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FILTERING COHOMOLOGY AND LIFTING VECTOR BUNDLES

BY

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ABSTRACT. For a module M over a local Cohen-Macaulay ring R we develop a (finite) sequence of presentations of M which facilitates the study of invariants arising from the cohomology modules of M. As an application we use this data, in case R is regular and M represents a vector bundle on the punctured spectrum of R with a vanishing cohomology module, to obtain bounds on how far M can be lifted as a vector bundle.

In an earlier article **[5]** we used the dual of a filtration of Auslander and Bridger **[1]** to obtain some results for nontrivial kth syzygies of rank k. In this article we establish a different method for filtering a module over a local Cohen-Macaulay ring, or more precisely a method for filtering a free presentation of a module (see discussion following Theorem 1.8). This filtration is accomplished through a finite sequence of presentations.

To be specific, let R be a local Cohen-Macaulay ring and let M be a finitely generated R-module such that $\operatorname{depth}(\operatorname{Ext}^i(M, R)) \geqslant i$ for all i less than or equal to the dimension of R. In Theorem 1.2 we establish an exact sequence, which we term a qth presentation of M,

$$0 \to \mathscr{K}_q(M) \to \mathscr{E}_q(M) \to M \to 0,$$

such that the projective dimension of $\mathscr{K}_q(M)$ is less than q and such that $\operatorname{Ext}^i(\mathscr{E}_q(M), R)$ is zero for i from one through q. In addition we establish uniqueness results (see Corollary 1.6 and Theorem 1.7) in the sense of stable isomorphism. As a consequence we recapture a variation of our main result in **[6]** without the assumptions on rank or the existence of maximal Cohen-Macaulay modules. We also show, if R is Gorenstein, that the intermediate cohomology of a module is always carried by a module of finite projective dimension (see Corollary 1.9).

In §2 we turn our attention to the lifting of vector bundles on the punctured spectra of regular local rings and achieve a substantial improvement over our results in **[5]**. In particular, if $\rho_i(M)$ denotes the rank of the ith syzygy of $\operatorname{Ext}^i(M, R)$ for $i \geqslant 0$, we show that a nontrivial vector bundle M with $\operatorname{Ext}^q(M, R)$ zero for some q in the range $1 \leqslant q \leqslant \dim R - 2$, cannot be lifted more than $\sum_{i=0}^{q-1}\rho_i(M)$ times. For a complete discussion of the projective case the reader should consult Barth and Van de Ven **[2]**, Sato **[12]** and Tyurin **[14]**. For additional information on this problem

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one should consult Hartshorne [7], our papers [4, 5] and the fundamental paper [9] by Horrocks.

1. The filtration. Serre in [13] and later Murthy in [11] use the following construction. Let M be a finitely generated R-module of projective dimension one. Then there is an exact sequence

$$0 \to R^n \to E \to M \to 0$$

in which the map $\operatorname{Hom}(R^n, R) \to \operatorname{Ext}^1(M, R)$ is surjective. One concludes that $\operatorname{Ext}^1(E, R)$ is zero and hence, since the projective dimension of E is at most one, that E is projective. If one drops the requirement that M has projective dimension one, then one can nevertheless construct such a sequence where $\operatorname{Ext}^1(E, R)$ is zero. This is the case $q = 1$ of our construction. We utilized this construction implicitly (for $q = 1$) in our article with Bruns [3, Lemma 2.1]. We present the full details here in order to give the flavor of our subsequent construction.

LEMMA 1.1. *Let R be a local ring and let M be a finitely generated R-module with* $\operatorname{Ext}^1(M, R)$ *generated by n elements. Then there is an exact sequence*

$$0 \to \mathscr{K}_1(M) \to \mathscr{E}_1(M) \to M \to 0$$

such that $\mathscr{K}_1(M)$ is isomorphic to R^n and such that $\operatorname{Ext}^1(\mathscr{E}_1(M), R)$ *is zero.*

PROOF. If $n = 0$ we take $\mathscr{E}_1(M) = M$ and $\mathscr{K}_1(M) = 0$. If $n > 0$ let $\xi_1, \ldots, \xi_n$ be a generating set for $\operatorname{Ext}^1(M, R)$ in which the class of ξ_i is represented by the extension $0 \to Re_i \to N_i \to M \to 0$. We form the direct sum of these extensions and the pullback diagram

$$\begin{array}{ccccccccc} 0 & \to & \oplus Re_i & \to & N & \to & M & \to & 0 \\ & & \| & & \downarrow & & \downarrow\Delta & & \\ 0 & \to & \oplus Re_i & \to & \oplus N_i & \to & M^n & \to & 0 \end{array}$$

where Δ is the diagonal map. We note that the pushout of the top row along the ith projection gives back ξ_i. Hence, the natural map $\operatorname{Hom}(\oplus Re_i, R) \to \operatorname{Ext}^1(M, R)$ is surjective and therefore $\operatorname{Ext}^1(N, R)$ is zero. Thus with $\mathscr{E}_1(M) = N$ and $\mathscr{K}_1(M) = \oplus Re_i$ we have the desired sequence.

THEOREM 1.2. *Let R be a Cohen-Macaulay local ring and let M be a finitely generated R-module. Assume that the grade of* $\operatorname{Ext}^i(M, R)$ *is at least i for each i up to the dimension of R. Then for each q between* 1 *and the dimension of R there is a short exact sequence*

$$0 \to \mathscr{K}_q(M) \to \mathscr{E}_q(M) \to M \to 0$$

such that the projective dimension of $\mathscr{K}_q(M)$ is less than q and $\operatorname{Ext}^i(\mathscr{E}_q(M), R)$ *is zero for* $1 \leqslant i \leqslant q$.

PROOF. We proceed by induction on q. The case $q = 1$ is equivalent to Lemma 1.1. Let

$$0 \to K \to F_{q-2} \to \cdots \to F_1 \to F_0 \to \mathscr{E}_{q-1}(M) \to 0$$

be an exact sequence with the F_i free R-modules and with $\mathscr{E}_{q-1}(M)$ already constructed with the desired properties. Then $\operatorname{Ext}^q(M, R)$ is isomorphic to $\operatorname{Ext}^q(\mathscr{E}_{q-1}(M), R)$. By a dimension shift we see that $\operatorname{Ext}^q(M, R)$ is isomorphic to $\operatorname{Ext}^1(K, R)$. We apply the case $q = 1$ to K and obtain a diagram

$$\begin{array}{ccccccccccc}
0 & \to & K & \to & F_{q-2} & \to \cdots \to & F_0 & \to & \mathscr{E}_{q-1}(M) & \to & 0 \\
 & & \uparrow & & & & & & & & \\
 & & \mathscr{E}_1(K) & & & & & & & & \\
 & & \uparrow & & & & & & & & \\
 & & \mathscr{K}_1(K) & & & & & & & & \\
 & & \uparrow & & & & & & & & \\
 & & 0 & & & & & & & &
\end{array}$$

where $\mathscr{K}_1(K)$ is R^n, the integer n being the number of generators of $\operatorname{Ext}^1(K, R)$. Dualizing with respect to R and using that $\operatorname{Ext}^i(\mathscr{E}_{q-1}(M), R) = 0$ for $1 \leqslant i \leqslant q - 1$, we obtain the diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & & & & & & & \\
 & & \downarrow & & & & & & & & \\
0 & \leftarrow & K^* & \leftarrow & F_{q-2}^* & \leftarrow \cdots \leftarrow & F_0^* & \leftarrow & \mathscr{E}_{q-1}(M)^* & \leftarrow & 0 \\
 & & \downarrow & & & & & & & & \\
 & & \mathscr{E}_1(K)^* & & & & & & & & \\
 & & \downarrow & & & & & & & & \\
 & & (R^n)^* & & & & & & & & \\
 & & \downarrow & & & & & & & & \\
 & & \operatorname{Ext}^1(K, R) & & & & & & & & \\
 & & \downarrow & & & & & & & & \\
 & & 0 & & & & & & & &
\end{array}$$

with exact row and column. Let V be the image of $\mathscr{E}_1(K)^*$ in $(R^n)^*$. Since the grade of $\operatorname{Ext}^1(K, R)$ is at least q we have that $\operatorname{Ext}^j(\operatorname{Ext}^1(K, R), R)$ is zero for $j = 1$ through $q - 1$. Thus $\operatorname{Ext}^i(V, R)$ is zero for $1 \leqslant i \leqslant q - 2$. Using the usual Cartan-Eilenberg construction for forming a projective resolution of the middle term of an exact sequence given resolutions of the ends, we obtain a diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \leftarrow & K^* & \leftarrow & F_{q-2}^* & \leftarrow \cdots \leftarrow & F_0^* & \leftarrow & \mathscr{E}_{q-1}(M)^* & \leftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \leftarrow & \mathscr{E}_1(K)^* & \leftarrow & F_{q-2}^* \oplus G_{q-2}^* & \leftarrow \cdots \leftarrow & F_0^* \oplus G_0^* & \leftarrow & W & \leftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \leftarrow & V & \leftarrow & G_{q-2}^* & \leftarrow \cdots \leftarrow & G_0^* & \leftarrow & Z & \leftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns and with the F_i^* and G_i^* free R-modules. We now dualize this diagram with respect to R and use that

$$0 \to K \to F_{q-2} \to \cdots \to F_0 \to \mathscr{E}_{q-1}(M) \to 0$$

is exact, that $V^* \cong R^n$ and that $\operatorname{Ext}^i(V, R) = 0$ for $1 \leqslant i \leqslant q - 2$. This yields the diagram

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \\
0 & \to & K & \to & F_{q-2} & \to & \cdots & \to & F_0 & \to & \mathscr{E}_{q-1}(M) & \to & 0 \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \\
0 & \to & \mathscr{E}_1(K) & \to & F_{q-2} \oplus G_{q-2} & \to & \cdots & \to & F_0 \oplus G_0 & \to & W^* & \to & C \to 0 \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \| \\
0 & \to & R^n & \to & G_{q-2} & \to & \cdots & \to & G_0 & \to & Z^* & \to & C \to 0 \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & & & 0 & & 0 & &
\end{array}$$

which has exact rows and columns. The cokernel C is isomorphic to $\operatorname{Ext}^q(\operatorname{Ext}^1(K, R), R)$ which is in turn isomorphic to $\operatorname{Ext}^q(\operatorname{Ext}^q(M, R), R)$. We let $\mathscr{E}_q(M)$ be the image of $F_0 \oplus G_0$ in W^* and let L be the image of G_0 in Z^*. Then the projective dimension of L is at most $q - 1$. We have an exact sequence

$$0 \to L \to \mathscr{E}_q(M) \to \mathscr{E}_{q-1}(M) \to 0.$$

Then $\operatorname{Ext}^q(\mathscr{E}_q(M), R) \cong \operatorname{Ext}^1(\mathscr{E}_1(K), R) = 0$ and $\operatorname{Ext}^i(\mathscr{E}_q(M), R) = 0$ for $1 \leqslant i \leqslant q$, by virtue of the dual exactness of the middle row of the above diagram. We let $\mathscr{K}_q(M)$ be the kernel of the map of $\mathscr{E}_q(M)$ onto M via the composition of $\mathscr{E}_q(M) \to \mathscr{E}_{q-1}(M) \to M$. Therefore there is an exact sequence $0 \to L \to \mathscr{K}_q(M) \to \mathscr{K}_{q-1}(M) \to 0$ and thus $\mathscr{K}_q(M)$ has projective dimension at most $q - 1$ as desired.

REMARK 1.3. (a) The above short exact sequence $0 \to \mathscr{K}_q(M) \to \mathscr{E}_q(M) \to M \to 0$ will hereafter be referred to as a qth presentation of M.

(b) If M has finite projective dimension or if M is locally free on the punctured spectrum of R or if R is Gorenstein, then the cohomology modules $\operatorname{Ext}^i(M, R)$ satisfy the grade assumptions of the theorem.

(c) If M has projective dimension $q < \infty$, then the sequence $0 \to \mathscr{K}_q(M) \to \mathscr{E}_q(M) \to M \to 0$ has $\mathscr{E}_q(M)$ free and thus is merely the beginning of a finite projective resolution of M. Thus the presentations $0 \to \mathscr{K}_q(M) \to \mathscr{E}_q(M) \to M \to 0$ begin with the Serre-Murthy exact sequence for $q = 1$ and end with a free presentation of M for q equal to the projective dimension of M.

(d) The properties of a qth presentation yield isomorphisms

$$\operatorname{Ext}^i\big(\mathscr{K}_q(M), R\big) \cong \operatorname{Ext}^{i+1}(M, R) \quad \text{for } 1 \leqslant i \leqslant q - 1,$$

and

$$\operatorname{Ext}^j(M, R) \cong \operatorname{Ext}^j\big(\mathscr{E}_q(M), R\big) \quad \text{for } j \geqslant q + 1.$$

(e) If R is Gorenstein and if M is a reflexive R-module, then both $\mathscr{E}_q(M)$ and $\mathscr{K}_q(M)$ are reflexive. This follows because the cokernel C is necessarily zero, since $\operatorname{Ext}^q(M, R)$ must have grade at least $q + 1$ in case M is reflexive.

The next results describe the uniqueness of the modules $\mathscr{E}_q(M)$ and $\mathscr{K}_q(M)$. The argument is in the same vein as many proofs of Schanuel's lemma.

THEOREM 1.4. *If* $0 \to \mathscr{K}_q(M) \to \mathscr{E}_q(M) \to M \to 0$ *and* $0 \to \mathscr{K}_q'(M) \to \mathscr{E}_q'(M) \to M \to 0$ *are both qth presentations of* M, *then* $\mathscr{E}_q(M) \oplus \mathscr{K}_q'(M)$ *is isomorphic to* $\mathscr{E}_q'(M) \oplus \mathscr{K}_q(M)$.

PROOF. We consider the following pullback diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathscr{K}_q'(M) & = & \mathscr{K}_q'(M) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{K}_q(M) & \to & P & \to & \mathscr{E}_q'(M) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{K}_q(M) & \to & \mathscr{E}_q(M) & \to & M & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Clearly it suffices to show the middle row and column of the above diagram are split exact sequences. This is an immediate consequence of the following lemma.

LEMMA 1.5. *Let R be a local ring and suppose A and B are finitely generated R-modules such that the projective dimension of A is less than k and* $\operatorname{Ext}^i(B, R) = 0$ *for* $1 \leqslant i \leqslant k$. *Then* $\operatorname{Ext}^i(B, A) = 0$ *for* $1 \leqslant i \leqslant k - \operatorname{pd} A$. ($\operatorname{pd} A$ *denotes the projective dimension of A.*)

PROOF. We proceed by induction on the projective dimension of A. The case $\operatorname{pd} A = 0$ is immediate.

Let $\operatorname{pd} A = n > 0$ and let $0 \to K \to F \to A \to 0$ be an exact sequence with F a finitely generated free R-module. Applying the functor $\operatorname{Hom}(B, -)$ and taking the long exact sequence in cohomology yields the exact sequence

$$\cdots \to \operatorname{Ext}^i(B, F) \to \operatorname{Ext}^i(B, A) \to \operatorname{Ext}^{i+1}(B, K) \to \cdots$$

for each $i \geqslant 1$. If $1 \leqslant i \leqslant k - \operatorname{pd} A$, then $2 \leqslant i + 1 \leqslant k - \operatorname{pd} K$. So by induction $\operatorname{Ext}^{i+1}(B, K) = 0$. Now $\operatorname{Ext}^i(B, F) = 0$ since F is free and $\operatorname{Ext}^i(B, R) = 0$. Thus $\operatorname{Ext}^i(B, A) = 0$.

We now prove a stronger uniqueness which you would expect for the analogue of a free resolution.

COROLLARY 1.6. *Let* $0 \to \mathscr{K}_q(M) \to \mathscr{E}_q(M) \to M \to 0$ *and* $0 \to \mathscr{K}_q'(M) \to \mathscr{E}_q'(M) \to M \to 0$ *be two qth presentations for M. Then there are free R-modules F, F', G and G' such that* $\mathscr{K}_q(M) \oplus F \cong \mathscr{K}_q'(M) \oplus F'$ *and* $\mathscr{E}_q(M) \oplus G \cong \mathscr{E}_q'(M) \oplus G'$.

PROOF. We already know by Theorem 1.4 that $\mathcal{K}_q(M) \oplus \mathcal{E}'_q(M) \cong \mathcal{K}'_q(M) \oplus \mathcal{E}_q(M)$. Further, it is enough to consider the case of R complete, since if two finitely generated R-modules are isomorphic over the completion they are already isomorphic. Let $K_1 \oplus \cdots \oplus K_n$, $E_1 \oplus \cdots \oplus E_m$, $K'_1 \oplus \cdots \oplus K'_u$ and $E'_1 \oplus \cdots \oplus E'_v$ be the decompositions of $\mathcal{K}_q(M)$, $\mathcal{E}_q(M)$, $\mathcal{K}'_q(M)$ and $\mathcal{E}'_q(M)$ respectively. By the Krull-Schmidt Theorem for complete local rings and the above isomorphism, we have that $n + v = m + u$ and that the indecomposables from each side of the isomorphism are pairwise isomorphic. Furthermore, we can assume that $n \leqslant u$ and that the pairing is arranged so as to pair as many K_i with K'_j as possible. If any K_i is not paired with a K'_j, then it is paired with an E_j. Hence it must have projective dimension less than q while $\operatorname{Ext}^s(K_i, R) = 0$ for $1 \leqslant s \leqslant q$. That is, in this case K_i must be free and therefore just R itself. Since the maximum number of K_i's were paired with the K'_j's we must have that $\mathcal{K}_q(M) \oplus F \cong \mathcal{K}'_q(M)$, where F is free. Similarly $\mathcal{E}_q(M) \oplus F \cong \mathcal{E}'_q(M)$, where F is the same free module since the rank of the largest free summand on either side of the isomorphism $\mathcal{K}_q(M) \oplus \mathcal{E}'_q(M) \cong \mathcal{K}'_q(M) \oplus \mathcal{E}_q(M)$ must be preserved.

If the qth presentation for a module M, $0 \to \mathcal{K}_q(M) \to \mathcal{E}_q(M) \to M \to 0$, has the property that there is a free R-module F with $\mathcal{K}_q(M) = K_q \oplus F$ and $\mathcal{E}_q(M) = E_q \oplus F$ in which the map $\mathcal{K}_q(M) \to \mathcal{E}_q(M)$ induces an isomorphism of F into F, then one may obtain a qth presentation of the form $0 \to K_q \to E_q \to M \to 0$, that is, one may remove the free R-module F. In such a situation we refer to $\mathcal{K}_q(M)$ and $\mathcal{E}_q(M)$ as having a common free summand. Clearly at least one qth presentation exists in which $\mathcal{K}_q(M)$ and $\mathcal{E}_q(M)$ have no common (nonzero) free summand. Such a qth presentation is called a minimal qth presentation. Our next theorem asserts that all qth presentations are obtained from minimal ones by adding a common free summand to $\mathcal{K}_q(M)$ and $\mathcal{E}_q(M)$.

THEOREM 1.7. *Let $0 \to \mathcal{K}_q(M) \to \mathcal{E}_q(M) \to M \to 0$ be a minimal qth presentation. If $0 \to K \to E \to M \to 0$ represents another qth presentation then K and E have a common free summand F such that $K \cong \mathcal{K}_q(M) \oplus F$ and $E \cong \mathcal{E}_q(M) \oplus F$.*

PROOF. It suffices to show, if K and E have no common free summand, then $K \cong \mathcal{K}_q(M)$ and $E \cong \mathcal{E}_q(M)$. By Corollary 1.6, we can write $\mathcal{K}_q(M) = K_1 \oplus G_1$ and $K = K_1 \oplus G_2$ so that K_1 has no nontrivial free summand and where G_1 and G_2 are free. Similarly, we have that $\mathcal{E}_q(M) = E_1 \oplus F_1$ and $E = E_1 \oplus F_2$, where E_1 has no nontrivial free summand and where F_1 and F_2 are free. Now consider the pullback diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K_1 \oplus G_2 & = & K_1 \oplus G_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & K_1 \oplus G_1 & \to & P & \to & E_1 \oplus F_2 & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & K_1 \oplus G_1 & \to & E_1 \oplus F_1 & \to & M & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

used in the proof of Theorem 1.4. The maps $K_1 \oplus G_1 \to F_1$ and $K_1 \oplus G_2 \to F_2$ necessarily have images in $\mathfrak{m}F_1$ and $\mathfrak{m}F_2$, respectively ($\mathfrak{m}$ denotes the maximal ideal of R), since there are no common free summands. Thus the induced map $F_1 \to F_2$, via the splitting of the middle column, must be surjective. By symmetry the induced map $F_2 \to F_1$ must also be surjective. Thus F_1 and F_2 are isomorphic. From this fact together with Theorem 1.4 we conclude that G_1 and G_2 are isomorphic. Thus $\mathscr{E}_q(M) \cong E$ and $\mathscr{K}_q(M) \cong K$.

In the following theorem we collect some properties of these presentations.

THEOREM 1.8. *Let R be a local Cohen-Macaulay ring and let M be a finitely generated R-module.*

(a) *If M has finite projective dimension less than or equal to q and if $0 \to \mathscr{K}_q(M) \to \mathscr{E}_q(M) \to M \to 0$ is a qth presentation of M, then $\mathscr{E}_q(M)$ is free and $\mathscr{K}_q(M)$ is the module of relations on M.*

(b) *IF* $\operatorname{Ext}^{q+1}(M, R) = 0$, *then a qth presentation of M is also a $(q+1)$st presentation.*

(c) *If $f\colon M \to N$ is a homomorphism of R-modules, then for each q, there is an induced commutative diagram:*

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{K}_q(M) & \to & \mathscr{E}_q(M) & \to & M & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow f & & \\ 0 & \to & \mathscr{K}_q(N) & \to & \mathscr{E}_q(N) & \to & N & \to & 0 \end{array}$$

(d) *If M represents a vector bundle on the punctured spectrum of R, then each qth presentation $0 \to \mathscr{K}_q(M) \to \mathscr{E}_q(M) \to M \to 0$ is an exact sequence of vector bundles.*

(e) *If for some $q \geqslant 2$ we have that $\mathscr{K}_q(M)$ is free, then* $\operatorname{Ext}^i(M, R) = 0$ *for $2 \leqslant i \leqslant q$.*

PROOF. Straightforward (see Remarks 1.3 and Lemma 1.5).

Let M be a module of finite projective dimension n over R. As noted above the presentation $0 \to \mathscr{K}_n(M) \to \mathscr{E}_n(M) \to M \to 0$ is a free presentation of M. Moreover, for $0 \leqslant i \leqslant j \leqslant n$ and taking $\mathscr{E}_0(M) = M$, $\mathscr{K}_0(M) = 0$, one may obtain commutative triangles (see beginning of the proof of Theorem 1.2)

$$\begin{array}{ccc} \mathscr{E}_n(M) & \to & \mathscr{E}_i(M) \\ \searrow & & \nearrow \\ & \mathscr{E}_j(M) & \end{array}$$

where the maps are surjective (however, the presentations are not in general minimal in this setup). One also obtains corresponding commutative triangles

$$\begin{array}{ccc} \mathscr{K}_n(M) & \to & \mathscr{K}_i(M) \\ \searrow & & \nearrow \\ & \mathscr{K}_j(M) & \end{array}$$

with surjective maps. In this fashion we obtain a filtration based on cohomology of a (nonminimal, in general) free presentation of M. We end this section with two corollaries resulting from this filtration.

COROLLARY 1.9. *Let R be a Gorenstein ring of dimension n and let M be a finitely generated R-module of depth d. Then* $\mathrm{Ext}^{i+1}(M, R) \cong \mathrm{Ext}^i(\mathcal{K}_{n-d}(M), R)$ *for* $1 \leqslant i \leqslant n-d-1$. *That is the cohomology modules* $\mathrm{Ext}^i(M, R)$, *for* $2 \leqslant i \leqslant n-d$, *are isomorphic to those of a module of finite projective dimension.*

PROOF. For R Gorenstein of dimension n and M of depth d, the only possible nonzero $\mathrm{Ext}^i(M, R)$ lie in the range $1 \leqslant i \leqslant n-d$. Then $\mathrm{Ext}^i(\mathcal{E}_{n-d}(M), R) = 0$ for $i > 0$ while $\mathrm{Ext}^i(\mathcal{K}_{n-d}(M), R)$ is isomorphic to $\mathrm{Ext}^{i+1}(M, R)$ for $i > 0$ (see 1.3(d)).

Our remaining corollary contains a weaker version of our result [**6**, Theorem 2.1]. This is moderately surprising since our proof there uses our syzygy theorem [**4**] which in turn uses Hochster's big Cohen-Macaulay modules [**8**], while the result below only uses Theorem 1.2. However, it should be stressed that we do not recover the full generality of [**6**, Theorem 2.1] but only the case of a vector bundle and without the minimality implicit in the earlier result.

In Corollary 1.10 as well as the next section we need to make use of Matlis duality. For an R-module L, the Matlis dual of L is $\mathrm{Hom}(L, I)$, where I is the injective envelope of the residue field of R. We use the notation L^v to indicate the dual module $\mathrm{Hom}(L, I)$. If L is an Artinian module, then L is naturally isomorphic to L^{vv}.

COROLLARY 1.10. *Let R be a regular local ring of dimension n with $n \geqslant 4$. Let M be a kth syzygy which is not a $(k+1)$th syzygy, for $2 \leqslant k \leqslant n-2$, which represents a vector bundle on the punctured spectrum of R. Then in the $(n-k-1)$th presentation $\mathcal{K}_{n-k-1}(M)$ is a $(k+2)$th syzygy and $\mathcal{E}_{n-k-1}(M)$ is a kth syzygy of* $\mathrm{Ext}^{k-1}(M^*, R)$.

REMARK. Unlike [**6**, Theorem 2.1] $\mathcal{E}_{n-k-1}(M)$ may not be the minimal kth syzygy of $\mathrm{Ext}^{k-1}(M^*, R)$.

PROOF. Since M represents a vector bundle on the punctured spectrum of R, each $\mathrm{Ext}^i(M, R)$ has finite length. The projective dimension of M is $n-k$. Thus by Theorem 1.2 and Theorem 1.8, $\mathcal{K}_{n-k-1}(M)$ and $\mathcal{E}_{n-k-1}(M)$ also represent vector bundles. Furthermore $\mathcal{K}_{n-k-1}(M)$ has projective dimension $n-k-2$ and $\mathcal{E}_{n-k-1}(M)$ has a unique nonvanishing Ext^i, namely $\mathrm{Ext}^{n-k}(\mathcal{E}_{n-k-1}(M), R)$. Hence $\mathcal{K}_{n-k-1}(M)$ is a $n-(n-k-2) = (k+2)$th syzygy and $\mathcal{E}_{n-k-1}(M)$ is a kth syzygy of $\mathrm{Ext}^{n-k}(\mathcal{E}_{n-k-1}(M), R)^v$ (see discussion of Horrocks and Matlis duality in §2) which is isomorphic to $\mathrm{Ext}^{n-k}(M, R)^v$. However, the duality of Horrocks [**9**] yields that $\mathrm{Ext}^{n-k}(M, R)^v = \mathrm{Ext}^{k-1}(M^*, R)$, which is nonzero because M is not a $(k+1)$th syzygy.

2. Lifting vector bundles. Throughout this section R will denote a regular local ring of dimension n with $n \geqslant 3$. We also assume that R contains a field in order that we may apply our results in [**4, 5**]. The R-module M will be finitely generated and reflexive and will denote a vector bundle on the punctured spectrum of R, that is, M_P is a free R_P-module for each nonmaximal prime ideal P of R. We shall use S to denote a regular local ring with parameter t such that $S/tS = R$.

If there is a finitely generated reflexive S-module M, which is a vector bundle on the punctured spectrum of S, such that $(M'/tM')^{**}$ is isomorphic to M (the double dual here is taken with respect to R), then we say that M lifts as a bundle to S. Horrocks [**10**] has asked if a bundle which lifts to arbitrarily high dimension must be trivial, that is, must the module M be free? (For an affirmative answer in the case of projective space see [**2**, **12**, **14**].) In [**5**] we have established this fact in case $\mathrm{Ext}^i(M, R)$ vanishes for $i = 1$ or $i = 2$. In this section we extend this result to include any i for $2 \leqslant i \leqslant n - 3$. Hence in view of the duality of Horrocks [**9**] and our result [**5**, Theorem 2.5] we establish an affirmative answer to Horrocks' question in case any cohomology $\mathrm{Ext}^i(M, R)$ vanishes for some i in the range $1 \leqslant i \leqslant n - 2$. Often we shall express these statements in terms of the sheaf cohomology of M on the punctured spectrum (viewing M as a vector bundle), and so we remind the reader of Horrocks' article [**9**] and specifically of the isomorphisms

$$H^i(M) \cong \mathrm{Ext}^i(M^*, R) \quad \text{and} \quad H^{n-j-1}(M^*) \cong H^j(M)^v,$$

where $(-)^v$ denotes the Matlis dual (see the end of §1 for a definition of Matlis duality).

Our technique is to show that a lifting of M together with the vanishing of $\mathrm{Ext}^i(M, R)$ for some i with $2 \leqslant i \leqslant n - 3$, induces a lifting of $\mathscr{K}_i(M)$. However, in this situation $\mathscr{K}_i(M)$ is at least a third syzygy and thus our result [**5**, Theorem 2.5] may be applied to $\mathscr{K}_i(M)$. Furthermore we provide bounds derived from intrinsic data on M as to how far one may expect to lift M as a bundle.

The notation $\overline{K}$ indicates the reduction of an S-module K modulo the parameter t, i.e., $\overline{K} = K/tK$.

LEMMA 2.1. *Suppose that the reflexive S-module L lifts M as a bundle to S. Let $0 \to \mathscr{K}_{q+1}(L) \to \mathscr{E}_{q+1}(L) \to L \to 0$ be a $(q+1)$th presentation of L over S, where $1 \leqslant q \leqslant n - 2$.*

(a) *If $\overline{\mathscr{K}_{q+1}(L)}$ and $\overline{\mathscr{E}_{q+1}(L)}^{**}$ (the double dual with respect to R) have a common free R-summand, then this free summand lifts to a common free S-summand of $\mathscr{K}_{q+1}(L)$ and $\mathscr{E}_{q+1}(L)$.*

(b) *If $\mathrm{Ext}^q_R(M, R) = H^q(M^*) = 0$ for some q with $1 \leqslant q \leqslant n - 3$, then $\mathrm{Ext}^i_S(L, S) = H^i(L^*) = 0$ for $i = q, q+1$ and the sequence*

$$0 \to \overline{\mathscr{K}_{q+1}(L)} \to \overline{\mathscr{E}_{q+1}(L)}^{**} \to M \to 0$$

is a qth presentation of M over R.

PROOF. Since $\mathscr{K}_{q+1}(L)$ necessarily has depth at least 4 (see Theorem 1.2) it follows that $\overline{\mathscr{K}_{q+1}(L)}$ is necessarily R-reflexive and that the sequence

$$0 \to \overline{\mathscr{K}_{q+1}(L)} \to \overline{\mathscr{E}_{q+1}(L)}^{**} \to M \to 0$$

is exact (note $\overline{L}^{**} \cong M$). In regard to part (b), the vanishing of the cohomology $H^i(L^*)$ for $i = q, q+1$ is just a result of the long exact sequence for cohomology (see [**5**, 1.6]). For the same reason it follows that $\operatorname{Ext}^i_R(\overline{\mathscr{E}_{q+1}(L)}^{**}, R) = 0$ for $1 \leqslant i \leqslant q$. Since $\operatorname{Ext}^i_S(L, S) = 0$ for $i = q, q+1$ it also follows that $\mathscr{K}_{q+1}(L)$ has projective dimension at most $q - 1$ over S. However, the projective dimension of $\mathscr{K}_{q+1}(L)$ over S is the same as the projective dimension of $\overline{\mathscr{K}_{q+1}(L)}$ over R. Part (b) now follows from the uniqueness result (Theorem 1.4). Returning to part (a), we observe that a common free summand F of $\overline{\mathscr{K}_{q+1}(L)}$ and $\overline{\mathscr{E}_{q+1}(L)}^{**}$ as described by the commutative triangle

$$\begin{array}{ccc} \overline{\mathscr{K}_{q+1}(L)} & \to & \overline{\mathscr{E}_{q+1}(L)}^{**} \\ \searrow & & \swarrow f \\ & F & \\ \swarrow & & \searrow \\ 0 & & 0 \end{array}$$

actually yields an identical one in which the R-double dual of $\overline{\mathscr{E}_{q+1}(L)}$ is removed. For the extension modules $\mathscr{E}_{q+1}(L)$ one always has an epimorphism $\operatorname{Hom}(\mathscr{E}_{q+1}(L), S) \to \operatorname{Hom}_R(\mathscr{E}_{q+1}(L), R)$, since one always has that the cohomology module $\operatorname{Ext}^1_S(\mathscr{E}_{q+1}(L), S) = 0$. Therefore the map $f\colon \overline{\mathscr{E}_{q+1}(L)} \to F$ lifts to an epimorphism $g\colon \mathscr{E}_{q+1}(L) \to G$, where $G/tG = F$. Further we note that g maps $\mathscr{K}_{q+1}(L)$ onto G, by Nakayama's Lemma, since f takes $\mathscr{K}_{q+1}(L)$ onto F. Thus part (a) is also established.

COROLLARY 2.2 (NOTATION AS ABOVE). *If $0 \to \mathscr{K}_q(M) \to \mathscr{E}_q(M) \to M \to 0$ and $0 \to \mathscr{K}_{q+1}(L) \to \mathscr{E}_{q+1}(L) \to L \to 0$ are minimal presentations over R and S, respectively, and if $H^q(M^*) = 0$, then $H^q(L^*) = 0$ and $\mathscr{K}_{q+1}(L)$ lifts $\mathscr{K}_q(M)$ as a bundle.*

PROOF. The proof is an immediate consequence of the uniqueness result Theorem 1.7 and Lemma 2.1. For in this situation, Lemma 2.1 guarantees (when $H^q(L^*) = 0$) that $0 \to \mathscr{K}_{q+1}(L) \to \mathscr{E}_{q+1}(L) \to L \to 0$ is a minimal $(q+1)$th presentation over S if and only if $0 \to \overline{\mathscr{K}_{q+1}(L)} \to \overline{\mathscr{E}_{q+1}(L)}^{**} \to M \to 0$ is a minimal qth presentation over R.

In order to apply our result [**5**, Theorem 2.5] we would like to obtain an estimate of the ranks of the $\mathscr{K}_q(M)$ in terms of more familiar invariants.

LEMMA 2.3. *Let $\rho_i(M)$ be the rank of the ith syzygy of $\operatorname{Ext}^i(M, R)$ for $i \geqslant 0$ and let $0 \to \mathscr{K}_q(M) \to \mathscr{E}_q(M) \to M \to 0$ be a minimal qth presentation for M. Then*

$$\operatorname{rank}\big(\mathscr{K}_q(M)\big) \leqslant \sum_{i=1}^{q} \rho_i(M) \quad \textit{for } q \geqslant 1.$$

PROOF. We induct on q. If $q = 1$, then $\mathscr{K}_1(M)$ is isomorphic to R^μ, where μ is the number of generators of $\operatorname{Ext}^1(M, R)$. However, since $\operatorname{Ext}^1(M, R)$ is torsion, then μ is also the rank of the first syzygy of $\operatorname{Ext}^1(M, R)$.

If $q > 1$, we appeal to the inductive description of the filtration (see discussion after Theorem 1.8 as well as the proof of Theorem 1.2) to obtain a commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & L & \to & \mathscr{K}_q(M) & \to & \mathscr{K}_{q-1}(M) & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & L & \to & \mathscr{E}_q(M) & \to & \mathscr{E}_{q-1}(M) & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & M & = & M & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

in which L has at most one nonzero cohomology, namely $\operatorname{Ext}^{q-1}(L, R)$. This property of L is inherent in the inductive step in the proof of Theorem 1.2 (particularly in view of the dual exactness properties of the complex G). While we may assume that the $(q-1)$th presentation in the above diagram is minimal we cannot assume the same about the qth presentation (middle column), since it may be necessary to add on a free module in order to keep the rows exact. However, the desired inequality will follow from the subsequent argument since the rank of the "minimal $\mathscr{K}_q(M)$" cannot be larger that that of $\mathscr{K}_q(M)$ in the diagram.

Returning to the module L we have that

$$\operatorname{Ext}^{q-1}(L, R) \cong \operatorname{Ext}^{q-1}(\mathscr{K}_q(M), R) \cong \operatorname{Ext}^q(M, R) \quad \text{for } q \geqslant 2.$$

Since $\operatorname{Ext}^1(\mathscr{E}_{q-1}(M), R) = 0$, the sequence $0 \to \mathscr{E}_{q-1}(M)^* \to \mathscr{E}_q(M)^* \to L^* \to 0$ is exact. As a consequence we may assume that L^* and therefore also L is constructed without nontrivial free summands. Thus, if L is nonzero, then L is a bundle with one nonzero cohomology module, $\operatorname{Ext}^{q-1}(L, R)$. Therefore L^* is a qth syzygy for $\operatorname{Ext}^{q-1}(L, R) = \operatorname{Ext}^q(M, R)$. But L and L^* have the same rank. It follows from the exact sequence $0 \to L \to \mathscr{K}_q(M) \to \mathscr{K}_{q-1}(M) \to 0$ that $\operatorname{rank} \mathscr{K}_q(M) = \operatorname{rank} L + \operatorname{rank} \mathscr{K}_{q-1}(M)$. Finally by induction we obtain the desired inequality $\operatorname{rank} \mathscr{K}_q(M) \leqslant \sum_{i=1}^{q} \rho_i(M)$, where $\mathscr{K}_q(M)$ is minimal.

We can now establish the bounds on the lifting of a vector bundle M having a vanishing cohomology module. We keep the notation of the previous discussion.

THEOREM 2.4. *Let M be a nonfree, reflexive R-module which represents a vector bundle on the punctured spectrum of R. If $H^q(M^*) \cong \operatorname{Ext}^q(M, R) = 0$ for some q with $1 \leqslant q \leqslant \dim R - 2$, then M can be lifted at most $\sum_{i=0}^{q-1} \rho_i(M)$ times.*

PROOF. If $q = 0$ or 1, then the result follows from our calculations in [**5**, Theorems 2.5 and 2.6]. So suppose that neither $\operatorname{Ext}^1(M, R)$ nor $\operatorname{Ext}^2(M, R)$ are zero, but that $\operatorname{Ext}^q(M, R)$ is zero for some $q > 2$. By duality with M^* and the case $q = 0$ we may assume that $q \leqslant \dim R - 4$. By Theorem 1.8(c), $\mathscr{K}_q(M)$ cannot be free. However, by Corollary 2.2, we have that $\mathscr{K}_q(M)$ lifts every time that M does. From Lemma 2.3, $\mathscr{K}_q(M)$ has rank less than or equal to $\sum_{i=0}^{q-1} \rho_i(M)$ (note $\rho_q(M) = 0$) and has depth at least dim $R - q + 1 \geqslant 3$. Again we appeal to [**5**, Theorem 2.5] and find that $\mathscr{K}_q(M)$ cannot be lifted more than $[\sum_{i=0}^{q-1}\rho_i(M)] - (\dim R - q + 1)$ times. Thus our argument is complete.

We remark that one could obtain finer bounds depending on which (or how many) $\operatorname{Ext}^q(M, R)$ vanish. However these can readily be obtained if needed by combining Theorems 2.5 and 2.6 of [**5**] together with Corollary 2.2 and Lemma 2.3 in individual cases. We also remark that M lifts if and only if M^* does, and moreover the cohomology modules are dual via Matlis duality. Thus one may restrict to the case of $\operatorname{Ext}^q(M, R) = 0$ for q at most $\frac{1}{2}(\dim R + 1)$. This suggests that perhaps the number $\sum_{i=0}^{h}\rho_i(M)$ is an upper bound for the number of times M can be lifted for general M, where h is the greatest integer in $\frac{1}{2}(\dim R + 1)$.

ADDED IN PROOF. We have recently learned that the existence of qth presentations (Theorem 1.2) and also Corollary 1.7 were first noted by Auslander and Bridger [**1**].

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