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FIXED POINTS AND CONJUGACY CLASSES OF REGULAR ELLIPTIC ELEMENTS IN Sp(3, Z)

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ABSTRACT. In this paper, we obtain 13 isolated fixed points (up to a $\mathrm{Sp}(3,\mathbf{Z})$-equivalence) and 86 conjugacy classes of regular elliptic elements in $\mathrm{Sp}(3,\mathbf{Z})$. Hence the contributions from regular elliptic conjugacy classes in $\mathrm{Sp}(3,\mathbf{Z})$ to the dimension formula computed via the Selberg trace formula can be computed explictly by the main theorem of [**4** or **5**].

Introduction. In [**6** and **7**], E. Gottschling studied the fixed points and their isotropy groups of finite order elements in $\mathrm{Sp}(2,\mathbf{Z})$. He finally obtained six $\mathrm{Sp}(2,\mathbf{Z})$-inequivalent isolated fixed points as follows:

(1) $$Z_1 = \mathrm{diag}[i, i],$$

(2) $$Z_2 = \mathrm{diag}[\rho, \rho], \qquad \rho = e^{\pi i/3},$$

(3) $$Z_3 = \mathrm{diag}[i, \rho],$$

(4) $$Z_4 = \frac{i}{\sqrt{3}}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

(5) $$Z_5 = \begin{bmatrix} \eta & (\eta-1)/2 \\ (\eta-1)/2 & \eta \end{bmatrix}, \qquad \eta = \frac{1}{3} + \frac{2\sqrt{2}i}{3},$$

(6) $$Z_6 = \begin{bmatrix} \omega & \omega+\omega^{-2} \\ \omega+\omega^{-2} & -\omega^{-1} \end{bmatrix}, \qquad \omega = e^{2\pi i/5}.$$

The isotropy subgroups at Z_i ($i = 1,2,3,4,5,6$) are groups of order 16, 36, 12, 12, 24 and 5, respectively.

By the argument of [**9**], these fixed points can be obtained from symplectic embeddings of

$$Q(i)\oplus Q(i), \quad Q(\rho)\oplus Q(\rho), \quad Q(i)\oplus Q(\rho),$$
$$Q(e^{\pi i/6}), \quad Q(e^{\pi i/4}), \quad Q(e^{2\pi i/5}),$$

into $M_4(Q)$. In this paper, we shall combine the reduction theory of symplectic matrices [**2, 3**] with the arguments of [**8, 9**] and obtain all $\mathrm{Sp}(3,\mathbf{Z})$-inequivalent isolated fixed point and conjugacy classes of regular elliptic elements in $\mathrm{Sp}(3,\mathbf{Z})$. A table for all representatives and their centralizer in $\mathrm{Sp}(3,\mathbf{Z})/\{\pm 1\}$ of regular elliptic conjugacy classes in $\mathrm{Sp}(3,\mathbf{Z})$ is given.

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1. Notations and basic results. Let **Z**, **Q**, **R** and **C** denote the ring of integers, the fields of rational, real and complex numbers, respectively. The real symplectic matrices of degree n,

$$\mathrm{Sp}(n,\mathbf{R}) = \left\{ M \in M_{2n}(\mathbf{R}) \,|\, {}^tMJM = J, J = \begin{bmatrix} 0 & E_n \\ -E_n & 0 \end{bmatrix} \right\},$$

act on the generalized half space H_n defined by

$$H_n = \left\{ Z \in M_n(\mathbf{C}) | Z = {}^tZ, \operatorname{Im} Z > 0 \right\}.$$

Here $M_{2n}(\mathbf{R})$ is the $2n \times 2n$ matrix ring over **R**, $M_n(\mathbf{C})$ is the $n \times n$ matrix ring over **C**, E_n is the identity of $M_n(\mathbf{C})$ and ${}^tZ$ is the transpose of Z.

A point Z_0 in H_n is called an isolated fixed point of $\mathrm{Sp}(3,\mathbf{Z})$ if there exists $M = [\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]$ in $\mathrm{Sp}(3,\mathbf{Z})$ such that Z_0 is the unique solution of the equation,

$$AZ + B = Z(CZ + D), \qquad Z \in H_n.$$

An element M of $\mathrm{Sp}(3,\mathbf{Z})$ is regular elliptic if M has an isolated fixed point (see [**4**]). Now suppose M is a regular elliptic element of $\mathrm{Sp}(3,\mathbf{Z})$; then by the discreteness of $\mathrm{Sp}(3,\mathbf{Z})$ and the property that $\mathrm{Sp}(3,\mathbf{Z})$ acts transitively on H_3, we conclude that

(1) M is an element of finite order,

(2) M is conjugate in $\mathrm{Sp}(3,\mathbf{R})$ to $[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}]$ with $A + Bi = \mathrm{diag}[\lambda_1, \lambda_2, \lambda_3]$, λ_i ($i = 1,2,3$) root of unity and $\lambda_i\lambda_j \neq 1$ for all i, j,

(3) the centralizer of M in $\mathrm{Sp}(3,\mathbf{Z})$ is a group of finite order.

By property (1), we see that the minimal polynomial of M is a product of different cyclotomic polynomials as follow: $X^2 + 1$, $X^2 - X + 1$, $X^2 + X + 1$, $X^4 + 1$, $X^4 - X^2 + 1$, $X^4 + X^3 + X^2 + X + 1$, $X^4 - X^3 + X^2 - X + 1$, $X^6 - X^3 + 1$, $X^6 + X^3 + 1$, $X^6 + X^5 + X^4 + X^3 + X^2 + X + 1$, $X^6 - X^5 + X^4 - X^3 + X^2 - X + 1$.

For our convenience, we identify $\mathrm{Sp}(n_1,\mathbf{R}) \times \mathrm{Sp}(n_2,\mathbf{R})$ as a subgroup of $\mathrm{Sp}(n_1 + n_2,\mathbf{R})$ via the embedding

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \to \begin{bmatrix} A & 0 & B & 0 \\ 0 & P & 0 & Q \\ C & 0 & D & 0 \\ 0 & R & 0 & S \end{bmatrix}.$$

Also, we consider the unitary group $U(n)$ as a maximal compact subgroup of $\mathrm{Sp}(n,\mathbf{R})$ via the identification $A + Bi \to [\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}]$.

2. Reducible cases. For each regular elliptic element M in $\mathrm{Sp}(3,\mathbf{Z})$, the ring $Q(M)$ is isomorphic to a direct sum of cyclotomic fields which have degree at most 6 since M is a semisimple element. The summand must be equal to one of the following:

$$Q[e^{\pi i/2}], \quad Q[e^{2\pi i/3}], \quad Q[e^{\pi i/4}], \quad Q[e^{2\pi i/5}],$$
$$Q[e^{\pi i/6}], \quad Q[e^{2\pi i/7}], \quad Q[e^{2\pi i/9}].$$

Now suppose the characteristic polynomial $P(X)$ of M is reducible over $\mathbf{Z}[X]$; then we obtain the following ten possible fixed points for M simply from fixed points of regular elliptic elements of $SL_2(\mathbf{Z})$ and $Sp(2,\mathbf{Z})$.

1. $Z_{01} = \mathrm{diag}[i, i, i]$, 2. $Z_{02} = [\rho, \rho, \rho]$,

3. $Z_{03} = \mathrm{diag}[\rho, i, i]$, 4. $Z_{04} = [i, \rho, \rho]$,

5. $$Z_{05} = \begin{bmatrix} i & 0 & 0 \\ 0 & \eta & (\eta-1)/2 \\ 0 & (\eta-1)/2 & \eta \end{bmatrix}, \quad \eta = \frac{1}{3} + \frac{2\sqrt{2}i}{3},$$

6. $$Z_{06} = \frac{i}{\sqrt{3}} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix},$$

7. $$Z_{07} \begin{bmatrix} i & 0 & 0 \\ 0 & \omega & \omega+\omega^{-2} \\ 0 & \omega+\omega^{-2} & -\omega^{-1} \end{bmatrix}, \quad \omega = e^{2\pi i/5},$$

8. $$Z_{08} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \eta & (\eta-1)/2 \\ 0 & (\eta-1)/2 & \eta \end{bmatrix},$$

9. $$Z_{09} = \frac{i}{\sqrt{3}} = \begin{bmatrix} 1+\bar{\rho} & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix},$$

10. $$Z_{10} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \omega & \omega+\omega^{-2} \\ 0 & \omega+\omega^{-2} & -\omega^{-1} \end{bmatrix}.$$

Let G_i (i = 01, 02, 03, 04, 05, 06, 07, 08, 09, 10) be the isotropy group of $Sp(3,\mathbf{Z})/\{\pm 1\}$ at Z_i (i = 01, 02, 03, 04, 05, 06, 07, 08, 09, 10), respectively. Then a direct calculation shows that the order of G_i (i = 01, 02,...,10) are 192, 648, 96, 144, 96, 48, 20, 144, 72, 30, respectively. By considering conjugacy classes in G_i (i = 01, 02,...,10), we get 72 conjugacy classes of regular elliptic elements of $Sp(3,\mathbf{Z})$ as shown in the table.

Now we shall show that every regular elliptic element with reducible characteristic polynomial is conjugate in $Sp(3,\mathbf{Z})$ to one of these 72 conjugacy classes. First we need

LEMMA 1. *Suppose $M \in Sp(n,\mathbf{Z})$ with characteristic polynomial $P(X)$ satisfying*

(1) *$P(X)$ is a product of two relative prime polynomials $P_1(X)$ and $P_2(x)$ with integral coefficients of degrees $2n_1$ and $2n_2$ ($n_1 + n_2 = n$), respectively,*

(2) $P_i(X) = X^{2n_i}P_i(1/X)$, $i = 1, 2$.

Then there exists $R \in Sp(n, Q)$ such that $R^{-1}MR = M_1 \times M_2 \in Sp(n_1, Q) \times Sp(n_2, Q)$. Furthermore, the characteristic polynomial of M_1 (resp. M_2) is $P_1(X)$ (resp. $P_2(X)$).

PROOF. (See Lemmas 1 and 2 of [2].)

LEMMA 2. *Let $M \in \mathrm{Sp}(n, \mathbf{Z})$. Suppose that there exists $R \in \mathrm{Sp}(n, Q)$ such that*

$$R^{-1}MR = \begin{bmatrix} A & 0 & B & * \\ * & {}^tU & * & * \\ C & 0 & D & * \\ 0 & 0 & 0 & U^{-1} \end{bmatrix}.$$

Then there exists $\tilde{R} \in \mathrm{Sp}(n, \mathbf{Z})$ such that $\tilde{R}^{-1}M\tilde{R}$ has the same form as $R^{-1}MR$.

PROOF. (See Satz 2 of [3].)

THEOREM 1. *Suppose M is a regular elliptic element of $\mathrm{Sp}(3, \mathbf{Z})$ with a reducible characteristic polynomial $P(X)$. Then M is conjugate in $\mathrm{Sp}(3, \mathbf{Z})$ to an element of $\bigcup_{i=01}^{10} G_i$.*

PROOF. Here we only prove three special cases, other cases follow with similar arguments.

(1) $P(X) = (X^2 + 1)^3$. A representative of M in $U(3)$ is $\mathrm{diag}[i, i, i]$. Thus M is conjugate in $\mathrm{Sp}(3, \mathbf{R})$ to $J = [\begin{smallmatrix} 0 & E \\ -E & 0 \end{smallmatrix}]$, i.e. there exists $L \in \mathrm{Sp}(3, \mathbf{R})$ such that $M = L^{-1}JL$. With the Iwasawa decomposition of $\mathrm{Sp}(3, \mathbf{R})$, we can write

$$L = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \begin{bmatrix} U & S\,{}^tU^{-1} \\ 0 & {}^tU^{-1} \end{bmatrix}, \qquad A + Bi \in U(3).$$

Since J commutes with $[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}]$, it follows

$$M = \begin{bmatrix} U & S\,{}^tU^{-1} \\ 0 & {}^tU^{-1} \end{bmatrix}^{-1} J = \begin{bmatrix} U & S\,{}^tU^{-1} \\ 0 & {}^tU^{-1} \end{bmatrix}.$$

This forces $U, {}^tU^{-1}, S \in \mathrm{GL}(3, \mathbf{Z})$. Hence M is conjugate in $\mathrm{Sp}(3, \mathbf{Z})$ to $J = [\begin{smallmatrix} 0 & E \\ -E & 0 \end{smallmatrix}]$.

(2) $P(X) = (X^2 - X + 1)^3$. A representative of M in $U(3)$ is $\mathrm{diag}[\rho, \rho, \rho]$ or $\mathrm{diag}[\rho^2, \rho^2, \rho^2]$. On the other hand, $Q(M) \cong Q(\rho)$ as fields and the class number of $Q(\rho)$ is 1 by Theorem 11.1 in Chapter 11 of [**10**]. Hence the number of conjugacy classes of regular elliptic elements with $X^2 - X + 1$ as minimal polynomial is 2 by [**8** or **9**]. Thus M is conjugate in $\mathrm{Sp}(3, \mathbf{Z})$ to $[\begin{smallmatrix} E & -E \\ E & 0 \end{smallmatrix}]$ or $[\begin{smallmatrix} E & -E \\ E & 0 \end{smallmatrix}]^2$.

(3) $P(X) = (X^2 + 1)(X^4 + X^3 + X^2 + X + 1)$. Note that M can be represented in $U(3)$ as

$$e[1/2, 2/5, 4/5] \text{ or } e[1/2, 4/5, 8/5] \text{ or } e[1/2, 6/5, 2/5] \text{ or } e[1/2, 8/5, 6/5].$$

($e[a, b, c]$ stands for $[e^{\pi i a}, e^{\pi i b}, e^{\pi i c}]$).

In particular, M^5 can be represented in $U(3)$ as $\mathrm{diag}[i, 1, 1]$ or $[-i, 1, 1]$ and has characteristic polynomial $(X^2 + 1)(X - 1)^4$. By Lemmas 1 and 2, there exists $R \in \mathrm{Sp}(3, \mathbf{Z})$ such that

$$R^{-1}M^5R = \begin{bmatrix} 0 & 0 & 1 & * \\ * & E_2 & * & * \\ -1 & 0 & 0 & * \\ 0 & 0 & 0 & E_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & -1 & * \\ * & E_2 & * & * \\ 1 & 0 & 0 & * \\ 0 & 0 & 0 & E_2 \end{bmatrix}$$

which is conjugate in $\mathrm{Sp}(3, \mathbf{Z})$ to

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \times E_4 \quad \text{or} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \times E_4.$$

Hence we may assume

$$R^{-1}M^5R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \times E_4 \quad \text{or} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \times E_4.$$

Note that the isolated fixed point of $R^{-1}MR$ is contained in the set of fixed points of $R^{-1}M^5R$, i.e. the set

$$Z = \begin{bmatrix} i & 0 & 0 \\ 0 & z_2 & z_{23} \\ 0 & z_{23} & z_3 \end{bmatrix}, \qquad \operatorname{Im} Z > 0.$$

Now it is easy to see that the isolated fixed point of $R^{-1}MR$ is $(\mathrm{SL}_2(\mathbf{Z}) \times \mathrm{Sp}(2,\mathbf{Z}))$-equivalent to Z_{07} and M is conjugate in $\mathrm{Sp}(3,\mathbf{Z})$ to an element of G_{07}. Q.E.D.

In the sections following, we shall determine conjugacy classes of regular elliptic elements of orders 9 and 7.

3. Symplectic embeddings of $Q(e^{2\pi i/9})$ and $Q(e^{2\pi i/7})$. For our convenience, we denote $e^{2\pi i/9}$ by ζ. Note that $Q(\zeta)$ is the splitting field of the cyclotomic polynomial $X^6 + X^3 + 1$ and contains the total real number field $Q(\zeta + \zeta^{-1})$ which is the splitting field of $X^3 - 3X + 1$. By a symplectic embedding of $Q(\zeta)$ into $M_6(Q)$, we mean an injection from $Q(\zeta)$ into $M_6(Q)$ such that ζ is mapped into a symplectic matrix M and $Q(\zeta) \cong Q(M)$ as fields [9].

LEMMA 3. *Let M be an element of* $\mathrm{Sp}(3,\mathbf{Z})$ *of order* 9. *Then M is conjugate in* $\mathrm{Sp}(3,\mathbf{R})$ *to one of the following*: $[\zeta, \zeta^4, \zeta^7]$, $[\zeta, \zeta^2, \zeta^4]$, $[\zeta, \zeta^2, \zeta^5]$, $[\zeta, \zeta^5, \zeta^7]$, $[\zeta^2, \zeta^4, \zeta^8]$, $[\zeta^2, \zeta^5, \zeta^8]$, $[\zeta^4, \zeta^7, \zeta^8]$, $[\zeta^5, \zeta^7, \zeta^8]$.

PROOF. The minimal polynomial of M is $X^6 + X^3 + 1$ which can be factored into

$$\begin{aligned}(X-\zeta)&(X-\zeta^2)(X-\zeta^4)(X-\zeta^5)(X-\zeta^7)(X-\zeta^8) \\ &= \left[X^2 + (\zeta+\zeta^{-1})X + 1\right]\left[X^2 - (\zeta^2+\zeta^{-2})X + 1\right]\left[X^2 - (\zeta^4+\zeta^{-4})X + 1\right].\end{aligned}$$

Hence M is conjugate in $\mathrm{Sp}(3,\mathbf{R})$ to

$$\begin{bmatrix} \cos\theta & \pm\sin\theta \\ \mp\sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} \cos 2\theta & \pm\sin 2\theta \\ \mp\sin 2\theta & \cos 2\theta \end{bmatrix} \times \begin{bmatrix} \cos 4\theta & \pm\sin 4\theta \\ \mp\sin 4\theta & \cos 4\theta \end{bmatrix}, \qquad \theta = \frac{2\pi}{9}.$$

Note that the above eight elements of $\mathrm{Sp}(3,\mathbf{R})$ are represented by the prescribed elements in $U(3)$ as in our lemma.

LEMMA 4. *The number of conjugacy classes of regular ellliptic elements of order* 9 *in* $\mathrm{Sp}(3,\mathbf{Z})$ *is* 8.

PROOF. The ideal class number of $Q(\zeta)$ is 1 by Theorem 11.1 of [9], hence the number of conjugacy classes of regular elliptic elements of order 9 is given by $[E_0 : N(E)]$, where

E: the group of units in $Q(\zeta)$,

E_0: the group of units in $Q(\zeta + \zeta^{-1})$,

$N(E) = \{u\bar{u} \mid u \in E\}$,

according to the argument of [8 or 9].

The group of units for cyclotomic fields is determined in Chapter 8 of [10]. Applying this to our case, we get $[E_0 : N(E)] = 8$ when the cyclotomic field is $Q(\zeta)$.

There are two conjugacy classes of elements of order 9 appearing in the isotropy group G_{02} of $Z_{02} = \operatorname{diag}[\rho, \rho, \rho]$. Indeed, if we let $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ with

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then it is a direct verification to show that

(1) M is an element of order 9.

(2) M can be represented in $U(3)$ as $[\zeta, \zeta^4, \zeta^7]$ or $[\zeta^2, \zeta^8, \zeta^5]$,

(3) $M^3 = \left[\begin{smallmatrix} 0 & -E \\ E & -E \end{smallmatrix}\right]$ has an isolated fixed point at Z_{02}.

Now we begin to look for the other six conjugacy classes of regular elliptic elements of order 9 in $\mathrm{Sp}(3, \mathbf{Z})$.

THEOREM 2. *Suppose α, β, γ are distinct roots of the equation $X^3 - 3X + 1 = 0$ (or more precisely, $\alpha = 2\cos 2\pi/9$, $\beta = 2\cos 4\pi/9$, $\gamma = 2\cos 8\pi/9$),*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \Omega = \frac{1}{3}\begin{bmatrix} -3 + \alpha + \alpha^2 & -3 + \beta + \beta^2 & -3 + \gamma + \gamma^2 \\ -1 + \alpha^2 & -1 + \beta^2 & -1 + \gamma^2 \\ 1 + \alpha & 1 + \beta & 1 + \gamma \end{bmatrix},$$

and

$$M = \begin{bmatrix} A & E \\ -E & 0 \end{bmatrix},$$

then

(1) *M is an element of order 9 in $\mathrm{Sp}(3, \mathbf{Z})$ and has an isolated fixed point at*

$$Z_{11} = -\tfrac{1}{2}A + i\Omega\left(E - \tfrac{1}{4}{}^t\Omega A^2\Omega\right)^{1/2}{}^t\Omega,$$

(2) *M is conjugate in $\mathrm{Sp}(3, \mathbf{R})$ to $[\zeta, \zeta^2, \zeta^4]$ of $U(3)$,*

(3) *the centralizer of M in $\mathrm{Sp}(3, \mathbf{Z})/\{\pm 1\}$ is a group of order 9.*

PROOF. (1) Since the characteristic polynomial of M is $X^6 + X^3 + 1$, it follows that M is an element of order 9 in $\mathrm{Sp}(3, \mathbf{Z})$. Note that $\frac{1}{3}{}^t[-3 + \alpha + \alpha^2, -1 + \alpha^2, 1 + \alpha]$ is the normalized eigenvector of A corresponding to the eigenvalue α. It follows that ${}^t\Omega A \Omega = \operatorname{diag}[\alpha, \beta, \gamma]$ and

$$\left(E - \tfrac{1}{4}{}^t\Omega A \Omega\right)^{1/2} = \operatorname{diag}\left[\left(1 - \alpha^2/4\right)^{1/2}, \left(1 - \beta^2/4\right)^{1/2}, \left(1 - \gamma^2/4\right)^{1/2}\right].$$

Now it is a direct verification to show that $AZ_{11} = Z_{11}A$ and $Z_{11}^2 + AZ_{11} + E = 0$. Thus $Z_{11} = -\frac{1}{2}A + i\Omega(E - \frac{1}{4}{}^t\Omega A^2\Omega)^{1/2}{}^t\Omega$ is a fixed point of M. But M has exactly one fixed point by Lemma 3, hence Z_{11} is the unique isolated fixed point of M.

(2) Let $R = \left[\begin{smallmatrix} \Omega & 0 \\ 0 & \Omega \end{smallmatrix}\right]$. Then $R \in \mathrm{Sp}(3, \mathbf{R})$ and

$$R^{-1}MR = \begin{bmatrix} \alpha & 1 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} \beta & 1 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} \gamma & 1 \\ -1 & 0 \end{bmatrix}.$$

Note that $R^{-1}MR$ is conjugate in $\mathrm{Sp}(3, \mathbf{R})$ to

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} \times \begin{bmatrix} \cos 3\theta & \sin 3\theta \\ -\sin 3\theta & \cos 3\theta \end{bmatrix}, \quad \theta = \frac{2\pi}{9},$$

because $\left[\begin{smallmatrix} 2\cos\mu & 1 \\ -1 & 0 \end{smallmatrix}\right]$ is conjugate in $SL_2(\mathbf{R})$ to $\left[\begin{smallmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{smallmatrix}\right]$. This proves our assertion in (2).

(3) Let $C(M, \mathbf{Z})$ be the centralizer of M in $\mathrm{Sp}(3,\mathbf{Z})/\{\pm 1\}$. Suppose γ is an element of $C(M, \mathbf{Z})$. Then

$$M(\gamma(Z_{11})) = \gamma(M(Z_{11})) = \gamma(Z_{11}).$$

Since Z_{11} is the only fixed point of M, this forces $\gamma(Z_{11}) = Z_{11}$.

Note that ${}^t\Omega Z_{11}\Omega = R(Z_{11}) = \mathrm{diag}[-\bar{\zeta}, -\bar{\zeta}^2, -\bar{\zeta}^4]$. Here $R = \left[\begin{smallmatrix} \Omega & 0 \\ 0 & \Omega \end{smallmatrix}\right]$ as in (2). From $\gamma(Z_{11}) = Z_{11}$, we get

$$R\gamma R^{-1}\left({}^t\Omega Z_{11}\Omega\right) = {}^t\Omega Z_{11}\Omega.$$

It follows that

$$R\gamma R^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \times \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix}$$

with

$$\begin{cases} -a\bar{\zeta} + b = c\bar{\zeta}^2 - d\bar{\zeta}, & ad - bc = 1, \\ -a'\bar{\zeta}^2 + b' = c'\bar{\zeta}^4 - d'\bar{\zeta}^2, & a'd' - b'c' = 1, \\ -a''\bar{\zeta}^4 + b'' = c''\bar{\zeta}^8 - d''\bar{\zeta}^4, & a''d'' - b''c'' = 1. \end{cases}$$

The general solution of a, b, c, d is given by

$$\begin{cases} a = \cos\theta - \cot\dfrac{2\pi}{9}\sin\theta, \quad b = -\sec\dfrac{2\pi}{9}\sin\theta, \\ \\ c = \sec\dfrac{2\pi}{9}\sin\theta, \quad d = \cos\theta + \cot\dfrac{2\pi}{9}\sin\theta, \end{cases} \qquad \theta \in \mathbf{R}.$$

The characteristic polynomial of $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is $X^2 - 2\cos\theta X + 1$, hence $2\cos\theta$ is an algebraic integer of degree 1 or 3. On the other hand, the fact that γ is an element of finite order implies $e^{i\theta}$ is a root of unity. Now we have the following cases:

Case I. If $2\cos\theta$ is an algebraic integer of degree 3, then the characteristic polynomial of γ is an irreducible polynomial of degree 6. Since M satisfies this case, the characteristic polynomial of γ is $X^6 + X^3 + 1$ or $X^6 - X^3 + 1$. This leads to the fact that $\theta = 2\pi/9$ or $4\pi/9$ or $8\pi/9$ and γ is one of the following elements: $\pm M$, $\pm M^2$, $\pm M^4$, $\pm M^5$, $\pm M^7$, $\pm M^8$.

Case II. If $2\cos\theta = \pm 1$, then γ is an element of order 3. Then $\gamma = \pm M^3$ or $\pm M^6$ by a direct calculation.

Case III. If $2\cos\theta = \pm 2$, then $\gamma = \pm E_6$.

Case IV. If $2\cos\theta = 0$, then

$$\gamma = R^{-1}\left\{\begin{bmatrix} -\cot\eta & -\sec\eta \\ \sec\eta & \cot\eta \end{bmatrix} \times \begin{bmatrix} -\cot 2\eta & -\sec 2\eta \\ \sec 2\eta & \cot 2\eta \end{bmatrix} \times \begin{bmatrix} -\cot 4\eta & -\sec 4\eta \\ \sec 4\eta & \cot 4\eta \end{bmatrix}\right\} R$$

with $\eta = 2\pi/9$. Such a γ is not an integral matrix.

By the above discussion, we conclude $C(M, \mathbf{Z})$ is a group of order 9 generated by M.

TABLE. Regular elliptic conjugacy classes of $\mathrm{Sp}(3,\mathbf{Z})$

Here $e[a,b,c]$ stands for $\mathrm{diag}[e^{\pi i a}, e^{\pi i b}, e^{\pi i c}]$, $P_1(X) = X^4 + X^3 + X^2 + X + 1$ and $P_2(X) = X^6 + X^5 + X^4 + X^3 + X^2 + X + 1$.

No.	Representative in $U(3)$	Minimal polynomial	Order of centralizer	No. of conjugates in isotropy group
1	$e[1/2,1/2,1/2]$	X^2+1	192	1
2	$e[1/2,1/4,5/4]$	$(X^2+1)(X^4+1)$	16	12
3	$e[1/2,3/4,7/4]$	$(X^2+1)(X^4+1)$	16	12
4	$e[1/6,5/6,9/6]$	X^6+1	6	32
5	$e[1/3,1/3,1/3]$	X^2-X+1	648	1
6	$e[2/3,1/3,1/3]$	X^4+X^2+1	216	3
7	$e[4/3,1/3,1/3]$	X^4+X^2+1	216	3
8	$e[2/3,2/3,1/3]$	X^4+X^2+1	216	3
9	$e[2/3,2/3,2/3]$	X^2+X+1	648	1
10	$e[5/3,2/3,2/3]$	X^4+X^2+1	216	3
11	$e[1/3,1/3,4/3]$	X^4+X^2+1	36	18
12	$e[2/3,2/3,5/3]$	X^4+X^2+1	36	18
13	$e[1/3,1/6,7/6]$	X^4+X^2+1	36	18
14	$e[2/3,1/6,7/6]$	X^4+X^2+1	36	18
15	$e[1/3,5/6,11/6]$	X^4+X^2+1	36	18
16	$e[2/3,5/6,11/6]$	X^4+X^2+1	36	18
17	$e[2/9,8/9,14/9]$	X^6+X^3+1	9	72
18	$e[4/9,10/9,16/9]$	X^6+X^3+1	9	72
19	$e[1/3,1/2,1/2]$	$(X^2-X+1)(X^2+1)$	96	1
20	$e[2/3,1/2,1/2]$	$(X^2+X+1)(X^2+1)$	96	1
21	$e[4/3,1/2,1/2]$	$(X^2+X+1)(X^2+1)$	96	1
22	$e[5/3,1/2,1/2]$	$(X^2-X+1)(X^2+1)$	96	1
23	$e[1/3,1/4,5/4]$	$(X^2-X+1)(X^4+1)$	24	4
24	$e[2/3,1/4,5/4]$	$(X^2+X+1)(X^4+1)$	24	4
25	$e[1/3,3/4,7/4]$	$(X^2-X+1)(X^4+1)$	24	4
26	$e[2/3,3/4,7/4]$	$(X^2+X+1)(X^4+1)$	24	4
27	$e[1/2,1/3,1/3]$	$(X^2+1)(X^2-X+1)$	144	1
28	$e[3/2,1/3,1/3]$	$(X^2+1)(X^2-X+1)$	144	1
29	$e[1/2,2/3,2/3]$	$(X^2+1)(X^2-X+1)$	144	1
30	$e[3/2,2/3,2/3]$	$(X^2+1)(X^2+X+1)$	144	1
31	$e[1/2,2/3,1/3]$	$(X^2+1)(X^4+X^2+1)$	72	2
32	$e[3/2,2/3,1/3]$	$(X^2+1)(X^4+X^2+1)$	72	2
33	$e[1/2,4/3,1/3]$	$(X^2+1)(X^4+X^2+1)$	72	2
34	$e[1/2,5/3,2/3]$	$(X^2+1)(X^4+X^2+1)$	72	2
35	$e[1/2,1/6,7/6]$	$(X^2+1)(X^4-X^2+1)$	24	6
36	$e[1/2,1/3,4/3]$	$(X^2+1)(X^4+X^2+1)$	24	6
37	$e[1/2,5/6,11/6]$	$(X^2+1)(X^4-X^2+1)$	24	6
38	$e[1/2,2/3,5/3]$	$(X^2+1)(X^4+X^2+1)$	24	6
39	$e[1/2,1/4,3/4]$	$(X^2+1)(X^4+1)$	16	6
40	$e[1/2,5/4,7/4]$	$(X^2+1)(X^4+1)$	16	6
41	$e[1/2,1/3,2/3]$	$(X^2+1)(X^4+X^2+1)$	24	2
42	$e[3/2,1/3,2/3]$	$(X^2+1)(X^2-X+1)$	24	2

TABLE (continued)

No.	Representative in $U(3)$	Minimal polynomial	Order of centralizer	No. of conjugates in isotropy group
43	$e[1/2,2/5,4/5]$	$(X^2+1)P_1(X)$	20	1
44	$e[1/2,4/5,8/5]$	$(X^2+1)P_1(X)$	20	1
45	$e[1/2,6/5,2/5]$	$(X^2+1)P_1(X)$	20	1
46	$e[1/2,8/5,6/5]$	$(X^2+1)P_1(X)$	20	1
47	$e[3/2,2/5,4/5]$	$(X^2+1)P_1(-X)$	20	1
48	$e[3/2,4/5,8/5]$	$(X^2+1)P_1(-X)$	20	1
49	$e[3/2,6/5,2/5]$	$(X^2+1)P_1(-X)$	20	1
50	$e[3/2,8/5,6/5]$	$(X^2+1)P_1(-X)$	20	1
51	$e[1/3,1/4,3/4]$	$(X^2-X+1)(X^4+1)$	24	6
52	$e[2/3,1/4,3/4]$	$(X^2-X+1)(X^4+1)$	24	6
53	$e[4/3,1/4,3/4]$	$(X^2+X+1)(X^4+1)$	24	6
54	$e[5/3,1/4,3/4]$	$(X^2+X+1)(X^4+1)$	24	6
55	$e[1/3,1/3,2/3]$	X^4+X^2+1	36	4
56	$e[2/3,1/3,2/3]$	X^4+X^2+1	36	4
57	$e[1/3,2/5,4/5]$	$(X^2-X+1)P_1(X)$	30	1
58	$e[2/3,2/5,4/5]$	$(X^2+X+1)P_1(X)$	30	1
59	$e[4/3,2/5,4/5]$	$(X^2+X+1)P_1(X)$	30	1
60	$e[5/3,2/5,4/5]$	$(X^2-X+1)P_1(X)$	30	1
61	$e[1/3,4/5,8/5]$	$(X^2-X+1)P_1(X)$	30	1
62	$e[2/3,4/5,8/5]$	$(X^2+X+1)P_1(X)$	30	1
63	$e[4/3,4/5,8/5]$	$(X^2+X+1)P_1(X)$	30	1
64	$e[5/3,4/5,8/5]$	$(X^2-X+1)P_1(X)$	30	1
65	$e[1/3,6/5,2/5]$	$(X^2-X+1)P_1(X)$	30	1
66	$e[2/3,6/5,2/5]$	$(X^2+X+1)P_1(X)$	30	1
67	$e[4/3,6/5,2/5]$	$(X^2+X+1)P_1(X)$	30	1
68	$e[5/3,6/5,2/5]$	$(X^2-X+1)P_1(X)$	30	1
69	$e[1/3,8/5,6/5]$	$(X^2-X+1)P_1(X)$	30	1
70	$e[2/3,8/5,6/5]$	$(X^2+X+1)P_1(X)$	30	1
71	$e[4/3,8/5,6/5]$	$(X^2+X+1)P_1(X)$	30	1
72	$e[5/3,8/5,6/5]$	$(X^2-X+1)P_1(X)$	30	1
73	$e[2/9,4/9,8/9]$	X^6+X^3+1	9	1
74	$e[4/9,8/9,16/9]$	X^6+X^3+1	9	1
75	$e[8/9,16/9,14/9]$	X^6+X^3+1	9	1
76	$e[10/9,2/9,4/9]$	X^6+X^3+1	9	1
77	$e[14/9,10/9,2/9]$	X^6+X^3+1	9	1
78	$e[16/9,14/9,10/9]$	X^6+X^3+1	9	1
79	$e[2/7,4/7,6/7]$	$P_2(X)$	7	1
80	$e[4/7,8/7,12/7]$	$P_2(X)$	7	1
81	$e[6/7,12/7,4/7]$	$P_2(X)$	7	1
82	$e[8/7,2/7,10/7]$	$P_2(X)$	7	1
83	$e[10/7,6/7,2/7]$	$P_2(X)$	7	1
84	$e[12/7,10/7,8/7]$	$P_2(X)$	7	1
85	$e[2/7,4/7,8/7]$	$P_2(X)$	7	3
86	$e[6/7,12/7,10/7]$	$P_2(X)$	7	3

With the same argument, we get the following result by simply replacing the role of $e^{2\pi i/9}$ by $e^{2\pi i/7}$.

LEMMA 5. *Let M be an element of* $\mathrm{Sp}(3, \mathbf{Z})$ *of order* 7. *Then M is conjugate in* $\mathrm{Sp}(3, \mathbf{R})$ *to one of the following* ($v = e^{2\pi i/7}$):

$$[v, v^2, v^3], \quad [v, v^2, v^4], \quad [v, v^4, v^5], \quad [v, v^3, v^5],$$
$$[v^2, v^3, v^6], \quad [v^2, v^4, v^6], \quad [v^4, v^5, v^6], \quad [v^3, v^5, v^6].$$

LEMMA 6. *The number of conjugacy classes of regular elliptic elements of order* 7 *in* $\mathrm{Sp}(3, \mathbf{Z})$ *is* 8.

THEOREM 3. *Suppose α, β, γ are distinct roots of the equation* $X^3 + X^2 - 2X + 1$ (*or more precisely*, $\alpha = 2\cos 2\pi/7$, $\beta = 2\cos 4\pi/7$, $\gamma = 2\cos 6\pi/7$),

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \quad \Omega' = \begin{bmatrix} \dfrac{\alpha + \alpha^2}{1 + 3\alpha} & \dfrac{\beta + \beta^2}{1 + 3\beta} & \dfrac{\gamma + \gamma^2}{1 + 3\gamma} \\ \dfrac{1 + 2\alpha}{1 + 3\alpha} & \dfrac{1 + 2\beta}{1 + 3\beta} & \dfrac{1 + 2\gamma}{1 + 3\gamma} \\ \dfrac{\alpha^2}{1 + 3\alpha} & \dfrac{\beta^2}{1 + 3\beta} & \dfrac{\gamma^2}{1 + 3\gamma} \end{bmatrix}$$

and

$$M = \begin{bmatrix} B & E \\ -E & 0 \end{bmatrix}.$$

Then

(1) *M is an element of order* 7 *in* $\mathrm{Sp}(3, \mathbf{Z})$ *and has an isolated fixed point at*

$$Z_{12} = -\tfrac{1}{2}B + i\Omega'\left(E - \tfrac{1}{4}{}^t\Omega' B^2 \Omega'\right)^{1/2} {}^t\Omega',$$

(2) *M is conjugate in* $\mathrm{Sp}(3, \mathbf{R})$ *to* $[v, v^2, v^3]$,

(3) *the centralizer of M in* $\mathrm{Sp}(3, \mathbf{Z})/\{\pm 1\}$ *is a group of order* 7 *generated by M.*

Note that $[v, v^2, v^4]$ and $[v^3, v^6, v^5]$ are exclusive in the set of all powers of $[v, v^2, v^3]$. To find all representatives for elliptic conjugacy classes of order 7, it suffices to get a representative which is conjugate in $\mathrm{Sp}(3, \mathbf{R})$ to $[v, v^2, v^4]$.

THEOREM 4. *Let B, Ω' be matrices as in Theorem* 3,

$$U = \mathrm{diag}[1, 1, -1] \quad \textit{and} \quad M = \begin{bmatrix} B & E + B \\ -(E + B)^{-1} & 0 \end{bmatrix}.$$

Then

(1) *M is an element of order* 7 *in* $\mathrm{Sp}(3, \mathbf{Z})$ *with isolated fixed point at*

$$Z_{13} = -\tfrac{1}{2}B(B + E) + i\Omega'\left[\left(E - \tfrac{1}{4}{}^t\Omega' B^2 \Omega'\right)^{1/2} {}^t\Omega'(B + E)\Omega U\right]^t \Omega',$$

(2) *M is conjugate in* $\mathrm{Sp}(3, \mathbf{R})$ *to* $[v, v^2, v^4]$,

(3) *the centralizer of M in* $\mathrm{Sp}(3, \mathbf{Z})/\{\pm 1\}$ *is a finite group of order* 7 *generated by M.*

PROOF. Since $\det(E + B) = 1$ and

$$'\Omega'(E+B)\Omega' = \operatorname{diag}\left[1 + 2\cos\frac{2\pi}{7},\, 1 + 2\cos\frac{4\pi}{7},\, 1 + 2\cos\frac{6\pi}{7}\right]$$

has signature $+, +, -$, it follows that $M \in \mathrm{Sp}(3, \mathbf{Z})$ and M is conjugate in $\mathrm{Sp}(3, \mathbf{R})$ to

$$M' = \begin{bmatrix} 2\cos\theta & 1 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} 2\cos 2\theta & 1 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} 2\cos 3\theta & -1 \\ 1 & 0 \end{bmatrix}, \qquad \theta = \frac{2\pi}{7}.$$

Indeed, if we let $R' = \Omega'\Lambda$ with

$$\Lambda = \operatorname{diag}\left[\left(1 + 2\cos\frac{2\pi}{7}\right)^{-1/2}, \left(1 + 2\cos\frac{2\pi}{7}\right)^{-1/2}, \left(-1 - 2\cos\frac{2\pi}{7}\right)^{-1/2}\right],$$

then $(R')^{-1}MR' = M'$. Hence (1) and (2) follow as a direct calculation. By a similar argument as in (3) of Theorem 2, we get (2).

By Theorems 1, 2, 3 and 4, we obtain the following table for conjugacy classes of regular elliptic elements in $\mathrm{Sp}(3, \mathbf{Z})$.

4. Application. Contributions from conjugacy classes of regular elliptic elements in $\mathrm{Sp}(n, \mathbf{Z})$ to the dimension formula for Siegel cusp forms of degree n and weight k **[4]** are given by

$$\sum |C(M, \mathbf{Z})|^{-1} \prod_{i=1}^{n} \bar{\lambda}_i^k \prod_{i \le j} (1 - \bar{\lambda}_i \bar{\lambda}_j)^{-1}.$$

Here the summation in M ranges over all conjugacy classes of regular elliptic elements in $\mathrm{Sp}(n, \mathbf{Z})$. M is conjugate in $\mathrm{Sp}(n, \mathbf{R})$ to $\left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right]$ with $A + Bi = \operatorname{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$, $\lambda_i\lambda_j \ne 1$ for all i, j and $C(M, \mathbf{Z})$ is the centralizer of M in $\mathrm{Sp}(3, \mathbf{Z})$. Applying this formula to the case $n = 3$, we get all contributions from 86 regular elliptic conjugacy classes in $\mathrm{Sp}(3, \mathbf{Z})$.

For the case $n = 1$ and $n = 2$, the contribution from a particular regular elliptic conjugacy class appears to be a residue of a generating function at a simple pole. For example, the contribution from the conjugacy class of regular elliptic elements of order 5 in $\mathrm{Sp}(2, \mathbf{Z})$ is given by

$$K = \tfrac{1}{25}\left[\omega^{-6k}(1 - \omega^{-2}) + \omega^{-2k}(1 - \omega^{-4}) + \omega^{-8k}(1 - \omega^{-6}) + \omega^{-4k}(1 - \omega^{-8})\right], \qquad \omega = e^{\pi i/5},$$

which is precisely the negative of the sum of residues of the function

$$\frac{1}{(1 - T^4)(1 - T^6)(1 - T^{10})(1 - T^{12})T^{k+1}}$$

at $T = e^{i\theta}$ with $\theta = \pm\pi/5, \pm 2\pi/5, \pm 3\pi/5, \pm 4\pi/5$ when k is even.

It is easy to see that the total contribution from conjugacy classes of elements of order 2 or 3 in $\mathrm{SL}_2(\mathbf{Z})$ is the negative of the sum of residues of the function

$$\frac{1}{(1 - T^4)(1 - T^6)T^{k+1}}$$

at $T = e^{i\theta}$ with $\theta = \pm\pi/2, \pm\pi/3, \pm 2\pi/3$ when k is even.

Note that

$$\frac{1}{(1-T^4)(1-T^6)} \quad \text{and} \quad \frac{1}{(1-T^4)(1-T^6)(1-T^{10})(1-T^{12})}$$

are well known to be generating functions of dimension formulas for modular forms of degree 1 and degree 2, respectively. It is hopeful to find a generating function of a dimension formula for modular forms of degree 3 by computing contributions from conjugacy classes of regular elliptic elements in $\mathrm{Sp}(3, \mathbf{Z})$. However, we can write down explictly the conjugacy classes of $\mathrm{Sp}(3, \mathbf{Z})$ simply by using our results in this paper and reduction theory in **[2, 3]**. Thus a dimension formula for Siegel cusps forms of degree 3 can be obtained by the Selberg trace formula and results of **[5]**.

REFERENCES

1. Z. I. Borevich and I. R. Shafarevich, *Number theory*, Academic Press, New York, 1966.
2. U. Christian, *A reduction theory for symplectic matrices*, Math. Z. **101** (1967), 213–244.
3. ______, *Zur Theorie der symplektischen Gruppen*, Acta Arith. **24** (1973), 61–85.
4. Minking Eie, *Contributions from conjugacy classes of regular elliptic elements in* $\mathrm{Sp}(n, \mathbf{Z})$ *to the dimension formula*, Trans. Amer. Math. Soc. **285** (1984), 403–410.
5. ______, *Siegel cusp forms of degree two and three*, Mem. Amer. Math. Soc. (to appear).
6. E. Gottschling, *Über die Fixpunkte der Siegelschen Modulgruppe*, Math. Ann. **143** (1961), 111–149.
7. ______, *Über die Fixpunktuntergruppen der Siegelschen Modulgruppe*, Math. Ann. **143** (1961), 399–430.
8. H. Midorikawa, *On the number of regular elliptic conjugacy classes in the Siegel modular group of degree* $2n$, Tokyo J. Math. **6** (1983), 25–28.
9. B. Steinle, *Fixpunktmannigfaltigkeiten symplektischen Matrizen*, Acta Arith. **20** (1972), 63–106.
10. Lawrence C. Washington, *Introduction to cyclotomic fields*, Springer-Verlag, New York, 1982.

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