TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 289, Number 1, May 1985

MONGE-AMPÈRE MEASURES ASSOCIATED TO EXTREMAL PLURISUBHARMONIC FUNCTIONS IN $\mathbf{C}^n$

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ABSTRACT. We consider the extremal plurisubharmonic functions L_E^* and U_E^* associated to a nonpluripolar compact subset E of the unit ball $B \subset \mathbf{C}^n$ and show that the corresponding Monge-Ampère measures $(dd^cL_E^*)^n$ and $(dd^cU_E^*)^n$ are mutually absolutely continuous. We then discuss the polynomial growth condition (L^*), a generalization of Leja's polynomial condition in the plane, and study the relationship between the asymptotic behavior of the orthogonal polynomials associated to a measure on E and the (L^*) condition.

1. Introduction. Two natural extremal plurisubharmonic functions can be associated to a compact subset E of the unit ball B in $\mathbf{C}^n$. One is the extremal function L_E^* [**S1**], which is the Green function for the unbounded component of $\mathbf{C} - E$ with pole at infinity when $n = 1$. The other is the relative extremal function $U_{E,B}^* \equiv U_E^*$ introduced by Siciak and studied by Bedford and Taylor [**BT**], which is an analogue of the one-variable harmonic measure of E relative to the unit disc. Both of these functions satisfy the complex Monge-Ampère equation $(dd^cu)^n = 0$ outside of E and hence both functions give rise to nonnegative measures supported in E.

In the case where the compact set E is regular, i.e., L_E^* and U_E^* are continuous, Nguyen Thanh Van and Zeriahi [**NZ**] have shown that the set E, together with the measure $\mu_E \equiv (dd^cU_E^*)^n$, satisfy the polynomial condition (L^*) which is a generalized version of Leja's polynomial condition in the complex plane. This result was used by Zeriahi [**Ze**] to show that certain classes of orthogonal polynomials associated with the measure μ_E exhibit asymptotic behavior similar to that of the extremal function L_E^*.

The main result of this paper shows that if E is a nonpluripolar compact subset of B, then the Monge-Ampère measures $\mu_E \equiv (dd^cU_E^*)^n$ and $\tilde{\mu}_E \equiv (dd^cL_E^*)^n$ are mutually absolutely continuous. As a corollary to this result, it follows that the pair $(E, \tilde{\mu}_E)$ satisfies the (L^*) condition for a regular compact set E. In addition, we show that the (L^*) condition can be formulated entirely in terms of properties of the carriers of the measure. This result gives the extension to $\mathbf{C}^n$ of a theorem of Ullman [**U2**] in the one-variable case.

2. Comparison of the Monge-Ampère measures. Given a domain Ω in $\mathbf{C}^n$, we let $P(\Omega)$ denote the class of plurisubharmonic functions on Ω. Let

$$L = \left\{u \in P(\mathbf{C}^n) \colon u(z) \leqslant \log|z| + O(1) \text{ as } |z| \to \infty\right\}$$

Received by the editors June 27, 1984.
1980 *Mathematics Subject Classification*. Primary 32F05, 31C10.

and for $E \subset B$ define

$$L_E(z) \equiv \sup\{u(z): u \in L, u \leqslant 0 \text{ on } E\}.$$

The uppersemicontinuous regularization $L_E^*(z) \equiv \overline{\lim}_{\xi \to z} L_E(\xi)$ is called the extremal function of E. By a result of Siciak [**S1**], either $L_E^* \equiv +\infty$, in which case E is pluripolar, or $L_E^* \in L$. In the latter case (see [**BT**]), then $L_E^*(z) = 0$ for $z \in E - Z$, where Z is a pluripolar set. If E is compact, then E is regular when $Z = \varnothing$. Furthermore, $(dd^c L_E^*)^n = 0$ in $\mathbf{C}^n - E$. Here, $d = \partial + \bar{\partial}$, $d^c = i(\bar{\partial} - \partial)$, so that $dd^c L_E^* = 2i\partial\bar{\partial} L_E^*$ and $(dd^c L_E^*)^n = dd^c L_E^* \wedge \cdots \wedge dd^c L_E^*$ (n times) where the last two formulas are to be interpreted as currents on $\mathbf{C}^n$. We refer the reader to [**Le**] for a definition of currents and to [**BT**] for a more complete discussion of the complex Monge-Ampère operator $(dd^c)^n$.

If Ω is a strictly pseudoconvex domain containing E, we define

$$U_E(\Omega, z) \equiv \sup\{u(z): u \in P(\Omega), u \leqslant 0 \text{ on } \Omega, u \leqslant -1 \text{ on } E\}$$

and call $U_E^*(\Omega, z) \equiv \overline{\lim}_{\xi \to z} U_E(\Omega, \xi)$ the relative extremal function of E (relative to Ω). It is easily seen that either $U_E^*(\Omega, \cdot) \equiv 0$, in which case E is pluripolar, or $U_E^*(\Omega, \cdot)$ is a nontrivial plurisubharmonic function in Ω taking values between -1 and 0. In the latter case, $U_E^*(\Omega, z) \to 0$ on $\partial\Omega$ and $U_E^*(\Omega, z) = -1$ on $E - Z$, where Z is pluripolar; $Z = \varnothing$ when E is regular. In addition, $(dd^c U_E^*(\Omega, \cdot))^n = 0$ in $\Omega - E$ if E is compact. Our first result shows that if E is not pluripolar and Ω_1, Ω_2 are two strictly pseudoconvex domains with $E \subset\subset \Omega_1 \subset \Omega_2$, and if Ω_1 is a Runge domain relative to Ω_2 (see [**Ho**, Theorem 4.3.3]), then $(dd^c U_E^*(\Omega_i, \cdot))^n$, $i = 1, 2$, are mutually absolutely continuous. This result is well known from classical potential theory in the case where $n = 1$ (see [**Ne**, Chapter 5]). For $n \geqslant 2$, the main ingredient in the proof is a comparison theorem for Monge-Ampère measures supported on E.

LEMMA 2.1. *Let E be a compact set in $\mathbf{C}^n$ with smooth boundary ∂E and nonempty interior E^0 such that E is the closure of E^0. Let $\omega \subset \mathbf{C}^n$ be a domain with $E \subset\subset \omega$. Suppose $u_1, u_2 \in P(\omega) \cap C(\bar{\omega})$ satisfy:*

(i) $u_1 = u_2 = 0$ *on* E;

(ii) $u_1 \geqslant u_2$ *on* ω;

(iii) $(dd^c u_1)^n = (dd^c u_2)^n = 0$ (*as measures*) *on* $\omega - E$; *and*

(iv) *there exists* $\eta > 0$ *with* $u_1 \geqslant u_2 + \eta$ *on* $\partial\omega$.

Then $\int_\omega \phi(dd^c u_1)^n \geqslant \int_\omega \phi(dd^c u_2)^n$ *for all* $\phi \in C_0^\infty(\omega)$ *with* $\phi \geqslant 0$.

PROOF. Without loss of generality, we assume E is connected. Given $\varepsilon > 0$, let $E_\varepsilon = \{z \in \omega: u_1(z) \leqslant u_2(z) + \varepsilon\}$. From (i) and the continuity of u_i on $\bar{\omega}$, $E \subset\subset E_\varepsilon$. By (iv), we also have that $E_\varepsilon \subset\subset \omega$ for ε sufficiently small. For any family of smoothings $\{u_i^\delta\}$ with $u_i^\delta \searrow u_i$ in a neighborhood of $\bar{\omega}$, we have $E \subset\subset E_{\varepsilon,\delta} \subset\subset \omega$ for sufficiently small δ, where

$$E_{\varepsilon,\delta} = \{z \in \omega: u_1^\delta(z) \leqslant u_2^\delta(z) + \varepsilon\}.$$

Also, since $u_i \in C(\bar{\omega})$, there exists $M > 0$ such that

(2.1) $$u_i^\delta \leqslant M \qquad (i = 1, 2) \quad \text{for } \delta \text{ sufficiently small.}$$

Given $\phi \in C_0^\infty(\omega)$ with $\phi \geqslant 0$,

$$\int_\omega \phi\left[\left(dd^c u_1^\delta\right)^n - \left(dd^c u_2^\delta\right)^n\right] = \int_\omega \phi\, dd^c\left(u_1^\delta - u_2^\delta\right) \wedge T_\delta,$$

where $T_\delta \equiv \Sigma_{k=0}^{n-1}(dd^c u_1^\delta)^{n-k-1} \wedge (dd^c u_2^\delta)^k$ is a positive, closed $(n-1, n-1)$-form with smooth coefficients. From two applications of Stokes' theorem, we have

(2.2) $$\begin{aligned}\int_\omega \phi\, dd^c\left(u_1^\delta - u_2^\delta\right) \wedge T_\delta &= \int_\omega \left(u_1^\delta - u_2^\delta\right) dd^c\phi \wedge T_\delta \\ &= \int_{E_{\varepsilon,\delta}} \left(u_1^\delta - u_2^\delta\right) dd^c\phi \wedge T_\delta \\ &\quad + \int_{\omega - E_{\varepsilon,\delta}} \left(u_1^\delta - u_2^\delta\right) dd^c\phi \wedge T_\delta.\end{aligned}$$

We first consider the integral over $E_{\varepsilon,\delta}$. Since $\phi \in C_0^\infty(\omega)$,

$$\left|\int_{E_{\varepsilon,\delta}} \left(u_1^\delta - u_2^\delta\right) dd^c\phi \wedge T_\delta\right| \leqslant c(\phi)\left\|u_1^\delta - u_2^\delta\right\|_{E_{\varepsilon,\delta}} \int_{E_{\varepsilon,\delta}} \beta \wedge T_\delta,$$

where $\|u_1^\delta - u_2^\delta\|_{E_{\varepsilon,\delta}} = \sup_{z \in E_{\varepsilon,\delta}} |u_1^\delta(z) - u_2^\delta(z)|$, $\beta = dd^c|z|^2$, and $c(\phi)$ is a constant depending only on ϕ and n. Choose a compact set K such that $E_{\varepsilon,\delta} \subset K \subset\subset \omega$ for ε, δ sufficiently small. Then, since T_δ is positive, $\beta \wedge T_\delta$ is a positive multiple of the volume form $\beta^n = \beta \wedge \cdots \wedge \beta$, and we have

$$\begin{aligned}\left|\int_{E_{\varepsilon,\delta}} \left(u_1^\delta - u_2^\delta\right) dd^c\phi \wedge T_\delta\right| &\leqslant c(\phi)\left\|u_1^\delta - u_2^\delta\right\|_{E_{\varepsilon,\delta}} \int_K \beta \wedge T_\delta \\ &\leqslant c(\phi)\left\|u_1^\delta - u_2^\delta\right\|_{E_{\varepsilon,\delta}} c' M^{n-1},\end{aligned}$$

where c' is a constant depending only on K and ω. This last inequality follows from (2.1) and the Chern-Levine-Nirenberg estimates on $(dd^c)^n$ for bounded plurisubharmonic functions (see [**CLN**, or **BT**, §2]). From the construction of the set $E_{\varepsilon,\delta}$, it follows that

$$\lim_{\varepsilon \to 0}\left\{\lim_{\delta \to 0}\left\|u_1^\delta - u_2^\delta\right\|_{E_{\varepsilon,\delta}}\right\} = 0$$

so that

$$\lim_{\varepsilon \to 0}\left\{\lim_{\delta \to 0}\left|\int_{E_{\varepsilon,\delta}} \left(u_1^\delta - u_2^\delta\right) dd^c\phi \wedge T_\delta\right|\right\} = 0.$$

For the integral over $\omega - E_{\varepsilon,\delta}$ in (2.2), note that for each $\delta > 0$, the set $E_{\varepsilon,\delta}$ has smooth boundary for almost all ε by Sard's theorem. Thus we can find a sequence of values of δ tending to 0 and a sequence of values of ε tending to 0 such that for each pair (ε, δ), the set $E_{\varepsilon,\delta}$ has smooth boundary. We can then apply Stokes' theorem to obtain

$$\begin{aligned}
\int_{\omega - E_{\varepsilon,\delta}} \left(u_1^\delta - u_2^\delta\right) dd^c\phi \wedge T_\delta &= -\int_{\omega - E_{\varepsilon,\delta}} d\left(u_1^\delta - u_2^\delta\right) \wedge d^c\phi \wedge T_\delta \\
&\quad - \int_{\partial E_{\varepsilon,\delta}} \left(u_1^\delta - u_2^\delta\right) d^c\phi \wedge T_\delta \\
&= -\int_{\omega - E_{\varepsilon,\delta}} d\phi \wedge d^c\left(u_1^\delta - u_2^\delta\right) \wedge T_\delta - \varepsilon\int_{\partial E_{\varepsilon,\delta}} d^c\phi \wedge T_\delta \qquad (2.3)\\
&= \int_{\omega - E_{\varepsilon,\delta}} \phi\left[\left(dd^c u_1^\delta\right)^n - \left(dd^c u_2^\delta\right)^n\right] \\
&\quad + \int_{\partial E_{\varepsilon,\delta}} \phi d^c\left(u_1^\delta - u_2^\delta\right) \wedge T_\delta - \varepsilon\int_{\partial E_{\varepsilon,\delta}} d^c\phi \wedge T_\delta .
\end{aligned}$$

Similar to before, we have

$$\left|\varepsilon\int_{\partial E_{\varepsilon,\delta}} d^c\phi \wedge T_\delta\right| \leqslant \varepsilon c(\phi)\int_K \beta \wedge T_\delta \leqslant \varepsilon c(\phi) c' M^{n-1}$$

which goes to zero as $\varepsilon \searrow 0$. We claim that the second term in (2.3) is nonnegative. This follows because $\phi \geqslant 0$, T_δ is a positive, closed $(n-1, n-1)$-form, and $u_1^\delta - u_2^\delta - \varepsilon$ is a defining function for the interior of $E_{\varepsilon,\delta}$ so that $d^c(u_1^\delta - u_2^\delta - \varepsilon) \wedge T_\delta = d^c(u_1^\delta - u_2^\delta) \wedge T_\delta \geqslant 0$ on $\partial E_{\varepsilon,\delta}$ [**Le**, p. 68]. For the first term in (2.3), since $E_\varepsilon \searrow E$ as $\varepsilon \searrow 0$, if we fix $\varepsilon > 0$, we can find $\delta_0 > 0$ such that $E_{\varepsilon/2} \subset E_{\varepsilon,\delta}$ for $\delta < \delta_0$. As $\delta \searrow 0$, the functions u_i^δ decrease to u_i on $\bar{\omega}$; hence the measures $(dd^c u_i^\delta)^n$ converge weakly to $(dd^c u_i)^n$, $i = 1, 2$ [**BT**, Theorem 2.1]. Thus

$$\begin{aligned}
0 \leqslant \lim_{\delta \to 0} \int_{\omega - E_{\varepsilon,\delta}} \phi\left(dd^c u_i^\delta\right)^n &\leqslant \lim_{\delta \to 0} \int_{\omega - E_{\varepsilon/2}} \phi\left(dd^c u_i^\delta\right)^n \\
= \int_{\omega - E_{\varepsilon/2}} \phi\left(dd^c u_i\right)^n = 0
\end{aligned}$$

by (iii) and the fact that $E \subset\subset E_{\varepsilon/2}$. Hence the sum of the integrals in (2.3) tends to a nonnegative number as δ and then ε decrease to zero so that from (2.2) we have

$$\lim_{\delta \to 0} \int_\omega \phi\left[\left(dd^c u_1^\delta\right)^n - \left(dd^c u_2^\delta\right)^n\right] = \int_\omega \phi\left[\left(dd^c u_1\right)^n - \left(dd^c u_2\right)^n\right] \geqslant 0$$

and the lemma is proved.

We will also make use of a domination principle of Bedford and Taylor for bounded plurisubharmonic functions [**BT**, Corollary 4.5].

LEMMA 2.2. *Let $\Omega \subset \mathbf{C}^n$ be open and bounded and suppose that $u, v \in P(\Omega) \cap L^\infty(\Omega)$ satisfy*:

(i) $\underline{\lim}_{\xi \to \partial\Omega}(u(\xi) - v(\xi)) \geqslant 0$, *and*

(ii) $\int_{\{u < v\}} (dd^c u)^n = 0$.

Then $u \geqslant v$ in Ω.

THEOREM 2.1. *Let $E \subset \mathbf{C}^n$ be a nonpluripolar compact set and let Ω_1, Ω_2 be two strictly pseudoconvex domains with $E \subset\subset \Omega_1 \subset \Omega_2$. Then*

$$\mu_1 \equiv \left(dd^c U_E^*(\Omega_1, \cdot)\right)^n \geqslant \mu_2 \equiv \left(dd^c U_E^*(\Omega_2, \cdot)\right)^n.$$

Furthermore, if (Ω_1, Ω_2) is a Runge pair, then there exists a constant $c > 0$ such that $c\mu_1 \leqslant \mu_2$. In particular, in this latter case, μ_1 and μ_2 are mutually absolutely continuous.

PROOF. For convenience, we work with the functions $u_i(z) \equiv U_E^*(\Omega_i, z) + 1$, $i = 1, 2$. Given ω open with $E \subset\subset \omega \subset \Omega_1$, we will show that there exists a constant $c > 0$ such that

$$c\int_\omega \phi(dd^c u_1)^n \leqslant \int_\omega \phi(dd^c u_2)^n \leqslant \int_\omega \phi(dd^c u_1)^n \tag{2.4}$$

for all $\phi \in C_0^\infty(\omega)$ with $\phi \geqslant 0$. It will follow from the proof that only the first inequality will require that (Ω_1, Ω_2) be a Runge pair. We first assume that E satisfies the conditions in Lemma 2.1 so that E is regular and u_1, u_2 are continuous. We then have that $u_1, u_2 \in C(\overline{\Omega}_1) \cap P(\Omega_1)$; also

(i) $u_1 = u_2 = 0$ on E

since E is regular. Furthermore,

(ii) $u_1 \geqslant u_2$ on Ω_1

by definition of the relative extremal functions; and

(iii) $(dd^c u_1)^n = (dd^c u_2)^n = 0$ on $\Omega_1 - E$.

Since $u_1 = 1$ on $\partial\Omega_1$ and $\max_{z \in \partial\Omega_1} u_2(z) = \alpha < 1$ by the maximum principle for plurisubharmonic functions, we have

(iv) $u_1 \geqslant u_2 + (1 - \alpha)$ on $\partial\Omega_1$.

Thus we can apply Lemma 2.1 to conclude that for any open set $\omega \subset \Omega_1$ with $E \subset\subset \omega$,

$$\int_\omega \phi(dd^c u_1)^n \geqslant \int_\omega \phi(dd^c u_2)^n$$

for all $\phi \in C_0^\infty(\omega)$ satisfying $\phi \geqslant 0$. In the other direction, we first note that since $E \subset\subset \Omega_i$, $i = 1, 2$, it follows from the pseudoconvexity of Ω_i that $\hat{E}^P_{\Omega_i} \subset\subset \Omega_i$, where

$$\hat{E}^P_{\Omega_i} \equiv \left\{z \in \Omega_i \colon u(z) \leqslant \sup_{z' \in E} u(z') \text{ for all } u \in P(\Omega_i)\right\}.$$

Since Ω_1 is a Runge domain relative to Ω_2, $\hat{E}^P_{\Omega_1} = \hat{E}^P_{\Omega_2}$ [**Ho**, Theorems 4.3.3 and 4.3.4]. From the definitions of $\hat{E}^P_{\Omega_i}$ and $U_E(\Omega_i, z)$, we have that $U_E(\Omega_i, z) > -1$ for $z \in \Omega_i - \hat{E}^P_{\Omega_i}$. Thus if we let $\eta = \min_{z \in \partial\Omega_1} u_2(z)$, we see that $\eta > 0$, so that $\eta u_1 \leqslant u_2$ on $\partial\Omega_1$ and $\eta u_1 = u_2 = 0$ on E. By (iii), it follows that

$$\int_{\{u_2 < \eta u_1\}} (dd^c u_2)^n = 0,$$

so that by Lemma 2.2 we conclude that $\eta u_1 \leqslant u_2$ in Ω_1. Given $0 < \varepsilon < \eta$, we then have that $(\eta - \varepsilon)u_1$, u_2 satisfy the hypotheses of Lemma 2.1 so that

$$\int_\omega \phi\left[dd^c(\eta - \varepsilon)u_1\right]^n \leqslant \int_\omega \phi(dd^c u_2)^n$$

for all $\phi \in C_0^\infty(\omega)$ with $\phi \geqslant 0$. Letting $\varepsilon \searrow 0$, we obtain

$$\eta^n \int_\omega \phi(dd^c u_1)^n \leqslant \int_\omega \phi(dd^c u_2)^n,$$

and (2.4) is proved with $c = \eta^n$ for compact sets E satisfying the conditions in Lemma 2.1.

For the general case, we take a sequence $\{E_k\}$ of compact sets as above which decrease to E. If we fix ω open with $E \subset\subset \omega \subset\subset \Omega_1$, we have $E_k \subset\subset \omega$ for k sufficiently large and we obtain

$$(\eta_k)^n \int_\omega \phi\left(dd^c u_1^k\right)^n \leqslant \int_\omega \phi\left(dd^c u_2^k\right)^n \leqslant \int_\omega \phi\left(dd^c u_1^k\right)^n$$

from (2.4), where $u_i^k(z) \equiv U_{E_k}^*(\Omega_i, z) + 1$, $\eta_k = \min_{z \in \partial\Omega_1} u_2^k(z)$, and $\phi \in C_0^\infty(\omega)$ satisfies $\phi \geqslant 0$. Since $u_i^k \nearrow u_i$ a.e. [**BT**, Proposition 6.4], the measures $(dd^c u_i^k)^n$ converge weakly to $(dd^c u_i)^n$ [**BT**, Theorem 2.1], $i = 1, 2$. Thus for ϕ as above,

$$\int_\omega \phi(dd^c u_2)^n = \lim_{k \to \infty} \int_\omega \phi\left(dd^c u_2^k\right)^n \leqslant \lim_{k \to \infty} \int_\omega \phi\left(dd^c u_1^k\right)^n = \int_\omega \phi(dd^c u_1)^n.$$

In addition, if we fix k_0, $\eta_k \geqslant \eta_{k_0}$ for $k \geqslant k_0$ so that

$$\int_\omega \phi(dd^c u_2)^n = \lim_{k \to \infty} \int_\omega \phi\left(dd^c u_2^k\right)^n \geqslant \lim_{k \to \infty} (\eta_k)^n \int_\omega \phi\left(dd^c u_1^k\right)^n$$
$$\geqslant \lim_{k \to \infty} (\eta_{k_0})^n \int_\omega \phi\left(dd^c u_1^k\right)^n = (\eta_{k_0})^n \int_\omega \phi(dd^c u_1)^n,$$

then (2.4) holds in the general case and the theorem is proved. We note that the constant in (2.4) can be taken as $c = (\min_{z \in \partial\Omega_1} u_2(z))^n$ when E is regular.

We now come to the main theorem of the paper.

THEOREM 2.2. *Let $E \subset B$ be a nonpluripolar compact set. Then the measures μ_E and $\tilde{\mu}_E$ are mutually absolutely continuous.*

PROOF. We first assume that E satisfies the conditions of Lemma 2.1. Let $a = \|L_E\|_B$. Then

$$u_1(z) \equiv U_E(z) + 1 \geqslant (1/a) L_E(z) \equiv u_2(z)$$

for $z \in B$ by definition of U_E. Furthermore, since ∂E is smooth, E is regular, so that $u_1 = u_2 = 0$ on E. Also, $(dd^c u_1)^n = (dd^c u_2)^n = 0$ outside of E. Thus, given $0 < \varepsilon < 1$, the functions u_1 and $(1 - \varepsilon)u_2$ satisfy the conditions (i)–(iv) of Lemma 2.1 and we have

$$\int_\omega \phi\, d\mu_E \geqslant \left(\frac{1-\varepsilon}{a}\right)^n \int_\omega \phi\, d\tilde{\mu}_E$$

for all $\phi \in C_0^\infty(\omega)$ with $\phi \geqslant 0$ where $E \subset\subset \omega \subset B$ and ω is open. Letting $\varepsilon \searrow 0$, we obtain

$$\int_\omega \phi\, d\mu_E \geqslant \left(\frac{1}{a}\right)^n \int_\omega \phi\, d\tilde{\mu}_E. \tag{2.5}$$

In the other direction, we first remark that by definition of the extremal function, if $E_1 \subset E_2$, then $L^*_{E_1}(z) \geqslant L^*_{E_2}(z)$ for all $z \in \mathbf{C}^n$. It is easy to show that the extremal function for a closed ball $\overline{B(a, r)} = \{z \in \mathbf{C}^n : |z - a| \leqslant r\}$ is
for a closed ball $\overline{B(a, r)} = \{z \in \mathbf{C}^n : |z - a| \leqslant r\}$ is

$$u(z) = \log^+ \frac{|z-a|}{r} = \max\left(\log \frac{|z-a|}{r}, 0\right);$$

thus, since $E \subset B$, $L_E(z) \geqslant \log^+ |z|$ for all z. In particular,

$$v_1(z) \equiv L_E(z)/\log 2 \geqslant 1 \quad \text{for } |z| = 2.$$

If we let $v_2(z) \equiv U_E(B(0,2), z) + 1$, then $v_1 = v_2 = 0$ on E; also, since $(dd^c v_1)^n$ is supported in E,

$$\int_{\{v_1 < v_2\}} (dd^c v_1)^n = 0.$$

Thus by Lemma 2.2, $v_1 \geqslant v_2$ on $B(0,2)$. Applying Lemma 2.1 to v_1 and $(1-\varepsilon)v_2$, we obtain

$$\frac{1}{(\log 2)^n} \int_\omega \phi \, d\tilde{\mu}_E \geqslant (1-\varepsilon)^n \int_\omega \phi (dd^c v_2)^n$$

for $\phi \in C_0^\infty(\omega)$ with $\phi \geqslant 0$. Letting $\varepsilon \searrow 0$, we obtain

$$\int_\omega \phi \, d\tilde{\mu}_E \geqslant (\log 2)^n \int_\omega \phi (dd^c v_2)^n.$$

We can apply Theorem 2.1, since B is a Runge domain relative to $B(0,2)$, to get

$$\int_\omega \phi (dd^c v_2)^n \geqslant \left(\min_{z \in \partial B} v_2(z)\right)^n \int_\omega \phi \, d\mu_E.$$

Thus from (2.5) and the above inequalities, we have

$$\left(\frac{1}{a}\right)^n \int_\omega \phi \, d\tilde{\mu}_E \leqslant \int_\omega \phi \, d\mu_E \leqslant \left[(\log 2)\left(\min_{z \in \partial B} v_2(z)\right)\right]^{-n} \int_\omega \phi \, d\tilde{\mu}_E \tag{2.6}$$

for all $\phi \in C_0^\infty(\omega)$ with $\phi \geqslant 0$, and the theorem is proved for E satisfying the conditions of Lemma 2.1. The general case follows as in the proof of Theorem 2.1 by choosing a sequence $\{E_k\}$ of compact sets as above which decrease to E. Applying (2.6) to E_k, $\tilde{\mu}_{E_K}$, μ_{E_K}, $a_k \equiv \|L_{E_k}\|_B$, and $v_2^k(z) \equiv U_{E_k}(B(0,2), z) + 1$, and then using the convergence theorems of Bedford and Taylor, the proof is complete.

3. Condition (L^*) and orthogonal polynomials. Given a compact set $E \subset \mathbf{C}^n$ and a nonnegative measure μ with support in E, we say that the pair (E, μ) satisfies condition (L^*) if for each family Φ of polynomials with $\sup_{p \in \Phi} |p(z)| < \infty$ μ-a.e. on E, and for each $\lambda > 1$, there exists a constant M and a neighborhood U of E such that

$$\|p\|_U \leqslant M \lambda^{\deg(p)} \quad \text{for all } p \in \Phi.$$

The condition (L^*) is an important concept in discussing the regularity of a compact set; for example, if there exists a nonnegative measure μ on E such that (E, μ)

satisfies (L^*), it follows that E is regular. On the other hand, Nguyen Thanh Van and Zeriahi have shown [**NZ**] that if E is regular, then the pair (E, μ_E) satisfies (L^*). From Theorem 2.2 and the definition of (L^*), it follows that, in addition, $(E, \tilde{\mu}_E)$ satisfies (L^*). We now give an alternate characterization of the condition (L^*).

THEOREM 3.1. *Let $E \subset B$ be a regular compact set and let μ be a measure on E. Then (E, μ) satisfies (L^*) if and only if for any Borel set $F \subset E$ with $\mu(F) = \mu(E)$, we have $L_F^* = L_E$.*

PROOF. We first assume that (E, μ) satisfies (L^*). Let $F \subset E$ be a Borel set with $\mu(F) = \mu(E)$. Since $F \subset E$, $L_F^* \geqslant L_E$; to prove the reverse inequality, it suffices to show that $L_F^* \leqslant 0$ on E. Suppose there exists a point $z_0 \in E$ with $L_F^*(z_0) > 0$. Let $F_0 = \{z \in F: L_F^*(z) = 0\}$. Then $F - F_0$ is a pluripolar set; thus, by a result of Siciak [**S2**, Theorem 1.6], there exists $v \in L$ with $v = -\infty$ on $F - F_0$. Without loss of generality, we may assume that $v \leqslant 0$ on E.

We claim that the set $N = \{z \in E: L_F^*(z) > L_E(z)\}$ is not pluripolar. For if it were, then by Proposition 3.11 of [**S1**], $L_{E-N}^* = L_E$. However, since $L_F^* \leqslant L_E = 0$ on $E - N$, it follows that $L_F^* \leqslant L_{E-N}^* = L_E$ in $\mathbf{C}^n$, contradicting our assumption that $L_F^*(z_0) > 0$. Thus N is not pluripolar and we may assume that $v(z_0) > -\infty$ and $L_F^*(z_0) = 5\delta > 0$. We now choose $\eta > 0$ sufficiently small so that:

$$(3.1) \qquad \begin{aligned} &\text{(i)} \quad 0 > \eta v(z_0) > -\delta, \text{ and} \\ &\text{(ii)} \quad (1-\eta) L_F^*(z_0) > 4\delta. \end{aligned}$$

Since L_F^*, $v \in L$, it follows that $(1-\eta)L_F^* + \eta v \in L$. For $m \in Z^+$ (positive integers), let $u_m = [(1-\eta)L_F^* + \eta v] * \chi_{1/m}$, where $\chi_{1/m}$ is a smooth, radially symmetric bump function supported in $|z| \leqslant 1/m$. Then

$$(3.2) \qquad u_m \searrow (1-\eta) L_F^* + \eta v \quad \text{as } m \to \infty;$$

$u_m \in P(\mathbf{C}^n) \cap C(\mathbf{C}^n)$; furthermore, $u_m \in L$ [**S1**, Proposition 1.2]. By an approximation theorem [**Fe**, Proposition 2, Chapter VII], for each m, there exists a sequence of positive integers $\{d_{m_k}\}$ and a family of polynomials $\{p_{m_k}\}$ with $\deg(p_{m_k}) = d_{m_k}$ such that

$$u_m(z) = \sup_k \frac{1}{d_{m_k}} \log |p_{m_k}(z)| \quad \text{for all } z \in E.$$

In particular, for each m, we can find a polynomial p_m of degree d_m with

$$(3.3) \qquad \begin{aligned} &\text{(i)} \quad (1/d_m)\log|p_m(z)| \leqslant u_m(z) \text{ for all } z \in E, \text{ and} \\ &\text{(ii)} \quad (1/d_m)\log|p_m(z_0)| > u_m(z_0) - \delta. \end{aligned}$$

Moreover, by taking powers of p_m if necessary, we may assume that $d_m \nearrow \infty$.

To get a contradiction to the (L^*) condition, we take $\Phi = \{e^{-\delta d_m} p_m\}$. We first show that

$$\sup_m e^{-\delta d_m} |p_m(z)| < \infty \quad \text{for all } z \in F.$$

For $z \in F - F_0$, $v(z) = -\infty$; thus $u_m(z) \leqslant 0$ for $m \geqslant m(z)$ by (3.2) so that

$$\sup_m e^{-\delta d_m} |p_m(z)| \leqslant e^{-\delta d_m} e^{d_m u_m(z)}$$

(by (3.3)(i)) is finite. For $z \in F_0$,

$$e^{-\delta d_m}|p_m(z)| \leqslant e^{-\delta d_m} e^{d_m u_m(z)}$$
$$\leqslant e^{-\delta d_m} e^{d_m[(1-\eta)L_F^*(z)+\eta v(z)]+d_m\sigma_m},$$

where $\sigma_m = \sigma_m(z) \to 0$ as $m \to \infty$ by (3.2). Since $v \leqslant 0$ on E and $L_F^* = 0$ on F_0, $(1-\eta)L_F^*(z) + \eta v(z) \leqslant 0$ if $z \in F_0$ and we obtain

$$e^{-\delta d_m}|p_m(z)| \leqslant e^{-d_m(\delta - \sigma_m)}.$$

Since $\sigma_m \to 0$ as $m \to \infty$, $\delta - \sigma_m$ is eventually positive and thus

$$\sup_m e^{-d_m\delta}|p_m(z)| < \infty \quad \text{for } z \in F_0.$$

We show that (L^*) is violated by proving that there is an $\varepsilon > 0$ such that

(3.4) $$e^{-d_m\delta}|p_m(z_0)| \geqslant m(1+\varepsilon)^{d_m} \quad \text{for } m \text{ sufficiently large.}$$

From (3.3)(ii), (3.2) and (3.1) we obtain

$$e^{-d_m\delta}|p_m(z_0)| \geqslant e^{-d_m\delta} e^{d_m[u_m(z_0)-\delta]}$$
$$\geqslant e^{-2d_m\delta} e^{d_m[(1-\eta)L_F^*(z_0)+\eta v(z_0)]}$$
$$\geqslant e^{d_m\delta} \geqslant (1+\delta)^{d_m} = (1+\varepsilon)^{d_m}((1+\delta)/(1+\varepsilon))^{d_m}.$$

Thus by choosing $\varepsilon < \delta$, we obtain (3.4).

For the converse, we assume that if $F \subset E$ is a Borel set with $\mu(F) = \mu(E)$, then $L_F^* = L_E$. Let Φ be a family of polynomials such that

$$\sup_{p\in\Phi} |p(z)| \equiv M(z) < \infty \quad \mu\text{-a.e. on } E.$$

Let $F = \{z \in E\colon M(z) < \infty\}$. Then $\mu(F) = \mu(E)$; furthermore, for each positive integer m, if we let $F_m = \{z \in E\colon M(z) \leqslant m\}$, then $F_m \nearrow F$, and since $M(z)$ is lowersemicontinuous, each F_m is compact. Thus F is a Borel set and hence $L_F^* = L_E$.

Since it is clear that for a polynomial p,

$$\frac{1}{\deg(p)} \log|p(z)| \in L,$$

it follows that for any $p \in \Phi$,

(3.5) $$\frac{1}{\deg(p)} \log \frac{|p(z)|}{m} \leqslant L_{F_m}(z) \leqslant L_{F_m}^*(z) \quad \text{for all } z \in \mathbf{C}^n.$$

Also, since L_E is continuous, given $\varepsilon > 0$,

$$U_\varepsilon = \{z \in \mathbf{C}^n\colon L_E(z) < \varepsilon\}$$

is an open neighborhood of E. The extremal functions $L_{F_m}^*$ decrease pointwise to $L_F^* = L_E$ on $\overline{U}_\varepsilon$; thus, by Dini's theorem, the functions $L_{F_m}^*$ converge uniformly to L_E on $\overline{U}_\varepsilon$ and we can find $m_0 = m_0(\varepsilon)$ such that

$$L_{F_m}^*(z) < 2\varepsilon \quad \text{for } z \in \overline{U}_\varepsilon \quad \text{if } m > m_0.$$

For such m,

$$\|p\|_{U_\varepsilon} = \sup_{z\in U_\varepsilon} |p(z)| \leqslant m(e^{2\varepsilon})^{\deg(p)} \quad \text{for all } p \in \Phi$$

by (3.5). Thus (E, μ) satisfies (L^*).

In the complex plane, Ullman [**U1**] calls a measure whose support $E = E(\mu)$ is an infinite set, a determined measure if for any Borel subset $F \subset E$ with $\mu(F) = \mu(E)$, the logarithmic capacity $C(F)$ of F equals $C(E)$. Since the function L_E^* is the Green function in one variable, Theorem 3.1 shows that for regular compact sets $E \subset \mathbf{C}$, (E, μ) satisfies (L^*) precisely when μ is determined. The condition that μ has infinite support is a density condition which insures the linear independence of the monomials $\{z^k\}$ in the space $L^2(\mu)$ of square-integrable functions with respect to μ and allows one to construct the orthogonal polynomials $\{p_k\}$ with respect to μ via the Hilbert-Schmidt process. The polynomials $p_k(z) = z^k + \cdots$ are the unique monic orthogonal polynomials of degree k $(k = 0, 1, 2, \dots)$ of minimal $L^2(\mu)$-norm among all monic polynomials of degree k. If we denote the $L^2(\mu)$-norm of p_k by N_k, Ullman [**U1**] has shown that if $C(E) > 0$ and μ is a determined measure, then

$$\lim_{k \to \infty} \left[\frac{|p_k(z)|}{N_k} \right]^{1/k} = e^{L_E(z)}$$

for z in the unbounded component of $\mathbf{C} - E$ except for a set of Lebesgue measure zero in the convex hull of E.

In $\mathbf{C}^n$, $n > 1$, let $E \subset B$ be compact and let μ be a measure on E. As a replacement for the monic normalization, we first put an ordering on the monomials $z^\alpha \equiv z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. Let α: $Z^+ \to (Z^+ \cup \{0\})^n$ be a bijective map on the positive integers such that $|\alpha(k)| \leqslant |\alpha(k+1)|$ for all $k \in Z^+$, where $|\alpha(k)| = \alpha_1 + \cdots + \alpha_n$ if $\alpha(k) = (\alpha_1, \dots, \alpha_n)$. Under certain conditions on the measure μ and the set E, the sequence of monomials $\{e_k(z) \equiv z^{\alpha(k)}\}$ is linearly independent in the space $L^2(\mu)$ and therefore we can construct orthogonal polynomials $\{A_k\}$ of the form

$$A_k(z) = e_k(z) + \sum_{j=1}^{k-1} a_j e_j(z), \qquad a_j \in \mathbf{C},$$

via the Hilbert-Schmidt process. Recall that if $E \subset \mathbf{C}^n$ is compact, the Silov boundary of E with respect to the uniform Banach algebra generated by the restrictions to E of the analytic polynomials in $\mathbf{C}^n$, denoted S_E, is the smallest closed subset of E such that for any polynomial p, there exists $z_0 \in S_E$ with $\|p\|_E = |p(z_0)|$.

PROPOSITION 3.1. *Let $E \subset \mathbf{C}^n$ be compact and suppose that E is not pluripolar. If μ is a measure on E such that $S_E \subset$ support of μ, then the monomials $\{e_k\}$ are linearly independent in $L^2(\mu)$.*

PROOF. By a result of Siciak [**S1**, Proposition 4.3], since E is not pluripolar, E is unisolvent, i.e., if p is a polynomial satisfying $p(z) = 0$ for $z \in E$, then $p \equiv 0$ on $\mathbf{C}^n$.

Suppose there exists a polynomial $p(z) = \sum_{j=1}^k c_j e_j(z)$, $p \not\equiv 0$, with $\int_E |p|^2 \, d\mu = 0$. Then $\|p\|_E = M > 0$ and we can find $z_0 \in S_E$ with $|p(z_0)| = M$. By continuity, we can find $r > 0$ such that $|p(z)| > M/2$ for $z \in B(z_0, r)$. Since $S_E \subset$ support of μ, $\mu(E \cap B(z_0, r)) > 0$ so that

$$0 = \int_E |p|^2 \, d\mu \geqslant \int_{E \cap B(z_0, r)} |p|^2 \, d\mu > \left(\frac{M}{2}\right)^2 \mu(E \cap B(z_0, r)) > 0,$$

which gives a contradiction. Thus $p \equiv 0$ on $\mathbf{C}^n$, i.e., $c_1 = \cdots = c_k = 0$, and the proposition is proved.

From the work of Nguyen Thanh Van and Zeriahi [**NZ**], if (E, μ) satisfies (L^*), then (E, μ) satisfies the density condition in Proposition 3.1 and we may construct the orthogonal polynomials $\{A_k\}$ associated with μ. Set $\nu_k \equiv [\int_E |A_k|^2 \, d\mu]^{1/2}$. By following the argument used by Zeriahi [**Ze**], we can now state a $\mathbf{C}^n$-version of Ullman's theorem.

THEOREM 3.2. *Let $E \subset B$ be a regular compact set and let μ be a finite measure on E such that (E, μ) satisfies (L^*). If $\{A_k\}$ is the sequence of orthogonal polynomials associated to μ and $\{\nu_k\}$ is the corresponding sequence of $L^2(\mu)$-norms, then*:

(i) $\underline{\lim}_{k\to\infty}[\|A_k\|_E/\nu_k]^{1/|\alpha(k)|} = 1$, *and*

(ii) $\overline{\lim}_{k\to\infty}[|A_k(z)|/\nu_k]^{1/|\alpha(k)|} = e^{L_E(z)}$ *for all $z \in \mathbf{C}^n - \hat{E}$, where $\hat{E}$ is the polynomial hull of E.*

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