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SUBSPACES OF BMO($\mathbf{R}^n$)

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ABSTRACT. We consider subspaces of BMO($\mathbf{R}^n$) generated by one singular integral transform. We show that the averages along x_j-lines of the jth Riesz transform of $g \in \mathrm{BMO} \cap L^2(\mathbf{R}^n)$ or $g \in L^\infty(\mathbf{R}^n)$ satisfy a certain strong regularity property. One consquence of this result is that such functions satisfy a uniform doubling condition on a.e. x_j-line. We give an example to show, however, that the restrictions to x_j-lines of the Riesz transform of $g \in \mathrm{BMO} \cap L^2(\mathbf{R}^n)$ do not necessarily have uniformly bounded BMO norm. Also, for a Calderón-Zygmund singular integral operator K with real and odd kernel, we show that $K(\mathrm{BMO}_c) \subseteq \overline{L^\infty + K(L_c^\infty)}$, where L_c^∞ and BMO_c are the spaces of L^∞ or BMO functions of compact support, respectively, and the closure is taken in BMO norm.

1. Introduction. Let BMO($\mathbf{R}^n$) be the set of complex-valued functions $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ such that

$$\|f\|_{\mathrm{BMO}} = \sup_Q \frac{1}{|Q|} \int_Q |f - f_Q| < \infty,$$

where the sup is taken over all cubes Q in $\mathbf{R}^n$ with sides parallel to the coordinate axes, $f_Q = (1/|Q|)\int_Q f$, and $|Q|$ is the Lebesgue measure of Q. For us, a singular integral operator K with kernel Ω is an operator of the form

$$Kf(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy,$$

where $\Omega_{S^{n-1}} \in C^\infty$, $\Omega(rx) = \Omega(x)$ for $r > 0$, and $\int_{S^{n-1}} \Omega(x)\, d\sigma(x) = 0$ for surface measure $d\sigma$ on $S^{n-1} = \{x \in \mathbf{R}^n\colon |x| = 1\}$. For $f \in \bigcup_{1 \leqslant p < \infty} L^p$, $Kf(x)$ exists a.e. (dx), and K is a bounded operator on L^p for $1 < p < \infty$. Associated with K is the Fourier multiplier m so that for $f \in L^2(\mathbf{R}^n)$, $(Kf)\hat{}(\xi) = m(\xi)\hat{f}(\xi)$, where the Fourier transform is defined by $\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{2\pi i x\cdot\xi}\, dx$; m satisfies $m(r\xi) = m(\xi)$ for $r > 0$ and $\xi \in \mathbf{R}^n - \{0\}$; and $m|_{S^{n-1}} \in C^\infty$. For $f \in L^\infty(\mathbf{R}^n)$, we define

$$\tilde{K}f(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \left(\frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(-y)}{|y|^n} \chi_{\{y:|y|>1\}}(y) \right) f(y)\, dy,$$

where χ_E denotes the characteristic function of the set E. For $f \in L^p \cap L^\infty$, $1 \leqslant p < \infty$, Kf and $\tilde{K}f$ differ only by a constant. The operator $\tilde{K}$ maps L^∞ into

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BMO boundedly. The jth Riesz transform R_j, $j \in \{1,\dots,n\}$, is the singular integral operator with kernel $\Omega_j(x) = C_n x_j/|x|$ and multiplier $m_j(\xi) = i\xi_j/|\xi|$ [**7**, p. 58].

Let $H^1(\mathbf{R}^n)$ be the set of $f \in L^1(\mathbf{R}^n)$ such that

$$\|f\|_{H^1} = \|f\|_{L^1} + \sum_{j=1}^{n} \|R_j f\|_{L^1} < \infty.$$

In [**1**], Fefferman and Stein proved that $(H^1)^* = \mathrm{BMO}$ (in an appropriate sense), and obtained by duality as a corollary that $\mathrm{BMO} = L^\infty + \sum_{j=1}^n \tilde{R}_j L^\infty$, with

$$\|f\|_{\mathrm{BMO}} \approx \inf\left\{ \sum_{j=0}^{n} \|g_j\|_{L^\infty} : f = g_0 + \sum_{j=1}^{n} \tilde{R}_j g_j, \text{ modulo constants} \right\}, \tag{1.1}$$

where $\approx$ means that the norms are equivalent. In [**8**], Uchiyama obtained (1.1) constructively for f of compact support, and proved that a certain condition on the multipliers of a set of singular integral operators $K_1,\dots,K_n$ implies that $\mathrm{BMO} = \sum_{j=1}^n \tilde{K}_j L^\infty$ with an appropriate equivalence of norms. This condition had been proven necessary by Janson [**5**].

Using in part techniques developed by Uchiyama in [**8**], we consider subspaces of BMO generated by one singular integral operator. The Riesz transforms are the most natural to consider in this context since one may expect that functions of the form $\tilde{R}_j g$ for $g \in L^\infty$, or $R_j g$ for $g \in \mathrm{BMO} \cap L^2$, when restricted to lines in the x_j direction, have regularity properties that functions in $\mathrm{BMO}(\mathbf{R}^n)$ do not necessarily have. Results of this type are proved in §3, Theorem 3.2 and §4, Corollary 4.3. (In §2, we state Uchiyama's results from [**8**] that are needed later.) Theorem 3.2 is obtained from Lemma 3.1, which characterizes $R_j(\mathrm{BMO} \cap L^2)$ in terms of a decomposition closely related to Uchiyama's decomposition of $\mathrm{BMO} \cap L^2$ [**8**, p. 224]. One direction of the characterization follows easily from Uchiyama's work; the other direction is obtained by solving the equation $R_j\varphi = b$ for φ under appropriate assumptions on b.

Corollaries of Theorem 3.2, proved in §4, state that if $f = R_j g$, $g \in \mathrm{BMO} \cap L^2$, or $f = \tilde{R}_j g$, $g \in L^\infty$, then f has a uniform doubling property on x_j-lines; namely

$$\left| \frac{1}{|I|} \int_I f(x_1,\dots,x_n)\, dx_j - \frac{1}{|J|} \int_J f(x_1,\dots,x_n)\, dx_j \right| \leq c\|g\|_{\mathrm{BMO}} \tag{1.2}$$

for almost all $(x_1,\dots,x_{j-1}, x_{j+1},\dots,x_n) \in \mathbf{R}^{n-1}$, whenever I and J are adjacent intervals in $\mathbf{R}$ of the same length. An example of Kahane [**6**] from another context shows that the uniform doubling property for a function f on x_j-lines is weaker than the uniform BMO property on x_j-lines. Both conditions restrict the singularities of f along x_j-lines to be at most logarithmic, but the BMO condition further controls the packing of singularities. Kahane's example is sketched in §4; it is then modified to show that $\exists G \in \mathrm{BMO} \cap L^2(\mathbf{R}^2)$ such that $\operatorname{ess\,sup}_{x_2} \|R_1 G(x_1, x_2)\|_{\mathrm{BMO}(x_1)} = \infty$. Hence (1.2) cannot be improved to obtain a uniform BMO condition on lines for the class $R_1(\mathrm{BMO} \cap L^2)$.

In §5 we show that if K is a singular integral operator with a real and odd kernel (in particular if K is a Riesz transform), then

$$K(\mathrm{BMO}_c) \subseteq \overline{L^\infty + K(L_c^\infty)}, \tag{1.3}$$

where L^∞_c and BMO_c are the spaces of functions of compact support in L^∞ and BMO, respectively, and the closure is taken in BMO norm. The proof is an adaptation of Uchiyama's main construction in **[8]**. This suggests that the action of K on BMO is in some sense close to the action of K on L^∞.

Finally, in §6 we make some remarks about related problems which are not yet solved.

NOTATION. The letter c denotes various constants depending possibly on n and, in §5, on K. All cubes Q, I, or J are assumed to be open and to have sides parallel to the coordinate axes; x_Q and $l(Q)$ denote the center and side length of Q, respectively. For $r > 0$, rQ is the cube with sides parallel to the axes having center x_Q and side length $rl(Q)$. For $j \in \{1, \ldots, n\}$, D_{x_j} is the derivative with respect to x_j, and for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, D^α is the differential operator $\partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For f continuous on $\mathbf{R}^n$, let

$$\|f\|_{\mathrm{Lip}\,1} = \sup_{x \neq y} (|f(x) - f(y)|/|x - y|),$$

and let $C^1(\mathbf{R}^n)$ be the space of continuously differentiable functions on $\mathbf{R}^n$. For $x, y \in \mathbf{R}^k$, $x \cdot y = \sum_{j=1}^k x_j y_j$.

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2. Background. We state several lemmas from Uchiyama **[8]** that will be used later. In fact, the following Lemmas U1–U6 are Lemmas 3.1–3.6, respectively, in **[8]**.

LEMMA U1. *Suppose $f \in \mathrm{BMO} \cap L^2(\mathbf{R}^n)$ and $\|f\|_{\mathrm{BMO}} \leqslant 1$. Then there exist functions $\{b_I(x)\}_I$ and $\{\lambda_I\}_I$, $\lambda_I \geqslant 0$, where I is taken over all dyadic cubes in $\mathbf{R}^n$, such that*

$$f(x) = \lim_{k \to \infty \text{ in } L^2} \sum_{2^{-k} \leqslant l(I) \leqslant 2^k} \lambda_I b_I(x), \tag{2.1}$$

$$\operatorname{supp} b_I \subseteq 3I, \quad \int b_I(x)\,dx = 0, \quad \|D^\alpha b_I\|_{L^\infty} \leqslant C_{|\alpha|} l(I)^{-|\alpha|}, \tag{2.2}$$

$$\sum_{I \subseteq J} \lambda_I^2 |I| \leqslant c|J| \quad \textit{for any cube } J \tag{2.3}$$

and

$$\sum_{\textit{all } I} \lambda_I^2 |I| = c\|f\|_{L^2}^2. \tag{2.4}$$

The last equality is not stated explicitly in **[8]**, but follows from the definition of the $\{\lambda_I\}_I$ and Plancherel's Theorem quite easily.

LEMMA U2. *For $\{\lambda_I\}_I$ satisfying* (2.3), $\lambda_I \geqslant 0$ $\forall I$, *set*

$$\eta_k(x) = \sum_{I:\, l(I) = 2^{-k}} \lambda_I \left(1 + 2^k |x - x_I|\right)^{-n-1},$$

and

$$\varepsilon_k(x) = \sum_{j=0}^{\infty} \left(\frac{2}{3}\right)^j \eta_{k-j}(x).$$

Then $\forall x, y$,

$$\eta_k(x) \leqslant \varepsilon_k(x) \leqslant c, \tag{2.5}$$

$$\eta_k(x) \leqslant c\left(2^k|x-y|+1\right)^{n+1}\eta_k(y), \tag{2.6}$$

$$\varepsilon_k(x) \leqslant c\left(2^k|x-y|+1\right)^{n+1}\varepsilon_k(y), \tag{2.7}$$

and if $l(I) = 2^{-p}$, *then*

$$\int_I \sum_{k=p}^{\infty} \varepsilon_k(x)\eta_k(x)\,dx \leqslant c|I|. \tag{2.8}$$

LEMMA U3. *Let j be a positive integer. Suppose* $\{b_I(x)\}_{I \text{ dyadic}}$ *satisfy*

$$\operatorname{supp} b_I \subseteq 2^j I, \quad \int b_I(x)\,dx = 0, \quad \|b_I\|_{\mathrm{Lip}\,1} \leqslant c\left(2^j l(I)\right)^{-1}. \tag{2.9}$$

If $\lambda_I \geqslant 0\ \forall I$ *and* $\beta > \alpha > 0$, *then*

$$\left\| \sum_{I:\,\alpha < l(I) < \beta} \lambda_I b_I(x) \right\|_{L^2} \leqslant c2^{jn}\left(\sum_I \lambda_I^2 |I|\right)^{1/2}. \tag{2.10}$$

LEMMA U4. *Suppose* j, $\{b_I\}_I$ *and* $\{\lambda_I\}_I$ *are as in Lemma* U3 *and* (2.3) *holds. For* $\alpha > 0$, *let* $f(x) = \sum_{l(I)<\alpha} \lambda_I b_I(x)$. *Then*

$$\|f\|_{\mathrm{BMO}} \leqslant C2^{jn}. \tag{2.11}$$

LEMMA U5. *Let I and* $p_I(x) \in C^1(\mathbf{R}^n)$ *be such that*

$$\int p_I(x)\,dx = 0, \tag{2.12}$$

$$|p_I(x)| \leqslant \left(1 + l(I)^{-1}|x - x_I|\right)^{-n-1} \tag{2.13}$$

and

$$|\nabla p_I(x)| \leqslant l(I)^{-1}\left(1 + l(I)^{-1}|x - x_I|\right)^{-n-2}. \tag{2.14}$$

Then $\exists\{\beta_{I,j}(x)\}_{j=0}^{\infty}$ *such that*

$$\|\beta_{I,j}\|_{\mathrm{Lip}\,1} \leqslant c\left(2^j l(I)\right)^{-1}, \quad \operatorname{supp}\beta_{I,j} \subseteq 2^j I, \quad \int \beta_{I,j}(x)\,dx = 0 \tag{2.15}$$

and

$$p_I(x) = \sum_{j=0}^{\infty} 2^{-j(n+1)}\beta_{I,j}(x). \tag{2.16}$$

If $\vec{p}_I = (p_{I,0}, \ldots, p_{I,m})$ *and each component of* $\vec{p}_I$ *is real valued and satisfies the conditions* (2.12)–(2.14), *and* $\vec{a} = (a_0, \ldots, a_m) \in \mathbf{R}^{m+1}$ *with* $|\vec{a}| = 1$ *satisfies* $\vec{p}_I(x) \cdot \vec{a} = 0\ \forall x$, *then we can take* $\vec{\beta}_{I,j} = (\beta_{I,j}^0, \ldots, \beta_{I,j}^m)$ *such that* (2.15) *is satisfied for each component of* $\vec{\beta}_{I,j}$, $\vec{p}_I(x) = \sum_{j=0}^{\infty} 2^{-j(n+1)}\vec{\beta}_{I,j}(x)$ *and* $\vec{\beta}_{I,j}(x) \cdot \vec{a} = 0\ \forall x, j$.

LEMMA U6. *Suppose K is a singular integral operator with multiplier m, and $b_I(x)$ satisfies* (2.2) *for some cube I. Then $p_I(x) = Kb_I(x)$ satisfies* (2.12),

$$|p_I(x)| \leqslant C(K)\left(1 + l(I)^{-1}|x - x_I|\right)^{-n-1} \tag{2.17}$$

and

$$|\nabla p_I(x)| \leqslant C(K) l(I)^{-1}\left(1 + l(I)^{-1}|x - x_I|\right)^{-n-2}, \tag{2.18}$$

where $|C(K)| \leqslant c \sup\{|D^\alpha m(\xi)| : |\xi| = 1,\ |\alpha| \leqslant C(n)\}$. (In fact, $C(n) = n + 4$ is sufficient.) Also it is enough to assume $\|D_{x_j} b_I\|_{\mathrm{Lip}\,1} \leqslant cl(I)^{-2}$ for $j = 1, \ldots, n$ instead of $\|D^\alpha b_I\|_{L^\infty} \leqslant c_{|\alpha|} l(I)^{-|\alpha|}$.

3. Properties of $R_1(\mathrm{BMO} \cap L^2)$. We characterize $R_1(\mathrm{BMO} \cap L^2)$ in terms of a decomposition similar to the decomposition of $\mathrm{BMO} \cap L^2$ in Lemma U1. For convenience we take $j = 1$ and write $x \in \mathbf{R}^n$ as $x = (x_1, x')$ for $x' \in \mathbf{R}^{n-1}$.

LEMMA 3.1. *Suppose $f = R_1 g$ for some $g \in \mathrm{BMO} \cap L^2$ with $\|g\|_{\mathrm{BMO}} \leqslant 1$. Then there exist functions $\{a_I(x)\}_I$ and $\{\lambda_I\}_I$, $\lambda_I \geqslant 0$, for I taken over all dyadic cubes, such that*

$$f(x) = \lim_{k \to \infty \text{ in } L^2} \sum_{2^{-k} \leqslant l(I) \leqslant 2^k} \lambda_I a_I(x), \tag{3.1}$$

$$|a_I(x)| \leqslant c\left(1 + l(I)^{-1}|x - x_I|\right)^{-n-1}, \tag{3.2}$$

$$|D^\alpha a_I(x)| \leqslant c_\alpha l(I)^{-|\alpha|}\left(1 + l(I)^{-1}|x - x_I|\right)^{-n-1-|\alpha|}, \tag{3.3}$$

$$\int a_I(x_1, x')\, dx_1 = 0 \quad \forall x' \in \mathbf{R}^{n-1}, \tag{3.4}$$

$$\sum_{I \subseteq J} \lambda_I^2 |I| \leqslant c|J| \quad \textit{for any cube } J \tag{3.5}$$

and

$$\sum_{\textit{all } I} \lambda_I^2 |I| < \infty. \tag{3.6}$$

Conversely, if f satisfies (3.1)–(3.6) *for some $\{a_I\}_I$ and $\{\lambda_I\}_I$, then $f = R_1 g$ for some $g \in \mathrm{BMO} \cap L^2$ with $\|g\|_{\mathrm{BMO}} \leqslant C$.*

PROOF. The forward direction follows easily from Uchiyama's lemmas. By Lemma U1,

$$g = \lim_{k \to \infty \text{ in } L^2} \sum_{2^{-k} \leqslant l(I) \leqslant 2^k} \lambda_I b_I$$

with each b_I satisfying (2.2), and (3.5) and (3.6) hold. Since R_1 is continuous on L^2, we obtain (3.1) by setting $a_I = R_1 b_I$. Then (3.2), and (3.3) for $|\alpha| = 1$, follow by Lemma U6; for $|\alpha| > 1$, $D^\alpha a_I$ is estimated by taking the derivatives inside the integral in the expression for a_I exactly as in the proof of Lemma U6. The cancellation property (3.4) holds because $\hat{a}_I(\xi) = i\xi_1|\xi|^{-1}\hat{b}_I(\xi)$.

The idea of the converse is to use (3.2)–(3.4) to write $a_I = R_1\varphi_I$ with appropriate control of φ_I. Let $h_I(x_1, x') = \int_{-\infty}^{x_1} a_I(t, x')\,dt$. We claim

$$|h_I(x)| \leqslant cl(I)\left(1 + l(I)^{-1}|x - x_I|\right)^{-n} \tag{3.7}$$

and

$$|D^\alpha h_I(x)| \leqslant c_\alpha l(I)^{1-|\alpha|}\left(1 + l(I)^{-1}|x - x_I|\right)^{-n-|\alpha|}. \tag{3.8}$$

To prove (3.7), we may assume $x_I = 0$. For x such that $x_1 < 0$, let k and m be positive integers such that $-(k+1)l(I) < x_1 \leqslant -kl(I)$ and $ml(I) \leqslant |x'| \leqslant (m+1)l(I)$. By (3.2),

$$|h_I(x)| \leqslant \int_{-\infty}^{x_1} |a_I(t, x')|dt \leqslant c\sum_{p=k}^{\infty} \frac{l(I)}{\left(1 + \sqrt{p^2 + m^2}\right)^{n+1}}$$

$$\leqslant \frac{cl(I)}{\left(1 + \sqrt{k^2 + m^2}\right)^n} \leqslant cl(I)\left(1 + l(I)^{-1}|x - x_I|\right)^{-n}.$$

If $x_1 > 0$, $h_I(x) = -\int_{x_1}^{\infty} a_I(t, x')\,dt$ by (3.4), so the same estimates hold. Then (3.8) follows from (3.3) and (3.4) similarly. Note that (3.7) and (3.8) imply that $h_I \in L^2$ and $|\nabla h_I| \in L^1 \cap L^2$.

Define $\varphi_I = -\sum_{j=1}^n R_j D_{x_j} h_I$. Then $\varphi_I \in L^2$ since $|\nabla h_I| \in L^2$. Note that $\hat{\varphi}_I(\xi) = -2\pi|\xi|\hat{h}_I(\xi)$; therefore $(R_1\varphi_I)\hat{\ }(\xi) = -2\pi i\xi_1\hat{h}_I(\xi) = \hat{a}_I(\xi)$, since $D_{x_1}h_I = a_I$. Hence $R_1\varphi_I = a_I$. If we can show that $\sum_I \lambda_I\varphi_I \in \mathrm{BMO} \cap L^2$, then we are done since

$$R_1\Big(\sum_I \lambda_I\varphi_I\Big) = \sum_I \lambda_I R_1\varphi_I = \sum_I \lambda_I a_I = f.$$

To see that $\sum_I \lambda_I\varphi_I \in \mathrm{BMO} \cap L^2$, we show that $D_{x_l}h_I$ satisfies the assumptions of Lemma U5 for $l = 1,\dots,n$. We have $\int D_{x_l}h_I(x)\,dx = 0$ since $|\nabla h_I| \in L^1$ and $\lim_{|x|\to\infty} h_I(x) = 0$ by (3.7); the estimates (2.13) and (2.14) follow from (3.8). Hence for each l, $\exists\{\beta_{I,j}(x)\}_{j=0}^{\infty}$ satisfying (2.15) and $D_{x_l}h_I(x) = \sum_{j=0}^{\infty} 2^{-j(n+1)}\beta_{I,j}(x)$. Therefore, using Lemma U3,

$$\Big\|\sum_I \lambda_I D_{x_l}h_I\Big\|_{L^2} \leqslant \sum_{j=0}^{\infty} 2^{-j(n+1)}\Big\|\sum_I \lambda_I\beta_{I,j}\Big\|_{L^2}$$

$$\leqslant c\sum_{j=0}^{\infty} 2^{-j(n+1)}2^{jn}\Big(\sum_I \lambda_I^2|I|\Big)^{1/2} < \infty,$$

by (3.6). Similarly, by Lemma U4,

$$\Big\|\sum_I \lambda_I D_{x_l}h_I\Big\|_{\mathrm{BMO}} \leqslant \sum_{j=0}^{\infty} 2^{-j(n+1)}\Big\|\sum_I \lambda_I\beta_{I,j}\Big\|_{\mathrm{BMO}} \leqslant c.$$

Therefore $\sum_I \lambda_I D_{x_l}h_I \in \mathrm{BMO} \cap L^2$ for each l; hence

$$\sum_I \lambda_I\varphi_I = -\sum_I \lambda_I\sum_{j=1}^n R_jD_{x_j}h_I = -\sum_{j=1}^n R_j\sum_I \lambda_I D_{x_j}h_I \in \mathrm{BMO} \cap L^2. \quad \square$$

The cancellation property of the Riesz transforms, reflected in condition (3.4), forces the x_1-averages of $f \in R_1(\mathrm{BMO} \cap L^2)$ to have a certain regularity. Let Q be a cube in $\mathbf{R}^n$, and write $Q = J \times L$ for $J \subseteq \mathbf{R}^1$ and $L \subseteq \mathbf{R}^{n-1}$. By Fubini's Theorem, the average $f_{J_y} = (1/|J|)\int_J f(t, y)\,dt$ is defined for almost $y \in L$.

THEOREM 3.2. *Suppose $f = R_1 g$ for some $g \in$ BMO $\cap L^2(\mathbf{R}^n)$ satisfying $\|g\|_{\text{BMO}} \leqslant 1$. For Q, J, and L as above, $\exists c > 0$ and a set $E \subseteq L$ of $(n-1)$-dimensional Lebesgue measure* 0 *such that*

$$|f_{J_y} - f_{J_{y'}}| \leqslant c \quad \forall y, y' \in L - E \tag{3.9}$$

and

$$|f_{J_y} - f_{J_{y'}}| \leqslant c\frac{|y-y'|}{l(Q)}\log\frac{l(Q)}{|y-y'|} \quad \textit{if } y, y' \in L - E \tag{3.10}$$

and $|y - y'| < \frac{1}{e} l(Q)$.

REMARKS. The number $\frac{1}{e}$ is chosen because the function $t \to t\log(\frac{1}{t})$ is increasing on $(0, \frac{1}{e})$. The proof uses only (3.1), (3.2), (3.3) for $\alpha = 1$, (3.4), and the estimate $\lambda_I \leqslant c$, which follows trivially from (3.5). An example of a function in BMO($\mathbf{R}^n$) which does not satisfy (3.10) is $f(x) = \log|x'|$.

PROOF OF THEOREM 3.2. We may assume $Q = [-1,1]^n$, so $J = [-1,1]$ and $L = [-1,1]^{n-1}$. For $y, y' \in L$ fixed, let $\varepsilon = |y - y'|$. It is sufficient to obtain $|f_{J_y} - f_{J_{y'}}| < c\varepsilon\log\frac{1}{\varepsilon}$ for $\varepsilon < \frac{1}{e}$, since (3.10) implies (3.9). By Lemma 3.1, $f = \sum_I \lambda_I a_I$ with (3.2)–(3.6). We also may assume that $\exists N$ so that $\lambda_I = 0$ whenever $l(I) > 2^N$ or $l(I) < 2^{-N}$. To see this, let $f_k = \sum_{2^{-k} \leqslant l(I) \leqslant 2^k} \lambda_I a_I$; then $\lim_{k\to\infty} f_k = f$ in L^2 and, for each k, f_k is continuous and the sum converges absolutely. Let

$$R_y^\delta = J \times \prod_{l=2}^{n} (y_l - \delta, y_l + \delta),$$

where $\delta > 0$ and $y = (y_1, \ldots, y_n)$, and similarly define $R_{y'}^\delta$. The result for f_k implies that the averages $(f_k)_{R_y^\delta}$ and $(f_k)_{R_{y'}^\delta}$ of f over R_y^δ and $R_{y'}^\delta$, respectively, satisfy $|(f_k)_{R_y^\delta} - (f_k)_{R_{y'}^\delta}| \leqslant c\varepsilon\log\frac{1}{\varepsilon}$ with c independent of k or δ. Since $\lim_{k\to\infty} f_k = f$ in L^2, $|f_{R_y^\delta} - f_{R_{y'}^\delta}| \leqslant c\varepsilon\log\frac{1}{\varepsilon}$ with c independent of δ. By Lebesgue's Theorem, $\lim_{\delta\to 0} f_{R_y^\delta} = f_{J_y}$ unless y is taken from a set $E \subseteq L$ of $(n-1)$-dimensional Lebesgue measure 0, and similarly for $f_{J'}$; hence (3.10) follows.

For $f = \sum_I \lambda_I a_I$ satisfying $\lambda_I = 0$ for $l(I) > 2^N$ or $l(I) < 2^{-N}$, write $f = f_1 + f_2 + f_3 + f_4$, where

$$f_1 = \sum_{l(I) \geqslant 1} \lambda_I a_I, \qquad f_2 = \sum_{\substack{l(I) < 1 \\ I \cap 3Q = \varnothing}} \lambda_I a_I, \quad f_3 = \sum_{\substack{l(I) \leqslant \varepsilon \\ I \subseteq 3Q}} \lambda_I a_I, \qquad f_4 = \sum_{\substack{\varepsilon < l(I) < 1 \\ I \subseteq 3Q}} \lambda_I a_I.$$

The functions f_1 and f_2 are estimated by Lipschitz estimates. By (3.3) and the fact that $\lambda_I \leqslant c$, if $t \in J$ we have

$$\begin{aligned}
|f_1(t, y) - f_1(t, y')| &\leqslant c\sum_{k=0}^{\infty} \sum_{l(I) = 2^k} |a_I(t, y) - a_I(t, y')| \\
&\leqslant c\sum_{k=0}^{\infty} \sum_{l(I) = 2^k} \sup_{x \in Q} |\nabla a_I(x)|\varepsilon \\
&\leqslant c\varepsilon \sum_{k=0} \sum_{q \in \mathbf{Z}^n} \frac{2^{-k}}{(1 + |q|)^{n+2}} \leqslant c\varepsilon.
\end{aligned}$$

Therefore $|(f_1)_{J_y} - (f_1)_{J_{y'}}| \leqslant c\varepsilon$. Similarly, if $t \in J$,

$$\left|f_2(t,y) - f_2(t,y')\right| \leqslant c \sum_{k=1}^{\infty} \sum_{\substack{l(I)=2^{-k} \\ I\cap 3Q=\varnothing}} \left|a_I(t,y) - a_I(t,y')\right|$$

$$\leqslant c\varepsilon \sum_{k=1}^{\infty} \sum_{\substack{l(I)=2^{-k} \\ I\cap 3Q=\varnothing}} \sup_{x\in Q} \frac{2^k}{\left(1 + 2^k|x - x_I|\right)^{n+2}}$$

$$\leqslant c\varepsilon \sum_{k=1}^{\infty} \sum_{\substack{q\in \mathbf{Z}^n \\ |q|\geqslant 2^k}} \frac{2^k}{\left(1 + |q|\right)^{n+2}} \leqslant c\varepsilon \sum_{k=1}^{\infty} 2^{-k} = c\varepsilon.$$

Therefore $|(f_2)_{J_y} - (f_2)_{J_{y'}}| \leqslant c\varepsilon$.

Using (3.4), we will obtain $|(f_3)_{J_y}| \leqslant c\varepsilon \log\frac{1}{\varepsilon}$ and $|(f_3)_{J_{y'}}| \leqslant c\varepsilon\log\frac{1}{\varepsilon}$. Let $I \subseteq 3Q$ be such that $l(I) = 2^{-k} \leqslant \varepsilon$, with $x_I = (x_{I_1}, x_I')$, $x_{I_1} \in \mathbf{R}$, satisfying $0 < x_{I_1} < 1$. Let $m \in \{0,1,2,\dots,2^k - 1\}$ and $p \in \mathbf{Z}$, $0 \leqslant p \leqslant 4\sqrt{n-1}\,(2^k - 1)$, be such that $m2^{-k} < 1 - x_{I_1} < (m+1)2^{-k}$ and $p2^{-k} < |x_I' - y| \leqslant (p+1)2^{-k}$. By (3.4), (3.2), and since $|J| = 2$,

$$\left|\frac{1}{|J|}\int_J a_I(t,y)\,dt\right| \leqslant 2\int_{-\infty}^{-1} |a_I(t,y)|dt + 2\int_1^{\infty} |a_I(t,y)|dt$$

$$\leqslant c\sum_{s=0}^{\infty} \frac{2^{-k}}{\left(1 + \sqrt{p^2 + (m+s)^2}\right)^{n+1}} \leqslant c\frac{2^{-k}}{\left(1 + \sqrt{p^2 + m^2}\right)^n}.$$

If I and p are as above except that $-1 < x_{I_1} < 0$, and $m \in \{0,1,\dots,2^k - 1\}$ is such that $m2^{-k} < x_{I_1} + 1 \leqslant (m+1)2^{-k}$, the same estimate is obtained by the same methods. If I and p are as above but $1 < |x_{I_1}| < 3$, say $m2^{-k} < x_{I_1} - 1 < (m+1)2^{-k}$ if $x_{I_1} > 1$, or $m2^{-k} < -1 - x_{I_1} < (m+1)2^{-k}$ if $x_{I_1} < -1$ for some $m \in \{0,1,2,\dots,2^{k+1} - 1\}$, then (3.4) is not used since (3.2) implies that

$$\left|\frac{1}{|J|}\int_J a_I(t,y)\,dt\right| \leqslant c\sum_{s=0}^{\infty} \frac{2^{-k}}{\left(1 + \sqrt{p^2 + (m+s)^2}\right)^{n+1}}$$

$$\leqslant c\frac{2^{-k}}{\left(1 + \sqrt{p^2 + m^2}\right)^n}.$$

Summing over all $I \subseteq 3Q$ with $l(I) \leqslant \varepsilon$, we obtain

$$\left|(f_3)_{J_y}\right| = \left|\sum_{k=[\log_2 1/\varepsilon]}^{\infty} \frac{1}{|J|}\int_J \sum_{\substack{l(I)=2^{-k} \\ I\subseteq 3Q}} \lambda_I a_I(t,y)\,dt\right|$$

$$\leqslant c \sum_{k=[\log_2 1/\varepsilon]}^{\infty} 2^{-k} \sum_{\substack{q\in\mathbf{Z}^n \\ |q|\leqslant 4\sqrt{n}\,2^k}} \frac{1}{\left(1 + |q|\right)^n}$$

$$\leqslant c \sum_{k=[\log_2 1/\varepsilon]}^{\infty} k2^{-k} \leqslant c\varepsilon\log\frac{1}{\varepsilon},$$

where we have used $\lambda_I \leqslant c$ again, and $[x]$ is the greatest integer in x. With the same estimate for $(f_3)_{J'}$, we obtain $|(f_3)_{J_y} - (f_3)_{J_{y'}}| < c\varepsilon \log\frac{1}{\varepsilon}$.

For f_4, (3.3) and (3.4) are both used. Suppose $\varepsilon < l(I) = 2^{-k} < 1$ and $I \subseteq 3Q$. Let $y^* = \frac{1}{2}(y + y')$. Let $p \in \mathbf{Z}$, $0 \leqslant p \leqslant 4\sqrt{n-1}(2^k - 1)$, be such that $p2^{-k} < |x'_I - y^*| \leqslant (p+1)2^{-k}$. Let $m \in \{0, 1, \ldots, 2^k - 1\}$ be such that $m2^{-k} < x_{I_1} - 1 < (m+1)2^{-k}$ if $0 < x_{I_1} < 1$, or such that $m2^{-k} < x_{I_1} + 1 < (m+1)2^{-k}$ if $-1 < x_{I_1} < 0$. By (3.4),

$$\left|\frac{1}{|J|}\int_J a_I(t, y')\,dt - \frac{1}{|J|}\int_J a_I(t, y)\,dt\right|$$

$$= \left|\frac{1}{|J|}\int_{-\infty}^{-1} a_I(t, y)\,dt - \frac{1}{|J|}\int_{-\infty}^{-1} a_I(t, y')\,dt\right.$$

$$\left. + \frac{1}{|J|}\int_1^{\infty} a_I(t, y)\,dt - \frac{1}{|J|}\int_1^{\infty} a_I(t, y')\,dt\right|$$

$$\leqslant 2\int_{-\infty}^{-1}|a_I(t, y) - a_I(t, y')|\,dt + 2\int_1^{\infty}|a_I(t, y) - a_I(t, y')|\,dt.$$

Therefore, by (3.3),

$$|(a_I)_{J_y} - (a_I)_{J_{y'}}| \leqslant 2\sum_{s=0}^{\infty} 2^{-k} \sup_{\substack{|x' - y^*| < \varepsilon/2 \\ (m+s)2^{-k} \leqslant |x_1 - x_{I_1}| \leqslant (m+s+1)2^{-k}}} |\nabla a_I(x)| \cdot \varepsilon$$

$$\leqslant c2^{-k}\varepsilon \sum_{s=0}^{\infty} \frac{2^k}{\left(1 + \sqrt{p^2 + (m+s)^2}\right)^{n+2}} \leqslant \frac{c\varepsilon}{\left(1 + \sqrt{p^2 + m^2}\right)^{n+1}}.$$

If I and p are as above but $1 < |x_{I_1}| < 3$, and $m \in \{0, 1, \ldots, 2^{k+1} - 1\}$ is such that $m2^{-k} < x_{I_1} - 1 < (m+1)2^{-k}$ if $x_{I_1} > 1$ or such that $m2^{-k} < -1 - x_{I_1} < (m+1)2^{-k}$ if $x_{I_1} < -1$, then (3.3) leads directly to the same estimates without use of (3.4). Therefore, since $\lambda_1 \leqslant c$,

$$\left|\frac{1}{|J|}\int_J \sum_{\substack{l(I) = 2^{-k} \\ I \subseteq 3Q}} \lambda_I a_I(t, y)\,dt - \frac{1}{|J|}\int_J \sum_{\substack{l(I) = 2^{-k} \\ I \subseteq 3Q}} \lambda_I a_I(t, y')\,dt\right|$$

$$\leqslant c\varepsilon \sum_{q \in \mathbf{Z}^n} \frac{1}{\left(1 + |q|\right)^{n+1}} = c\varepsilon.$$

Therefore

$$|(f_4)_{J_y} - (f_4)_{J_{y'}}| \leqslant c\varepsilon\#\{k : \varepsilon < 2^{-k} < 1\} = c\varepsilon \log\tfrac{1}{\varepsilon}.$$

Theorem 3.2 follows from the estimates on f_1, f_2, f_3 and f_4. □

4. Corollaries and examples. Theorem 3.2 implies that functions in $R_j(\mathrm{BMO} \cap L^2)$ satisfy a doubling condition on almost every x_j-line. For convenience we take $j = 1$ and write $x \in \mathbf{R}^n$ as (x_1, x') for $x_1 \in \mathbf{R}$, $x' \in \mathbf{R}^{n-1}$.

COROLLARY 4.1. *Suppose* $f = R_1 g$ *for some* $g \in \mathrm{BMO} \cap L^2(\mathbf{R}^n)$. *Then* $\exists c > 0$ *and a set* $E \subseteq \mathbf{R}^{n-1}$ *of* $(n-1)$*-dimensional Lebesgue measure* 0 *such that for* $x' \notin E$ *and* I_1 *and* I_2 *adjacent intervals in* $\mathbf{R}$ *with* $|I_1| = |I_2|$,

$$(4.1)\qquad \left| \frac{1}{|I_1|} \int_{I_1} f(x_1, x')\,dx_1 - \frac{1}{|I_2|} \int_{I_2} f(x_1, x')\,dx_1 \right| \leqslant c\|g\|_{\mathrm{BMO}}.$$

PROOF. We can assume $\|g\|_{\mathrm{BMO}} = 1$. Assume first that I_1 and I_2 are dyadic intervals. Let F be the union over all dyadic cubes of the exceptional sets obtained for f in Theorem 3.2. For $x' \notin F$, let $L \subseteq \mathbf{R}^{n-1}$ be a closed dyadic cube containing x' such that $|I_1| = l(L)$, and write $Q_1 = I_1 \times L$, $Q_2 = I_2 \times L$. By Lemma 3.2, since $x' \notin F$, $|f_{I_1^{x'}} - f_{I_1^s}| \leqslant c$ for a.e. $s \in L - F$. Therefore $|f_{I_1^{x'}} - f_{Q_1}| \leqslant c$, and similarly $|f_{I_2^{x'}} - f_{Q_2}| \leqslant c$. Since Q_1 and Q_2 are adjacent cubes in $\mathbf{R}^n$ and $f \in \mathrm{BMO}(\mathbf{R}^n)$, $|f_{Q_1} - f_{Q_2}| \leqslant c|f|_{\mathrm{BMO}} \leqslant c$ [**3**, p. 223]. Therefore $|f_{I_1^{x'}} - f_{I_2^{x'}}| \leqslant c$ for $x' \notin F$. To obtain a single exceptional set E for I_1, I_2 not necessarily dyadic, apply this result to all dilations and translations of f by rational coordinates. □

EXAMPLE 4.2. There exists $f \in \mathrm{BMO} \cap L^2(\mathbf{R}^n)$ which does not satisfy (4.1). In $\mathbf{R}^2$, for example, for each dyadic square I, let f_I be a function satisfying $0 \leqslant f_I(x) \leqslant 1$, $\operatorname{supp} f_I \subseteq 3I$, $\|f_I\|_{\mathrm{Lip}\,1} \leqslant cl(I)^{-1}$ and $f_I(x) = 1\ \forall x \in I$. Let $A_k = \{I \text{ dyadic: } l(I) = 2^{-k},\ 0 < x_{I_1} < 1, \text{ and } \bar{I} \cap \{x : x_2 = 0\} \neq \varnothing\}$. Let $f = \Sigma_{k=0}^{\infty} \Sigma_{I \in A_k} f_I$. Then it is not difficult to show that $f \in \mathrm{BMO} \cap L^2(\mathbf{R}^2)$, but

$$\lim_{x_2 \to 0} \left(\int_0^1 f(x_1, x_2)\,dx_1 - \int_1^2 f(x_1, x_2)\,dx_1 \right) = \infty.$$

For details, see [**2**].

COROLLARY 4.3. *If* $f = \tilde{R}_1 g$ *for some* $g \in L^\infty$ *such that* $\|g\|_{L^\infty} \leqslant 1$, *then* (3.9), (3.10) *and* (4.1) *hold.*

PROOF. The estimates hold if $f = \tilde{R}_1 g_m$, where $g_m(x) = g(x)\chi_{\{x : |x| \leqslant m\}}(x)$ since $g_m \in L^2 \cap L^\infty$ so that $R_1 g_m$ and $\tilde{R}_1 g_m$ differ by a constant. The result follows since, by a calculation in [**2**, p. 48], if K is any compact set in $\mathbf{R}^n$,

$$\lim_{m \to \infty} \left(\sup_{x \in K} |\tilde{R}_1(g - g_m)(x)| \right) = 0. \quad \square$$

A proof of Corollary 4.3 based on duality with singular measures is included in [**2**].

EXAMPLE 4.4. If $f(x) = \log\log(1/|x'|)$ for $|x'| < 1/e$ and $f(x) = 0$ otherwise, then $f \in \mathrm{BMO}(\mathbf{R}^n)$ but f does not satisfy (3.9). Although $f \notin L^\infty$, f belongs to the BMO closure of L^∞. Therefore $L^\infty + R_1 L^\infty$, $L^\infty + R_1(\mathrm{BMO} \cap L^2)$ and $L^\infty + R_1(L_c^\infty)$ are not closed subspaces of BMO. This observation is relevant to §5 below. The space $L^\infty + R_1 L^\infty$ is not dense in BMO in BMO norm; a proof due to Paul Koosis is included in [**2**].

EXAMPLE 4.5 (KAHANE'S EXAMPLE). Kahane's example [**6**] can be used to show that there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of functions on $(0,1)$ satisfying a uniform doubling condition but such that $\lim_{n\to\infty}\|f_n\|_{\mathrm{BMO}(0,1)} = \infty$. For $m_1, m_2, \ldots, m_n \in \{1, 2, 3, 4\}$, define

$$I_{m_1\cdots m_n} = \left(\frac{m_1 - 1}{4} + \frac{m_2 - 1}{4^2} + \cdots + \frac{m_n - 1}{4^n},\right.$$
$$\left.\frac{m_1 - 1}{4} + \cdots + \frac{m_{n-1} - 1}{4^{n-1}} + \frac{m_n}{4^n}\right),$$

and $Q_{m_1\cdots m_n} = I_{m_1\cdots m_n} \times [0, 4^{-n}]$. Let $f_1(x_1, x_2)$ be defined by

$$f_1 = \chi_{Q_1} - \chi_{Q_2} - \chi_{Q_3} + \chi_{Q_4}.$$

With $\operatorname{sgn} t = t/|t|$ if $t \neq 0$ and $\operatorname{sgn} 0 = 0$, define f_{n+1} inductively by

$$f_{n+1} = f_n + \sum_{m_1, \ldots, m_n = 1}^{4} a_{m_1\cdots m_n},$$

where

$$a_{m_1\cdots m_n} = \operatorname{sgn} f_n\left(-\chi_{Q_{m_1\cdots m_n 1}} + \chi_{Q_{m_1\cdots m_n 2}} + \chi_{Q_{m_1\cdots m_n 3}} - \chi_{Q_{m_1\cdots m_n 4}}\right).$$

Regard f_n now as restricted to $(0, 1)$. Then clearly

$$\|f_n\|_{L^1} = 1 \quad \forall n = 1, 2, 3, \ldots.$$

If $\alpha_n = |\operatorname{supp} f_n|$, then α_n decreases to 0 as $n \to \infty$. That α_n decreases is clear; that $\lim_{n\to\infty} \alpha_n = 0$ can be seen by the fact that a simple random walk in one dimension is recurrent. Since $\int_0^1 f_n(x)\,dx = 0\ \forall n$, and since

$$\|f\|_{\mathrm{BMO}} \approx \sup_Q \left(\frac{1}{|Q|}\int_Q |f - f_Q|^2\right)^{1/2},$$

we obtain $\lim_{n\to\infty}\|f_n\|_{\mathrm{BMO}(0,1)} = \infty$ since

$$\|f_n\|_{\mathrm{BMO}(0,1)} \geqslant c\|f_n\|_{L^2(0,1)} \geqslant c\frac{\|f_n\|_{L^1(0,1)}}{\sqrt{\alpha_n}} = \frac{c}{\sqrt{\alpha_n}}. \tag{4.2}$$

If I_1 and I_2 are adjacent intervals of the same length in $\mathbf{R}$, then $|(f_n)_{I_1} - (f_n)_{I_2}| \leqslant c$, with c independent of I_1, I_2, or n, by the main estimate in Kahane's example: see [**6**, p. 190]. □

EXAMPLE 4.6. There exists $G \in \mathrm{BMO} \cap L^2(\mathbf{R}^2)$ such that

$$\operatorname*{ess\,sup}_{x_2} \|R_1 G(x_1, x_2)\|_{\mathrm{BMO}(x_1)} = \infty. \tag{4.3}$$

We construct $R_1 G$ by symmetrizing and smoothing the functions $f_n(x_1, x_2)$ of Example 4.5 in the x_2 direction.

Regarding f_n above as a function of (x_1, x_2) again, let $\tilde{f}_n(x_1, x_2)$ be the function which agrees with $f_n(x_1, x_2)$ for $x_2 \geqslant 0$ and satisfies $f_n(x_1, -x_2) = f_n(x_1, x_2)$ $\forall (x_1, x_2)$. Define $\tilde{a}_{m_1\ldots m_n}$ similarly related to $a_{m_1\ldots m_n}$. Since

$$\|\tilde{f}_{n+1} - \tilde{f}_n\|_{L^2(\mathbf{R}^2)} \leqslant c2^{-n},$$

$\tilde{f} = \lim_{n \to \infty} \tilde{f}_n$ exists in $L^2(\mathbf{R}^2)$ and a.e. The following sequence of steps can be carried out to show that $\tilde{f} \in \mathrm{BMO}(\mathbf{R}^2)$:

(a) Since the measure of the supports of $\tilde{f}_{n+1} - \tilde{f}_n$ decrease rapidly,

$$\frac{1}{|Q|} \int_Q |\tilde{f} - \tilde{f}_Q| \leqslant c$$

for any 4-adic square Q.

(b) If Q_1 and Q_2 are 4-adic squares of the same side length with $\overline{Q}_1 \cap \overline{Q}_2 \neq \varnothing$, then $|\tilde{f}_{Q_1} - \tilde{f}_{Q_2}| \leqslant c$. For Q_1 and Q_2 horizontally adjacent, this follows from the doubling condition for f_n. For Q_1 and Q_2 vertically adjacent, this follows from the construction.

(c) An easy observation, noted by Varopoulous, shows that (a) and (b) imply $\tilde{f} \in \mathrm{BMO}(\mathbf{R}^2)$.

In this example, we wish to replace the functions $\tilde{a}_{m_1 \ldots m_n}$ with versions which are smoothed out in the x_2 direction. Let $\psi \in C^\infty(\mathbf{R})$ be a fixed function satisfying $0 \leqslant \psi(x) \leqslant 1$ $\forall x$, supp $\psi \subseteq (-2, 2)$, and $\psi(x) = 1$ if $x \in [-1, 1]$. For $m_1, \ldots, m_n \in \{1, 2, 3, 4\}$, let

$$\varphi_{m_1 \ldots m_n}(x_1, x_2) = \chi_{I_{m_1 \ldots m_n}}(x_1) \psi(4^n x_2).$$

Let $g_1 = \varphi_1 - \varphi_2 - \varphi_3 + \varphi_4$, and define $g_{n+1}(x_1, x_2)$ inductively by

$$g_{n+1} = g_n + \sum_{m_1 \ldots m_n = 1}^{4} b_{m_1 \ldots m_n},$$

where

$$b_{m_1 \ldots m_n} = \operatorname{sgn} g_n(x_1, 0)\big(-\varphi_{m_1 \ldots m_n 1} + \varphi_{m_1 \ldots m_n 2} + \varphi_{m_1 \ldots m_n 3} - \varphi_{m_1 \ldots m_n 4}\big).$$

Note that $\operatorname{sgn} g_n(x_1, 0) = \operatorname{sgn} f_n(x_1, 0)$ for f_n in Example 4.5; in particular,

$$\operatorname{supp}\big(b_{m_1 \ldots m_n} - \tilde{a}_{m_1 \ldots m_n}\big) \subseteq I_{m_1 \ldots m_n} \times \big\{x_2 \colon 4^{-n-1} \leqslant |x_2| \leqslant 2 \cdot 4^{-n-1}\big\}.$$

Hence if $g = \lim_{n \to \infty} g_n$, then $g - \tilde{f} \in L^\infty(\mathbf{R}^2)$ by the disjointness of the supports of the different $g_n - \tilde{f}_n$. Therefore $g \in \mathrm{BMO} \cap L^2(\mathbf{R}^2)$ since $\tilde{f} \in \mathrm{BMO} \cap L^2(\mathbf{R}^2)$.

We assert that $g = R_1 G$ for some $G \in \mathrm{BMO} \cap L^2$; we solve for G by the method of Lemma 3.1. Let

$$h_0(x_1, x_2) = \int_{-\infty}^{x_1} g_1(t, x_2)\, dt$$

and

$$h_{m_1 \ldots m_n}(x_1, x_2) = \int_{-\infty}^{x_1} b_{m_1 \ldots m_n}(t, x_2)\, dt.$$

Note that $h_{m_1 \ldots m_n}(x_1, x_2) = 0$ if $x_1 \notin I_{m_1 \ldots m_n}$ (set $I_0 = (0, 1)$), or if $|x_2| > 2 \cdot 4^{-n-1}$. Further,

$$\begin{aligned}
&\big|D_{x_2} h_{m_1 \ldots m_n}(x_1, x_2)\big| \\
&\quad = \left|D_{x_2}\big(\psi(4^n x_2)\big) \int_{-\infty}^{x_1} \big(\chi_{I_{m_1 \ldots m_n 1}}(t) - \chi_{I_{m_1 \ldots m_n 2}}(t) - \chi_{I_{m_1 \ldots m_n 3}}(t) + \chi_{I_{m_1 \ldots m_n 4}}(t)\big)\, dt\right| \\
&\quad \leqslant \psi'(4^n x_2) \cdot 4^n \cdot 4^{-n-1} \leqslant C.
\end{aligned}$$

Also

$$\operatorname{supp} D_{x_2} h_{m_1 \ldots m_n} \subseteq I_{m_1 \ldots m_n} \times \left\{ x_2 : 4^{-n-1} \leqslant |x_2| \leqslant 2 \cdot 4^{-n-1} \right\},$$

since $\psi(x_2) = 1$ if $x_2 \in [-1, 1]$. Therefore

$$H \equiv D_{x_2} h_0 + \sum_{n=1}^{\infty} \sum_{m_1, \ldots, m_n = 1}^{4} D_{x_2} h_{m_1 \ldots m_n} \in L_c^\infty .$$

By Fourier transform we obtain $g_1 = R_1(-R_1 g_1 - R_2 D_{x_2} h_0)$ and

$$b_{m_1 \ldots m_n} = R_1\left(-R_1 b_{m_1 \ldots m_n} - R_2 D_{x_2} h_{m_1 \ldots m_n}\right).$$

Therefore

$$g = g_1 + \sum_{n=1}^{\infty} \sum_{m_1 \ldots m_n = 1}^{4} b_{m_1 \ldots m_n} = R_1(-R_1 g - R_2 H).$$

Since $g, H \in \mathrm{BMO} \cap L^2$, $G = -R_1 g - R_2 H \in \mathrm{BMO} \cap L^2$ and $R_1 G = g$. Note that $\tilde{f}(x_1, x_2) = f_n(x_1, x_2)$ for $4^{-n-1} < x_2 < 4^{-n}$, and hence by (4.2)

$$\operatorname*{ess\,sup}_{x_2} \|\tilde{f}(x_1, x_2)\|_{\mathrm{BMO}(x_1)} = \infty .$$

Since $R_1 G - \tilde{f} \in L^\infty$, (4.3) follows. $\square$

5. A density theorem. Throughout this section K is a fixed singular integral operator with kernel Ω which is real and odd; it follows [7, p. 39] that the multiplier m for K is odd and pure imaginary. We will show that

$$K(\mathrm{BMO}_c) \subseteq \overline{L^\infty + K(L_c^\infty)} ,$$

where the closure is taken in BMO norm. To fix notation for this section, I always denotes a dyadic cube, $\vec{f}$ denotes a vector-valued function with two components, $\vec{f} = (f_0, f_1)$, and we define

$$|\vec{f}(x)| = \left(f_0^2(x) + f_1^2(x)\right)^{1/2}, \qquad \|\vec{f}\| = \|f_0\| + \|f_1\|$$

for any function space norm $\|\cdot\|$, and $\vec{K} \cdot \vec{f} = f_0 + K f_1$. Also define

$$|\nabla \vec{f}(x)| = |\nabla f_0(x)| + |\nabla f_1(x)|$$

and

$$\int \vec{f}(x)\, dx = \left(\int f_0(x)\, dx, \int f_1(x)\, dx \right).$$

Our method is based closely on Uchiyama's construction in [8]. Uchiyama obtains uniformly bounded solutions, which satisfy a certain orthogonality condition, to a certain inversion problem. We are not able to obtain uniformly bounded solutions to the corresponding inversion problem in our case. Hence, some additional estimates are necessary.

First we state explicitly the assumptions we require regarding the inversion problem. We say that a collection of real-valued functions $\{b_I(x)\}_I$ satisfies $(*)$ if there exist constants $c > 0$ and $l > 0$ such that for all I,

(i) for any $\vec{a} = (a_0, a_1) \in \mathbf{R}^2$ with $a_1 \neq 0$, the equations

$$\vec{K} \cdot \vec{u}_I = b_I \tag{5.1}$$

and

$$\vec{a} \cdot \vec{u}_I(x) = \vec{0} \quad \forall x \tag{5.2}$$

have a solution $\vec{u}_I = (u_{0_I}, u_{1_I})$, $\vec{u}_I\colon \mathbf{R}^n \to \mathbf{R}^2$, satisfying

$$|\vec{u}_I(x)| \leqslant c\left(1 + \left|\frac{a_0}{a_1}\right|^l\right)\left(1 + l(I)^{-1}|x - x_I|\right)^{-n-1}, \tag{5.3}$$

$$|\nabla \vec{u}_I(x)| \leqslant c\left(1 + \left|\frac{a_0}{a_1}\right|^l\right) l(I)^{-1}\left(1 + l(I)^{-1}|x - x_I|\right)^{-n-2} \tag{5.4}$$

and

$$\int \vec{u}_I(x)\, dx = \vec{0}; \tag{5.5}$$

and

(ii) the equation $K(u_{1_I}) = b_I$ has a solution $u_{1_I}\colon \mathbf{R}^n \to \mathbf{R}$ satisfying

$$\left|u_{1_I}(x)\right| \leqslant c\left(1 + l(I)^{-1}|x - x_I|\right)^{-n-1}, \tag{5.6}$$

$$\left|\nabla u_{1_I}(x)\right| \leqslant c l(I)^{-1}\left(1 + l(I)^{-1}|x - x_I|\right)^{-n-2} \tag{5.7}$$

and

$$\int u_{1_I}(x)\, dx = 0. \tag{5.8}$$

THEOREM 5.1. *Suppose* $f(x) = \lim_{k\to\infty} \sum_{2^{-k} \leqslant l(I) \leqslant 2^k} \lambda_I b_I(x)$ *(convergence in* L^2*) with* $\lambda_I \geqslant 0$ $\forall I$, *where the real valued functions* $\{b_I(x)\}_I$ *satisfy* $(*)$, *and for some* $c > 0$,

$$\sum_{I \subseteq J} \lambda_I^2 |I| \leqslant c|J| \quad \textit{for all cubes } J, \tag{5.9}$$

$$\sum_{\textit{all } I} \lambda_I^2 |I| < \infty, \tag{5.10}$$

$$\lim_{M\to\infty} \Bigg\| \sum_{l(I) > 2^M} \lambda_I b_I \Bigg\|_{\mathrm{BMO}} = 0 \tag{5.11}$$

and

$$\exists \text{ a cube } S \text{ such that } \lambda_I = 0 \text{ if } 3I \cap S = \varnothing. \tag{5.12}$$

Then $f \in \overline{L^\infty + K(L_c^\infty)}$.

Before proving Theorem 5.1, we mention some consequences.

COROLLARY 5.2. *For* K *as above*, $K(\mathrm{BMO}_c) \subseteq \overline{L^\infty + K(L_c^\infty)}$, *where the closure is taken in* BMO *norm.*

PROOF. Suppose $g \in \mathrm{BMO}_c$. We may assume that g is real valued. By Lemma U1, $g = \sum_I \lambda_I a_I$ with (5.9) and (5.10). The functions a_I satisfy $\operatorname{supp} a_I \subseteq 3I$, $\|D^\alpha a_I\|_{L^\infty} \leqslant c_{|\alpha|} l(I)^{-|\alpha|}$ and $\int a_I(x)\,dx = 0$ $\forall I$. That (5.12) and the condition $\lim_{M\to\infty} \|\sum_{l(I)>2^M} \lambda_I a_I\|_{\mathrm{BMO}} = 0$ are also obtained by the decomposition in Lemma U1 for functions of compact support is proved in [**8**, pp. 232 and 234]. Set $b_I = Ka_I$; since K is bounded on L^2 and BMO, $Kg = \sum \lambda_I K a_I = \sum \lambda_I b_I$ in L^2. Each b_I is real valued since the $\{a_I\}_I$ obtained by Lemma U1 are real valued (since g is real valued), and since K has a real-valued kernel.

We need only show that $\{b_I\}_I$ satisfies $(*)$. Suppose $a_1 \neq 0$. Then (5.1) and (5.2) can be solved by setting

$$u_{0_I}(x) = \left[\left(\frac{-a_1}{m(\xi)a_0 - a_1}\right)\hat{b}_I(\xi)\right]^{\vee}(x) \tag{5.13}$$

and

$$u_{1_I}(x) = \left[\left(\frac{a_0}{m(\xi)a_0 - a_1}\right)\hat{b}_I(\xi)\right]^{\vee}(x), \tag{5.14}$$

where $^\vee$ is the inverse Fourier transform. (This is Uchiyama's solution reduced to our case.) Since $\hat{b}_I(\xi) = m(\xi)\hat{a}_I(\xi)$, the u_{0_I} and u_{1_I} are obtained from a_I by the Fourier multipliers

$$m_0(\xi) = \frac{-a_1 m(\xi)}{m(\xi)a_0 - a_1} \quad \text{and} \quad m_1(\xi) = \frac{-a_0 m(\xi)}{m(\xi)a_0 - a_1},$$

respectively, which are C^∞ on S^{n-1} (since $m(\xi)$ is pure imaginary and $a_1 \neq 0$), and homogeneous of degree 0 (since m is). In fact, since $m(-\xi) = \overline{m(\xi)}$, the same is true for m_0 and m_1 and hence u_{0_I} and u_{1_I} may be taken real valued. By [**7**, p. 75], $\exists \alpha_0, \alpha_1 \in \mathbf{R}$ and singular integral operators T_0 and T_1 such that $u_{i_I} = \alpha_i a_I + T_i a_I$, $i = 0, 1$. An easy calculation and the fact that m is pure imaginary show that $|D^\alpha m_i(\xi)| \leqslant c_{|\alpha|}(1 + |a_0/a_1|^{|\alpha|})$ $\forall \xi \in S^{n-1}$. Also $|\alpha_i| \leqslant \|m_i\|_{L^\infty(S^{n-1})} \leqslant C$ for $i = 0, 1$. Hence by Lemma U6, the solutions $\vec{u}_I$ of (5.1) and (5.2) satisfy (5.3), (5.4) and (5.5) for $l = C(n)$. Solving $K(u_{1_I}) = b_I$ is trivial since $b_I = Ka_I$, so that $u_{1_I} = a_I$ satisfies (5.6), (5.7) and (5.8). $\square$

COROLLARY 5.3. *If* $f(x) = \sum_I \lambda_I b_I(x)$ *in* L^2 *with* $\lambda_I \geqslant 0$ $\forall I$, *and if* (5.9)–(5.12) *hold and each* b_I *satisfies*

$$\operatorname{supp} b_I \subseteq 3I, \quad \|D_{x_j} b_I\|_{\mathrm{Lip}\,1} \leqslant c l(I)^{-2}, \quad \textit{and} \int b_I(x', x'')\,dx' = 0 \quad \forall x'', \tag{5.15}$$

where $x' = (x_1, \ldots, x_k)$ *and* $x'' = (x_{k+1}, \ldots, x_n)$, *then*

$$f \in \overline{L^\infty + \sum_{j=1}^{k} R_j(L_c^\infty)},$$

where the closure is taken in BMO *norm.*

PROOF. We may assume each b_I is real valued. If $k = 1$, we need only to show that $(*)$ holds for $K = R_1$ and $\{b_I\}_I$ under the assumption that $\int b_I(x_1, x'')\,dx_1 = 0$ $\forall x''$, I. For $a_1 \neq 0$, (5.1) and (5.2) can be solved by (5.13) and (5.14) with properties

(5.3), (5.4) and (5.5) as above. We solve $R_1 u_{1_I} = b_I$ as in Lemma 3.1 by setting $h_I(x_1, x') = \int_{-\infty}^{x_1} b_I(t, x')\, dt$. Then by (5.15), $\operatorname{supp} h_I \subseteq 3I$. Set

$$u_{1_I} = -\left(R_1 b_I + \sum_{j=2}^{n} R_j D_{x_j} h_I \right);$$

then $R_1 u_{1_I} = b_I$ as in Lemma 3.1. Since $\operatorname{supp} D_{x_j} h_I \subseteq 3I$, $\int D_{x_j} h_I(x)\, dx = 0$ and $\|D_{x_p} D_{x_j} H_I\|_{\mathrm{Lip}\,1} \leqslant cl(I)^{-2}$, $\forall j, p \in \{1, \ldots, n\}$, we obtain (5.6)–(5.8) by Lemma U6.

In general, it is sufficient to apply the case above once we note that if b_I satisfies (5.15), then $b_I = \sum_{p=1}^{k} b_{I_p}$, where each b_{I_p} satisfies $\operatorname{supp} b_{I_p} \subseteq 3I$, $\|D_{x_j} b_{I_p}\|_{\mathrm{Lip}\,1} \leqslant cl(I)^{-2}\ \forall_j$, and

$$\int b_{I_p}(x_1, \ldots, x_{p-1}, x_p, x_{p+1}, \ldots, x_n)\, dx_p = 0$$

$\forall (x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_n)$. To see this, suppose $I = \prod_{j=1}^{n} I_j$ and let $\alpha(x_1)$ be a function satisfying $\operatorname{supp} \alpha(x_1) \subseteq 3I_1$, $(1/|I_1|) \int \alpha(x_1)\, dx_1 = 1$ and $\|d^p \alpha / dx_1^p\|_{L^\infty} \leqslant c_p l(I)^{-p}$ for $p = 1, 2, 3 \ldots$. Let $\tilde{x} = (x_2, \ldots, x_n)$ and define $\beta(\tilde{x}) = (1/|I_1|) \int b_I(x_1, \tilde{x})\, dx_1$. Set $b_I^*(x) = \alpha(x_1)\beta(\tilde{x})$ and $b_{I_1} = b_I - b_I^*$. Then it follows from the definitions that $\int b_{I_1}(x_1, \tilde{x})\, dx_1 = 0\ \forall \tilde{x}$, $\operatorname{supp} b_{I_1} \subseteq 3I$ and $\|D_{x_j} b_{I_1}\|_{\mathrm{Lip}\,1} \leqslant cl(I)^{-2}$ for $j = 1, \ldots, n$. Similarly $\operatorname{supp} b_I^* \subseteq 3I$, $\|D_{x_j} b_I^*\|_{\mathrm{Lip}\,1} \leqslant cl(I)^{-2}$ for $j = 1, \ldots, n$, and

$$\begin{aligned}\int b_I^*(x_1, \ldots, x_n)\, dx_2 \cdots dx_k &= \alpha(x_1) \int \beta(\tilde{x})\, dx_2 \cdots dx_k \\ &= \frac{\alpha(x_1)}{|I_1|} \int b_I(x)\, dx_1 \cdots dx_k = 0 \quad \forall x_1.\end{aligned}$$

Continuing similarly with x_2 and b_I^*, after $k-1$ more steps we obtain $b_I = \sum_{p=1}^{k} b_{I_p}$ with the desired properties. □

Since the proof of Theorem 5.1 is an adaptation of the proof of the Main Lemma in [**8**, pp. 231–238], we follow the organization and terminology of Uchiyama's proof quite closely. Estimates which are essentially the same as Uchiyama's will only be stated or sketched. More detail is given in [**2**].

PROOF OF THEOREM 5.1. Let $\varepsilon > 0$ be given. We will obtain $g_0 \in L^\infty$, $g_1 \in L_c^\infty$ such that $\|f - g_0 - Kg_1\|_{\mathrm{BMO}} < \varepsilon$. We may assume $S = [-1, 1]^n$. Define $\{\eta_k\}_{k=-\infty}^{\infty}$ and $\{\varepsilon_k\}_{k=-\infty}^{\infty}$ as in Lemma U2 for the $\{\lambda_I\}_I$ given in Theorem 5.1. Let $l > 0$ be the number given by $(*)$. A sufficiently large positive integer M, and sufficiently large positive numbers R and p will be determined later, depending on ε. We inductively construct the functions $\{\vec{g}_k(x)\}_{k=-M-1}^{\infty}$, $\{\varphi_k(x)\}_{k=-M}^{\infty}$, $\{\vec{\beta}_{I,j}(x)\}_{j=0, l(I) \leqslant 2^M}^{\infty}$, $\{\vec{\psi}_k(x)\}_{k=-M}^{\infty}$ and $\{\vec{\zeta}_k(x)\}_{k=-M}^{\infty}$, satisfying

$$\operatorname{supp} \vec{\beta}_{I,j} \subseteq 2^j I, \quad \left|\nabla \vec{\beta}_{I,j}(x)\right| \leqslant cR^{l/p}\left(2^j l(I)\right)^{-1}, \quad \int \vec{\beta}_{I,j}(x)\, dx = \vec{0}, \tag{5.16}$$

$$\vec{K} \cdot \sum_{j=0}^{\infty} 2^{-j(n+1)} \vec{\beta}_{I,j} = b_I \quad \forall I \text{ such that } l(I) \leqslant 2^M, \tag{5.17}$$

$$\vec{\varphi}_k = \vec{\psi}_k + \vec{\zeta}_k \quad \text{for } -M \leqslant k < \infty, \tag{5.18}$$

(5.19)

$$\left|\vec{\psi}_k(x) - \vec{\psi}_k(y)\right| \leqslant c\left(2^{(n+2)M} R^{(2l/p-1)} + R^{-1/p}\right) 2^k |x - y| \quad \text{if } |x - y| \leqslant 2^{-k},$$

(5.20) $$\left|\vec{\psi}_k(x)\right| \leqslant c2^{(n+2)M}R^{(2l/p-1)}\varepsilon_k(x)\eta_k(x),$$

(5.21) $$\operatorname{supp}\vec{\psi}_k \subseteq \left\{x\colon |x| \leqslant 2\sqrt{n}\max(2^{M-k},1)\right\},$$

(5.22) $$\operatorname{supp}\vec{\zeta}_k \subseteq \left\{x\colon |x| \leqslant 2\sqrt{n}\max(2^{M-k},1)\right\},$$

(5.23) $$\vec{g}_{-M-1} = (R,0),$$

(5.24) $$|\vec{g}_k(x)| = R \quad \forall x, \forall k \geqslant -M-1,$$

(5.25) $$\vec{g}_k(x) - \vec{g}_{k-1}(x) = \sum_{l(I)=2^{-k}} \lambda_I \sum_{j=0}^{M} 2^{-j(n+1)}\vec{\beta}_{I,j}(x) - \vec{\varphi}_k(x)$$

and

(5.26) $$|\vec{g}_k(x) - \vec{g}_k(y)| \leqslant c_0 R^{l/p}\varepsilon_k(x)2^k|x-y| \quad \text{if } |x-y| \leqslant 2^{-k}.$$

As a consequence of the induction, we will obtain

(5.27) $$\sum_{j=-M}^{k} \vec{\zeta}_j(x) \quad \text{converges in } L^2 \text{ as } k \to \infty$$

and

(5.28) $$\left\| \sum_{j=-M}^{\infty} \vec{\zeta}_j(x) \right\|_{\mathrm{BMO}} \leqslant cR^{-1/p}.$$

First we assume the construction and complete the proof of the theorem. By (5.25) and (5.18), if $k \geqslant -M$

(5.29) $$\begin{aligned}\vec{g}_k(x) - \vec{g}_{-M-1}(x) = {} & \sum_{j=0}^{\infty} 2^{-j(n+1)} \sum_{I\colon 2^M \geqslant l(I) \geqslant 2^{-k}} \lambda_I \vec{\beta}_{I,j} \\ & - \sum_{j=M+1}^{\infty} 2^{-j(n+1)} \sum_{I\colon 2^M \geqslant l(I) \geqslant 2^{-k}} \lambda_I \vec{\beta}_{I,j} \\ & - \sum_{j=-M}^{k} \vec{\psi}_j - \sum_{j=-M}^{k} \vec{\zeta}_j.\end{aligned}$$

By (5.10), (5.16) and Lemma U3, the first two terms on the right-hand side of (5.29) converge in L^2 as $k \to \infty$. By (5.27) the last term in (5.29) converges in L^2 as $k \to \infty$. By (5.20), (5.21), (5.9) and Lemma U2, $\sum_{j=-M}^{k} \vec{\psi}_j$ converges in L^1 as $k \to \infty$. Since $\|\vec{g}_k - \vec{g}_j\|_{L^\infty} \leqslant cR$ $\forall k, j \geqslant -M-1$, $\vec{g}_k - \vec{g}_{-M-1}$ converges in L^2 as $k \to \infty$ by the argument in [**8**, p. 233]. Hence $\sum_{j=-M}^{k} \vec{\psi}_j$ converges in L^2 as $k \to \infty$ as well. Set

$$\vec{g}(x) = \vec{g}_{-M-1}(x) + \lim_{k \to \infty \text{ in } L^2} (\vec{g}_k(x) - \vec{g}_{-M-1}(x)).$$

Then $|\vec{g}(x)| = R$ $\forall x \in \mathbf{R}^n$. By (5.12), (5.16), (5.21), (5.29) and (5.23),

$$\operatorname{supp}(\vec{g} - (R,0)) \subseteq \left\{x\colon |x| \leqslant 2\sqrt{n}\,2^{2M}\right\}.$$

Hence $g_1 \in L_c^\infty$. By (5.29) and (5.17),

$$\vec{K}\cdot\vec{g}_k = \sum_{2^M \geqslant l(I) \geqslant 2^{-k}} \lambda_I b_I - \vec{K}\cdot\left\{\sum_{j=M+1}^{\infty} 2^{-j(n+1)} \sum_{I:\, 2^M \geqslant l(I)\geqslant 2^{-k}} \lambda_I \vec{\beta}_{I,j} + \sum_{j=-M}^{k} \vec{\psi}_j + \sum_{j=-M}^{k} \vec{\zeta}_j\right\}.$$

Therefore,

$$\vec{K}\cdot\vec{g} = f - \sum_{l(I) > 2^M} \lambda_I b_I - \vec{K}\cdot\left\{\sum_{j=M+1}^{\infty} 2^{-j(n+1)} \sum_{I:\, 2^M \geqslant l(I)} \lambda_I \vec{\beta}_{I,j} + \sum_{j=-M}^{\infty} \vec{\psi}_j + \sum_{j=-M}^{\infty} \vec{\zeta}_j\right\}.$$

By (5.11), $\lim_{M\to\infty}\|\sum_{l(I)>2^M} \lambda_I b_I\|_{\mathrm{BMO}} = 0$. By the boundedness of K on BMO, and by (5.16) and Lemma U4, we obtain, as in [**8**, p. 237],

$$\left\|\vec{K}\cdot \sum_{j=M+1}^{\infty} 2^{-j(n+1)} \sum_{I:\, 2^M \geqslant l(I)} \lambda_I \vec{\beta}_{I,j}\right\|_{\mathrm{BMO}} \leqslant c_1 2^{-M} R^{l/p}.$$

As in [**8**, p. 238], from (5.19) and (5.20) we obtain

$$\left\|\vec{K}\cdot \sum_{j=-M}^{\infty} \vec{\psi}_j\right\|_{\mathrm{BMO}} \leqslant c_2\left(2^{(n+2)M} R^{(2l/p-1)} + R^{-1/p}\right).$$

By (5.28), $\|\vec{K}\cdot\sum_{j=-M}^{\infty}\vec{\zeta}_j\|_{\mathrm{BMO}} \leqslant c_3 R^{-1/p}$. Let $c_4 = \max(c_1, c_2, c_3, 1)$. We will choose R sufficiently large and p so that $R^{1/p} = 8\varepsilon^{-1}c_4$. Choose M so that $\|\sum_{l(I)>2^M}\lambda_I b_I\|_{\mathrm{BMO}} < \varepsilon/4$, and so that $c_1 2^{-M} R^{l/p} = c_1 2^{-M}(8\varepsilon^{-1}c_4)^l < \varepsilon/4$. Now choose R sufficiently large so that

$$c_2 2^{(n+2)M} R^{(2l/p-1)} = c_2 2^{(n+2)M}\left(8\varepsilon^{-1}c_4\right)^{2l} R^{-1} < \varepsilon/8.$$

Finally pick p so that $R^{1/p} = 8\varepsilon^{-1}c_4$; then $c_2 R^{-1/p} < \varepsilon/8$ and $c_3 R^{-1/p} < \varepsilon/8$. Then $\|\vec{K}\cdot\vec{g} - f\|_{\mathrm{BMO}} < \varepsilon$, with $g_0 \in L^\infty$, $g_1 \in L_c^\infty$. For future reference, note that our choices allow us to assume $R^{1/p} > 1$ and $R^{2l/p} \ll R$.

We begin the proof of the construction. Define $\vec{g}_{-M-1}$ by (5.23). Assume $\{\vec{g}_p\}_{p=-M-1}^{k-1}$, $\{\vec{\varphi}_p\}_{p=-M}^{k-1}$, $\{\vec{\beta}_{I,j}\}_{j=0,2^{-k+1}\leqslant l(I)\leqslant 2^M}^{\infty}$, $\{\vec{\psi}_p\}_{p=-M}^{k-1}$ and $\{\vec{\zeta}_p\}_{p=-M}^{k-1}$, satisfying (5.16)–(5.26), have been constructed. Let I be a dyadic cube with $l(I) = 2^{-k}$. Let $\vec{a}(I) = (a_0(I), a_I(I)) = \vec{g}_{k-1}(x_I)$. Let

$$A_k = \left\{I\colon l(I) = 2^{-k} \text{ and } |a_0(I)/a_1(I)| \leqslant R^{1/p}\right\}.$$

Let $B_k = \{I\colon l(I) = 2^{-k}$ and either $a_1(I) = 0$ or $|a_0(I)/a_1(I)| > R^{1/p}\}$.

For $I \in A_k$, apply $(*)$(i), to I, b_I and $\vec{a} = \vec{a}(I)$ to obtain $\vec{u}_I\colon \mathbf{R}^n \to \mathbf{R}$ satisfying (5.1)–(5.5). Since $|a_0(I)/a_1(I)| \leqslant R^{1/p}$, by Lemma U5, $\exists\{\vec{\beta}_{I,j}(x)\}_{j=0}^{\infty}$ satisfying $\operatorname{supp}\vec{\beta}_{I,j} \subseteq 2^j I$, $\|\vec{\beta}_{I,j}\|_{\mathrm{Lip}\,1} \leqslant cR^{l/p}(2^j l(I))^{-1}$, $\int\vec{\beta}_{I,j}(x)\,dx = \vec{0}$, $\vec{a}(I)\cdot\vec{\beta}_{I,j}(x) = 0\ \forall x$, and $\vec{u}_I(x) = \sum_{j=0}^{\infty} 2^{-j(n+1)}\vec{\beta}_{I,j}(x)$. Note that therefore $|\vec{\beta}_{I,j}(x)| \leqslant cR^{l/p}\ \forall x$.

For $I \in B_k$, apply (*)(ii), to solve $Ku_{1I} = b_I$ with u_{1I} satisfying (5.6)–(5.8). By Lemma U5, $\exists\{\beta^1_{I,j}(x)\}_{j=0}^{\infty}$ satisfying $\operatorname{supp}\beta^1_{I,j} \subseteq 2^j I$, $\|\beta^1_{I,j}\|_{\mathrm{Lip}\,1} \leqslant c(2^j l(I))^{-1}$ and $\int\beta^1_{I,j}(x)\,dx = 0$, such that

$$u_{1I} = \sum_{j=0}^{\infty} 2^{-j(n+1)}\beta^1_{I,j}(x).$$

Define $\vec{\beta}_{I,j}(x) = (0, \beta^1_{I,j})$. Note that $|\vec{\beta}_{I,j}(x)| \leqslant c\ \ \forall x$, and $|\vec{a}(I)\cdot\vec{\beta}_{I,j}(x)| = |a_1(I)\beta^1_{I,j}(x)| \leqslant cR^{1-1/p}$, since $|a_1(I)| \leqslant |a_0(I)|R^{-1/p}$ and $I \in B_k$, and $|a_0(I)| \leqslant |\vec{g}_{k-1}(x_I)| = R$ by (5.24).

Note that the $\vec{\beta}_{I,j}$ defined in the above two cases satisfy (5.16) and (5.17). Define

$$\vec{h}_k(x) = \sum_{I:\, l(I)=2^{-k}} \lambda_I \sum_{j=0}^{M} 2^{-(n+1)j}\vec{\beta}_{I,j}(x),$$

$$\vec{\gamma}_k(x) = \vec{g}_{k-1}(x) + \vec{h}_k(x), \qquad \vec{g}_k(x) = R\frac{\vec{\gamma}_k(x)}{|\vec{\gamma}_k(x)|}$$

and

$$\begin{aligned}\vec{\varphi}_k(x) &= \vec{g}_{k-1}(x) + \vec{h}_k(x) - \vec{g}_k(x)\\ &= \left[\vec{g}_{k-1}(x) + \vec{h}_k(x)\right]\left[1 - \frac{R}{\left|\vec{g}_{k-1}(x) + \vec{h}_k(x)\right|}\right]\\ &= \frac{\vec{\gamma}_k(x)}{|\vec{\gamma}_k(x)|}\left[|\vec{\gamma}_k(x)| - R\right].\end{aligned} \tag{5.30}$$

Then (5.24) and (5.25) hold. Note that $\vec{h}_k(x) = \vec{0}$ and hence, by (5.30) and (5.24), $\vec{g}_k(x) = \vec{0}$, if $|x| > 2\sqrt{n}\max(2^{M-k}, 1)$, by (5.12) and the fact that $\operatorname{supp}\vec{\beta}_{I,j} \subseteq 2^j I$. From $|\vec{\beta}_{I,j}(x)| \leqslant cR^{l/p}$ we obtain

$$\left|\vec{h}_k(x)\right| \leqslant cR^{l/p}\eta_k(x) \tag{5.31}$$

as in [8, p. 235]. Similarly, from $|\nabla\vec{\beta}_{I,j}(x)| \leqslant cR^{l/p}(2^j l(I))^{-1}$, we obtain

$$\left|\vec{h}_k(x) - \vec{h}_k(y)\right| \leqslant c_5 R^{l/p}\eta_k(x)|x-y| \quad \text{if } |x-y| \leqslant 2^{-k}. \tag{5.32}$$

Therefore, if $|x-y| \leqslant 2^{-k}$, using (5.26) inductively, we obtain

$$\begin{aligned}|\vec{\gamma}_k(x) - \vec{\gamma}_k(y)| &\leqslant |\vec{g}_{k-1}(x) - \vec{g}_{k-1}(y)| + \left|\vec{h}_k(x) - \vec{h}_k(y)\right|\\ &\leqslant \left\{(c_0/2)\varepsilon_{k-1}(x) + c_5\eta_k(x)\right\}R^{l/p}2^k|x-y|\\ &\leqslant \tfrac{3}{4}c_0\varepsilon_k(x)R^{l/p}2^k|x-y|,\end{aligned}$$

if we choose $c_0 > \frac{4}{3}c_5$. Since $|\vec{h}_k(x)| \leqslant cR^{l/p} \ll R = |\vec{g}_{k-1}(x)|$, we obtain

$$|\vec{g}_k(x) - \vec{g}_k(y)| \leqslant \tfrac{4}{3}|\vec{\gamma}_k(x) - \vec{\gamma}_k(y)| \leqslant c_0\varepsilon_k(x)R^{l/p}2^k|x-y|$$

for $|x-y| \leqslant 2^{-k}$. Hence (5.26) holds.

We consider now $\vec{\varphi}_k(x) = \vec{\gamma}_k(x)[|\vec{\gamma}_k(x)| - R]/|\vec{\gamma}_k(x)|$. Write

$$\begin{aligned}|\vec{\gamma}_k(x)| - R &= \left|\vec{g}_{k-1}(x) + \vec{h}_k(x)\right| - R\\ &= R\left(1 + \frac{2\vec{g}_{k-1}(x)\cdot\vec{h}_k(x)}{R^2} + \frac{\left|\vec{h}_k(x)\right|^2}{R^2}\right)^{1/2} - R\\ &= R\left(1 + \frac{\vec{g}_{k-1}(x)\cdot\vec{h}_k(x)}{R^2} + \frac{\left|\vec{h}_k(x)\right|^2}{2R^2}\right) - R + E_1(x)\\ &= \frac{\vec{g}_{k-1}(x)\cdot\vec{h}_k(x)}{R} + E_2(x),\end{aligned}$$

where the last two equalities define E_1 and E_2. Since $|\sqrt{1+a} - 1 - \frac{1}{2}a| \leqslant a^2$ for $|a| < \frac{1}{2}$, using (5.24) we obtain

(5.33)

$$|E_2(x)| \leqslant \frac{\left|\vec{h}_k(x)\right|^2}{2R} + cR\left(\frac{\left|\vec{g}_{k-1}(x)\cdot\vec{h}_k(x)\right|}{R^2} + \frac{\left|\vec{h}_k(x)\right|^2}{R^2}\right)^2 \leqslant c\frac{\left|\vec{h}_k(x)\right|^2}{R}.$$

Hence

$$\vec{\varphi}_k(x) = \frac{\vec{\gamma}_k(x)}{|\vec{\gamma}_k(x)|}\,\frac{\vec{g}_{k-1}(x)\cdot\vec{h}_k(x)}{R} + \frac{\vec{\gamma}_k(x)}{|\vec{\gamma}_k(x)|}E_2 \equiv (\mathrm{I})_k + (\mathrm{II})_k.$$

We write

$$\begin{aligned}(\mathrm{I})_k &= \frac{\vec{g}_{k-1}(x) + \vec{h}_k(x)}{\left|\vec{g}_{k-1}(x) + \vec{h}_k(x)\right|}\,\frac{\vec{g}_{k-1}(x)\cdot\vec{h}_k(x)}{R}\\ &= \frac{\vec{g}_{k-1}(x)\cdot\vec{h}_k(x)}{R}\,\frac{\vec{h}_k(x)}{\left|\vec{g}_{k-1}(x) + \vec{h}_k(x)\right|} + \frac{\vec{g}_{k-1}(x)\cdot\vec{h}_k(x)}{R}\,\frac{\vec{g}_{k-1}(x)}{R}\\ &\quad + \frac{\vec{g}_{k-1}(x)\cdot\vec{h}_k(x)}{R}\vec{g}_{k-1}(x)\left[\frac{1}{\left|\vec{g}_{k-1}(x) + \vec{h}_k(x)\right|} - \frac{1}{R}\right]\\ &\equiv (\mathrm{III})_k + (\mathrm{IV})_k + (\mathrm{V})_k.\end{aligned}$$

Now

$$\begin{aligned}(\mathrm{IV})_k &= \frac{\vec{g}_{k-1}(x)}{R}\left(\frac{1}{R}\sum_{I:\,l(I)=2^{-k}}\lambda_I\sum_{j=0}^{M}2^{-j(n+1)}\vec{g}_{k-1}(x)\cdot\vec{\beta}_{I,j}(x)\right)\\ &= \frac{\vec{g}_{k-1}(x)}{R^2}\sum_{I:\,l(I)=2^{-k}}\lambda_I\sum_{j=0}^{M}2^{-j(n+1)}\left(\vec{g}_{k-1}(x) - \vec{g}_{k-1}(x_I)\right)\cdot\vec{\beta}_{I,j}(x)\\ &\quad + \frac{1}{R^2}\sum_{I:\,l(I)=2^{-k}}\lambda_I\sum_{j=0}^{M}2^{-j(n+1)}\left(\vec{g}_{k-1}(x) - \vec{g}_{k-1}(x_I)\right)\vec{g}_{k-1}(x_I)\cdot\vec{\beta}_{I,j}(x)\\ &\quad + \frac{1}{R^2}\sum_{I:\,l(I)=2^{-k}}\lambda_I\sum_{j=0}^{M}2^{-j(n+1)}\vec{g}_{k-1}(x_I)\vec{g}_{k-1}(x_I)\cdot\vec{\beta}_{I,j}(x)\\ &\equiv (\mathrm{VI})_k + (\mathrm{VII})_k + (\mathrm{VIII})_k.\end{aligned}$$

Define $\vec{\psi}_k = (\mathrm{II})_k + (\mathrm{III})_k + (\mathrm{V})_k + (\mathrm{VI})_k + (\mathrm{VII})_k$, and $\vec{\zeta}_k = (\mathrm{VIII})_k$. Then (5.18) holds. By (5.12) and the fact that $\operatorname{supp} \vec{\beta}_{I,j} \subseteq 2^j I$, (5.22) holds. By (5.30), $\vec{\varphi}_k(x) = \vec{0}$ if $|x| > 2\sqrt{n}\max(2^{M-k}, 1)$, and so (5.21) holds also.

We now prove (5.20). By (5.33), (5.31) and (2.5),

$$|(\mathrm{II})_k| = |E_2(x)| \leqslant \frac{c}{R}\left|\vec{h}_k(x)\right|^2 \leqslant cR^{(2l/p-1)}\eta_k^2(x)$$
$$\leqslant cR^{(2l/p-1)}\varepsilon_k(x)\eta_k(x).$$

Similarly,

$$|(\mathrm{III})_k| \leqslant \frac{c}{R}\left|\vec{h}_k(x)\right|^2 \leqslant |cR^{(2l/p-1)}\varepsilon_k(x)\eta_k(x)$$

and

$$|(\mathrm{V})_k| \leqslant \left|\vec{g}_{k-1}(x)\cdot\vec{h}_k(x)\right| \frac{\left|R - \left|\vec{g}_{k-1}(x) + \vec{h}_k(x)\right|\right|}{R\left|\vec{g}_{k-1}(x) + \vec{h}_k(x)\right|}$$
$$\leqslant \frac{\left|\vec{g}_{k-1}(x)\cdot\vec{h}_k(x)\right|^2}{R^2\left|\vec{g}_{k-1}(x) + \vec{h}_k(x)\right|} + \frac{\left|\vec{g}_{k-1}(x)\cdot\vec{h}_k(x)\right|\,|E_2|}{R\left|\vec{g}_{k-1}(x) + \vec{h}_k(x)\right|}$$
$$\leqslant \frac{c}{R}\left|\vec{h}_k(x)\right|^2 \leqslant cR^{(2l/p-1)}\varepsilon_k(x)\eta_k(x),$$

by our above estimates on $|\vec{\gamma}_k(x)| - R$ and E_2. To estimate $(\mathrm{VI})_k$ and $(\mathrm{VII})_k$, we note, as in [**8**, p. 235], that if $l(I) = 2^{-k}$ and $\vec{\beta}_{I,j}(x) \neq \vec{0}$, $0 \leqslant j \leqslant M$, then $|x - x_I| \leqslant c2^{M-k}$, and

(5.34) $$\left|\vec{g}_{k-1}(x) - \vec{g}_{k-1}(x_I)\right| \leqslant c2^{(n+1)M}R^{l/p}\varepsilon_{k-1}(x)2^{k-1}|x - x_I|.$$

Therefore, for the same x and I,

(5.35) $$\left|\left(\vec{g}_{k-1}(x) - \vec{g}_{k-1}(x_I)\right)\cdot\vec{\beta}_{I,j}(x)\right| \leqslant c2^{(n+2)M}R^{l/p}\varepsilon_{k-1}(x)\left|\vec{\beta}_{I,j}(x)\right|$$

since $|x - x_I| \leqslant c2^{M-k}$ for these x and I. Therefore

$$|(\mathrm{VI})_k| \leqslant \frac{c}{R}\sum_{I:\, l(I)=2^{-k}} \lambda_I \sum_{j=0}^{M} 2^{-j(n+1)}R^{l/p}2^{(n+2)M}\varepsilon_{k-1}(x)\left|\vec{\beta}_{I,j}(x)\right|$$
$$\leqslant c2^{(n+2)M}R^{(2l/p-1)}\varepsilon_{k-1}(x)\eta_k(x),$$

since $|\vec{\beta}_{I,j}(x)| \leqslant cR^{l/p}$, by the method used to obtain (5.31). Similarly, using (5.34) instead of (5.35),

$$|(\mathrm{VII})_k| \leqslant \frac{c}{R}\sum_{I:\, l(I)=2^{-k}} \lambda_I \sum_{j=0}^{M} 2^{-j(n+1)}2^{(n+1)M}R^{l/p}\varepsilon_{k-1}(x)2^{k-1}|x - x_I|\left|\vec{\beta}_{I,j}(x)\right|$$
$$\leqslant c2^{(n+2)M}R^{(2l/p-1)}\varepsilon_{k-1}(x)\sum_{j=0}^{M} 2^{-(n+1)j} \sum_{\substack{I:\, l(I)=2^{-k} \\ \operatorname{dist}(x,\, I)\leqslant 2^{j-k}}} \lambda_I$$
$$\leqslant c2^{(n+2)M}R^{(2l/p-1)}\varepsilon_{k-1}(x)\eta_k(x),$$

as in (5.31). Hence (5.20) holds, since $\varepsilon_{k-1} \leqslant \frac{3}{2}\varepsilon_k$.

To establish (5.19), we prove corresponding estimates for $\vec{\varphi}_k$ and $\vec{\zeta}_k$ and use (5.18). Following [**8**, p. 236], for $|x-y| \leqslant 2^{-k}$, write

$$\begin{aligned}\vec{\varphi}_k(x) - \vec{\varphi}_k(y) &= |\vec{\gamma}_k(x)|^{-1}\big(|\vec{\gamma}_k(x)| - R\big)\big(\vec{\gamma}_k(x) - \vec{\gamma}_k(y)\big)\\ &\qquad + \vec{\gamma}_k(y) R \frac{|\vec{\gamma}_k(x)| - |\vec{\gamma}_k(y)|}{|\vec{\gamma}_k(x)||\vec{\gamma}_k(y)|}\\ &\equiv (\tilde{\mathrm{I}})_k + (\widetilde{\mathrm{II}})_k.\end{aligned}$$

Since $\big||\vec{\gamma}_k(x)| - R\big| \leqslant cR^{l/p}$, $|\vec{\gamma}_k(x) - \vec{\gamma}_k(y)| \leqslant c_0 \varepsilon_k(x) R^{l/p} 2^k |x-y|$ and $|\vec{\gamma}_k(x)| > R/2$ (since $|\vec{h}_k(x)| \ll R$), we obtain

$$|(\tilde{\mathrm{I}})_k| \leqslant cR^{(2l/p-1)} \varepsilon_k(x) 2^k |x-y| \quad \text{for } |x-y| \leqslant 2^{-k}.$$

As in [**8**, p. 237],

$$\begin{aligned}\big|(\widetilde{\mathrm{II}})_k\big| &\leqslant 2\Big||\vec{g}_{k-1}(x) + \vec{h}_k(x)| - |\vec{g}_{k-1}(y) + \vec{h}_k(x)|\Big|\\ &\qquad + 2\Big||\vec{g}_{k-1}(y) + \vec{h}_k(x)| - |\vec{g}_{k-1}(y) + \vec{h}_k(y)|\Big|\\ &\equiv (\widetilde{\mathrm{III}})_k + (\widetilde{\mathrm{IV}})_k.\end{aligned}$$

Now

$$\begin{aligned}(\widetilde{\mathrm{III}})_k &= \Bigg| R\left(1 + \frac{2\vec{g}_{k-1}(x)\cdot\vec{h}_k(x)}{R^2} + \frac{|\vec{h}_k(x)|^2}{R^2}\right)^{1/2}\\ &\qquad - R\left(1 + \frac{2\vec{g}_{k-1}(y)\cdot\vec{h}_k(x)}{R^2} + \frac{|\vec{h}_k(x)|^2}{R^2}\right)^{1/2}\Bigg|\\ &\leqslant \frac{c}{R}\big|\big(\vec{g}_{k-1}(x) - \vec{g}_{k-1}(y)\big)\cdot \vec{h}_k(x)\big|\\ &\leqslant cR^{(2l/p-1)}\varepsilon_{k-1}(x)2^k|x-y|\eta_k(x) \leqslant cR^{(2l/p-1)}\varepsilon_k(x)2^k|x-y|,\end{aligned}$$

by (5.26), (5.31) and a Lipschitz estimate for the function $f(t) = (1+t)^{1/2}$ for t sufficiently small. Similarly,

$$\begin{aligned}\big|(\widetilde{\mathrm{IV}})_k\big| &= \Bigg| R\left(1 + \frac{2\vec{g}_{k-1}(y)\cdot\vec{h}_k(x)}{R^2} + \frac{|\vec{h}_k(x)|^2}{R^2}\right)^{1/2}\\ &\qquad - R\left(1 + \frac{2\vec{g}_{k-1}(y)\cdot\vec{h}_k(y)}{R^2} + \frac{|\vec{h}_k(y)|^2}{R^2}\right)^{1/2}\Bigg|\\ &\leqslant \frac{c}{R}\big|\vec{g}_{k-1}(y)\cdot\big(\vec{h}_k(x) - \vec{h}_k(y)\big)\big| + \frac{c}{R}\Big||\vec{h}_k(x)|^2 - |\vec{h}_k(y)|^2\Big|\\ &\equiv |(\tilde{\mathrm{V}})_k| + \big|(\widetilde{\mathrm{VI}})_k\big|.\end{aligned}$$

By (5.32) and (2.5) we have $|\widetilde{(\mathrm{VI})}_k| \leqslant cR^{(2l/p-1)}2^k|x-y|$. Now

$$|(\tilde{\mathrm{V}})_k| \leqslant \frac{c}{R}\left|\sum_{I:\, l(I)=2^{-k}} \lambda_I \sum_{j=0}^{M} 2^{-j(n+1)}\big(\vec{g}_{k-1}(y)-\vec{g}_{k-1}(x_I)\big)\cdot\big(\vec{\beta}_{I,j}(x)-\vec{\beta}_{I,j}(y)\big)\right|$$

$$+\frac{c}{R}\left|\sum_{I:\, l(I)=2^{-k}} \lambda_I \sum_{j=0}^{M} 2^{-j(n+1)}\vec{g}_{k-1}(x_I)\cdot\big(\vec{\beta}_{I,j}(x)-\vec{\beta}_{I,j}(y)\big)\right|$$

$$\equiv \widetilde{(\mathrm{VII})}_k + \widetilde{(\mathrm{VIII})}_k.$$

If $\vec{\beta}_{I,j}(x)$ and $\vec{\beta}_{I,j}(y)$ are not both $\vec{0}$, then $|y-x_I| \leqslant c\cdot 2^{M-k}$, since $|x-y|\leqslant 2^{-k}$; so by applying (5.34) we obtain

$$\left|\widetilde{(\mathrm{VII})}_k\right| \leqslant cR^{(l/p-1)}2^{(n+2)M}\sum_{I:\, l(I)=2^{-k}} \lambda_I \sum_{j=0}^{M} 2^{-j(n+1)}\left|\vec{\beta}_{I,j}(x)-\vec{\beta}_{I,j}(y)\right|$$

$$\leqslant cR^{(2l/p-1)}2^{(n+2)M}\eta_k(x)2^k|x-y|,$$

by the calculation in estimate (5.32). Since $\vec{g}_{k-1}(x_I)\cdot\vec{\beta}_{I,j}(x)=0$ $\forall x$ whenever $I\in A_k$, we have

$$\left|\widetilde{(\mathrm{VIII})}_k\right| = \frac{c}{R}\left|\sum_{I\in B_k}\lambda_I\sum_{j=0}^{M}2^{-j(n+1)}a_1(I)\big(\beta^1_{I,j}(x)-\beta^1_{I,j}(y)\big)\right|,$$

since $\vec{\beta}_{I,j}=(0,\beta^1_{I,j})$ for $I\in B_k$. Since $|a_1(I)|\leqslant R^{1-1/p}$ and $\|\beta^1_{I,j}\|_{\mathrm{Lip}\,1}\leqslant c(2^jl(I))^{-1}$ for $I\in B_k$, and by (2.5) and the proof of (5.32),

$$\left|\widetilde{(\mathrm{VIII})}_k\right| \leqslant cR^{-1/p}\sum_{j=0}^{M}2^{-j(n+1)}\sum_{\substack{I:\, l(I)=2^{-k}\\ \mathrm{dist}(x,I)\leqslant 2^{j-k}}}\lambda_I|x-y|$$

$$\leqslant cR^{-1/p}\eta_k(x)2^k|x-y| \leqslant cR^{-1/p}2^k|x-y| \quad \text{for } |x-y|\leqslant 2^{-k}.$$

Hence

$$|\vec{\varphi}_k(x)-\vec{\varphi}_k(y)| \leqslant c\big(2^{(n+2)M}R^{(2l/p-1)}+R^{-1/p}\big)2^k|x-y|$$

if $|x-y|\leqslant 2^{-k}$.

Whenever $|x-y|\leqslant 2^{-k}$, we have

$$|\vec{\zeta}_k(x)-\vec{\zeta}_k(y)| = \frac{1}{R^2}\left|\sum_{I\in B_k}\lambda_I\sum_{j=0}^{M}2^{-j(n+1)}\vec{g}_{k-1}(x_I)\vec{g}_{k-1}(x_I)\cdot\big(\vec{\beta}_{I,j}(x)-\vec{\beta}_{I,j}(y)\big)\right|$$

$$\leqslant cR^{-1/p}\sum_{I\in B_k}\lambda_I\sum_{j=0}^{M}2^{-j(n+1)}\left|\beta^1_{I,j}(x)-\beta^1_{I,j}(y)\right|$$

$$\leqslant cR^{-1/p}2^k|x-y|,$$

since $\vec{g}_{k-1}(x_I)\cdot\vec{\beta}_{I,j}(x)=0$ for $I\in A_k$, and for $I\in B_k$, $\vec{g}_{k-1}(x_I)=(a_0(I),a_I(I))$, with $|a_1(I)|\leqslant R^{1-1/p}$ and $\vec{\beta}_{I,j}=(0,\beta^1_{I,j})$. The estimates for $\vec{\psi}_k$ and $\vec{\zeta}_k$ give (5.19), by (5.18).

We now verify (5.27) and (5.28). We have, for $k > q \geqslant -M$,

$$\left\| \sum_{s=q+1}^{k} \vec{\zeta}_s \right\|_{L^2} = \frac{1}{R^2} \left\| \sum_{I \in \bigcup_{s=q+1}^{k} B_s} \lambda_I \sum_{j=0}^{M} 2^{-j(n+1)} \vec{g}_{s-1}(x_I) a_1(I) \beta_{I,j}^1 \right\|_{L^2}$$

$$\leqslant cR^{-1/p} \sum_{j=0}^{M} 2^{-j(n+1)} \left\| \sum_{I \in \bigcup_{s=q+1}^{k} B_s} \frac{\vec{g}_{s-1}(x_I) a_1(I)}{R^{2-1/p}} \lambda_I \beta_{I,j}^1 \right\|_{L^2}.$$

Note that $|\vec{g}_{s-1}(x_I) a_1(I)/R^{2-1/p}| \leqslant c$ for $I \in \bigcup_{s=q+1}^{k} B_s$, so by Lemma U3

$$\left\| \sum_{s=q+1}^{k} \vec{\zeta}_s \right\|_{L^2} \leqslant cR^{-1/p} \left(\sum_{I \in \bigcup_{s=q+1}^{k} B_s} \lambda_I^2 |I| \right)^{1/2}.$$

By (5.10), then, (5.27) holds. Similarly, by Lemma U4,

$$\left\| \sum_{s=-M}^{\infty} \vec{\zeta}_s \right\|_{\mathrm{BMO}} \leqslant cR^{-1/p} \sum_{j=0}^{M} 2^{-j(n+1)} \left\| \sum_{I \in \bigcup_{s=-M}^{\infty} B_s} \frac{\vec{g}_{s-1}(x_I) a_1(I)}{R^{2-1/p}} \lambda_I \beta_{I,j}^1 \right\|_{\mathrm{BMO}}$$

$$\leqslant cR^{-1/p}.$$

This establishes (5.28) and hence Theorem 5.1. □

6. Problems and remarks. The work of Janson [5] and Uchiyama [8] gives the complete result that a collection $\{K_j\}_{j=1}^{p}$ of singular integral operators satisfy $\sum_{j=1}^{p} \tilde{K}_j L^\infty = \mathrm{BMO}$ if and only if

$$\operatorname{rank} \begin{pmatrix} m_1(\xi) & \cdots & m_p(\xi) \\ m_1(-\xi) & & m_p(-\xi) \end{pmatrix} = 2 \quad \forall \xi \in S^{n-1}. \tag{6.1}$$

However, little is known about the subspace $\sum_{j=1}^{p} \tilde{K}_j L^\infty$ of BMO when (6.1) fails. For the Riesz transforms, Corollary 4.3 implies that any $f \in \sum_{j=1}^{p} \tilde{R}_j L^\infty$, $p < n$, satisfies a regularity condition on appropriate p-dimensional subspaces of $\mathbf{R}^n$ that is not satisfied for all BMO functions. One would hope to obtain some similar interpretation, whenever (6.1) fails, of the fact that $\sum_{j=1}^{p} \tilde{K}_j L^\infty \neq \mathrm{BMO}$. The study of the Riesz transforms was facilitated by their cancellation property; to proceed in the general case, one would presumably require a geometrical interpretation of the failure of (6.1).

However, even for the Riesz transforms the results are far from complete. Our regularity conditions hold for $f \in R_1(\mathrm{BMO} \cap L^2)$ as well as $\tilde{R}_1 L^\infty$; boundedness played a role in Corollary 4.3 only in removing the L^2 restriction. Further, only the condition $\lambda_I \leqslant c$, rather than the full strength of the BMO packing condition (3.5), is used in proving Theorem 3.2. Hence our regularity conclusions hold for larger classes of functions and cannot characterize $R_1(\mathrm{BMO} \cap L^2)$ or $\tilde{R}_1 L^\infty$. The decomposition result in Lemma 3.1 is a complete, but not explicitly geometrical, characterization of $R_1(\mathrm{BMO} \cap L^2)$; it is not clear whether a complete geometrical characterization is to be expected. Example 4.6 shows that we cannot expect much stronger regularity along lines.

Since our regularity results hold for $R_1(\mathrm{BMO} \cap L^2)$ as well as $\tilde{R}_1 L^\infty$, it is natural to ask if there are additional geometrical properties that hold for $\tilde{R}_1 L^\infty$ but not for $R_1(\mathrm{BMO} \cap L^2)$. Corollary 5.2 suggests that any such distinction may be subtle. Explicit examples of functions $f \in L^\infty + R_1(\mathrm{BMO} \cap L^2)$ such that $f \notin L^\infty + \tilde{R}_1 L^\infty$ would be helpful, but are not apparent since our only criterion for excluding f from $L^\infty + \tilde{R}_1 L^\infty$ is the failure of our regularity condition, which holds for $L^\infty + R_1(\mathrm{BMO} \cap L^2)$ as well. Our function $G \in \mathrm{BMO} \cap L^2(R^2)$ in Example 4.6 is relevant: if $R_1 G \in L^\infty + \tilde{R}_1 L^\infty$, then a uniform BMO condition on x_1-lines does not hold for $L^\infty + \tilde{R}_1 L^\infty$; if not, then $R_1 G$ distinguishes $R_1(L^2 \cap \mathrm{BMO})$ from $L^\infty + \tilde{R}_1 L^\infty$. Either case will give information as soon as it is determined which case holds.

A further question regarding each subspace $Y \subseteq \mathrm{BMO}$ under consideration is the determination of $\mathrm{dist}(f, Y) = \inf\{\|f - g\|_{\mathrm{BMO}}\colon g \in Y\}$ for $f \in \mathrm{BMO}$. When $Y = L^\infty$, Garnett and Jones [4] have complete results. For K as in §5, Corollary 5.2 is obviously equivalent to the statement that

$$\mathrm{dist}\big(f, L^\infty + K(L_c^\infty)\big) = \mathrm{dist}\big(f, L^\infty + K(\mathrm{BMO}_c)\big).$$

For the Riesz transforms, our conditions on lines should play a role in this problem.

As noted above, some of our results must apply in greater generality than presented here. Since the method of the decomposition of BMO (Lemma U1) appears quite general, one suspects that further information about the action of specific operators on appropriate spaces can be obtained by similar methods.

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