosen to be the first intersection with V_0 along $F_t^{-m}(V_1)$. Then $F_t^m(B_n)$ is a vertical strip near q_1, and $F_t^n(B_n) = F_t^k \circ F_t^m(B)$ is a thin nonlinear box near q_0 which is parallel to $\gamma_t^{\mathrm{u}} \subset W^{\mathrm{u}}(P_t, F_t)$.

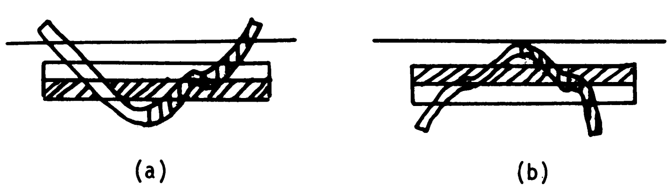

FIGURE 3

For N large enough there is an $\varepsilon > 0$ such that

(1) $F_t^n(B_n) \cap B_n = \varnothing$ for either (a) $t = t_0$ or, respectively, (b) $t = t_0 - \varepsilon$;

(2) for either (a) $t = t_0 + \varepsilon$ or, respectively, (b) $t = t_0$, $F_t^n(B_n)$ crosses B_n

(i) from top to bottom and back again as in Figure 3(a), or

(ii) from bottom to top and back again as in Figure 3(b);

(3) $F_t^n(B_n)$ never intersects the sides of B_n for $t_0 - \varepsilon \leqslant t \leqslant t_0 + \varepsilon$; and

(4) the images of the top and bottom of B_n by F_t^n never intersect B_n for $t_0 - \varepsilon \leqslant t \leqslant t_0 + \varepsilon$.

The crossings are not necessarily monotone, and there can be several crossings. In any case there is a substrip B_n' of B_n,

$$B_n' = \{(x, y): |x - x_0| \leqslant \delta_1^s, y_n' \leqslant y \leqslant y_n''\} \subset B_n,$$

such that $F_t^n(x, y_n')$ lies above B_n and $F_t^n(x, y_n'')$ lies below B_n for (a) $t = t_0 + \varepsilon$ or, respectively, (b) $t = t_0$. Since F_t^n has no fixed points on the boundary of B_n', it easily follows that the Lefschetz index of F_t^n relative to B_n' is $+1$, $I(B_n', F_t^n) = I(B_n') = 1$. (See [**7** or **5**] for a discussion of the index. It is locally much like the index of an equilibrium point for a vector field or differential equation as discussed in many differential equations books.) If the fixed points of F_t^n in B_n' are isolated, then the sum of their indices is 1. In any case since P_t is dissipative, F_t^n restricted to B_n decreases area, so the possible fixed points of F_t^n in B_n' are as follows:

Type	*Eigenvalues*	*Index*
weak twisted saddle	$\lambda_1 \leqslant -1, \lvert\lambda_2\rvert < 1$	1
sink	$\lvert\lambda_1\rvert < 1, \lvert\lambda_2\rvert < 1$	1
center-stable	$\lambda_1 = 1, \lvert\lambda_2\rvert < 1$	-1, 0, or 1
saddle	$\lambda_1 > 1, \lvert\lambda_2\rvert < 1$	-1

All the fixed points, except possibly the center-stable points, are isolated. If there are no center-stable points then for $t = t_0 + \varepsilon$ or $t = t_0$,

$$1 = I(B_n', F_t^n) = \sum \{ I(q, F_t^n): q \text{ is a fixed point of } F_t^n \}.$$

Thus, there must be either a sink or a weak twisted saddle point. (The adjective weak refers to the fact that $\lambda_1 = -1$ is allowed.) If there is a sink, then the proof is finished. Otherwise, there is a weak twisted periodic saddle point. As the parameter t varies from $t_0 + \varepsilon$ to t_0 (respectively, t_0 to $t_0 - \varepsilon$), the twisted saddle fixed point of F_t^n in B_n disappears without crossing the boundary of B_n because of properties (3) and (4) above. By applying [**12**, Proposition 3.3] it follows that, for some intermediate parameter value $t = t_n$, there is a sink.

If F_t^n has some center-stable fixed points, then the following argument shows there must be a quasi-sink for some parameter value. Let S be the set of center-stable fixed points of F_t^n in B_n'. For any q in S one of the eigenvalues $|\lambda_2| < 1$, so by the Center Manifold Theorem [**3**] there is a neighborhood W of q and a C^1 curve C such that all the fixed points of F_t^n in W lie on C. By extending the curves when the neighborhoods overlap, the endpoints of C can be chosen off S. The extended curves C cannot be closed curves because they are invariant and F_t^n decreases area and the interior would be invariant. The endpoints of the curves C either (i) both move inward so $I(C, F_t^n) = 1$, (ii) both move outward so $I(C, F_t^n) = -1$, or one end moves inward and the other outward so $I(C, F_t^n) = 0$. The set S can be covered by a finite number of neighborhoods W_i with corresponding curve segments C_i. Then

$$\begin{aligned} 1 &= I(B_n', F_t^n) \\ &= \sum I(C_i, F_t^n) + \sum \{ I(q, F_t^n) \colon q = F_t^n(q) \text{ is not center-stable} \}. \end{aligned}$$

Since the sum is 1, one of the terms must be 1. If there is a sink or twisted saddle, then the argument is similar to the above. Otherwise, there is a curve segment C_i with index 1. If $C_i \cap S$ is an interval of fixed points, then it is attracting, and so a quasi-sink. Otherwise, there are a finite or a countable number of open subcurves $\{J_{ij}\}$ in C_i in which the points are moved by F_t^n. For any such J_{ij} the points are all moved in one direction on C_i. Then one-half of $C_i - J_{ij}$ is an attracting set, A_n^j. By picking the attracting sets A_n^j from $A_n^{j-1} - J_{ij}$ sequentially, $A_n = \bigcap_j A_n^j$ is a quasi-sink. By construction, the set A_n is closed and connected, so it is either a point or a curve segment, and it is made up entirely of points of period n.

Property 3 of the Theorem follows because the boxes B_n can be made to converge to $\gamma_{t_0}^{\mathrm{u}} \cap \gamma_{t_0}^{\mathrm{s}}$ by letting δ_0^{s} depend on n.

3. Orientation reversing case. In this section P_t is still assumed to be a fixed point of F_t. If F_t preserves the orientation of $W^{\mathrm{s}}(P_t, F_t)$, let V_1 be as before. If it reverses orientation of $W^{\mathrm{s}}(P_t, F_t)$, let

$$V_1 \equiv \{ (x, y) \colon |y - y_1| \leqslant \delta_1^{\mathrm{u}}, |x| \leqslant \delta_1^{\mathrm{s}} \}.$$

Similarly, if F_t reverses the orientation of $W^{\mathrm{u}}(P_t, F_t)$, change the definition of V_0 to

$$V_0 \equiv \{ (x, y) \colon |x - x_0| \leqslant \delta_0^{\mathrm{s}}, |y| \leqslant \delta_0^{\mathrm{u}} \}.$$

Let $B_n \equiv \text{component}(V_0 \cap F_t^{-m}(V_1))$, where $m = n - k$. If F_t preserves the orientation of $W^{\mathrm{u}}(P_t, F_t)$, then B_n is on the same side of $W^{\mathrm{u}}(P_t, F_t)$; if it reverses orientation, then B_n is on different sides depending on the parity of n. In the same

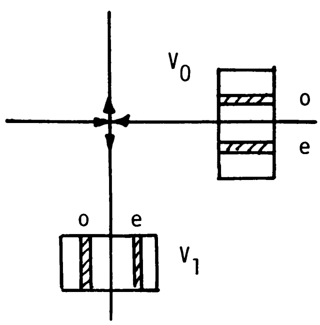

FIGURE 4

way, $F_t^m(B_n)$ is either on one side of $W^{\mathrm{u}}(P_t, F_t)$ or both sides depending on whether or not F_t preserves the orientation of $W^{\mathrm{s}}(P_t, F_t)$. The same applies to $F_t^n(B_n)$ and the sides of γ_t^{u}.

The rest of the proof is similar to §2. If only one of the orientations is reversed, it is not hard to see that the t_n which gives a quasi-sink of period n will be on different sides of t_0 depending on the parity of n. The same statement is true if both orientations are reversed, but more care is needed in the argument. In the case depicted in Figure 5(a–i,ii), it is clear the t_n are on opposite sides of t_0 depending on the parity of n. In the case depicted in Figure 5(b–i), it is harder. Because for $t = t_0$ the distance from $F_t^n(B_n)$ to γ_t^{u} is proportional to μ^m, the distance from B_n to γ_t^{s} is proportional to λ^{-m}, and $\mu^m < \lambda^{-m}$, $F_t^n(B_n)$ and B_n intersect for n odd and do not intersect for n even. Thus, in this case the quasi-sinks for n odd occur for $t_n < t_0$ and for n even for $t_n > t_0$. All other cases are similar to one of these two. This completes the argument.

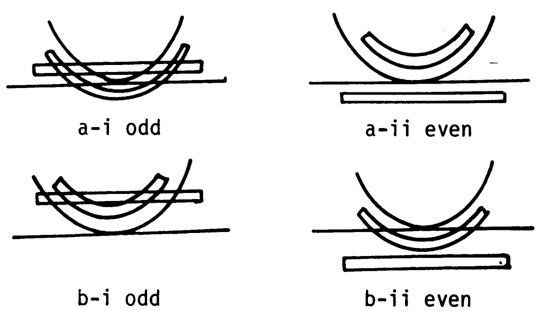

FIGURE 5

4. Periodic case. When P_t has period a, k is chosen so that $F_t^k(q_1) = q_0$ with $q_1 \in W^{\mathrm{u}}_{\mathrm{loc}}(P_t, F_t)$. Note, q_1 is on the local unstable manifold for P_t and not another point of the orbit of P_t. The sets V_0 and V_1 are constructed near q_0 and q_1, respectively, as before. Let $B_n = \text{component}(V_0 \cap F_t^{-na}(V_1))$. As before, for $b = b_n = na + k$, F_t^b has a quasi-sink in B_n for a suitable chosen parameter value t.

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DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60201