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CHAOS, PERIODICITY, AND SNAKELIKE CONTINUA

BY

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ABSTRACT. The results of this paper relate the dynamics of a continuous map f of the interval and the topology of the inverse limit space with bonding map f. These inverse limit spaces have been studied by many authors, and are examples of what Bing has called "snakelike continua". Roughly speaking, we show that when the dynamics of f are complicated, the inverse limit space contains indecomposable subcontinua. We also establish a partial converse.

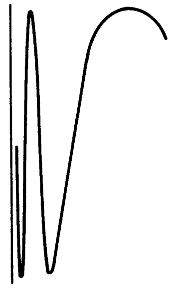
Introduction. Let I be a closed interval, and let $f\colon I \to I$ be a continuous function. Associated with f is the inverse limit space $(I, f) = \{(x_0, x_1, \ldots) | f(x_{n+1}) = x_n\}$. With a natural topology, (I, f) is a compact, connected, metric space, and is an example of what Bing **[Bi]** has called a snakelike continuum. In this paper we will investigate the relationship between behavior of the orbits $\{f^n(x) | n \geqslant 0\}$ of points of I under f, and the topological properties of the space (I, f). These examples suggest some of the ideas which we will explore.

EXAMPLE 1. Let $I = [0,1]$, and define $f\colon I \to I$ by

$$f(t) = \begin{cases} \frac{3}{2}t & \text{if } 0 \leqslant t \leqslant \frac{2}{3}, \\ \frac{5}{3} - t & \text{if } \frac{2}{3} \leqslant t \leqslant 1. \end{cases}$$

It can be verified that (i) if $x \in I$, then $\{f^n(x) | n \geqslant 0\}$ is not dense in I, and (ii) f has points of period 1 and 2, but no points of period $n \geqslant 3$. Now let X be the subspace of the plane defined by

$$X = \{(x, y) | 0 < x \leqslant 1 \text{ and } y = \sin(1/x)\} \cup \{(x, y) | x = 0 \text{ and } -1 \leqslant y \leqslant 1\}.$$



$X \sim (I, f)$

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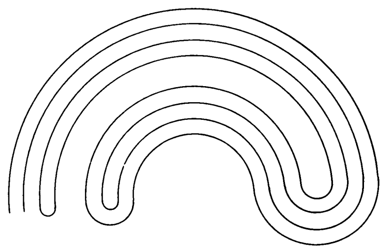
It can be verified that X is homeomorphic with (I, f) and has the following property: if C is a nondegenerate subcontinuum of X, then C is the union of two of its proper subcontinua.

EXAMPLE 2. Let $I = [0, 1]$, and let $f\colon I \to I$ be defined by

$$f(t) = \begin{cases} 2t & \text{if } 0 \leqslant t \leqslant \frac{1}{2}, \\ 2 - 2t & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

It can be verified that (i) there is a point $x \in I$ such that $\{f^n(x) | n \geqslant 0\}$ is dense in I, and (ii) for each positive integer k, f has points of period k. Now let X be the subspace of the plane described as follows: let C be the standard Cantor set on the x-axis. Let K_0 be the union of all semicircles in the upper half-plane with endpoints on C, which are symmetric with the line $x = \frac{1}{2}$. For each positive integer i, let K_i be the union of all semicircles in the lower half-plane with endpoints on C, which are symmetric about the line $x = 5/3^i \cdot 2$. Then X is $\bigcup_{i=0}^{\infty} K_i$.

$X \sim (I, f)$



It can be verified that X is homeomorphic with (I, f) and has the following property: X is not the union of two of its proper subcontinua.

Definitions and terminology. If a and b are distinct real numbers we will let $[a, b]$ denote the smallest closed interval containing both a and b, and let (a, b) denote the associated open interval. We will generically let I be a closed interval and will be considering continuous functions $f\colon I \to I$. All of the functions which we will consider are continuous.

If $f\colon I \to I$ and $x \in I$, then the *orbit* of x under f is $\{y |$ for some integer n, $n \geqslant 0$, $y = f^n(x)\}$. We will be interested in functions $f\colon I \to I$ for which there is a point x whose orbit is dense in I. (In **[AY]** a function f is defined to be chaotic if there is a point whose orbit is dense and if every point is unstable. As a corollary to Lemma 2 we show that, for functions on the interval, the existence of a dense orbit implies that every point is unstable.)

If $f\colon I \to I$ and $x \in I$, the statement that x *has period* n, means that n is a positive integer, $f^n(x) = x$, and if k is an integer, $1 \leqslant k < n$, then $f^k(x) \neq x$.

Associated with $f\colon I \to I$ is the compact, connected metric space $(I, f) = \{(x_0, x_1, \ldots) | f(x_i) = x_{i-1}\}$ with metric

$$d((x_0, x_1, \ldots), (y_0, y_1, \ldots)) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}.$$

(I, f) is an example of what Bing **[Bi]** has called a *snakelike continuum*. The reason for this terminology is that for each $\varepsilon > 0$, there is a finite open covering $\{g_1, g_2, \ldots, g_n\}$ of (I, f) such that (i) diam $g_i < \varepsilon$, and (ii) $g_i \cap g_j \neq \varnothing$ iff $j = i - 1$, i, or $i + 1$. We will denote elements of (I, f) by subbarred letters, as $\underline{x} = (x_0, x_1, \ldots)$. The *projection maps* π_n of (I, f) onto I given by $\pi_n(\underline{x}) = x_n$ are continuous. If H is a *subcontinuum* (compact, connected subspace) of (I, f) we will let H_n denote $\pi_n(H)$. Note that H_n is a closed interval or point, and that $f(H_{n+1}) = H_n$.

If $f\colon I \to I$, then f induces a homeomorphism $\hat{f}\colon (I, f) \to (I, f)$ by $\hat{f}((x_0, x_1, \ldots)) = (f(x_0), x_0, x_1, \ldots)$. Notice that $f \circ \pi_n = \pi_n \circ \hat{f}$, $\pi_n = \pi_{n+1} \circ \hat{f}$, and $f \circ \pi_{\pi+1} = \pi_n$.

Here are important facts about snakelike continua which we will utilize. Suppose S is snakelike; then (i) the intersection of any collection of subcontinua of S is a subcontinuum of S, and (ii) if H is a subcontinuum of S, then $S - H$ has at most two components (see **[Bi]**).

If S is a continuum, the statement that S is *indecomposable* means that S is not the union of two of its proper subcontinua. Here are two conditions each of which is equivalent to the indecomposability of S (see **[HY**, pp. 139–141]).

(1) If H is a subcontinuum of S, then H contains no open set in S.

(2) S contains three distinct points x, y and z such that S is irreducible between each pair of these points. (*Irreducibility* between x and y means that no proper subcontinuum of S contains both x and y.)

We will utilize the following important construction due to Bing **[Bi]**. Suppose that (I, f) contains no indecomposable subcontinuum with interior. For each $\underline{x} \in (I, f)$ let $g_{\underline{x}}$ be the intersection of all subcontinua of (I, f) that contain interiorly a subcontinuum that contains $\underline{x}$ in its interior. Then $g_{\underline{x}}$ is a subcontinuum of (I, f). Furthermore, the sets $g_{\underline{x}}$ partition (I, f) and if we let $G = \{g_{\underline{x}} | \underline{x} \in (I, f)\}$ with the quotient topology, then G is an arc (i.e. homeomorphic with I). Moreover, f induces a homeomorphism $\hat{\hat{f}}$ of G onto G given by $\hat{\hat{f}}(g_{\underline{x}}) = g_{\hat{f}(\underline{x})}$. Bing also shows that, for each $\underline{x} \in (I, f)$, $g_{\underline{x}}$ does not have interior.

If A is a set, we will let both $\bar{A}$ and cl(A) denote the closure of A and let int A denote the interior of A.

THEOREM 1. *Suppose that k and n are integers, $k \geqslant 0$, $n \geqslant 1$, and that f has a point of period $2^k(2n + 1)$; i.e., of period not a power of 2. Then (I, f) has an indecomposable subcontinuum that is invariant under $\hat{f}^{2^{k+1}}$.*

PROOF. If follows from Sarkovskii's Theorem **[S, N]** and the hypothesis that f has a point of period $(2^{k+1})(3)$. Then $f^{2^{k+1}}$ has a point of period 3. Now, for each positive integer j, the spaces (I, f) and (I, f^j) are homeomorphic. One such homeomorphism is $(x_0, x_1, \ldots, x_j, \ldots) \to (x_0, x_j, x_{2j}, \ldots)$. Using this fact, we need only show that if $g\colon I \to I$ is continuous and g has a point of period 3, then (I, g) has an indecomposable subcontinuum that is invariant under $\hat{g}$. Let g be such a function and let x be a point of period 3. Then we have the point $\underline{x} = (x, g^2(x), g(x), x, \ldots)$ of (I, g) with $\hat{g}^3(\underline{x}) = \underline{x}$ and $\hat{g}(\underline{x}) \neq \underline{x}$.

Now let S be the intersection of all subcontinua of (I, g) which contain $\{\underline{x}, \hat{g}(\underline{x}), \hat{g}^2(\underline{x})\}$. Since $\{\underline{x}, \hat{g}(\underline{x}), \hat{g}^2(\underline{x})\}$ is contained in both $\hat{g}(S)$ and $\hat{g}^{-1}(S)$, it follows that $S = g(S)$. So S is a subcontinuum of (I, g) which is invariant under $\hat{g}$.

We will next show that S is irreducible between each pair of $\underline{x}$, $\hat{g}(\underline{x})$ and $\hat{g}^2(\underline{x})$. It will follow that S is indecomposable. Suppose, for example, that S is reducible between $\hat{g}(\underline{x})$ and $\hat{g}^2(\underline{x})$. Then there is a proper subcontinuum H of S which contains $\hat{g}(\underline{x})$ and $\hat{g}^2(\underline{x})$. Now, because H is proper, $\underline{x} \notin H$, and it follows that there is an integer N such that if $n > N$, then $\pi_n(\underline{x}) \notin H_n$. However, for every third n, $\pi_n(\underline{x})$ is between $\pi_n(\hat{g}(\underline{x}))$ and $\pi_n(\hat{g}^2(\underline{x}))$, which are in the closed interval H_n. This contradiction establishes Theorem 1.

In what follows, if $x \in I$, and s and k are integers, $s \geqslant 1$, $k \geqslant 0$, we let $A_{s,k}(x) = A_{s,k}$ be $\{f^{sn+k}(x) | n \geqslant 0\}$.

LEMMA 2. *Suppose that f has a dense orbit, and that x is a point whose orbit is dense, i.e. $A_{1,0}$ is dense in I. Then one of the following occurs.*

(i) *$A_{2,0}$ is dense in I, in which case $A_{s,k}$ is dense in I for each $s \geqslant 1$, $k \geqslant 0$, or*

(ii) *$A_{2,0}$ is not dense in I, in which case $I = \overline{A}_{2,0} \cup \overline{A}_{2,1}$, $\overline{A}_{2,0}$ and $\overline{A}_{2,1}$ are closed intervals which intersect in a point, and $f(\overline{A}_{2,0}) = \overline{A}_{2,1}$, $f(\overline{A}_{2,1}) = \overline{A}_{2,0}$. Moreover, for each $k \geqslant 1$, $A_{2k,0}$ is dense in $\overline{A}_{2,0}$ and $A_{2k,1}$ is dense in $\overline{A}_{2,1}$.*

PROOF. Let s be an integer, $s \geqslant 1$, and for each integer r, $0 \leqslant r \leqslant s-1$, let $B_r = \overline{A}_{s,r}$. Then since $\bigcup_{r=0}^{s-1} A_{s,r} = A_{1,0}$, it follows that $\bigcup_{r=0}^{s-1} B_r = I$. From this we see that there is an r, $0 \leqslant r \leqslant s-1$, so that B_r has nonempty interior.

Next, notice that if J is a closed subinterval of I, then $f(J)$ is a closed interval, because if $f(J)$ is a point, then for some integer n, $f^n(x)$ is periodic. From this remark, and the fact that $f(B_r) \subset B_{r+1}$ (mod s) it follows that, for each integer i, $0 \leqslant i \leqslant r-1$, B_i has nonempty interior.

We next show that if the interiors of B_i and B_j intersect, then these interiors are identical. For, if int $B_i \cap$ int $B_j \neq \varnothing$, then there is a positive integer n so that $f^{2n+i}(x) \in$ int $B_i \cap$ int B_j, and there is a sequence $n_1, n_2, n_3, \ldots$ of positive integers such that $f^{n_k s+j}(x) \to f^{sn+i}(x)$. Then for every integer $l > 0$ we have $f^{s(n_k+l)+j}(x) \to f^{s(n+l)+i}(x)$. From this it follows that $\text{cl}\{f^{sn+i}(x), f^{s(n+1)+i}(x), \ldots\} \subset B_j$ and hence that

$$B_i \subset B_j \cup \{f^i(x), f^{s+i}(x), \ldots, f^{(n-1)s+i}(x)\}.$$

From this we see that int $B_i \subset$ int B_j. A similar argument shows that int $B_j \subset$ int B_i, and hence int $B_i =$ int B_j.

Now let $G = \{g |$ for some r, $0 \leqslant r \leqslant s-1$, g is a component of int $B_r\}$. Notice that G is a collection of disjoint open intervals whose union is dense in I. Since G is countable we list G as $\{g_1, g_2, \ldots\}$. Now for each $g_i \in G$ let $C_i = \overline{g_i}$. Then C_i is a closed interval, $f(C_i)$ is a closed interval by an earlier remark, and there is an r, $0 \leqslant r \leqslant s-1$, such that $f(C_i) \subset B_r$. Then int $f(C_i) \subset$ int B_r and there is an integer k so that $f(C_i) \subset C_k$. Because x has a dense orbit, we see that if i and k are integers which are subscripts of elements of G, then there is a positive integer j so that $f^j(C_i) \subset C_k$. Since we have this transitivity, it follows that G is finite. Thus we may list G as $\{g_1, g_2, \ldots, g_n\}$ and their closures as $C_1, C_2, \ldots, C_n$. Notice that because of the above transitivity condition, the set $\{C_1, C_2, \ldots, C_n\}$ is permuted by f.

We next show that $n \leqslant 2$. Let y be a fixed point of f. Now, if $y \in \operatorname{int} C_i$, then $f(C_i) = C_i$, which is impossible unless $n = 1$. Similarly, if y is an endpoint of I, then $n = 1$. If y is a common endpoint of C_i and C_j, then $f(C_i) = C_j$ and $f(C_j) = C_i$ which is impossible unless $n = 2$. Notice that the integer n depends on s. In what follows we will refer to n as $n(s)$.

We now verify the conclusion. First, assume that $A_{2,0}$ is dense in I. Let s be an integer, $s \geqslant 1$, and suppose that $n(s) = 2$. Then, there are closed intervals C_1 and C_2 with $C_1 \cup C_2 = I$, $C_1 \cap C_2 = \{\text{pt}\}$, $f(C_1) = C_2$ and $f(C_2) = C_1$. Assuming that $x \in C_2$ we see that, for each j, $f^{2j}(x) \in C_2$ and hence $A_{2,0} \cap \operatorname{int} C_1 \neq \varnothing$. This contradicts the fact that $A_{2,0}$ is dense in I, and hence $n(s) = 1$. Then, for each r, $0 \leqslant r \leqslant s-1$, $B_r = I$. Then $\overline{A}_{s,r} = I$, and so $A_{s,r}$ is dense in I. From this, we see that for any integer $k \geqslant 0$, $A_{s,k}$ is dense in I.

Next, assume that $A_{2,0}$ is not dense in I. Let $s = 2$. Since $A_{2,0}$ is not dense, $B_0 \neq I$ and so $n(2) = 2$. Now let j be an integer, $j \geqslant 1$. Then, for each integer l, $A_{2j,l} \subset A_{2,l}$ and since $\overline{A}_{2,0} \neq I$, we have $n(2j) = 2$. Now, notice that the intervals C_1 and C_2 which we construct for $s = 2j$ are independent of j. This is because their common endpoint is the only fixed point for the function f. Then, assuming $x \in C_2$, we have $C_2 = \overline{A}_{2,0}$, $C_1 = \overline{A}_{2,1}$, and, for each integer $k \geqslant 1$, $\overline{A}_{2k,0} = C_2$ and $\overline{A}_{2k,1} = C_1$. This establishes Lemma 2.

The following result is known [N], but we include it for completeness.

Corollary. *Suppose that f has a dense orbit. Then the set of periodic points of f is dense in I.*

Proof. Let V be an open interval in I. Let x be a point of V whose orbit is dense in I. If $\{f^{2n}(x) | n \geqslant 0\}$ is not dense in I, we may assume from Lemma 1 that $V \subset \operatorname{cl}\{f^{2n}(x) | n \geqslant 0\}$. Let j be an integer such that $f^j(x) \in V$. We may assume that $x < f^j(x)$. Let $g\colon I \to I$ be the function $g = f^j$. From Lemma 2, it follows that $\{g^k(x) | k \geqslant 0\}$ is dense in V. Now let l be the smallest positive integer such that $g^l(g(x)) < g(x)$. Then $g^l(x) = g^{l-1}(g(x)) \geqslant g(x) > x$ and $g^l(g(x)) < g(x)$. So $g^l(x) > x$ and $g^l(g(x)) < g(x)$. Consequently g^l has a fixed point y, $x < y < g(x)$. Since $g^l(y) = y$, $f^{kl}(y) = y$ and, since $y \in V$, V contains a periodic point of f.

Definition If $y \in I$, the statement that y is *topologically stable* means that if $\varepsilon > 0$, then there is a $\delta > 0$ such that if $z \in I$ and $|y - z| < \delta$ then for each positive integer n, $|f^n(y) - f^n(z)| < \varepsilon$. If y is not topologically stable, then y is called *topologically unstable*.

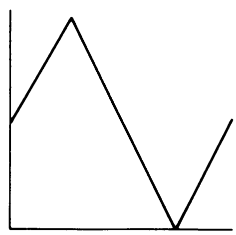
Corollary. *Suppose that f has a dense orbit. Then every point of I is topologically unstable.*

Proof. Suppose that $y \in I$ and that y is topologically stable. Let x be a point of I whose orbit under f is dense. We first show that the orbit of y is dense. Suppose that U is an open interval in I and, for each n, $f^n(y) \notin U$. Let V be the open interval which is the open middle third of U. Let $\varepsilon = \frac{1}{3} \operatorname{diam} U$. Then, since y is topologically stable, there is a $\delta > 0$ such that if $|z - y| < \delta$ then, for each n, $|f^n(y) - f^n(z)| < \varepsilon$. In particular, if $|z - y| < \delta$ then, for each n, $f^n(z) \notin V$. Now since x has a dense orbit, there is a j such that $|f^j(x) - y| < \delta$. Then there is an integer $k > j$ such that $f^k(x) \in V$. But then $f^{k-j}(f^j(x)) \in V$ and this is a contradiction. Therefore, the orbit of y is dense.

Now it follows from Lemma 2 that there is a positive number ε and a subinterval C of I such that $\operatorname{diam} C > 3\varepsilon$, and, for each positive integer n, $\{f^{kn}(y)|k \geqslant 0\}$ is dense in C. Now choose δ such that if $|z - y| < \delta$, then for each j, $|f^j(z) - f^j(y)| < \varepsilon$. Using the previous corollary, let t be a periodic point such that $|t - y| < \delta$. Let n be the period of t. Then, for each k, $|f^{kn}(t) - f^{kn}(y)| < \varepsilon$, so $|t - f^{kn}(y)| < \varepsilon$. But then $\{f^{kn}(y)|k \geqslant 0\}$ is not dense in C. This establishes the Corollary.

Example 3. Let $I = [0, 1]$, and let $f\colon I \to I$ be defined by

$$f(x) = \begin{cases} 2x + \frac{1}{2}, & 0 \leqslant x \leqslant \frac{1}{4}, \\ -2x + \frac{3}{2}, & \frac{1}{4} \leqslant x \leqslant \frac{3}{4}, \\ 2x - \frac{3}{2}, & \frac{3}{4} \leqslant x \leqslant 1. \end{cases}$$



Then f has the following properties. (i) There is a number x such that $\{f^n(x)|n \geqslant 0\}$ is dense in I, and (ii) if $y \in I$, then $\{f^{2n}(y)|n \geqslant 0\}$ is not dense in I.

It can be shown that (I, f) is homeomorphic with Example 2 together with its reflection through the origin.

Theorem 3. *Suppose that $x \in I$, and that x has a dense orbit under f. Then one of the following occurs.*

(a) $\{f^{2n}(x)|n \geqslant 0\}$ *is dense in I, in which case (I, f) is indecomposable, or*

(b) $\{f^{2n}(x)|n \geqslant 0\}$ *is not dense in I, in which case there are proper subcontinua H and K of (I, f) such that* (i) *H and K are indecomposable*, (ii) $H \cup K = (I, f)$, (iii) *$H \cap K$ is a point*, (iv) $\hat{f}(H) = K$, *and* (v) $\hat{f}(K) = H$.

Proof. First, assume that $\{f^{2n}(x)|n \geqslant 0\}$ is dense in I. Assume further that (I, f) has no indecomposable subcontinuum with interior. Then Bing's construction [**Bi**] applies to yield a homeomorphism $\hat{\hat{f}}\colon G \to G$ of the arc G onto itself. Since x has a dense orbit, the function f is onto, and by choosing inverse images we may construct the point $\underline{x} = (x, f^{-1}(x), \ldots)$ of (I, f). It is clear that $\{\hat{f}^n(\underline{x})|n \geqslant 0\}$ is dense in (I, f). From this it follows that $\{\hat{\hat{f}}^n(g_{\underline{x}})|n \geqslant 0\}$ is dense in G. This is impossible since $\hat{\hat{f}}$ is a homeomorphism and G is an arc. Therefore there is a subcontinuum S of (I, f) such that S is indecomposable and has interior.

We will next show that $S = (I, f)$. First, suppose that for each positive integer k, $\operatorname{int}(\hat{f}^k(S)) \cap \operatorname{int} S = \varnothing$. As above, let $\underline{x} = (x, f^{-1}(x), f^{-2}(x), \ldots)$. The previous assumption makes it impossible for $\{\hat{f}^n(\underline{x}) | n \geqslant 0\}$ to be dense in (I, f). It follows that there is a positive integer k such that $\operatorname{int}(\hat{f}^k(S)) \cap \operatorname{int} S \neq \varnothing$. Then $\hat{f}^k(S) \cap S$ is a subcontinuum of both $\hat{f}^k(S)$ and S which has interior, and, since S is indecomposable, $\hat{f}^k(S) = S$. Then, for each positive integer j, $\hat{f}^{jk}(S) = S$. Now let l be a positive integer such that $\hat{f}^l(\underline{x}) \in S$. From Lemma 2 we have $\{f^{j(kl)}(x) | j \geqslant 1\}$ in dense in I and hence $\{\hat{f}^{j(kl)}(\underline{x}) | j \geqslant 1\}$ is dense in (I, f). Since $\hat{f}^{j(kl)}(\underline{x}) \in S$ we see that $S = (I, f)$.

Next, we consider the case where $\{f^{2n}(x) | n \geqslant 0\}$ is not dense in I. By Lemma 2, there are closed subintervals C_1 and C_2 such that $I = C_1 \cup C_2$, $C_1 \cap C_2 = \{p\}$, $f(C_1) = C_2$ and $f(C_2) = C_1$. Now let

$$H = \{\underline{y} | \underline{y} \in (I, f), y_{2n} \in C_1 \text{ and } y_{2n+1} \in C_2 \text{ if } n \geqslant 0\}$$

and

$$K = \{\underline{y} | \underline{y} \in (I, f), y_{2n+1} \in C_1 \text{ and } y_{2n} \in C_2 \text{ if } n \geqslant 0\}.$$

Then H and K are subcontinua of (I, f), $H \cup K = (I, f)$, $H \cap K = (p, p, p, \ldots)$, $\hat{f}(H) = K$ and $\hat{f}(K) = H$. In order to see that K is indecomposable, consider the function $h = f^2\colon C_2 \to C_2$. Then, assuming that $x \in C_2$, it follows from Lemma 2 that both $\{h^n(x) | n \geqslant 0\}$ and $\{h^{2n}(x) | n \geqslant 0\}$ are dense in C_2. By the first part of this theorem, (C_2, h) is indecomposable. The correspondence

$$(y, h^{-1}(y), h^{-2}(y), \ldots) \leftrightarrow (y, f(h^{-1}(y)), h^{-1}(y), f(h^{-2}(y)), h^{-2}(y), \ldots)$$

is a homeomorphism between (C_2, h) and K. Therefore K is indecomposable and, as $H = \hat{f}(K)$, H is indecomposable. This establishes Theorem 3.

Definition. Suppose that y is a fixed point of f. This statement that x *is homoclinic to the fixed point* y means that $x \neq y$ and there is a choice of inverse images $f^{-1}(x), f^{-2}(x), \ldots$ such that both $f^n(x) \to y$ and $f^{-n}(x) \to y$. If y is a periodic point of f with period s, then the statement that x *is homoclinic to* y means that x is homoclinic to the fixed point y under f^s.

The next result can be obtained from Theorem 1 and **[Bl]**. We include a direct proof.

Theorem 4. *If f has a point homoclinic to a periodic point, then (I, f) contains an indecomposable subcontinuum.*

Proof. Since, for each positive integer s, (I, f) is homeomorphic with (I, f^s), we will assume that f has a point homoclinic to a fixed point. Let y be a fixed point and let $x \neq y$, together with a choice of inverse images, be such that $f^n(x) \to y$ and $f^{-n}(x) \to y$. In (I, f) let $\underline{y} = (y, y, y, \ldots)$ and $\underline{x} = (x, f^{-1}(x), f^{-2}(x), \ldots)$. Then $\hat{f}(\underline{y}) = \underline{y}$, $\hat{f}^n(\underline{x}) \to \underline{y}$ and $\hat{f}^{-n}(\underline{x}) \to \underline{y}$.

Now let S be the intersection of all subcontinua of (I, f) which contain $\{\underline{y}\} \cup \{\hat{f}^n(\underline{x}) | -\infty < n < \infty\}$. Since both $\hat{f}(S)$ and $\hat{f}^{-1}(S)$ contain $\{\underline{y}\} \cup \{\hat{f}^n(\underline{x}) | -\infty < n < \infty\}$, we see that $\hat{f}(S) = S$.

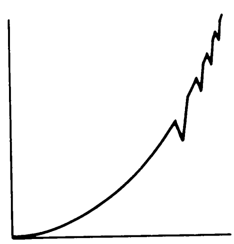
Now, for each $n \geqslant 0$, $\pi_n \circ \hat{f} = f \circ \pi_n$, and so $f(\pi_n(S)) = \pi_n(\hat{f}(S)) = \pi_n(S)$. Thus $\pi_n(S)$ is invariant under f. Now let $J = \pi_0(S)$. Then $f(J) = J$ and $S = (J, f)$.

Suppose now that $S = (J, f)$ contains no indecomposable subcontinuum with interior. Then Bing's construction **[Bi]** yields an arc G, and $\hat{\hat{f}}: G \to \hat{G}$ is a homeomorphism. Now $\hat{\hat{f}}^n(g_{\underline{x}}) \to g_{\underline{y}}$ and $\hat{\hat{f}}^{-n}(g_{\underline{x}}) \to g_{\underline{y}}$. Since G is an arc and $\hat{\hat{f}}$ is a homeomorphism, this is impossible unless, for each j, $\hat{\hat{f}}^j(g_{\underline{x}}) = g_{\underline{y}}$. This implies that, for each j, $g_{\hat{f}^j(\underline{x})} = g_{\underline{y}}$. But then $g_{\underline{y}}$ contains $\{\underline{y}\} \cup \{\hat{f}^j(\underline{x}) | -\infty < j < \infty\}$ and hence $g_{\underline{y}} = S$. But then G is degenerate, and this is impossible.

Therefore S contains an indecomposable subcontinuum with interior, and (I, f) contains an indecomposable subcontinuum.

DEFINITION. If $f: I \to I$ is continuous, then the statement that f is *organic* means that if $\underline{x} \in (I, f)$, $\underline{y} \in (I, f)$ and (I, f) is irreducible from $\underline{x}$ to $\underline{y}$, then there is a positive integer n such that $f^n([\pi_n(\underline{x}), \pi_n(\underline{y})]) = I$. The statement that f is *inorganic* means that f is not organic.

EXAMPLE 4. The accompanying figure is a sketch of a function which is inorganic.



In **[H]**, Henderson shows that (I, f) is a pseudo-arc, a particular snakelike continuum which is hereditarily indecomposable. Notice that f has no points of period greater than one.

LEMMA 5. *Let $I = [a, b]$ and suppose that $f: I \to I$ is continuous and onto. Further, suppose that (I, f) is irreducible between $\underline{x} = (x_0, x_1, \ldots)$ and $\underline{y} = (y_0, y_1, \ldots)$. Then if c and d are numbers, $a < c < d < b$, then there is an integer N such that if $n > N$, then $[c, d] \subset f^n([x_n, y_n])$.*

PROOF. Recall that $[x_n, y_n]$ is the smallest closed interval containing x_n and y_n.

First, notice that if $n_2 > n_1$, then $f^{n_1}([x_{n_1}, y_{n_1}]) \subset f^{n_2}([x_{n_2}, y_{n_2}])$. This is because $f^{n_2-n_1}(x_{n_2}) = x_{n_1}$ and $f^{n_2-n_1}(y_{n_2}) = y_{n_1}$. Then we have $[x_0, y_0] \subset f([x_1, y_1]) \subset f^2([x_2, y_2]) \subset \cdots$. Now if k is an integer, $k \geqslant 0$, let J_k be $\operatorname{cl}(\bigcup_{n \geqslant k} f^{n-k}([x_n, y_n]))$. Then for each k, J_k is a closed subinterval of I, and $f(J_{k+1}) = J_k$.

Now let J be the subcontinuum of (I, f) defined by $J = \{(z_0, z_1, z_2, \ldots) | z_k \in J_k$ and $f(z_{k+1}) = z_k\}$. Since $\underline{x}$ and $\underline{y}$ belong to J, and (I, f) is irreducible from $\underline{x}$ to $\underline{y}$, it follows that $J = (I, f)$. Now, since f is onto, $J_0 = I$. Then $I = \operatorname{cl}(\bigcup_{n=0}^{\infty} f^n([x_n, y_n]))$, and since $f^n([x_n, y_n]) \subset f^{n+1}([x_{n+1}, y_{n+1}])$, the conclusion follows.

LEMMA 6. *Suppose that $I = [a, b]$ and that $f: I \to I$ is continuous. If there are numbers p and q, $a < p < b$, $a < q < b$, and integers r and s such that $f^r(p) = a$, $f^s(q) = b$, then f is organic.*

PROOF. Suppose that (I, f) is irreducible between $\underline{x} = (x_0, x_1, \ldots)$ and $\underline{y} = (y_0, y_1, \ldots)$ It follows from the argument given in Lemma 5 that there is an integer

N_r such that if $n > N_r$, then $p \in f^{n-r}([x_n, y_n])$, and an integer N_s such that if $n > N_s$, then $q \in f^{n-s}([x_n, y_n])$. Then, if $n > N_r$, $a \in f^n([x_n, y_n])$, and if $n > N_s$, then $b \in f^n([x_n, y_n])$.

Now if $n > N_r + N_s$, then $I = f^n([x_n, y_n])$, and so f is organic.

THEOREM 7. *If $f\colon I \to I$ is organic, and (I, f) is indecomposable, then f has a periodic point whose period is not a power of 2.*

PROOF. Since (I, f) is indecomposable, there are three points $\underline{x} = (x_0, x_1, \ldots)$, $\underline{y} = (y_0, y_1, \ldots)$ and $\underline{z} = (z_0, z_1, \ldots)$ in (I, f), such that (I, f) is irreducible between any two of them. Because f is organic, there is a positive integer n such that $f^n([x_n, y_n]) = f^n([x_n, z_n]) = f^n([y_n, z_n]) = I$. We will assume that the notation is chosen so that $x_n < y_n < z_n$. Now since $[x_n, y_n] \subset f^n([y_n, z_n])$, there is a closed subinterval J_1 of $[y_n, z_n]$ such that $f^n(J_1) = [x_n, y_n]$. Now there is a closed subinterval J_2 of $[y_n, z_n]$ such that $f^n(J_2) = J_1$. Notice that $y_n \notin J_1 \cap J_2$. Finally, let J_3 be a closed subinterval of $[x_n, y_n]$ such that $f^n(J_3) = J_2$.

Now $J_3 \subset f^{3n}(J_3)$, and so there is a point p of J_3 such that $f^{3n}(p) = p$.

Now suppose $f^n(p) = p$. Then $p = y_n$ and $f^n(y_n) = f^{2n}(y_n) = f^{3n}(y_n) = y_n$. But then $y_n \in J_1 \cap J_2$, which is a contradiction. Thus the points p, $f^n(p)$ and $f^{2n}(p)$ are distinct.

Let s be the period of p. Then $f^{3n}(p) = p$ and it follows that 3 divides s. Therefore, s is not a power of 2. This establishes Theorem 7.

LEMMA 8. *Suppose $f\colon I \to I$ is continuous and onto. Suppose that J is a proper closed subinterval of I and, for each $n \geqslant 1$, $f^{-n}(J)$ is an interval. Then (I, f) is decomposable.*

PROOF. Let $H = \{\underline{x} \mid \underline{x} \in (I, f)$ and $\pi_0(\underline{x}) \in J\}$. Since J is proper and f is onto, H is a proper subset of (I, f). Since, for each n, $f^{-n}(J)$ is an interval, $H_n = \pi_n(H) = f^{-n}(J)$, and so H is a subcontinuum of (I, f). Now let $U = \pi_0^{-1}(\operatorname{int} J)$. Then U is open in (I, f) and $U \subset H$. Therefore H is a proper subcontinuum of (I, f) with interior, and it follows that (I, f) is decomposable.

The following lemma is well known.

LEMMA 9. *Let $f\colon I \to I$ be continuous, $h\colon I \to I$ be a homeomorphism, and $f_1 = h \circ f \circ h^{-1}$. Then (I, f) and (I, f_1) are homeomorphic.*

PROOF. Define $H\colon (I, f) \to (I, f_1)$ by $H((x_0, x_1, \ldots)) = (h(x_0), h(x_1), \ldots)$. It is clear that H is a homeomorphism.

DEFINITION. If $f\colon I \to I$ is continuous, then the statment that f has *finitely many turning points* means that there is a finite set $\{a_0, a_1, \ldots, a_l\}$, $a = a_0 < a_1 < \cdots < a_l = b$ in $I = [a, b]$ such that f is monotone on $[a_{i-1}, a_i]$ for $i = 1, 2, \ldots, l$.

THEOREM 10. *Suppose that $f\colon I \to I$ is continuous, onto, and has finitely many turning points. Then if (I, f) is indecomposable, f has a periodic point whose period is not a power of 2.*

PROOF. If f is organic, then the conclusion follows from Theorem 7. We will show that if f is inorganic, then f has a point of period 3. Suppose that f is inorganic. Then it follows from Lemma 6 that either

(i) $f^{-1}(\{a\}) = \{a\}$,

(ii) $f^{-1}(\{b\}) = \{b\}$, *or*

(iii) $f(a) = b, f(b) = a$ and $f^{-1}(\{a, b\}) = \{a, b\}$.

Now if (ii) holds, define $h\colon I \to I$ by $h(x) = (a + b) - x$, and let $f_1 = h \circ f \circ h^{-1}$. Then $f_1^{-1}(\{a\}) = \{a\}$. It follows from Lemma 8 that $(I, f) \sim (I, f_1)$, and so the hypotheses of the theorem hold for f_1.

If (iii) holds let $f_1 = f^2$. Then $f_1^{-1}(\{a\}) = \{a\}$ and, since $(I, f) \sim (I, f^2)$, the hypotheses of the theorem hold for f_1.

It follows from the preceding discussion that we may assume that $f^{-1}(\{a\}) = \{a\}$. Let the turning points of f be $a = a_0 < a_1 < \cdots < a_l = b$. We will consider two cases.

Case 1. $f^{-1}(\{b\}) = \{b\}$. Now suppose that there is a number c in (a, b) such that if $x \in [a, c]$, then $f(x) > x$. Since $f^{-1}(\{a\}) = \{a\}$, we may assume that c is chosen so that $f([c, b]) \cap [a, c] = \varnothing$. Then, for each $n \geqslant 0$, $f^{-n}([a, c])$ is an interval, and it follows from Lemma 8 that (I, f) is decomposable. Therefore for each $c \in (a, b)$ there is an $x \in (a, c)$ such that $f(x) \leqslant x$. Similarly, for each $c \in (a, b)$ there is an $x \in (c, b)$ such that $f(x) \geqslant x$. It follows that there are fixed points for f in (a, b).

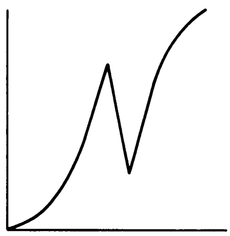
Now let $\mathscr{C} = \{c \mid a < c < b$ and $f([a, c]) = [a, c]\}$. We will show that $\mathscr{C}$ is nonempty. Let q be a point of (a, b) such that $f(q) = q$. Then either $f([a, q]) = [a, q]$ or there is a turning point a_i, $a < a_i < q$, with $f(a_i) > a_i$. But in this case there is a point q_1, $a < q_1 < a_1$, with $f(q_1) = q_1$. Since there are only finitely many turning points there is a number c_0, $a < c_0 < b$, such that $f(c_0) = c_0$, and $f([a, c_0]) = [a, c_0]$.

Now since $f([a, c_0]) = [a, c_0]$ and (I, f) is indecomposable, it follows from Lemma 8 that $f([c_0, b]) \cap [a, c_0] \neq \varnothing$. Let a_{n_0} be the smallest turning point such that $f(a_{n_0}) < c_0 < a_{n_0}$. Notice that $a_{n_0} \neq b$. Now, there is a q, $a_{n_0} < q < b$, such that $f(q) = q$. Let c_1 be the smallest such q. Now either $f([a, c_1]) = [a, c_1]$, or there is a number x, $c_0 < x < c_1$, with $f(x) > c_1$. If $x > a_{n_0}$, there is a q, $a_{n_0} < q < x$, such that $f(q) = q$, which contradicts the choice of c_1. Therefore $c_0 < x < a_{n_0}$. Again, there is a q, $x < q < a_{n_0}$, such that $f(q) = q$. Now, we have $[c_0, c_1] \subset f([c_0, q])$ and $[c_0, c_1] \subset f([q, c_1])$, and using the same argument as in the proof of Theorem 7, we have a point of period 3. Thus we have the conclusion of the theorem, or $c_1 \in \mathscr{C}$. If $c_1 \in \mathscr{C}$ we repeat the argument, replacing c_0 with c_1. Continuing this way we get $a_{n_1}, c_2, a_{n_3}, c_3, \ldots$ with $a_{n_0} < a_{n_1} < a_{n_2} < \cdots$. Since there are only finitely many turning points, the process will end in a finite number of steps with a point of period 3.

Case 2. There is a point $p \in (a, b)$ with $f(p) = b$. We proceed as in Case 1. We have $\mathscr{C} \cap (a, p) \neq \varnothing$. Choose c_0 such that $f(c_0) = c_0$ and $f([a, c_0]) = [a, c_0]$. Let a_{n_0} be as in Case 1, except now it might be that $a_{n_0} = b$. If $a_{n_0} = b$, there is a q, $p < q < b$, such that $f(q) = q$. Then $[c_0, a_{n_0}] \subset f([c_0, q])$ and $[c_0, q] \subset f([q, a_{n_0}])$. As before, it follows that f has a point of period 3. In fact, if $a_{n_0} > p$, the same result holds.

Thus we may assume that $a_{n_0} < p$. We may then find c_1, $a_{n_0} < c_1 < p$, and proceed as in Case 1. Thus f has a point of period 3.

EXAMPLE 5. Let f be as sketched in the accompanying figure. Then it can be shown that (I, f) is indecomposable and that f is inorganic. It follows from the argument in Theorem 10 that f has a point of period 3.



COROLLARY 11. *Suppose that f: $I \to I$ is continuous and has finitely many turning points. Then, there is an integer $l \geqslant 0$ and an indecomposable subcontinuum of (I, f) which is invariant under $\hat{f}^{2^l}$ if and only if f has a periodic point whose period is not a power of* 2.

PROOF. Theorem 1 shows that if f has the required periodic point, then (I, f) has the required indecomposable subcontinuum.

Suppose then that S is an indecomposable subcontinuum of (I, f) which is invariant under $\hat{f}^{2^l}$. Let $g = f^{2^l}$. Then (I, f) is homeomorphic with (I, g). Let S_1 be the image of S under the natural homeomorphism. Then S_1 is invariant under $\hat{g}$. Let $J = \pi_0(S_1)$. Then J is invariant under g, and g has finitely many turning points in J. We now apply Theorem 10 to g: $J \to J$ and find that g has a periodic point whose period is not a power of 2, and it follows that f has a periodic point whose period is not a power of 2.

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