

ON CONTRACTIONS OF SEMISIMPLE LIE GROUPS

BY

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ABSTRACT. A limiting formula is given for the representation theory of the Cartan motion group associated to a Riemannian symmetric pair (G, K) in terms of the representation theory of G .

Introduction. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and H a closed subgroup with subalgebra $\mathfrak{h}$. The coset space G/H is called reductive [9, p. 389] if $\mathfrak{h}$ admits an $\text{Ad}_G(H)$ invariant complement $\mathfrak{m}$ in $\mathfrak{g}$; i.e. a subspace $\mathfrak{m} \subset \mathfrak{g}$ such that

- (i) $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ (direct sum),
- (ii) $\text{Ad}_G(H)\mathfrak{m} \subset \mathfrak{m}$.

In this case we can form the semidirect product $\mathfrak{m} \rtimes H$ with respect to the adjoint action of H on $\mathfrak{m}$. In this paper we shall restrict ourselves to the case where G is semisimple with finite centre and (G, H) is a Riemannian symmetric pair [10, p. 209]. Hence H is contained in the fixed point set H_σ of an analytic involution σ of G , it contains the identity component $(H_\sigma)_e$ and $\text{Ad}_G(H)$ is compact. Following custom, we write K and $\mathfrak{k}$ rather than H and $\mathfrak{h}$ in this instance. It is well known that K is compact [10, p. 252] and connected if G is noncompact. Furthermore $\mathfrak{k}$ is the $+1$ eigenspace of $d\sigma_e$. We make the natural choice for $\mathfrak{m}$, namely the -1 eigenspace V of $d\sigma_e$. When G is noncompact $d\sigma_e$ is a Cartan involution. Then V is usually denoted $\mathfrak{p}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is called a Cartan decomposition [10, p. 182]. When G is compact one can choose a real form $\mathfrak{g}_0$ of the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$, and a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{p}$ such that $V = i\mathfrak{p}$, i.e. $\mathfrak{g} = \mathfrak{k} + i\mathfrak{p}$ [10, V, §2].

The semidirect product $V \rtimes K$, in the situation described above, is called the *Cartan motion group* associated to the pair (G, K) . The idea of relating the representation theories of $V \rtimes K$ and G has been prevalent in the literature (cf. notably [11, 13, 18]). In particular, $V \rtimes K$ is a contraction of the Lie group G in the sense of [11], and there has been some interest in understanding the relationship between the representation theories of $V \rtimes K$ and G that this implies (cf. particularly the footnote on p. 343 of [17]). The aim of this paper is to give such a precise relationship. A key ingredient involved is the global counterpart of the contraction of the Lie algebra of G to that of $V \rtimes K$. It is the family of smooth maps

$$\begin{aligned} \pi_\lambda: \quad V \rtimes K &\rightarrow G \\ (vk) &\rightarrow \exp_G(\lambda v)k. \end{aligned}$$

These are homomorphisms to within $O(\lambda)$ as $\lambda \rightarrow 0$.

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In §2 we give a brief discussion of contractions of Lie groups and indicate how the Kirillov-Kostant method of orbits suggests the relationship between the representation theories of a Lie group and a contraction of it. In §3 we give the description of the unitary irreducible representations of $V \rtimes K$ by Mackey's semidirect product analysis [19]. In particular we consider the family $\rho_{\psi, \eta}$ of irreducible representations which are called the generic irreducible representations of $V \rtimes K$. These are parametrised by a linear function ψ on a maximal abelian subalgebra α of V , and an irreducible representation η of $M \subset K$, where M is the centraliser of $A = \exp_G \alpha$ in K . They form a set of full Plancherel measure in the unitary dual of $V \rtimes K$ [14].

The irreducible unitary principal series representations of a noncompact semisimple Lie group G , with finite centre, can be given exactly the same parametrisation as the generic irreducible representations of $V \rtimes K$. We describe this in §4 and give our first main theorem, Theorem 1, which shows how the sequence $\sigma_{n\psi, \eta}$ of principal series representations approximates $\rho_{\psi, \eta}$ in a weak sense when composed with the global contraction map $\pi_{1/n}$.

In §5 we prove the analogue of Theorem 1 in the compact case (Theorem 2). Once again we show how a family of irreducible representations of the compact group G can be parametrised by a linear function $\psi \in \alpha^*$ and an irreducible representation η of M . This is a version of the Borel Weil theorem. However, the ψ 's which actually give rise to representations of G are not arbitrary elements of α^* but form a cone of a lattice in it. Hence, if $\psi \in \alpha^*$ is arbitrary, and we wish to approximate the irreducible representation $\rho_{\psi, \eta}$, we will not usually have available the sequence of irreducible representations of G of the form $\sigma_{n\psi, \eta}$, as in the noncompact case. Instead we must choose a sequence $\psi_1, \psi_2, \dots$ from the cone of the lattice in α^* so that $\psi_n/n \rightarrow \psi$. Another problem is that, in contrast with the noncompact case, M and A may intersect, and this forces compatibility requirements between ψ and η which affect the choice of the sequence $\psi_1, \psi_2, \dots$. Theorem 2 shows how, with the appropriate choice of sequence, $\sigma_{\psi_n, \eta}$ composed with $\pi_{1/n}$ approximates $\rho_{\psi, \eta}$ in a weak sense. The special case of this theorem where $G = SO(n+1)$ and $K = SO(n)$ was proved in [5]. An infinitesimal version for spherical functions was proved in [4].

In §6 we describe how to extend our results to the case of nongeneric irreducible representations $\rho_{\psi, \eta}$ of $V \rtimes K$. In this case η is an irreducible representation of the stabiliser K_ψ of ψ under the coadjoint action of K on V . We replace M by K_ψ and A by $\exp \alpha_\psi$, where $\alpha_\psi \subset \alpha$ is the centraliser in α of the Lie algebra $\mathfrak{k}_\psi$ of K_ψ . In the noncompact case we replace the principal series representations of G by the appropriate H -series (H being a Cartan subgroup) and the arguments of Theorem 1 go through much the same to give complete results for the nongeneric case. If G is compact, then intersection of K_ψ with A causes difficulties in repeating the arguments of Theorem 2. This leaves the results of the compact case a little incomplete.

2. Deformations and contractions of Lie groups.

(2.1) DEFINITION. Let $\mathfrak{g}$ and $\mathfrak{g}_1$ be two Lie algebras with the same underlying vector space $\mathcal{U}$. We say that $\mathfrak{g}_1$ is a deformation of $\mathfrak{g}$ if there is a set $(\phi_\lambda)_{\lambda \in \mathbf{R}^+} \subseteq \mathcal{GL}(\mathcal{U})$ so that $\lim_{\lambda \rightarrow 0} \phi_\lambda^{-1}[\phi_\lambda x, \phi_\lambda y]_{\mathfrak{g}} = [x, y]_{\mathfrak{g}_1}$ for all $x, y \in \mathcal{U}$.

This definition goes back to Saletan. A number of results about deformations have been proved, for example [21, 22]. However, the definition seems too general to be useful in our context. Our main interest lies in a special case of this notion, a contraction. Before we pass to consideration of contractions let us record

(2.2) LEMMA. Suppose $\mathfrak{g}$ is a Lie algebra with underlying vector space $\mathcal{U}$ and $(\phi_\lambda)_{\lambda \in \mathbf{R}^+} \subseteq \mathcal{GL}(\mathcal{U})$ is such that for all $x, y \in \mathcal{U}$, $[x, y]_1 = \lim_{\lambda \rightarrow 0} \phi_\lambda^{-1}[\phi_\lambda x, \phi_\lambda y]$ exists. Then $[\]_1$ is a Lie bracket on $\mathcal{U}$ and the resulting Lie algebra $\mathfrak{g}_1$ is a deformation of $\mathfrak{g}$.

(2.3) DEFINITION. Let $\mathfrak{g}$ be a Lie algebra with underlying vector space $\mathcal{U}$ and let $\mathfrak{k}$ be a Lie subalgebra. Let V denote a subspace of $\mathcal{U}$ complementary to $\mathfrak{k}$. Thus $x \in \mathcal{U}$ can be uniquely written $x = x_{\mathfrak{k}} + x_V$. For $\lambda \in \mathbf{R}^+$, define a map $\phi_\lambda \in \mathcal{GL}(\mathcal{U})$ by $\phi_\lambda(x) = x_{\mathfrak{k}} + \lambda x_V$.

The proof of the following lemma is computational and is left to the reader.

(2.4) LEMMA. Let $x, y \in \mathcal{U}$. Then $\lim_{\lambda \rightarrow 0} \phi_\lambda^{-1}[\phi_\lambda x, \phi_\lambda y]$ exists and is equal to $[x, y]_1 = [x_{\mathfrak{k}}, y_{\mathfrak{k}}] + ([x_{\mathfrak{k}}, y]_V + ([x, y_{\mathfrak{k}}])_V)$. Thus $[\]_1$ is a Lie bracket on $\mathcal{U}$ and the resulting Lie algebra is a deformation of $\mathfrak{g}$.

A deformation arising in this way is called a *contraction with respect to $\mathfrak{k}$* . Notice that $\mathfrak{g}_1$ contains V as an abelian ideal and that $\mathfrak{k} \cong \mathfrak{g}_1/V$.

In the special case where V is $\text{ad}_{\mathfrak{k}}$ invariant (i.e. $[\mathfrak{k}, V] \subseteq V$) we have $[x, y]_1 = [x_{\mathfrak{k}}, y_{\mathfrak{k}}] + [x_{\mathfrak{k}}, y_V] + [x_V, y_{\mathfrak{k}}]$.

Contractions were introduced by İnönü and Wigner [13], who wished to study relationships between the representation theories of the two groups. Indeed, [13] contains a number of examples, for specific groups, of limiting behaviour for special functions, which are special cases of our main theorems. Other special cases have been worked out, see for example [16, 20], or Chapter 10 of [6].

We now globalize the above definitions by replacing $\mathfrak{g}$ by a Lie group G and $\mathfrak{k}$ by a Lie subgroup K .

(2.5) DEFINITION. Let G be a Lie group and K a Lie subgroup, reductive in G . Thus, we can write $\mathfrak{g} = \mathfrak{k} \oplus V$, where the decomposition is Ad_K invariant ($\text{Ad}(K)V \subseteq V$). Let $V \rtimes K$ denote the semidirect product of V by K relative to this action. Define $\pi_\lambda: V \rtimes K \rightarrow G$ by $\pi_\lambda(vk) = \exp_G(\lambda v)k$ for each $\lambda \in \mathbf{R}^+$.

(2.6) PROPOSITION. The Lie algebra of $V \rtimes K$ is precisely $\mathfrak{g}_1$ of Lemma (2.4). Further, the differential of π_λ at the identity is precisely the map ϕ_λ of (2.3).

Thus, we are justified in making the following definition.

(2.7) DEFINITION. The semidirect product $V \rtimes K$ is called a *contraction of G with respect to K* , and the family $(\pi_\lambda)_{\lambda \in \mathbf{R}^+}$ of maps $V \rtimes K \rightarrow G$ is called the *family of contraction maps*.

Our canonical example of a contraction is that outlined in the introduction, where (G, K) is a Riemannian symmetric pair and the contraction of G with respect to K is the associated Cartan motion group. It seems, however, that some of the methods

used here might extend to further examples, for instance where K is noncompact. Indeed, there are infinitesimal results in this setting (cf. [11]).

The Kirillov-Kostant method of orbits [2] gives further evidence of a definite relationship in general between the irreducible representations of a Lie group and a contraction of it; indeed the method of orbits inspired us to the theorems of this paper. One considers the orbits of a Lie group G under the coadjoint action on the dual $\mathfrak{g}^*$ of its Lie algebra. If G_ψ is the stabiliser of $\psi \in \mathfrak{g}^*$, one calls the orbit through ψ *integral* if there is a character χ of G_ψ such that $d\chi_e = 2\pi i\psi$ on $\mathfrak{g}_\psi$. One then forms the G -module induced from χ . The set of such G -modules so obtained (from various possible choices of χ) depends only on the coadjoint orbit through ψ and not ψ itself. From further geometrical data, namely a choice of polarisation on the orbit [2, 7], one can produce irreducible representations of G .

The simplest example will now serve to make our point. Let $G = SO(3)$ and K be the subgroup leaving, say, the z -axis fixed. The Lie algebra $\mathfrak{so}(3)$ is identified with $\mathbf{R}^3$ and the family

$$\begin{aligned}\phi_\lambda: \quad \mathbf{R}^3 &\rightarrow \mathbf{R}^3 \\ (x, y, z) &\mapsto (\lambda x, \lambda y, z)\end{aligned}$$

defines a contraction of $\mathfrak{so}(3)$ with respect to $\mathfrak{k}$. The contracted group is the Euclidean motion group of $\mathbf{R}^2$, i.e. $M_2 = \mathbf{R}^2 \rtimes SO(2)$. Each circular cylinder with axis the z -axis is an integral orbit of the coadjoint action of M_2 on $\mathfrak{m}_2^* \equiv \mathbf{R}^3$. Each sphere of integer radius with centre $0 \in \mathbf{R}^3$ is an integral coadjoint orbit of $SO(3)$ on $\mathfrak{so}(3)^* = \mathbf{R}^3$. In these cases, despite the possible choices of characters and polarisations, each orbit produces a unique irreducible unitary representation of the group concerned. Let ρ_R be the representation of $M(2)$ deriving from the cylinder of radius $R > 0$, and σ_n the representation of $SO(3)$ deriving from the sphere of radius $n \in \mathbf{N}$. If we let $\lambda_n = R/n$, then the image of the sphere of radius n under ϕ_{λ_n} is a cigar shape (for $\lambda_n < 1$) of height n , which is tangent to the cylinder of radius R along their intersection with the x - y plane. The sequence of images for $n = 1, 2, \dots$ is a sequence of cigar shapes which converges, in a sense, to the cylinder. Loosely speaking one may say that, upon following through the Kirillov-Kostant construction, the associated representations exhibit a similar convergence; that the sequence σ_n , distorted by the values π_{λ_n} of the global contraction, converge in a sense to ρ_R .

In general, if $\bar{G}$ is a contraction of a Lie group G , then to each integral coadjoint orbit of $\bar{G}$ we expect to find a sequence of integral coadjoint orbits of G converging to it under the adjoint of the contracting family, as in the example above. We would further expect to produce a convergence of associated irreducible representations. Theorems 1 and 2 of this paper realize these ideas in the case of the contraction of a Riemannian symmetric pair to its associated Cartan motion group. It has proved more convenient in this case to expound the representation theory in the more conventional form, rather than to persist with the coadjoint orbit description.

3. Irreducible representations of Cartan motion groups. Let (G, K) be a Riemannian symmetric pair. We summarise the description of the unitary dual of the associated Cartan motion group $V \rtimes K$ via Mackey's normal subgroup analysis [18,

19]. We consider $V \rtimes K$ as the group generated by V and K subject to the relations

$$kvk^{-1} = \text{Ad}(k)v \quad \forall v \in V, k \in K.$$

Any character χ of the vector group V can be uniquely expressed as $e^{i\psi}$ for some $\psi \in V^*$, and this identifies the character group $\hat{V}$ with the dual space V^* . For Mackey's analysis one must consider the action of K on $\hat{V}$ defined by

$$k \cdot \chi(v) \equiv \chi(k^{-1}vk) = e^{i\psi}(\text{Ad}(k^{-1})v)$$

which clearly coincides with the coadjoint action of K on V^* . For any $\psi \in V^*$ let K_ψ be the stabiliser of ψ under the coadjoint action of K , and let η be an irreducible unitary representation of K_ψ on some Hilbert space H_η . Then $e^{i\psi} \otimes \eta$ is an irreducible representation of the subgroup $V \rtimes K_\psi$ on H_η , namely

$$e^{i\psi} \otimes \eta(vm) \equiv e^{i\psi(v)}n(m) \quad \forall v \in V, m \in K_\psi.$$

The representation of $V \rtimes K$ induced from $e^{i\psi} \otimes \eta$ gives rise to an irreducible unitary representation which we denote by $\rho_{\psi, \eta}$. Every irreducible unitary representation of $V \rtimes K$ occurs in this way [19, Theorems A and B, pp. 42, 43].

By its definition $\rho_{\psi, \eta}$ acts on a space of functions $f: V \rtimes K \rightarrow H_\eta$ satisfying

$$e^{i\psi(v)}\eta(m)f(gvm) = f(g) \quad \forall v \in V, m \in K_\psi, g \in V \rtimes K.$$

Consequently,

$$f(vk) = f(k \text{Ad}(k^{-1})v) = e^{-i\psi(\text{Ad}(k^{-1})v)} \cdot f(k)$$

and any such f is uniquely determined by its restriction to K . For the representation of $\rho_{\psi, \eta}$ one considers only those functions whose restrictions to K lie in $L^2(K, H_\eta)$. These restrictions comprise the closed subspace $H_{\psi, \eta}$ of $L^2(K, H_\eta)$ whose functions f satisfy

$$\eta(m)f(km) = f(k) \quad \forall m \in K_\psi, k \in K.$$

In this way the representation space of $\rho_{\psi, \eta}$ inherits a Hilbert space structure on which $\rho_{\psi, \eta}$ acts unitarily. Note that

$$\begin{aligned} \rho_{\psi, \eta}(vk)f(k_0) &\equiv f(k^{-1}(-v)k_0) = f(k^{-1}k_0 \text{Ad}(k_0^{-1})(-v)) \\ &= e^{i\psi(\text{Ad}(k_0^{-1})v)}f(k^{-1}k_0) \end{aligned}$$

which allows one to define $\rho_{\psi, \eta}$ directly on $\mathcal{H}_{\psi, \eta}$.

In order for two irreducible representations ρ_{ψ_1, η_1} and ρ_{ψ_2, η_2} to be unitarily equivalent, it is necessary and sufficient that ψ_1 and ψ_2 lie in the same coadjoint orbit of K , and that η_1 and η_2 be unitarily equivalent (note that K_{ψ_1} and K_{ψ_2} are conjugate subgroups in this case). Because K is compact we may endow V with an $\text{Ad}(K)$ -invariant inner product $\langle \cdot, \cdot \rangle$ (for example, the Killing form restricted to V) by which we identify V with V^* , and the adjoint with the coadjoint action of K . Let $\alpha \subset V$ be a maximal abelian subalgebra of V . Every adjoint orbit of K in V intersects α [10, p. 247]. Hence every irreducible unitary representation of $V \rtimes K$ has the form $\rho_{\psi, \eta}$, where ψ has the form $v \mapsto \langle H, v \rangle$ for some $H \in \alpha$. Such a ψ can be regarded as

an element of $\mathfrak{a}^*$ which has been extended uniquely to V by making it 0 on the orthogonal complement of $\mathfrak{a}$. In this sense the unitary dual of $V \rtimes K$ is described by the representations $\rho_{\psi, \eta}$ with $\psi \in \mathfrak{a}^*$.

For $\psi \in V^*$ of the form $v \mapsto \langle H, v \rangle$, K_ψ is the same as the stabiliser of H under the adjoint action of K . Let $A \equiv \exp_G \mathfrak{a}$ and M be the centraliser of $\mathfrak{a}$ in K . Except on certain hyperplanes, the root hyperplanes, the stabiliser of $H \in \mathfrak{a}$ is M [10, p. 263]. We call such H 's, the corresponding ψ 's and associated irreducible representations $\rho_{\psi, \eta}$ generic. Because the nongeneric elements of $\mathfrak{a}$ have Lebesgue measure zero, one can show that the nongeneric irreducible representations in the unitary dual of $V \rtimes K$ have Plancherel measure zero [14]. In what follows we shall be chiefly concerned with the generic case.

4. The noncompact case. Let (G, K) be a Riemannian symmetric pair of the noncompact type. Let $\mathfrak{a} \subset V$ be a maximal abelian subalgebra of V , $A = \exp_G \mathfrak{a}$, $G = KAN$ an Iwasawa decomposition, and M the centraliser of A in K [10, p. 270]. If $\gamma \in \mathfrak{a}^* \otimes \mathbb{C}$ and η is an irreducible unitary representation of M , we define the representation $e^\gamma \otimes \eta$ of MAN by

$$e^\gamma \otimes \eta: man \mapsto e^{\gamma(\log a)} \eta(m) \quad \forall m \in M, a \in A, n \in N,$$

where $\log a \in \mathfrak{a}$ is defined by $a = \exp_G(\log a)$. A principal series representation of G is one induced from a representation $e^\gamma \otimes \eta$ of MAN . As such, it is realised on a space of functions $F: G \rightarrow H_\eta$, where H_η is the representation space of η , and satisfying

$$e^{\gamma(\log a)} \eta(m) f(gman) = f(g) \quad \forall g \in G, man \in MAN.$$

By the Iwasawa decomposition, such functions are clearly determined by their restrictions to K .

In general, principal series representations do not give rise to unitary representations. Where they do their representation spaces consist of the functions satisfying the above condition, and whose restrictions to K lie in $L^2(K, H_\eta)$. These restrictions comprise the subspace of $L^2(K, H_\eta)$ whose functions f satisfy

$$\eta(m) f(km) = f(k) \quad \forall k \in K, m \in M.$$

The representation space is thus endowed with a Hilbert space structure. A principal series representation acts unitarily on this Hilbert space if and only if $\gamma = \mu + i\psi$, where $\psi \in \mathfrak{a}^*$ and $\mu \in \mathfrak{a}^*$ is a particular linear map; its form will not concern us, but for the sake of completeness we note that [23] $\mu = \sum_{\alpha \in P^+} m_\alpha \alpha$, where P^+ is the set of positive restricted roots and m_α is the multiplicity of α [10, p. 264]. By $\sigma_{\psi, \eta}$ we denote the unitary principal series representation of G induced from $e^{(\mu + i\psi)} \otimes \eta$, and by $H_{\psi, \eta}$ its representation space.

To any generic irreducible unitary representation of $V \rtimes K$, $\rho_{\psi, \eta}$, we can associate the sequence $\{\sigma_{n\psi, \eta}\}_1^\infty$ of unitary principal series representations of G . We shall show how this sequence approximates $\rho_{\psi, \eta}$. For this we will need the global contraction

$$\begin{aligned} \pi_\lambda: \quad V \rtimes K &\rightarrow G \\ vk &\rightarrow \exp_G(\lambda v)k \end{aligned}$$

and the function

$$s_\psi: G \rightarrow \mathbb{C}$$

$$kan \rightarrow e^{-i\psi(\log a)}.$$

Obviously s_ψ is K invariant and has the value 1 on K . Furthermore, if $f \in H_{n\psi, \eta}$ then $s_\psi f \in H_{(n+1)\psi, \eta}$, and $s_\psi f$ has the same restriction to K as f . Thus multiplication by s_ψ identifies $H_{n\psi, \eta}$ with $H_{(n+1)\psi, \eta}$ in a way that identifies functions with the same restriction to K .

(4.1) LEMMA. *If $H \in \mathfrak{a}$ and $X \in \mathfrak{k} + \mathfrak{n}$, then*

$$s_\psi(\exp_G(H + tX)) = s_\psi(\exp_G H) + O(t^2).$$

PROOF. Write $X = X_1 + X_2$, where $X_1 \in \mathfrak{k}$ and $X_2 \in \mathfrak{n}$. Both $t \rightarrow \exp_G(H + tX)$ and $t \rightarrow \exp_G tX_1 \cdot \exp_G H \cdot \exp_G tX_2$ have the same tangent vector at $t = 0$, so that $t \rightarrow s_\psi(\exp_G(H + tX))$ and $t \rightarrow s_\psi(\exp_G tX_1 \cdot \exp_G H \cdot \exp_G tX_2)$ have the same derivative at $t = 0$. By the definition of s_ψ the second function is constant and so has derivative zero. The lemma follows by Taylor's theorem.

(4.2) COROLLARY. *Let $\psi \in \mathfrak{a}^*$ and extend it to V via the Killing form, i.e. represent $\psi(-) = \langle H, - \rangle$ for $H \in \mathfrak{a}$. Then $s_\psi^n(\exp_G -v/n) = e^{i\psi(v)} + O(1/n)$.*

PROOF. Let $v = v_k + v_a + v_n$, where $v_k \in \mathfrak{k}$, $v_a \in \mathfrak{a}$, $v_n \in \mathfrak{n}$. By the lemma

$$s_\psi(\exp_G -v/n) = s_\psi(\exp_G -v_a/n) + O(1/n^2).$$

By the definition of s_ψ

$$s_\psi(\exp_G -v_a/n) = e^{+i\psi(v_a/n)}.$$

Hence

$$s_\psi^n(\exp_G -v_a/n) = e^{i\psi(v_a)} + O(1/n).$$

Since $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ are orthogonal under the Killing form, $\psi(v_a) = \psi(v)$.

(4.3) THEOREM 1. *With the notation above, define*

$$\bar{\sigma}_{n\psi, \eta}(vk): f \mapsto \sigma_{n\psi, \eta}[\pi_{1/n}(vk)](s_\psi^n f)$$

$\forall vk \in V \rtimes K$ and $f \in H_{0, \eta}$. Then

$$\lim_{n \rightarrow \infty} \sigma_{n\psi, \eta}(vk)(f)|_K = \rho_{\psi, \eta}(vk)(f|_K).$$

The limit is in the sense of $L^2(K, H_\eta)$.

PROOF. For $f \in H_{0, \eta}$, $vk \in V \rtimes K$ and $k_0 \in K$ we have

$$\begin{aligned} \bar{\sigma}_{n\psi, \eta}(vk)f(k_0) &\equiv \sigma_{n\psi, \eta}[\pi_{1/n}(vk)](s_\psi^n f)(k_0) \\ &= s_\psi^n f\left(\left(\exp_G\left(\frac{v}{n}\right)k\right)^{-1}k_0\right) \\ &= s_\psi^n\left(k^{-1}k_0 \exp_G - \text{Ad}\left(k_0^{-1}\right)\frac{v}{n}\right)f\left(k^{-1}k_0 \exp_G - \text{Ad}\left(k_0^{-1}\right)\frac{v}{n}\right). \end{aligned}$$

By the K -invariance of s_ψ and the corollary above

$$s_\psi''\left(k^{-1}k_0 \exp_G - \text{Ad}(k_0^{-1})\frac{v}{n}\right) = e^{i\psi(\text{Ad}(k_0^{-1})v)} + O\left(\frac{1}{n}\right).$$

Hence

[*]

$$\begin{aligned} \bar{\sigma}_{n\psi,\eta}(vk)f(k_0) &= e^{i\psi(\text{Ad}(k_0^{-1})v)}f(k^{-1}k_0) \\ &\quad + e^{i\psi(\text{Ad}(k_0^{-1})v)}\left[f\left(k^{-1}\exp_G\left(-\frac{v}{n}\right)k_0\right) - f(k^{-1}k_0)\right] + O\left(\frac{1}{n}\right). \end{aligned}$$

Since

$$\rho_{\psi,\eta}(vk)f(k_0) = e^{i\psi(\text{Ad}(k_0^{-1})v)}f(k^{-1}k_0),$$

we obtain

$$\|\bar{\sigma}_{n\psi,\eta}(vk)(f) - \rho_{\psi,\eta}(vk)(f|_K)\| = \|\sigma_{0,\eta}\left(\exp_G\left(\frac{v}{n}\right)\right)f|_K - f\| + O\left(\frac{1}{n}\right),$$

where $\|\cdot\|$ is the norm on $L^2(K, H_\eta)$. The theorem follows by the strong continuity of $\sigma_{0,\eta}$.

(4.4) COROLLARY. *If $f \in H_{0,\eta}$ is a smooth function, then*

$$\|\bar{\sigma}_{n\psi,\eta}(vk)(f)|_K - \rho_{\psi,\eta}(vk)(f|_K)\| = O(1/n)$$

with the estimate uniform on compact subsets of $V \rtimes K$.

PROOF. For such an f

$$f\left(k^{-1}\exp_G\frac{-v}{n}k_0\right) = f(k^{-1}k_0) + O\left(\frac{\|v\|}{n}\right)$$

by the mean value theorem, and the smoothness of f guarantees uniformity of the estimate on K . Hence equation [*] in Theorem 1 can be replaced by

$$\bar{\sigma}_{n\psi,\eta}(vk)f(k_0) = e^{i\psi(\text{Ad}(k_0^{-1})v)}f(k^{-1}k_0) + O(1/n)$$

and the corollary follows.

5. The compact case. We shall describe the irreducible representations of a compact connected Lie group by a form of the Borel-Weil-Bott theorem [25, Chapter 6]. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}_\mathbb{C} \equiv \mathfrak{g} \otimes \mathbb{C}$ its complexification. We let $\mathfrak{g}$ and $\mathfrak{g}_\mathbb{C}$ act on $C^\infty(G)$ as left invariant vector fields; i.e. for $X \in \mathfrak{g}$ and $f \in C^\infty(G)$

$$(i) \quad Xf(g) = \frac{d}{ds}f(g \exp sX)|_{s=0} \quad \forall g \in G$$

and for $X \in \mathfrak{g}_\mathbb{C}$

$$(ii) \quad Xf = X_1f + iX_2f \quad \text{where } X = X_1 + iX_2, X_1, X_2 \in \mathfrak{g}.$$

Fix a maximal torus T with Lie algebra $\mathfrak{t}$. The adjoint action of T on $\mathfrak{g}_\mathbb{C}$ decomposes into a sum of characters $e^{i\alpha}$, $\alpha \in \mathfrak{t}^*$. The α 's which occur are called the roots of $\mathfrak{g}_\mathbb{C}$ with respect to T , and the one-dimensional representation space $\mathfrak{g}_\alpha \subset \mathfrak{g}_\mathbb{C}$ of $e^{i\alpha}$ is

called the root space of α . The characters occur in pairs $e^{i\alpha}$, $e^{-i\alpha}$, and by suitably ordering t [12, p. 47], one chooses one out of each pair α , $-\alpha$ of roots as being the set Φ^+ of positive roots. Let $\mathfrak{n}$ be the sum of the positive root spaces, i.e.

$$(iii) \quad \mathfrak{n} = \sum_{\alpha \in \Phi^+} \mathfrak{g}_{\alpha}.$$

For any character χ of T , the subspace of $C^\infty(G)$ satisfying

$$(iv) \quad \chi(t)f(gt) = f(g) \quad \forall t \in T, g \in G,$$

$$(v) \quad Xf = 0 \quad \forall X \in \mathfrak{n},$$

where $f \in C^\infty(G)$, is invariant under the left regular representation. The restriction of it to this subspace is called the representation of G induced holomorphically from χ , and we denote it $\mathcal{H}_\chi$. Condition (iv) is the usual induced representation condition, and (v) is called the holomorphy condition.

The Borel-Weil-Bott theorem asserts that every irreducible representation of G has the form $\mathcal{H}_\chi$ for some unique χ , and that in general $\mathcal{H}_\chi$ is either irreducible or zero. Furthermore, let us use the Killing form $(\cdot, \cdot)$ of $\mathfrak{g}$ restricted to $\mathfrak{t}$ to represent each root α in the form $(t_\alpha, -)$ for some $t_\alpha \in \mathfrak{t}$. $\mathcal{H}_\chi$ is nonzero if and only if

$$(vi) \quad d\chi_e(t_\alpha) \leq 0 \quad \forall \alpha \in \Phi^+.$$

We shall now convert the Borel-Weil-Bott description of the irreducible representations of a compact connected semisimple Lie group G into a form resembling the principal series of a noncompact such group. Let (G, K) be a Riemannian symmetric pair of compact type and, as before, let $\mathfrak{a} \subset V$ be a maximal abelian subalgebra of V , $A = \exp_G \mathfrak{a}$ and M the centralizer of A in K . Since A is a torus, its centralizer in G is connected [10, p. 287], and in fact has Lie algebra $\mathfrak{m} + \mathfrak{a}$ [10, p. 263]. The centralizer must therefore be $M_e \cdot A$ [10, p. 271], where $M_e = \exp_G \mathfrak{m}$ is the identity component of M . Hence $M \cdot A = M_e \cdot A$ and $M = M_e \cdot (M \cap A)$. Note that since $M \cap A$ is abelian, compact and has Lie algebra $\mathfrak{m} \cap \mathfrak{a} = 0$ it is a finite abelian group. Furthermore, if σ is the involution of (G, K) and $m \in M \cap A$, then $\sigma(m) = m$ because $m \in K$, and $\sigma(m) = m^{-1}$ because $m \in V$, hence $M \cap A$ consists of involutions. We summarise these well-known arguments in the following proposition.

(5.1) PROPOSITION. *The centraliser of A (in G) is MA and this equals $M_e A$, where M_e is the identity component of M . Hence $M = M_e \cdot (M \cap A)$, where $M \cap A$ is a finite abelian group of involutions.*

(5.2) PROPOSITION. *The irreducible representations of MA are $\eta \tilde{\otimes} e^{i\psi}$, where $\psi \in \mathfrak{a}^*$ satisfies*

$$e^{i\psi(H)} = 1 \quad \text{whenever } \exp_G(H) = e,$$

where η is an irreducible representation of M_e such that $\eta(a)$ is multiplication by $e^{i\psi(a)}$ for each $a \in M_e \cap A$, and where $\eta \tilde{\otimes} e^{i\psi}$ is defined by

$$\eta \tilde{\otimes} e^{i\psi}(ma) = e^{i\psi(a)}\eta(m) \quad \forall m \in M_e, a \in A.$$

PROOF. Since A centralises MA , an irreducible representation restricted to A must act as scalar multiplication by a character. Since, by Proposition 1, $MA = M_e A$, its restriction to M_e must also be irreducible. The characters of A have the form

$$(vii) \quad \exp_G(H) \mapsto e^{i\psi(H)} \quad \forall H \in \mathfrak{a},$$

where $\psi \in \mathfrak{a}^*$ satisfies

$$(viii) \quad e^{i\psi(H)} = 1 \quad \text{whenever } \exp_G(H) = e.$$

An irreducible representation of MA therefore has the form stated in the proposition.

The realisations of the irreducible representations of M_e and G given by the Borel-Weil-Bott theorem are defined by choices of maximal tori and of positive root systems with respect to these. We want to make compatible such choices in the following sense.

(5.3) DEFINITION. Let T_1 be a maximal torus of M_e . Then $T_1 \equiv T_1 A$ is a maximal torus of both MA and G .

Let Φ^+ be a set of positive roots of $\mathfrak{g}_e$ with respect to T ; whence $\alpha \in \mathfrak{L}^*$ is a root of $(\mathfrak{m} + \mathfrak{a})_e$ if and only if it is a root of $\mathfrak{g}_e$ and $\mathfrak{g}_\alpha \subset \mathfrak{m}$. Define

$$(ix) \quad P_- = \{ \alpha \in \Phi^+ \mid \mathfrak{g}_\alpha \subset \mathfrak{m}_e \},$$

$$(x) \quad P_+ = \{ \alpha \in \Phi^+ \mid \mathfrak{g}_\alpha \not\subset \mathfrak{m}_e \}.$$

Then P_- is a positive root system for M_e with respect to T_1 , in the sense that $\alpha|_{\mathfrak{t}_1}$, for each $\alpha \in P_-$ form such a positive root system.

We say that T_1 and T are compatible choices of maximal tori for M_e and M and that Φ^+ and P_- are compatible choices of positive root systems with respect to them.

In terms of compatibly chosen maximal tori and positive root systems we define the representation of G holomorphically induced from the irreducible representations of MA .

(5.4) DEFINITION. Let $\eta \tilde{\otimes} e^{i\psi}$ be an irreducible representation of MA .

Let T_1 and T be compatible maximal tori of M_e and G , and let Φ^+ be a system of positive roots of G with respect to T , and let P_- and P_+ be defined as in Definition 1.

Let H_η be the representation space of η defined by T_1 and P_1 . Hence there is a unique character χ_1 of T_1 such that H_η is the space of smooth functions for M_e satisfying

$$(xi) \quad \chi(t_1)f(mt_1) = f(m) \quad \forall t_1 \in T_1, m \in M_e,$$

$$(xii) \quad Xf = 0 \quad \forall X \in \mathfrak{n}_-,$$

where

$$(xiii) \quad \mathfrak{n}_- \equiv \sum_{\alpha \in P_-} \mathfrak{g}_\alpha.$$

On the space $C^\infty(G, H_\eta)$ of smooth functions $F: G \rightarrow H_\eta$ we let $\mathfrak{g}_e$ act as left invariant vector fields, and for each $X \in \mathfrak{g}_e$ define ∇_X by

$$(xiv) \quad \nabla_X F(g)(m) \equiv [\text{Ad}(m)X] \cdot F(g)(m) \quad \forall m \in M_e, g \in G.$$

Let

$$(xv) \quad n_+ \equiv \sum_{\alpha \in P_+} \mathfrak{g}_\alpha.$$

The representation $\sigma_{\psi, \eta}$ of G , holomorphically induced from $\eta \tilde{\otimes} e^{i\psi}$, is the restriction of the left regular representation on $C^\infty(G, H_\eta)$ to the subspace of functions F satisfying

$$(xvi) \quad \eta \tilde{\otimes} e^{i\psi}(ma)F(gma) = F(g) \quad \forall g \in G, m \in M_e, a \in A,$$

$$(xvii) \quad \nabla_X F = 0 \quad \forall X \in n_+.$$

There is a natural identification between the representations $\sigma_{\psi, \eta}$ described above and those holomorphically induced from characters of T . For suppose that $\eta \tilde{\otimes} e^{i\psi}$ is an irreducible representation of MA and η is induced holomorphically from the character χ_1 of T_1 . Then χ_1 and $e^{i\psi}$ agree on $T_1 \cap A$. Indeed, because $M_e \cap A$ is central in M_e , and T_1 is maximal abelian, $M_e \cap A = T_1 \cap A$. Moreover $\eta(a)$ is required to act as multiplication by $e^{i\psi}(a)$ for each $a \in M_e \cap A$, but (xi) shows that it must act as multiplication by $\chi_1(a)$. Since $e^{i\psi}$ and χ_1 agree on $T_1 \cap A$, they define a character χ of $T = T_1 A$ by

$$\chi(t_1 a) = \chi_1(t_1) e^{i\psi}(a) \quad \forall t_1 \in T, a \in A.$$

We then have

(5.5) PROPOSITION. *Let $\eta \tilde{\otimes} e^{i\psi}$ be an irreducible representation of MA . Let T_1 and T be compatible maximal tori of M_e and G , let Φ^+ be a system of positive roots of G with respect to T , and let P_- and P_+ be defined as in Definition 1. Let χ_1 be the character of T_1 from which η is holomorphically induced, using P_- to define the holomorphy condition. Define the character χ of T by*

$$(xviii) \quad \chi(t_1 a) = \chi_1(t_1) e^{i\psi}(a) \quad \forall t_1 \in T_1, a \in A.$$

For each $F \in C^\infty(G, H_\eta)$ define $\hat{F}$ by

$$(xix) \quad \hat{F}: g \mapsto F(g)(e) \quad \forall g \in G$$

and for each $f \in C^\infty(G)$ define $\check{f}$ by

$$(xx) \quad \check{f}(g): m \mapsto f(gm) \quad \forall m \in M_e, g \in G.$$

Then $F \mapsto \check{F}$ defines a morphism of representations $\sigma_{\psi, \eta} \mapsto \mathcal{H}_\chi$ whose inverse is $f \mapsto \check{f}$.

It follows that each $\sigma_{\psi, \eta}$ is either irreducible or zero, and that each irreducible representation of G occurs in this form for a unique choice of η and ψ .

In order to formulate an analogue of Theorem 1 for a compact Riemannian symmetric pair we require K -invariant functions to play the role of s_ψ (see §4). We require these to belong to the representation spaces of certain of the representations $\sigma_{\psi, \eta}$. An irreducible representation of G containing a K -invariant vector is called a K -class 1 representation. In respect of these we have

(5.6) THEOREM (CARTAN [3], SEE ALSO [15]). *An irreducible representation $\sigma_{\psi, \eta}$ of G is K -class 1 if and only if $\eta = 1$, the trivial representation of M . The subspace of K -invariant vectors is one-dimensional.*

As to the existence of irreducible representations of the form $\sigma_{\psi,1}$, we have the following well-known result.

(5.7) PROPOSITION. *There is a basis $\phi_1, \dots, \phi_l$ of $\mathfrak{a}^*$ for which $\sigma_{\phi_1,1}, \dots, \sigma_{\phi_l,1}$ are nonzero (hence irreducible) representations of G .*

PROOF. If α is a root of $\mathfrak{g}_{\mathbb{C}}$ with respect to T , let $\bar{\alpha}$ be its restriction to $\mathfrak{a}$. Note that $e^{i\bar{\alpha}}$ is the restriction to A of the character $e^{i\alpha}$ of T . Recall that $\bar{\alpha}$ is called a restricted root, is called positive if $\alpha \in P_+$ and simple if it is positive and not the sum of two other positive restricted roots. If $\beta_1, \dots, \beta_l$ is the set of all simple restricted roots, then they are a basis for $\mathfrak{a}$ [10, p. 292]. We define $\phi_i = 2\beta_i$ for $i = 1, \dots, l$. By the remark above, for each $i = 1, \dots, l$, $e^{i\phi_i}$ is a well-defined character of A , and since $e^{i\phi_i} = (e^{i\beta_i})^2$ and $M \cap A$ consists of involutions, it agrees with the trivial representation 1 of M on $M \cap A$. Hence $\sigma_{\phi_i,1}$ is well defined.

We claim that, for each $i = 1, \dots, l$, $\sigma_{\phi_i,1}$ is nonzero. Indeed, $\sigma_{\phi_i,1} = \mathcal{H}_{\chi_i}$, where χ_i is the character of T whose restriction to T_1 is 1 and to A is $e^{i\phi_i}$. Hence $-id\chi_{ie}$ vanishes on $\mathfrak{t}_1$ and equals ϕ_i on $\mathfrak{a}$. If $\beta_i = \bar{\alpha}_i$ and α_i is represented by t_{α_i} under the Killing form, it follows easily that $id\chi_{ie} = (a_{\alpha_i}, -)$, where $t_{\alpha_i} = t'_{\alpha_i} + a_{\alpha_i}$ with $t'_{\alpha_i} \in \mathfrak{t}_1$, $a_{\alpha_i} \in \mathfrak{a}$. For any positive root α , represented by t_{α} under the Killing form, we then have [10, p. 291]

$$-id\chi_{ie}(t_{\alpha}) = (a_{\alpha_i}, a_{\alpha}) \leq 0,$$

where $t_{\alpha} = t'_{\alpha} + a_{\alpha}$, $t_{\alpha} \in \mathfrak{t}_1$, $a_{\alpha} \in \mathfrak{a}$. Hence $\mathcal{H}_{\chi_i}$, and thus $\sigma_{\phi_i,1}$, is nonzero.

From now on, we fix the basis $\phi_1, \dots, \phi_l$ of $\mathfrak{a}^*$ as in Proposition 5.7 above.

(5.8) DEFINITION. For each $i = 1, \dots, l$ let s_i be the unique K -invariant vector of $\sigma_{\phi_i,1}$ whose value on K is 1.

(5.9) LEMMA. *For each $k = 1, \dots, l$ and $v \in V$*

$$|s_k(\exp_G - tv) - e^{it\phi_k(v)}| = O(t^2).$$

PROOF. Regarding each $v \in V$ as a tangent vector to G at e , the result follows by Taylor's theorem, once we prove

$$v \cdot s_k = i\phi_k(v) \quad \forall v \in V.$$

Let $V = \mathfrak{a} + q_0$, where q_0 is the orthogonal complement of $\mathfrak{a}$ under the Killing form, and let $q = q_0 \otimes \mathbb{C}$. By [10, p. 288], q is spanned by $X_{\alpha} - \theta X_{\alpha}$, $\alpha \in P_+$, where $X_{\alpha} \in \mathfrak{g}_{\alpha}$. Since $X_{\alpha} + \theta X_{\alpha}$ lies in $\mathfrak{k}_{\mathbb{C}}$, and s_k is K -invariant

$$(X_{\alpha} + \theta X_{\alpha}) \cdot s_k = 0 \quad \forall \alpha \in P_+.$$

By the holomorphy condition defining $\sigma_{\psi_k,1}$, and since $\nabla_X s(e) = Xs(e)$, we have $X_{\alpha} \cdot s = 0 \quad \forall \alpha \in P_+$. Hence

$$v \cdot s_k = 0 = i\phi_k(v) \quad \forall v \in q_0.$$

If $v \in \mathfrak{a}$, then by the induced representation condition

$$s_k(\exp_G - tv) = e^{it\phi_k(v)}s(e) = e^{i\phi_k(v)}$$

so that $v \cdot s_k = i\phi_k(v)$. This completes the proof.

(5.10) DEFINITION. For each $c = (c_1, \dots, c_l) \in \mathbf{N}^l$ define $s^c \equiv s_1^{c_1} \cdot s_2^{c_2} \cdots s_l^{c_l}$.

(5.11) LEMMA. Let $\psi = \gamma_1 \phi_1 + \cdots + \gamma_l \phi_l$ with $\gamma = (\gamma_1, \dots, \gamma_l) \in \mathbf{R}^l$. Let $\{c_n\}_1^\infty$ be a sequence in $\mathbf{N}^l$ such that

$$c_{kn}/n = \gamma_k + O(1/n) \quad \text{where } c_n = (c_{n1}, \dots, c_{nl}).$$

Then $|s^{c_n}(\exp_G v/n) - e^{i\psi(v)}| = O(1/n)$.

PROOF. By Lemma 5.9

$$|\ln s_k(\exp_G - v/n) - i\phi_k(v)| = O(1/n^2)$$

and so, since $c_{kn} = O(n)$

$$|c_{kn} \ln s_k(\exp_G - v/n) - ic_{kn}\phi_k(v)/n| = O(1/n)$$

and

$$|c_{kn} \ln s_k(\exp_G - v/n) - i\gamma_k \phi_k(v)| = O(1/n).$$

Adding these for $k = 1, \dots, l$ gives

$$|\ln s^{c_n}(\exp_G - v/n) - i\psi(v)| = O(1/n)$$

from which the lemma follows.

We now proceed to formulate the theorem describing how each generic irreducible representation $\rho_{\psi, \eta}$ of $V \rtimes K$ is approximated by a sequence of irreducible representations of G . First note that if η is an irreducible representation of M , then, as observed above, its restriction to $M \cap A$ acts as multiplication by a character of $M \cap A$. This character can be extended to a character of A . Indeed, since $M \cap A$ consists of involutions it is easy to describe such extensions explicitly. Hence, given η , there exists a character $e^{i\psi_0}$ of A such that $\sigma_{\psi_0, \eta}$ is defined (although it may be zero). Secondly, by the definition of $\phi_1, \dots, \phi_l$ and [10, p. 323], every coadjoint orbit of K in V^* not only passes through α^* , but contains a ψ such that $\psi = \gamma_1 \phi_1 + \cdots + \gamma_l \phi_l$ with $\gamma_k \geq 0$, $k = 1, \dots, l$. Hence, for each irreducible representation $\rho_{\psi, \eta}$ of $V \rtimes K$ we may assume ψ to have this form. Moreover, $\rho_{\psi, \eta}$ is generic if and only if $\gamma_k > 0$, $k = 1, \dots, l$.

(5.12) THEOREM 2. Let $\rho_{\psi, \eta}$ be a generic irreducible representation of $V \rtimes K$, where $\psi = \gamma_1 \phi_1 + \cdots + \gamma_l \phi_l$ with $\gamma_k > 0$, $k = 1, \dots, l$. Let $\{c_n\}_1^\infty$ be a sequence of l -tuples of nonnegative integers such that

$$c_{nk}/n = \gamma_k + O(1/n), \quad k = 1, \dots, l, \text{ where } c_n = (c_{n1}, \dots, c_{nl}).$$

Let $\psi_0 \in \alpha^*$ be such that $\sigma_{\psi_0, \eta}$ is defined and let $\psi_n = \psi_0 + c_{n1}\phi_1 + \cdots + c_{nl}\phi_l$. Let $\mathcal{H}_n$ be the representation space of $\sigma_{\psi_n, \eta}$. Then,

(i) $\mathcal{H}_n \neq 0$ for large enough n .

(ii) For each $p \in \mathbf{N}$, $n \geq p$ and $f \in \mathcal{H}_p$, $s^{c_n - c_p} \cdot f \in \mathcal{H}_n$. Define, for $f \in \mathcal{H}_p$

$$\bar{\sigma}_{\psi_n, \eta}(vk)(f) \equiv \sigma_{\psi_n, \eta}(\pi_{1/n}(vk))(s^{c_n - c_p} f).$$

Then

(iii) $|\bar{\sigma}_{\psi_n, \eta}(vk)(f)(k_0) - \rho_{\psi, \eta}(vk)(f)(k_0)| = O(1/n)$
and hence

$$\|\bar{\sigma}_{\psi_n, \eta}(vk)(f)(k_0) - \rho_{\psi, \eta}(vk)(f)(k_0)\| = O(1/n)$$

in $L^2(K, H_\eta)$.

(iv) If $\mathcal{H}_{n|_K}$ denotes the functions which are restrictions to K of those in $\mathcal{H}_n$, then $\mathcal{H}_{n|_K} \subset \mathcal{H}_{n+1|_K}$ and $\bigcup_{n=1}^\infty \mathcal{H}_{n|_K}$ is dense in $H_{\psi, \eta}$, the representation space of $\rho_{\psi, \eta}$.

PROOF. (i) Let η be holomorphically induced from χ_1 . Then $\sigma_{\psi_n, \eta} = \mathcal{H}_\chi$, where χ is defined on $T = T_1 A$ by

$$\chi(t_1 a) = \chi_1(t_1) e^{i\psi_n(a)} \quad \forall t_1 \in T, a \in A.$$

Now for any positive root α of $\mathfrak{g}_\mathbb{C}$ with respect to T , $-id\chi_e(t'_\alpha) = -id\chi_{1e}(t'_\alpha) + \psi_n(a_\alpha)$, where $t_\alpha \in \mathfrak{t}$ represents α under the Killing form, and $t_\alpha = t'_\alpha + a_\alpha$ with $t'_\alpha \in \mathfrak{t}_1$ and $a_\alpha \in \mathfrak{a}$. If $\alpha \in P_-$, then $a_\alpha = 0$ and $-id\chi_{1e}(t'_\alpha) \leq 0$ because η is nontrivial. If $\alpha \in P_+$, then, by definition of ϕ_k , $\phi_k(a_\alpha) \leq 0$, $k = 1, \dots, l$, and not all of these are zero. Hence, because $\gamma_k > 0$, $k = 1, \dots, l$, $\psi_n(a_\alpha) \rightarrow -\infty$ as $n \rightarrow \infty$, and $-id\chi_e(t_\alpha) < 0$ for n large enough. It follows that $\sigma_{\psi_n, \eta}$ is nonzero for n large enough.

(ii) Let $f \in \mathcal{H}_p$ and consider $s_k \cdot f$. Note that by the definition of $\sigma_{\phi_k, 1}$, s_k is a function $G \rightarrow \mathbb{C}$ so that the product makes sense. We show that $s_k f$ belongs to the representation space of $\sigma_{\psi_p + \phi_k, \eta}$. Repeated application of this result for $k = 1, \dots, l$ gives (ii). Now $s_k f, G \rightarrow H_\eta$, and for $ma \in MA$ and $g \in G$

$$\begin{aligned} e^{i(\psi_p + \phi_k)}(a) \eta(m) s_k f(gma) &= e^{i\phi_k}(a) s(gma) e^{i\psi_p}(a) \eta(m) f(gma) \\ &= s_k(g) f(g) = s_k f(g). \end{aligned}$$

Moreover, by the holomorphy condition in Definition 5.4

$$\nabla_X s_k f = \nabla_X s_k \cdot f + s_k \cdot \nabla_X f = 0 \quad \forall X \in \mathfrak{n}_+.$$

Hence $s_k f \in \sigma_{\psi_p + \phi_k, \eta}$.

(iii)

$$\begin{aligned} \bar{\sigma}_{\psi_n, \eta}(vk)(f)(k_0) &= (s^{c_n - c_p} \cdot f) \left(k^{-1} \exp_G \left(-\frac{v}{n} \right) k_0 \right) \\ &= s^{c_n - c_p} \left(k^{-1} k_0 \exp_G \left(-\text{Ad}(k_0^{-1}) \frac{v}{n} \right) \right) f \left(k^{-1} k_0 \exp_G \left(-\text{Ad}(k_0^{-1}) \frac{v}{n} \right) \right) \\ &= s^{c_n - c_p} \left(\exp_G \left(-\text{Ad}(k_0^{-1}) \frac{v}{n} \right) \right) f \left(k^{-1} k_0 \exp_G \left(-\text{Ad}(k_0^{-1}) \frac{v}{n} \right) \right). \end{aligned}$$

Now

$$\left| s^{c_n - c_p} \left(\exp_G \left(-\text{Ad}(k_0^{-1}) \frac{v}{n} \right) \right) - e^{i\psi(\text{Ad}(k_0^{-1})v)} \right| = O\left(\frac{1}{n}\right)$$

by Lemma 5.11, and

$$\left| f \left(k^{-1} k_0 \exp_G \left(-\text{Ad}(k_0^{-1}) \frac{v}{n} \right) \right) - f(k^{-1} k_0) \right| = O\left(\frac{v}{n}\right)$$

by the mean value theorem. Therefore we have

$$|\bar{\sigma}_{\psi_n, \eta}(vk)(f)(k_0) - \rho_{\psi, \eta}(v)(f|_K)(k_0)| = O(1/n).$$

Since $s^{c_n - c_p}$ and f are real analytic, the $O(1/n)$ estimates are uniform on K . Hence by integration over K , we obtain

$$\|\sigma_{\psi_n, \eta}(vk)(f)|_K - \rho_{\psi, \eta}(vk)(f|_K)\| = O(1/n).$$

(iv) Since $s^{c_n - c_p}$ has the value 1 on K , $s^{c_n - c_p}f$ has the same restriction to K as f for each $f \in \mathcal{H}_p$. Hence for every $n \geq p$, $\mathcal{H}_p|_K \subseteq \mathcal{H}_n|_K$. From (iii) it follows that for each $vk \in V \rtimes K$ and $f \in \mathcal{H}_p$, $\rho_{\psi, \eta}(vk)(f|_K)$ lies in the $L^2(K, H_\eta)$ closure of $\bigcup_{n=1}^\infty \mathcal{H}_n|_K$. By (i) this is nonzero, and so carries a nonzero subrepresentation of $\rho_{\psi, \eta}$. Since $\rho_{\psi, \eta}$ is irreducible, the closure in $L^2(K, H_\eta)$ of $\bigcup_{n=1}^\infty \mathcal{H}_n|_K$ must be $H_{\psi, \eta}$.

6. The nongeneric case. Recall that the irreducible unitary representations of the Cartan motion group $V \rtimes K$ have the form $\rho_{\psi, \eta}$ (see §3), where $\psi \in \mathfrak{a}^*$ and η is an irreducible unitary representation of K_ψ , the stabilizer of ψ under the coadjoint action of K on V . Generically $K_\psi = M$, the stabilizer of $\mathfrak{a}$ under the adjoint action. But certainly one can have $K_\psi \subsetneq M$. We call this the nongeneric case and we discuss the corresponding extensions of Theorems 1 and 2, with certain exceptions in respect of Theorem 2. Note that nongeneric refers to the representations of $V \rtimes K$ and not necessarily to the approximating ones from G .

Fix a maximal torus T_1 of K_ψ with Lie algebra $\mathfrak{t}_1$ and let $\mathfrak{a}_1$ be the centralizer of $\mathfrak{t}_1$ in $\mathfrak{a}$. Then $\mathfrak{h}_c = (\mathfrak{t}_1 + \mathfrak{a}) \otimes \mathfrak{c}$ is a Cartan subalgebra of $\mathfrak{g}_c$ and we let Φ be the set of roots of $\mathfrak{g}_c$ with respect to $\mathfrak{h}_c$. The centralizer of $\mathfrak{a}_{1c}$ in $\mathfrak{g}_c$ is

$$(i) \quad \mathfrak{m}_{1c} = \sum_{\alpha \in \Phi, \alpha(\mathfrak{a}_1) = 0} \mathfrak{g}_\alpha.$$

If $A_1 = \exp_G \mathfrak{a}_1$ one can show that the centraliser of A_1 in G has the form $M_1 A_1$, where M_1 is a reductive θ -stable subgroup of G with Lie algebra $\mathfrak{m}_1$, the real part of $\mathfrak{m}_{1c}$ (under the conjugation of $\mathfrak{g}_c$ with respect to $\mathfrak{g}$).

If (G, K) is a Riemannian symmetric pair of noncompact type, then [24] $M_1 \cap A_1 = e$, and $K_\psi = M_1 \cap K$ is a maximal compact subgroup of M_1 . Choosing a set Φ^+ of positive roots of $\mathfrak{g}_c$ with respect to $\mathfrak{h}_c$ we define, as in §4

$$(ii) \quad P_- = \{ \alpha \in \Phi^+ \mid \alpha(H) = 0 \ \forall H \in \mathfrak{a}_1 \},$$

$$(iii) \quad P_+ = \{ \alpha \in \Phi^+ \mid \alpha \notin P_- \},$$

$$(iv) \quad \mathfrak{n}_+ = \sum_{\alpha \in P_+} \mathfrak{g}_\alpha.$$

Let $\mathfrak{n}$ be the real part of $\mathfrak{n}_c$ and $N = \exp_G \mathfrak{n}$. Then $P = M_1 A_1 N$ is a parabolic subgroup of G , and we define an irreducible representation $\eta' \otimes e^\gamma$ of P by

$$(v) \quad \eta' \otimes e^\gamma(man) = e^\gamma(a) \eta'(m) \quad \forall man \in M_1 A_1 N,$$

where η' is a discrete series representation of M_1 , $\gamma \in \mathfrak{a}_1^*$ and e^γ is the map $a \mapsto e^{\gamma(\log a)}$. Let H be the Cartan subgroup which is the stabiliser of $\mathfrak{h}_c$ under the adjoint action of G . An H -series representation of G is, by definition, one induced from a representation $\eta \otimes e^\gamma$ of P . As such, it is realised on a space of functions $f: G \rightarrow H_\eta$ satisfying

$$(vi) \quad \eta' \otimes e^\gamma(p) f(gp) = f(g) \quad \forall g \in G, p \in P.$$

An H -series representation gives rise to a unitary representation if and only if $\gamma = \mu + i\phi$ with $\phi \in \mathfrak{a}_1^*$ and μ a particular linear map in $\mathfrak{a}^*$. In this case, one can take for the representation space the functions satisfying 6.4 and

$$(vii) \quad \int_K \|f(k)\|^2 d\mu < \infty.$$

We denote this H -series representation by $\sigma_{\phi, \eta'}$ and its representation space by $\mathcal{H}_{\phi, \eta'}$.

Given an irreducible representation η of K_ψ we choose a discrete series representation η' of M_1 whose restriction to K_ψ contains η , or more correctly, has η as a quotient. This can always be done. One can, for example, note that discrete series representations are spinor induced from the irreducible representations of a maximal compact subgroup [1] and apply a form of Frobenius reciprocity. Alternatively one can choose the discrete series representation of M_1 whose highest weight is the same as that of η , and apply the formula of Blattner-Hecht-Schmidt to prove containment [8]. These give a natural choice of η' . In general there will be many choices, as exemplified in the simplest case with $G = SL(2, \mathbf{R})$, $K = SO(2)$. Let Q denote a projection of $\eta'_{|K}$ onto η .

As in §4, define $s_\psi(k \exp_G v) = e^{-i\psi(v)}$. Then $s_\psi f \in \mathcal{H}_{(n+1)\psi, \eta'}$ for any $f \in \mathcal{H}_{n\psi, \eta'}$. Let

$$(viii) \quad (\Psi f)(k) \equiv Q(f(k)).$$

The map Ψ generalises the mere restriction of functions to K in the generic case. We have

$$(ix) \quad \Psi(f) \in H_{(n+1)\psi, \eta} \quad \text{and} \quad \Psi(s_\psi f) = \Psi(f).$$

One can now follow the proof of Theorem 1 to obtain the following generalisation to the nongeneric case.

THEOREM 1'. *With the notation above, define*

$$(x) \quad \bar{\sigma}_{n\psi, \eta'}(vk): f \mapsto \sigma_{n\psi, \eta'}[\pi_{1/n}(vk)](s_\psi^n f)$$

$\forall vk \in V \rtimes K$ and $f \in \mathcal{H}_{0, \eta'}$. Then

$$(xi) \quad \lim_{n \rightarrow \infty} \Psi[\bar{\sigma}_{n\psi}(vk)f] = \rho_{\psi, \eta}(\Psi f).$$

Moreover, $\Psi(\mathcal{H}_{n\psi, \eta'}) = H_{\psi, \eta} \forall n \in \mathbf{N}$ and the limit above is in the sense of $L^2(K, H_\eta)$.

Suppose now that (G, K) is a Riemannian symmetric pair of the compact type. In this case we note that $K_\psi A_1$ is a closed subgroup of maximal rank, and we should like to use it as a replacement for MA in the generic case. In contrast to this case, $K_\psi A_1$ is in general disconnected, and this causes difficulties in carrying through the arguments of §5. Using the root system P_+ defined by (iii) above, one can define holomorphic induction from irreducible representations of $K_\psi A_1$. Since A_1 centralises K_ψ , such an irreducible representation has the form $\eta \otimes e^{i\phi}$, where η is an irreducible representation of K_ψ and $e^{i\phi}$ is a character of A_1 . If $\sigma_{\phi, \eta}$ denotes the representation of G holomorphically induced from $\eta \otimes e^{i\phi}$, one can show by the Cartan Theorem that there is a basis $\phi_1, \dots, \phi_m$ of $\mathfrak{a}_1^*$ for which $\sigma_{\phi_i, 1}$, $i = 1, \dots, m$, are

irreducible K -class 1 representations of G . One now arrives at the problem that, for an arbitrary η , it is not clear that one can find a character $e^{i\phi_0}$ of A_1 for which $\sigma_{\phi_0, \eta}$ is nonzero. One cannot guarantee, therefore, that the sequence of representations $\sigma_{\phi_n, \eta}$, which would be defined by analogy with Theorem 2, is nontrivial.

This difficulty is circumvented in the generic case because $MA = M_e A$, and an irreducible representation of MA is uniquely determined by one of M_e and a character of A . The arguments leading to Theorem 2 can be followed almost verbatim in the nongeneric case, where, likewise, an irreducible representation of $K_\psi A_1$ is determined by one of K_ψ and a character of A_1 . In this case one can relate representations holomorphically induced from $K_\psi A_1$ to those holomorphically induced from the connected subgroup $K_\psi A_1$.

We can show that $K_\psi = K_\psi \cdot (K \cap A)$, and that, for a suitable subgroup $F \subset K \cap A$, $K_\psi = K_\psi \rtimes F$. By Mackey's semidirect product analysis an irreducible representation η of K_ψ is determined by an irreducible representation $\bar{\eta}$ of K_ψ and a character of the subgroup F whose elements a stabilise $\bar{\eta}$ in the sense that $k \mapsto \bar{\eta}(aka^{-1})$ is equivalent to $\bar{\eta}$. If this subgroup of F lies in A_1 , then we may extend the character of the subgroup to all of A_1 to obtain a character $e^{i\phi_0}$ say, and show from this that the sequence $\sigma_{\phi_n, \eta}$ is nontrivial. It is only in this case that we can prove the nongeneric analogue of Theorem 2.

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