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ANALYTIC UNIFORMLY BOUNDED REPRESENTATIONS OF $SU(1, n + 1)$

BY

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ABSTRACT. By analytically continuing suitably normalized spherical principal series, a family of uniformly bounded representations of $SU(1, n + 1)$, all of which act on the same Hilbert space $L^2(\mathbf{R}^{2n+1})$, is constructed which is parametrized by complex numbers s lying in the strip $-1 < \mathrm{Re}(s) < 1$. The proper normalization of the principal series representations involves the intertwining operators of equivalent principal series representations. These intertwining operators are first analyzed using Fourier analysis on the Heisenberg group.

1. Introduction. In a series of papers [**7–9**], Kunze and Stein constructed uniformly bounded representations of several real and complex semisimple Lie groups. The construction of these representations is interesting since it involves various intertwining operator techniques. Moreover, such representations have important uses. First, they offer a unified view of the complementary and principal series representations. In this picture, the complementary series is parametrized by a bounded interval of the real line while the principal series is parametrized by the imaginary axis of the complex plane. Secondly, these uniformly bounded representations may be used to obtain estimates which establish such L^p convolution theorems as the so-called Kunze-Stein phenomenon. However, M. Cowling [**1, 2**] proved in general that such convolution theorems may be established without using uniformly bounded representations. Various of these subsequent results have been extended to some other semisimple Lie groups by P. Sally [**14**], Lipsman [**11**] and Wilson [**16**].

Let G denote the group $SU(1, n + 1)$, $n > 0$. The unitary spherical principal series representation of G is a family of representations $T(\cdot, s)$, where s is a purely imaginary complex number. The purpose of this paper is to construct a family of representations $S(\cdot, s)$ for $-1 < \mathrm{Re}(s) < 1$ satisfying:

(1) $S(\cdot, s)$ is unitarily equivalent to $T(\cdot, s)$ when $\mathrm{Re}(s) = 0$,
(2) $s \to S(g, s)$ is analytic in $-1 < \mathrm{Re}(s) < 1$ for each $g \in G$,
(3) $\sup_{g \in G} \|S(g, s)\| < \infty$ for each s,
(4) $S(\cdot, s)$ is unitary and satisfies $S(\cdot, s) = S(\cdot, -s)$ when $-1 < s < 1$.

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In order to construct uniformly bounded representations of G, we analytically continue suitably normalized spherical principal series representations. The operators used to normalize the principal series are given formally by singular integral operators and are closely related to the intertwining operators of equivalent principal series. For the Lorentz groups $SO(1, n+1)$, Wilson **[16]** amply demonstrates the important role Euclidean Fourier analysis plays in studying such intertwining operators. For the semisimple group $SU(1, n+1)$, we find that understanding these intertwining operators is possible by invoking the (noncommutative) Fourier analysis of the Heisenberg group H_n. The Plancherel theorem for H_n allows us to view each intertwining operator as a left multiplication operator on a Hilbert space of Hilbert-Schmidt valued functions. One advantage to this group theoretic approach is demonstrated by the natural way each of these left multiplication operators, at least in a pointwise manner, is diagonalized. The explicit computation of these diagonal entries is then what makes our problem tractable. Indeed, from these diagonal entries one determines the meromorphic properties of the intertwining operators. In addition, these diagonal entries permit us to construct uniformly bounded representations of $SU(1, n+1)$ which are equivalent to the complementary series.

It is important to point out that M. Cowling **[1, 2]** has also constructed a Hilbert space H_z and a uniformly bounded representation R_z of $SU(1, n+1)$ for each z in $|\mathrm{Re}(z)| < n+1$. In contrast, the construction given here yields uniformly bounded representations $S(\cdot, s)$ all of which act on the same Hilbert space $L^2(H_n)$. The significance of constructing all representations on the same Hilbert space is that the analyticity condition (2) as well as the symmetry condition (4) can be established, thus paralleling the earlier results of Kunze and Stein. Furthermore, M. Cowling **[1]** also computes the Fourier transform of the intertwining operators using complex variable techniques. A more natural and direct approach to this calculation is taken which illustrates the canonical way special functions arise in the representation theory of H_n. The desired transform is obtained here by computing a certain integral over the unit sphere and then applying a known Bessel function integral identity.

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The author would also like to thank the referee for valuable suggestions concerning points of clarity and exposition.

2. Harmonic analysis on the Heisenberg group. In this section we state, without proof, some of the basic results concerning the Fourier analysis of the Heisenberg group.

For $n \geqslant 1$, we define the Heisenberg group H_n to be the set $\mathbf{R} \times \mathbf{C}^n$ with group product given by

$$(t, z)(t', z') = \left(t + t' + \mathrm{Im}(z|z'), z + z'\right),$$

where $t, t' \in \mathbf{R}$ and $z, z' \in \mathbf{C}^n$. This is a two-step nilpotent group whose Haar measure is given by ordinary Lebesgue measure on $\mathbf{R}^{2n+1}$.

Let $\mathbf{R}'$ denote the nonzero real numbers. For each $\alpha \in \mathbf{R}'$, we define $L^2_\alpha(H_n)$ to be the Hilbert space of all measurable functions f on H_n which satisfy

$$f\big((t, z)(t', 0)\big) = \exp(-\pi i \alpha t') f(t, z), \qquad t, t' \in \mathbf{R}, z \in \mathbf{C},$$

and

$$\|f\|_2^2 = \int_{\mathbf{C}^n} |f(0, z)|^2 \, dz < \infty.$$

Let $L^\alpha(h)\colon L^2_\alpha(H_n) \to L^2_\alpha(H_n)$ denote left translation by $h \in H_n$ which leaves $L^2_\alpha(H_n)$ invariant. For $f \in L^1(\mathbf{C}^n)$, we define a continuous operator $L^\alpha(f)$ by

$$L^\alpha(f) = \int_{\mathbf{C}^n} f(z) L^\alpha(0, z) \, dz.$$

It is clear that for any $f \in L^1(\mathbf{C}^n) \cap L^2(\mathbf{C}^n)$ and $g \in L^2_\alpha(H_n)$,

$$(L^\alpha(f) g)(h) = \int_{\mathbf{C}^n} f(z) g\big((0, z)^{-1} h\big) \, dz.$$

We refer to the right side here as the convolution of f and g with respect to $\mathbf{C}^n$ and denote it by $(f * g)(h)$. For any $f, g \in L^2_\alpha(H_n)$, $f * g$ defines a continuous bounded function on H_n.

Let $p_\alpha \in L^2_\alpha(H_n)$ be the function

$$p_\alpha(t, z) = |\alpha|^n \exp\Big(-\pi\big(i\alpha t + \tfrac{1}{2}|\alpha|\, |z|^2\big)\Big), \qquad (t, z) \in H_n.$$

For $\alpha < 0$, let $C(\alpha)$ denote the linear space of all square integrable functions over $\mathbf{C}^n$ of the form $p_\alpha(t, z) H(z)$, where $z \to H(z)$ is an entire function on $\mathbf{C}^n$. If $\alpha > 0$, then we define $C(\alpha)$ to be the set of complex conjugates of elements in $C(-\alpha)$. It is known that $C(\alpha)$ is a closed left-invariant subspace of $L^2_\alpha(H_n)$ and that the restriction of left translation to $C(\alpha)$ is an irreducible unitary representation of H_n which is essentially the standard Fock model for unitary representations of H_n.

In the following result, let $\mathscr{H}(C(\alpha))$ denote the Hilbert space of all Hilbert-Schmidt operators on $C(\alpha)$, $\alpha \in \mathbf{R}'$.

THEOREM 2.1. *For $f \in L^1(\mathbf{C}^n) \cap L^2(\mathbf{C}^n)$, let $S^\alpha(f)$ denote the restriction of $|\alpha|^{n/2} L^\alpha(f)$ to $C(\alpha)$. Then S^α extends to a unitary transformation of $L^2(\mathbf{C}^n)$ onto $\mathscr{H}(C(\alpha))$. In particular,*

$$\|f\|_2^2 = |\alpha|^n \operatorname{tr}(L^\alpha(f)^* L^\alpha(f)),$$

where $f \in L^2(\mathbf{C}^n)$ and the trace is taken with respect to a basis of $C(\alpha)$.

PROOF. We refer to Theorem 1.6 of [4] for details. □

For $\alpha \in \mathbf{R}'$ and $f \in L^1(H_n) \cap L^2(H_n)$, we define the continuous operator $\Lambda^\alpha(f) = \int_{\mathbf{R}\times\mathbf{C}^n} f(t, z) L^\alpha(t, z) \, dt \, dz$ on $L^2_\alpha(H_n)$. If $\Lambda^\alpha(f)$ is restricted to the subspace $C(\alpha)$, then it is known that this restriction defines a Hilbert-Schmidt operator on $C(\alpha)$ such that

$$\|f\|_2^2 = \int_{\mathbf{R}} \operatorname{tr}(\Lambda^\alpha(f)^* \Lambda^\alpha(f)) |\alpha|^n \, d\alpha/2, \tag{2.1}$$

where the trace is taken with respect to a basis of $C(\alpha)$. Let $L^2(\mathbf{R}'; \mathrm{HS})$ denote the space of measurable Hilbert-Schmidt valued functions F defined on $\mathbf{R}'$ such that $F(\alpha) \in \mathscr{H}(C(\alpha))$, a.e. $\alpha \in \mathbf{R}'$, and $\int_{\mathbf{R}} \operatorname{tr}(F(\alpha)^* F(\alpha))|\alpha|^n \, d\alpha/2 < \infty$. For $f \in L^1(H_n) \cap L^2(H_n)$, let $\Lambda(f)$ denote the function whose value at $\alpha \in \mathbf{R}'$ is $\Lambda^\alpha(f)$. The Plancherel theorem for H_n states that the map $f \to \Lambda(f)$ extends to a unitary transformation of $L^2(H_n)$ onto $L^2(\mathbf{R}'; \mathrm{HS})$.

Let $\mathbf{N}$ denote the nonnegative integers. If $z \in \mathbf{C}^n$ and $\gamma \in \mathbf{N}^n$, then we shall adopt the usual notational convention for z^γ. For each $\gamma \in \mathbf{N}^n$ and $(t, z) \in H_n$, set

$$f_\gamma^\alpha(t, z) = \begin{cases} z^\gamma p_\alpha(t, z) & \text{if } \alpha < 0, \\ \bar{z}^\gamma p_\alpha(t, z) & \text{if } \alpha > 0. \end{cases}$$

The span of the set $B(\alpha) = \{ f_\gamma^\alpha \colon \gamma \in \mathbf{N}^n \}$ is dense in $C(\alpha)$. Furthermore, Proposition 1.4.8 of **[13]** states that $B(\alpha)$ is an orthogonal set. By normalizing the elements of $B(\alpha)$, we obtain a maximal orthonormal set whose elements we shall again denote by f_γ^α.

3. Diagonal operators. The purpose of this section is to construct strongly continuous (analytic) operators on $L^2(\mathbf{R}'; \mathrm{HS})$ given a sequence $\{ q^\gamma(s) \colon \gamma \in \mathbf{N}^n \}$ of continuous (analytic) functions defined in some subset of $\mathbf{C}$. Initially we shall define operators on $C(\alpha)$ which are diagonal with respect to the basis $B(\alpha)$ and then proceed to lift these to operators on $L^2(\mathbf{R}'; \mathrm{HS})$. On the other hand, we shall investigate conditions under which certain operators defined on $C(\alpha)$ may be diagonalized with respect to $B(\alpha)$.

Let $a > 0$ and suppose $\{ q^\gamma(s) \colon 0 \leqslant \operatorname{Re}(s) < a, \gamma \in \mathbf{N}^n \}$ is a sequence of continuous functions such that for each $\varepsilon \in (0, a)$, there exists a number K_ε satisfying

$$|q^\gamma(s)| \leqslant K_\varepsilon \tag{3.1}$$

for all $\gamma \in \mathbf{N}^n$ and $0 \leqslant \operatorname{Re}(s) \leqslant \varepsilon$. Set $S(\varepsilon) = \{ s \in \mathbf{C} \colon 0 \leqslant \operatorname{Re}(s) < \varepsilon \}$. For each $s \in S(\varepsilon)$, we define an operator $G(s)$ on $C(\alpha)$ by setting $G(s) f_\gamma^\alpha = q^\gamma(s) f_\gamma^\alpha$, $\gamma \in \mathbf{N}^n$, and extending $G(s)$ linearly. It follows from (3.1) that $G(s)$ is a bounded operator on $C(\alpha)$.

LEMMA 3.1. *The operator $G(s)$ initially defined on $C(\alpha)$ extends to a bounded operator on $L^2(\mathbf{R}'; \mathrm{HS})$, which we again denote by $G(s)$. Furthermore, for each $F \in L^2(\mathbf{R}'; \mathrm{HS})$ the function $s \to G(s)F$ is continuous in the strip $S(a)$ and $\|G(s)F\|_2 \leqslant K_\varepsilon \|F\|_2$ for all $s \in S(\varepsilon)$ ($\varepsilon \in (0, a)$) and $F \in L^2(\mathbf{R}'; \mathrm{HS})$.*

PROOF. The proof is straightforward and is omitted.

We now consider a sequence $\{ b^\gamma(s) \colon \gamma \in \mathbf{N}^n \}$ of functions which are analytic in the symmetric strip $|\operatorname{Re}(s)| < a$, $a > 0$. We assume that for each $\varepsilon \in (0, a)$, there exists a number K_ε such that

$$|b^\gamma(s)| \leqslant K_\varepsilon \tag{3.2}$$

for all $\gamma \in \mathbf{N}^n$ and s satisfying $|\operatorname{Re}(s)| < a - \varepsilon$. We define an operator on $C(\alpha)$ by setting $B(s) f_\gamma^\alpha = b^\gamma(s) f_\gamma^\alpha$, $\gamma \in \mathbf{N}^n$, and extending $B(s)$ linearly. Then for each s satisfying $|\operatorname{Re}(s)| < a$, $B(s)$ is continuous on $C(\alpha)$.

LEMMA 3.2. *The operator $B(s)$ initially defined on $C(\alpha)$ extends to a bounded operator on $L^2(\mathbf{R}'; \mathrm{HS})$, which we again denote by $B(s)$. Moreover, for each $F, G \in L^2(\mathbf{R}'; \mathrm{HS})$ the function $s \to (B(s)F|G)$ is analytic in the strip $|\mathrm{Re}(s)| < a$.*

PROOF. The proof is straightforward and is omitted.

We now consider certain convolution operators on $C(\alpha)$. In an effort to calculate the values of these operators on the basis $B(\alpha)$, we shall rely upon some basic facts concerning integration over the unit sphere in $\mathbf{C}^n$. These facts may be found in §1.4 of [**13**].

As before, $\mathbf{C}^n$ is considered topologically as the Euclidean space $\mathbf{R}^{2n}$ equipped with ordinary Lebesgue measure. Set $S = S_{2n-1} = \{z \in \mathbf{C}^n: |z| = 1\}$ and let $O(2n)$ be the group of all linear transformations on $\mathbf{R}^{2n}$ mapping S onto S. Let σ denote the positive Borel $O(2n)$-invariant measure on S such that $\sigma(S) = 1$. Then for all Borel measurable f

$$\int_{\mathbf{C}^n} f(z)\, dz = 2\pi^n/\Gamma(n) \int_0^\infty r^{2n-1}\, dr \int_S f(r\zeta)\, d\sigma(\zeta). \tag{3.3}$$

For $k \in \mathbf{N}$ and $a \in (-1, \infty)$, we define the Laguerre polynomial $L_k^a(x)$ by

$$L_k^a(x) = \sum_{j=0}^{k} \frac{\Gamma(k+a+1)}{\Gamma(j+a+1)} \frac{(-x)^j}{j!(k-j)!}. \tag{3.4}$$

Note, if $a = 0$, then $L_k^0(x) = \sum_{j=0}^k c(k, j)(-x)^j/j!$, where $c(k, j) = k!/(k-j)!j!$.

LEMMA 3.3. *If $\beta \in \mathbf{R}$ and $k \in \mathbf{N}$, then*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (w - ze^{i\theta})^k \exp(\beta e^{-i\theta} w\bar{z})\, d\theta = w^k L_k^0(\beta |z|^2)$$

for all $w, z \in \mathbf{C}$.

PROOF. For fixed $\beta \in \mathbf{R}$, $k \in \mathbf{N}$ and $z \in \mathbf{C}$, the integral in the lemma is an entire function f of w. By differentiating under the integral, one easily sees that $D^j f(0) = 0$, $j \neq k$. On the other hand,

$$\begin{aligned} D^k f(0) &= \frac{1}{2\pi} \sum_{m=0}^{k} c(k, m) \frac{k!}{(k-m)!} \int_{-\pi}^{\pi} (-ze^{i\theta})^{k-m} (\beta e^{-i\theta} \bar{z})^{k-m}\, d\theta \\ &= \sum_{m=0}^{k} c(k, m) \frac{k!}{(k-m)!} (-\beta |z|^\alpha)^{k-m} = k! L_k^0(\beta |z|^\alpha). \end{aligned}$$

The lemma now follows by expanding f about $w = 0$. □

LEMMA 3.4. *Let K be a compact subgroup of $O(2n)$. Then*

$$\int_S f(\zeta)\, d\sigma(\zeta) = \int_S d\sigma(\zeta) \int_K f(g\zeta)\, dg,$$

where f is continuous on S and dg denotes Haar measure on K.

PROOF. This is 1.4.2 of [**13**]. □

For $k = (k_1, \ldots, k_n) \in \mathbf{N}^n$, we set $|k| = k_1 + \cdots + k_n$ and $k! = k_1! \cdots k_n!$.

LEMMA 3.5. *If* $k = (k_1, \ldots, k_n) \in \mathbf{N}^n$, $\beta \in \mathbf{R}$, $r > 0$ *and* $w = (w_1, \ldots, w_n) \in \mathbf{C}^n$, *then*

$$\int_S (w - r\zeta)^k \exp(\beta(w|r\zeta))\, d\sigma(\zeta) = w^k c(|k| + n - 1, n - 1)^{-1} L_{|k|}^{n-1}(\beta r^2).$$

PROOF. Let $U(1) = \{z \in \mathbf{C}\colon |z| = 1\}$ act on S by sending $(\zeta_1, \ldots, \zeta_n) \in S$ to $(\zeta_1, \ldots, e^{i\theta}\zeta_n)$. Under this action, $U(1)$ may be identified with a compact subgroup of $O(2n)$. For continuous f, we have by Lemma 3.4

$$\int_S f(\zeta)\, d\sigma(\zeta) = \int_S d\sigma(\zeta) \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\zeta', e^{i\theta}\zeta_n)\, d\sigma,$$

where $\zeta' = (\zeta_1, \ldots, \zeta_{n-1})$. If $w' = (w_1, \ldots, w_{n-1})$ and $k' = (k_1, \ldots, k_{n-1})$, then by Lemma 3.3 we have

$$\begin{aligned}
&\int_S (w - r\zeta)^k \exp\big(\beta\big(w \cdot \overline{r\zeta}\big)\big)\, d\sigma(\zeta) \\
&\quad = \int_S d\sigma(\zeta)(w' - r\zeta')^{k'} \exp\Big(\beta\Big(w' \cdot \overline{r\zeta'}\Big)\Big) \\
&\qquad \times \frac{1}{2\pi} \int_{-\pi}^{\pi} \big(w_n - re^{i\theta}\zeta_n\big)^{k_n} \exp\big(\beta e^{-i\theta} w_n r\bar{\zeta}_n\big)\, d\theta \\
&\quad = w_n^{k_n} \int_S (w' - r\zeta')^{k'} \exp\Big(\beta\Big(w' \cdot \overline{r\zeta'}\Big)\Big) L_{k_n}^0\Big(\beta r^2 |\zeta_n|^2\Big)\, d\sigma(\zeta).
\end{aligned}$$

In the same manner, let $U(1)$ act on each of the remaining $n - 1$ variables of S to obtain

$$\begin{aligned}
&\int_S (w - r\zeta)^k \exp\big(\beta w \cdot \overline{r\zeta}\big)\, d\sigma(\zeta) \\
&\quad = w^k \int_S L_{k_1}^0\Big(\beta r^2 |\zeta_1|^2\Big) \cdots L_{k_n}^0\Big(\beta r^2 |\zeta_n|^2\Big)\, d\sigma(\zeta).
\end{aligned} \tag{3.5}$$

If we let $L_{k_i}^0(x) = \sum_{m_i=0}^{k_i} c(k_i, m_i)(-x)^{m_i}/m_i!$, then the integral on the right side of (3.5) is

$$\sum_{|m| \leq |k|} c(k_1, m_1) \cdots c(k_n, m_n) \frac{\left(-\beta r^2\right)^{|m|}}{m!} \int_S |\zeta^m|^2\, d\sigma(\zeta). \tag{3.6}$$

From 1.4.8 of **[13]**,

$$\int_S |\zeta^m|^2\, d\sigma(\zeta) = \frac{(n-1)!m!}{(n - 1 + |m|)!}.$$

By cancelling factorials, and using the fact $\sum_{|m|=j} c(k_1, m_1) \cdots c(k_n, m_n) = c(|k|, j)$, (3.6) becomes

$$\sum_{j=0}^{|k|} c(|k|, j)(-\beta r^2)^j (n-1)!/(n-1+j)!.$$

From (3.4) we see

$$(3.7) \quad L_{|k|}^{n-1}(x) = \frac{1}{|k|!}\sum_{j=0}^{|k|} c(|k|, j)\frac{\Gamma(|k|+n)}{\Gamma(j+n)}(-x)^j$$
$$= c(|k|+n-1, n-1)\sum_{j=0}^{|k|} c(|k|, j)\,\frac{(n-1)!}{(j+n-1)!}(-x)^j.$$

The lemma now follows from (3.5) and (3.7). □

A function $\phi: \mathbf{C}^n \to \mathbf{C}$ is said to be radial if $\phi(gz) = \phi(z)$ for all $g \in O(2n)$ and $z \in \mathbf{C}^n$. If ϕ is radial, then there exists a function $F: \mathbf{R} \to \mathbf{C}$ such that $\phi(z) = F(|z|)$ for all $z \in \mathbf{C}^n$.

THEOREM 3.6. *Let E denote the linear span of $B(\alpha)$ ($\alpha \neq 0$). Assume $\phi(z) = F(|z|)$ is a radial Borel function on $\mathbf{C}^n$ such that*

$$\int_0^\infty F(r)e^{-\beta r^2/2}L_m^{n-1}(\beta r^2)r^{2n-1}\,dr < \infty$$

for all $m \in \mathbf{N}$ and $\beta > 0$. Then the function $f \to \phi * f$, $f \in E$, is a densely defined linear operator such that

$$\phi * f_\gamma^\alpha = \left(d_\gamma\int_0^\infty F(r)e^{-\pi|\alpha|r^2/2}L_{|\gamma|}^{n-1}(\pi|\alpha|r^2)r^{2n-1}\,dr\right)f_\gamma^\alpha$$

for all $f_\gamma^\alpha \in B(\alpha)$. Here the number d_γ depends only on γ and n.

PROOF. We consider first the case $\alpha < 0$. For $z, w \in \mathbf{C}^n$, we have by the definition of f_γ^α

$$(3.8) \quad f_\gamma^\alpha\big((0,z)^{-1}(0,w)\big) = |\alpha|^n(w-z)^\gamma e^{\pi i\alpha\,\mathrm{Im}(z|w) - \pi|\alpha|\,|w-z|^2/2}$$
$$= |\alpha|^n e^{-\pi|\alpha|\,|w|^2/2}(w-z)^\gamma e^{-\pi|\alpha|\,|z|^2/2}e^{\pi|\alpha|(w|z)}.$$

Then

$$(\phi * f_\gamma^\alpha)(0,w) = \int_{\mathbf{C}^n}\phi(z)f_\gamma^\alpha\big((0,z)^{-1}\cdot(0,w)\big)\,dz$$
$$= p_\alpha(0,w)\int_{\mathbf{C}^n}\phi(z)e^{-\pi|\alpha|\,|z|^2/2}(w-z)^\gamma e^{\pi|\alpha|(w|z)}\,dz.$$

By (3.3), this last integral may be rewritten as

$$\frac{2n\pi^n}{n!}\int_0^\infty F(r)e^{-\pi|\alpha|r^2/2}r^{2n-1}\,dr\int_S (w-r\zeta)^\gamma e^{\pi|\alpha|(w|r\zeta)}\,d\sigma(\zeta).$$

We apply Lemma 3.5 to obtain

$$(\phi * f_\gamma^\alpha)(0,w) = d_n w^\gamma p_\alpha(0,w)\int_0^\infty F(r)e^{-\pi|\alpha|r^2/2}L_{|\gamma|}^{n-1}(\pi|\alpha|r^2)r^{2n-1}\,dr,$$

where

$$d_n = \frac{2n\pi^n}{n!}c(|\gamma|+n-1, n-1)^{-1}.$$

Since $w^\gamma p_\alpha(0,w) = f_\gamma^\alpha(0,w)$, the proof is now complete in the case $\alpha < 0$.

To prove the theorem for $\alpha > 0$, one uses the fact that the conjugate of $f_\gamma^{-\alpha}$ is f_γ^{α} and relies on the validity of the case $\alpha < 0$. □

4. Homogeneous norms on the Heisenberg group. We define a one-parameter group of dilations $\{\delta_r: r > 0\}$ on H_n by setting

$$\delta_r(t, z) = (r^2 t, rz), \qquad (t, z) \in H_n.$$

We further define a homogeneous norm $\|\cdot\|$ on H_n to be a continuous function $h \to \|h\|$ such that for all $h \in H_n$

(i) $\|\delta_r h\| = r\|h\|$ for all $r > 0$,

(ii) $\|h\| \geqslant 0$,

(iii) $\|h\| = 0 \Leftrightarrow h = 0$.

In this section we shall study the particular homogeneous norm defined by

$$\|(t,z)\|_0 = \left(t^2 + \tfrac{1}{4}|z|^4\right)^{1/4}, \qquad (t, z) \in H_n.$$

The set $\{h \in H_n: \|h\|_0 = 1\}$ is clearly compact. Consequently, if $\|\cdot\|$ is any homogeneous norm on H_n, then there exists a constant $C \geqslant 1$ such that $C^{-1}\|h\|_0 \leqslant \|h\| \leqslant C\|h\|_0$ for all $h \in H_n$. This shows all homogeneous norms are equivalent.

LEMMA 4.1. *For all $h, k \in H_n$ we have $\|hk\|_0 \leqslant \|h\|_0 + \|k\|_0$.*

PROOF. We refer to Cygan [3].

Let $\Sigma_n = \{h \in H_n: \|h\|_0 = 1\}$ and $S_{2n-1} = \{z \in \mathbf{C}^n: |z| = 1\}$. Each nonzero $h \in H_n$ may be written uniquely as $h = \delta_r \eta$, where $r = \|h\|_0$ and $\eta \in \Sigma_n$. Furthermore, for each $(t, z) \in \Sigma_n$, there exists a unique $(\phi, u) \in [0, \pi] \times S_{2n-1}$ such that $(t, z) = (\cos\phi, (2\sin\phi)^{1/2}u)$. By the change of variables formula,

(4.1)

$$\int_{H_n} f(t, z)\, dt\, dz$$
$$= \int_0^\infty \int_0^\pi \int_{S_{2n-1}} f\left(\delta_r\left(\cos\phi, (2\sin\phi)^{1/2}u\right)\right) 2^n \sin^{n-1}\phi\, r^{Q-1}\, d\sigma(u)\, d\phi\, dr,$$

where each of the measures is Lebesgue measure restricted to the respective sets and $Q = 2n + 2$. For simplicity, we shall denote an element of Σ_n by η and set $d\sigma(\eta) = 2^n \sin^{n-1}\phi\, d\sigma(u)\, d\phi$. Then (4.1) assumes the familiar form

$$\int_{H_n} f(t, z)\, dt\, dz = \int_0^\infty \int_{\Sigma_n} f(\delta_r \eta) r^{Q-1}\, d\sigma(\eta)\, dr. \tag{4.2}$$

We shall refer to r and η as the homogeneous polar coordinates induced by $\|\cdot\|_0$, and to $Q = 2n + 2$ as the homogeneous degree of H_n.

LEMMA 4.2. *Let $\xi, \eta \in \Sigma_n$ and $t > 0$. Then for $t \neq 1$ and $0 < \lambda < Q$,*

$$\int_{\Sigma_n} \left\|\xi \cdot (\delta_t \eta)^{-1}\right\|_0^{-\lambda} d\sigma(\eta) \leqslant A|1 - t|^{-\lambda/Q},$$

where A is a number depending only upon λ and n.

PROOF. Let $\xi = (\cos\theta, (2\sin\theta)^{1/2}w)$ and $\eta = (\cos\phi, (2\sin\phi)^{1/2}u)$, where $\theta, \phi \in [0, \pi]$ and $w, u \in S_{2n-1}$. First observe that

$$\left\|\xi \cdot (\delta_t\eta)^{-1}\right\|_0^Q = \left|e^{-i\theta} - t^2e^{i\phi} - 2t(\sin\theta\sin\phi)^{1/2}(-iw|u)\right|^{Q/2}.$$

Let $y = -iw$ and $z = 2t(\sin\theta\sin\phi)^{1/2}/(e^{-i\theta} - t^2e^{i\phi})$. We may then write

$$\left\|\xi \cdot (\delta_t\eta)^{-1}\right\|_0^Q = \left|e^{-i\theta} - t^2e^{i\phi}\right|^{n+1}\left|1 - (zy|u)\right|^{n+1}. \tag{4.3}$$

We wish to show that for $t \neq 1$, $|z| < 1$. But $t \neq 1$ implies

$$0 < \left|t^2 - e^{i(\theta-\phi)}\right|^2 \tag{4.4}$$

for any $\theta, \phi \in [0, \pi]$. On the other hand, $|z| < 1$ is equivalent to

$$4t^2\sin\theta\sin\phi < t^4 - 2t^2\cos(\theta + \phi) + 1. \tag{4.5}$$

By using elementary trigonometric identities, we see (4.5) reduces to (4.4).

From the proof of Proposition 1.4.10 of **[13]**, we know that there exists a constant K', depending only on n, such that

$$\int_{S_{2n-1}} \frac{d\sigma(u)}{\left|1 - (zy|u)\right|^{n+1}} \leqslant K'\left(1 - |z|^2\right)^{-1}, \qquad t \neq 1. \tag{4.6}$$

Furthermore, note that

$$\left|e^{-i\theta} - t^2e^{i\phi}\right|^2\left(1 - |z|^2\right) = \left|t^2 - e^{i(\phi-\theta)}\right|^2. \tag{4.7}$$

Then from (4.3), (4.6) and (4.7) we obtain

$$\begin{aligned}
&\int_{\Sigma_n} \left\|\xi \cdot (\delta_t\eta)^{-1}\right\|_0^{-Q} d\sigma(\eta)\\
&\quad = \int_0^\pi \frac{2^n(\sin\phi)^{n-1}}{\left|e^{-i\phi} - t^2e^{i\phi}\right|^{n+1}}\,d\phi \int_{S_{2n-1}} \frac{d\sigma(u)}{\left|1 - (zy|u)\right|^{n+1}}\\
&\quad \leqslant \int_0^\pi \frac{K'2^n(\sin\phi)^{n-1}}{\left|e^{-i\theta} - t^2e^{i\phi}\right|^{n+1}\left(1 - |z|^2\right)}\,d\phi\\
&\quad = K'2^n\int_0^\pi \left(\frac{\sin\phi}{\left|e^{-i\theta} - t^2e^{i\phi}\right|}\right)^{n-1} \frac{1}{\left|t^2 - e^{i(\phi-\theta)}\right|^2}\,d\phi.
\end{aligned}$$

The set $\{\sin\phi/|e^{-i\theta} - t^2e^{i\phi}|\colon \theta, \phi \in [0, \pi], t \in \mathbf{R}\}$ is easily seen to be bounded. In addition,

$$\int_0^\pi \left|t^2 - e^{i(\phi-\theta)}\right|^{-2} d\phi \leqslant 2\pi|1 - t|^{-1}$$

provided $0 < t \neq 1$. We conclude

$$\int_{\Sigma_n} \left\|\xi \cdot (\delta_t\eta)^{-1}\right\|_0^{-Q} d\sigma(\eta) \leqslant K''|1 - t|^{-1} \tag{4.8}$$

for all positive $t \neq 1$. An application of Jensen's inequality to (4.8) shows

$$\left\{\int_{\Sigma_n} \left\|\xi\cdot(\delta_t\eta)^{-1}\right\|_0^{-\lambda}\sigma(\Sigma_n)^{-1}\,d\sigma(\eta)\right\}^{Q/\lambda}$$
$$\leqslant \int_{\Sigma_n} \left\|\xi\cdot(\delta_t\eta)^{-1}\right\|_0^{-Q}\sigma(\Sigma_n)^{-1}\,d\sigma(\eta)$$
$$\leqslant \sigma(\Sigma_n)^{-1}K''|1-t|^{-1}.$$

The lemma now follows by taking the appropriate power of this last inequality. □

In the following result, let p' denote the conjugate of p and dh denote Lebesgue measure on H_n.

THEOREM 4.3. *Let* $1 < p < \infty$, $0 < \lambda < Q$, $\alpha < Q/p'$, $\beta < Q/p$ *and* $Q = \alpha + \beta + \lambda$. *For any* $f \in L^p(H_n)$ *and* $g \in L^{p'}(H_n)$,

$$\left|\int_{H_n}\int_{H_n}\frac{f(h)g(v)}{\|h\|_0^{\alpha}\left\|v\cdot(h)^{-1}\right\|_0^{\lambda}\|v\|_0^{\beta}}\,dh\,dv\right| \leqslant K\|f\|_p\|g\|_{p'},$$

where K *is a number depending only on* n, p, α, λ *and* β.

PROOF. This is the Heisenberg version of Theorem B_2^* of [**15**] for the case $p = q$. In fact, the proof of Theorem B_2^* (pp. 505–510) provides a proof in the Heisenberg case once the necessary changes are made. For example, Lemma (2.1) of [**15**] remains valid provided n is replaced by Q and polar coordinates in Euclidean space are replaced by homogeneous polar coordinates in H_n. Lemma (2.2) of [**15**] also remains true once we set

$$\Delta(t,\xi,\eta) = \left\|\xi\cdot(\delta_t\eta)^{-1}\right\|_0$$

and replace inequality (2.8) of [**15**] with the inequality in Lemma 4.3. Mutatis mutandis, the rest of the proof of Theorem B_2^* remains valid (when $p = q$) and thus establishes the theorem.

REMARKS ON THE PROOF. (1) From [**15**, p. 508], we must be able to use the inequality

$$\left|\,\|v\|_0 - \|h\|_0\right| \leqslant \left\|v\cdot(h)^{-1}\right\|_0, \qquad v, h \in H_n.$$

Now this is a consequence of Lemma 4.1.

(2) As the proof of Theorem B_2^* shows, the numbers $K = K(\alpha,\beta,\lambda)$ are uniformly bounded as α, β and λ vary within compact subsets of their respective domains.

(3) The central idea of the proof is to partition $H_n \times H_n$ into three disjoint sets

$$S_1 = \left\{(h,v)\colon \|h\|_0 \leqslant \tfrac{1}{2}\|v\|_0\right\},\quad S_2 = \left\{(h,v)\colon \|v\|_0 \leqslant \tfrac{1}{2}\|h\|_0\right\},$$
$$S_3 = \left\{(h,v)\colon \tfrac{1}{2} < \|h\|_0/\|v\|_0 < 2\right\}$$

and separately consider each of the integrals

$$I_i = \int \int_{S_i} \frac{f(h)g(v)}{\|h\|_0^{\alpha} \|v \cdot (h)^{-1}\|_0^{\lambda} \|v\|_0^{\beta}} \, dv \, dh, \qquad i = 1, 2, 3.$$

Now in order to establish the desired boundedness of the integrals I_1 and I_2, it is necessary to assume only $0 < \lambda$, $\alpha < Q/p'$, $\beta < Q/p$ and $\alpha + \beta + \lambda = Q$. I_3 is bounded using only $0 < \lambda < Q$ and $\alpha + \beta + \lambda = Q$.

5. Convolution operators and their Fourier transforms. The purpose of this section is to define and compute the Fourier transform of certain convolution operators. These operators are defined on $L^2(H_n)$ and will later be used to normalize $T(\cdot, s)$.

Let D denote the dense subset of $L^2(H_n)$ consisting of all bounded Baire functions $f\colon H_n \to \mathbf{C}$ with compact support. For $s \in \mathbf{C}$, let ϕ_s be the function defined on nonzero elements of H_n by

$$\phi_s(h) = \|h\|_0^{s-Q}, \qquad Q = 2n + 2. \tag{5.1}$$

We define an operator $A(s)$ on D by

$$(A(s)f)(h) = \frac{1}{g(s)} \int_{H_n} \phi_s(v) f(v^{-1}h) \, dv, \tag{5.2}$$

where $g(s) = 2\pi^{n+1-s/2}\Gamma(s/2)/\Gamma(\frac{1}{4}(Q - s))^2$ and dv denotes Lebesgue measure on $\mathbf{R}^{2n+1}$. By reproducing the arguments in Lemma 4.1 of **[11]**, it may be shown that if $s \in \mathbf{C}$ satisfies $0 < \operatorname{Re}(s) < Q/2$ and $f \in D$, then the integral in (5.2) converges absolutely for all $h \in H_n$ and defines an element of $L^2(H_n)$.

If $\operatorname{Re}(s) < 2n$ and $\alpha \in \mathbf{R}$, then for each nonzero $z \in \mathbf{C}^n$ the function $t \mapsto \hat{\phi}_s(t, z)$ is integrable. We define $\hat{\phi}_s(\alpha, z) = \int_{\mathbf{R}} e^{\pi i \alpha t} \phi_s(t, z) \, dt$ for $\alpha \in \mathbf{R}$ and $\operatorname{Re}(s) < 2n$. We observe that the Fourier transform of ϕ_s is convolution on the left by $\hat{\phi}_s(\alpha, z)$ and this convolution operator is diagonalized by $B(\alpha)$. In order to find the diagonal entries, we first write $\hat{\phi}_s(\alpha, z)$ as a multiple of a Bessel function.

Let $\mathbf{C}^-$ denote the complex plane cut along the negative real axis. For $\nu \in \mathbf{C}$ and $z \in \mathbf{C}^-$, let $K_\nu(z)$ denote the modified Bessel function of the third kind. From **[10**, p. 119**]** we have the integral representation

$$K_\nu(z) = \frac{1}{2}(z/2)^\nu \int_0^\infty e^{-t-(z^2/4t)} t^{-\nu-1} \, dt, \qquad |\arg z| < \pi/4. \tag{5.3}$$

LEMMA 5.1. *If* $\operatorname{Re}(s) = \sigma < 2n$, *then for all nonzero* $z \in \mathbf{C}^n$ *and* $\alpha \neq 0$ *we have*

$$\hat{\phi}_s(\alpha, z) = k(\alpha, s)|z|^{(s-2n)/2} K_{(s-2n)/4}\left(\pi|\alpha|\,|z|^2/2\right),$$

where $k(\alpha, s) = 2\pi^{(\alpha-s)/4}|\alpha|^{(2n-s)/4}/\Gamma(\frac{1}{4}(Q - s))$.

PROOF. For $z \neq 0$, we have

$$\begin{aligned} \hat{\phi}_s(\alpha, z) &= \int_{\mathbf{R}} \left(t^2 + |z|^4/4\right)^{(s-Q)/4} e^{\pi i|\alpha|t} \, dt \\ &= 2^{n-s/2}|z|^{s-2n} \int_{\mathbf{R}} (t^2 + 1)^{(s-Q)/4} e^{i\pi|\alpha|\,|z|^2 t/2} \, dt. \end{aligned} \tag{5.4}$$

From the identity $\int_0^\infty e^{-pr}r^{z-1}\,dr = p^{-z}\Gamma(z)$, $\mathrm{Re}(p) > 0$, $\mathrm{Re}(z) > 0$ and Fubini's theorem this last integral is

$$\Gamma\left(\tfrac{1}{4}(Q-s)\right)^{-1}\int_0^\infty e^{-r}r^{(Q-s)/4}\frac{dr}{r}\int_{\mathbf{R}} e^{i\pi|\alpha|\,|z|^2t/2}e^{-rt^2\,dt}.$$

It is known that the integral with respect to t here is $\pi^{1/2}r^{-1/2}e^{-(\pi\alpha|z|^2/2)^2/4r}$. From (5.3) we then see that the integral in (5.4) may be written as

$$j(s)|\alpha|^{(2n-s)/4}|z|^{(2n-s)/2}\Gamma\left(\tfrac{1}{4}(Q-s)\right)K_{\nu(s)}\left(\pi|\alpha|\,|z|^2/2\right),$$

where $j(s) = \pi^{1/2+(2n-s)/4}2^{1-n+s/2}$ and $\nu(s) = \frac{1}{4}(s-2n)$. The lemma now follows by combining factors. □

Let $E_m(x) = e^{-x/2}L_m^{n-1}(x)$ for $x \in \mathbf{R}$. Theorem 3.6 leads us to consider integrals of the form

$$I_m(s) = \int_0^\infty \hat{\phi}_s(\alpha, r)E_m(\pi|\alpha|r^2)r^{2n-1}\,dr, \tag{5.5}$$

where $\mathrm{Re}(s) < 2n$, $\alpha \neq 0$. If we make the change of variables $x = r^2$ and then apply the transformation $x \to (\pi|\alpha|)^{-1}x$, we obtain from (5.5)

$$I_m(s) = k(s)|\alpha|^{-s/2}\int_0^\infty x^{(s+2n)-1/4}e^{-x/2}K_{\nu(s)}(x/2)L_m^{n-1}(x)\,dx, \tag{5.6}$$

where $k(s) = \pi^{(1-s)/2}/\Gamma(\frac{1}{4}(Q-s))$.

If $\mathrm{Re}(\mu) > 0$ and $\mathrm{Re}(\mu) > |\mathrm{Re}(\nu)|$ then we have from [**12**, p. 107]

$$\int_0^\infty x^{\mu-1}e^{-x}K_\nu(x)\,dx = 2^{-\mu}\pi^{1/2}\frac{\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{\Gamma(\mu+1/2)}. \tag{5.7}$$

Furthermore, from a theorem of Vandermonde we know that if $a, b \in \mathbf{C}$ such that $0 < \mathrm{Re}(s) < \mathrm{Re}(b)$, then for all $m \in \mathbf{N}$

$$\sum_{k=0}^{m}\frac{\Gamma(a+k)(-1)^k}{\Gamma(b+k)k!(m-k)!} = \frac{\Gamma(a)\Gamma(b-a+m)}{\Gamma(b-a)\Gamma(b+m)\Gamma(m+1)}. \tag{5.8}$$

LEMMA 5.2. *If $I_m(s)$ is the integral defined by* (5.5), $0 < \mathrm{Re}(s) < 2n$, *then*

$$I_m(s) = b(\alpha, s)\frac{\Gamma((Q-s)/4+m)}{\Gamma((Q+s)/4+m)},$$

where

$$b(\alpha, s) = \pi^{1-s/2}|\alpha|^{-s/2}\frac{\Gamma(m+n)\Gamma(s/2)}{\Gamma(m+1)\Gamma((Q-s)/4)^2}.$$

PROOF. Let $I(s,k) = \int_0^\infty x^{(s+2n)/4+k-1}e^{-x/2}K_{\nu(s)}(x/2)\,dx$, $k \in \mathbf{N}$, $0 < \mathrm{Re}(s) < 2n$ and $\nu(s) = \frac{1}{4}(s-2n)$. Since $\mathrm{Re}(s) > 0$, we know from (5.7) that

$$I(s,k) = \pi^{1/2}\Gamma(k+s/2)\Gamma(k+n)/\Gamma\left(\tfrac{1}{4}(s+2n)+k+\tfrac{1}{2}\right).$$

Now let $d(\alpha, s) = \pi^{(1-s)/2}|\alpha|^{-s/2}/\Gamma(\frac{1}{4}(Q-s))$. By combining constants, we have from (5.6)

$$I_m(s) = d(\alpha, s) \sum_{k=0}^{m} \frac{\Gamma(m+n)(-1)^k}{\Gamma(k+n)k!(m-k)!} I(s, k)$$

$$= \pi^{1/2} d(\alpha, s) \sum_{k=0}^{m} \frac{\Gamma(m+n)\Gamma(k+s/2)(-1)^k}{\Gamma((s+2n)/4+k+1/2)k!(m-k)!}.$$

From (5.8) we finally obtain

$$I_m(s) = \frac{d(\alpha,s)\pi^{1/2}\Gamma(m+n)\Gamma(s/2)\Gamma((Q-s)/4+m)}{\Gamma(m+1)\Gamma((Q-s)/4)\Gamma((Q+s)/4+m)}.$$

The proof is now completed by combining constants. □

LEMMA 5.3. *Let ε be a number such that $0 < \varepsilon < Q$. If $0 \leqslant \mathrm{Re}(s) < Q - \varepsilon$, then for all $m \in \mathbf{N}$*

$$\left| \frac{\Gamma((Q-s)/4+m)}{\Gamma((Q+s)/4+m)} \right| \leqslant K/\varepsilon,$$

where K is a number depending only on n.

PROOF. This estimate follows from standard arguments involving Stirling's formula [**10**]. We refer also to a similar result in [**16**]. □

Let $q^\gamma(s) = \Gamma(\frac{1}{4}(Q-s) + |\gamma|)/\Gamma(\frac{1}{4}(Q+s) + |\gamma|)$ for $0 \leqslant \mathrm{Re}(s) < Q$. For each $\alpha \neq 0$, we define the linear operator $G(s)$ on $C(\alpha)$ by setting $G(s)f_\gamma^\alpha = q^\gamma(s)f_\gamma^\alpha$. By Lemmas 3.1 and 3.2, $G(s)$ defines a bounded left multiplication operator on $L^2(\mathbf{R}'; \mathrm{HS})$ which is strongly continuous in $0 \leqslant \mathrm{Re}(s) < Q$, analytic in $0 < \mathrm{Re}(s) < Q$ and unitary when $\mathrm{Re}(s) = 0$. We shall abuse the language and again denote this left multiplication operator by $G(s)$.

Let Δ denote the linear subspace of $L^2(\mathbf{R}'; \mathrm{HS})$ consisting of those F such that

$$\|F\|_\infty = \operatorname{ess\,sup}\left\{ \|F(\alpha)\|_{\mathrm{HS}} |\alpha|^{n/2} : \alpha \in \mathbf{R}' \right\} < \infty.$$

It is clear from Theorem 2.1 that $f \in D$ implies $\Lambda(f) \in \Delta$. By the Plancherel theorem is follows that Δ is dense in $L^2(\mathbf{R}'; \mathrm{HS})$.

For $s \in \mathbf{C}$ we define the operator $N(s)$ on Δ by $(N(s)F)(\alpha) = |\alpha|^{-s/2}F(\alpha)$. By dominated convergence, it follows that the $L^2(\mathbf{R}'; \mathrm{HS})$-valued function $s \to N(s)F$ is strongly continuous in $0 \leqslant \mathrm{Re}(s) < 1$ and is analytic in $0 < \mathrm{Re}(s) < 1$. We define $\hat{A}(s)$ on Δ by $\hat{A}(s) = G(s)N(s)$, $0 \leqslant \mathrm{Re}(s) < 1$. The function $s \to \hat{A}(s)F$, $F \in \Delta$, is then strongly continuous in $0 \leqslant \mathrm{Re}(s) < 1$. Furthermore, one may show that for $F \in \Delta$ and F' in a suitable dense subspace of $L^2(\mathbf{R}'; \mathrm{HS})$ the function $s \to (\hat{A}(s)F|F')$ extends to a meromorphic function on $\mathbf{C}$ with simple poles occurring in the set $\{Q + 4k: k \in \mathbf{N}\}$.

THEOREM 5.4. *For $f \in D$ the function $s \to A(s)f$ extends to a continuous function in $0 \leqslant \mathrm{Re}(s) < 1$. Furthermore, if $\mathrm{Re}(s) = 0$, then $A(s)$ is isometric and extends to a unitary operator on $L^2(H_n)$ such that $A(s)^{-1} = A(-s)$.*

PROOF. Let X denote the characteristic function of the set $\{h \in H_n: \|h\|_0 \leqslant 1\}$. Let $g_s = X\phi_s$ and $k_s = (1 - X)\phi_s$. Since the functions g_s and k_s lie in $L^1(H_n)$ and $L^2(H_n)$, respectively, their Fourier transforms exist. For $f \in D$ and $0 < \mathrm{Re}(s) < Q/2$, we then have $\Lambda(A(s)f) = (\Lambda(g_s) + \Lambda(k_s))\Lambda(f)$. From Lemma 5.2 we know explicitly the nonzero matrix entries of $\Lambda(g_s) + \Lambda(k_s)$, $0 < \mathrm{Re}(s) < Q/2$. It now follows that $\Lambda(A(s)f) = \hat{A}(s)\Lambda(f)$.

We extend the definition of $A(s)$ to $\mathrm{Re}(s) = 0$ by setting $A(s) = \Lambda^{-1}\hat{A}(s)\Lambda$ for $0 \leqslant \mathrm{Re}(s) < 1$. The theorem is then a consequence of the properties of $\hat{A}(s)$. □

6. Principal series representations of $SU(1, n+1)$. Let $\mathbf{C}^{k\times j}$ denote the space of $k \times j$ matrices with entries in the field of complex numbers. If x and y are in $\mathbf{C}^{1\times j}$, we set $(x|y) = xy^*$, where y^* denotes the conjugate transpose of y. For $n \geqslant 1$, let $SU(1, n+1)$ denote the group of matrices in $\mathbf{C}^{(n+2)\times(n+2)}$ which leave invariant the quadratic form $-x_1^2 + x_2^2 + \cdots + x_{n+2}^2$ and have determinant 1. For convenience, put $m = n + 2$. Let $E \in \mathbf{C}^{m\times m}$ be the matrix

$$E = \begin{bmatrix} 0 & & \cdots & 0 & 1 \\ & & & & 0 \\ \vdots & & I_n & & \vdots \\ 0 & & & & \\ 1 & 0 & \cdots & & 0 \end{bmatrix}$$

and set $G = \{x \in \mathbf{C}^{m\times m}: xEx^* = E, \det(x) = 1\}$. G is then a group which is conjugate to $SU(1, n+1)$.

Let A be the subgroup of G consisting of all matrices of the form $a(r) = \mathrm{diag}(r, 1, \ldots, 1, r^{-1})$, $r > 0$. We define subgroups M and N of G by

$$M = \left\{ m(w, u) = \begin{bmatrix} w & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & u & \vdots \\ 0 & & \cdots & w \end{bmatrix} : u \in U(n), \det u = w^{-2} \right\}$$

and

$$N = \left\{ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\bar{z}^t & & I_n & \vdots \\ -|z|^2/2 - it & & z & 1 \end{bmatrix} : t \in \mathbf{R}, z \in \mathbf{C}^{1\times n} \right\}.$$

Evidently, both M and A normalize N. It is known that the subgroup $B = MAN$ is closed.

Let $p \in G$ be the matrix

$$p = \begin{bmatrix} 0 & & \cdots & 1 \\ \cdot & -1 & & 0 \\ \cdot & & & \cdot \\ \cdot & & I_{n-1} & \cdot \\ 0 & & & \cdot \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

The matrix p is a representative of the nontrivial element of the Weyl group for G. If we set $V = pNp^{-1}$, then an element of V has the form

$$\begin{bmatrix} 1 & -z_1 & z_2 & \cdots & z_n & -|z|^2/2 - it \\ 0 & & & & & \bar{z}_1 \\ \vdots & & & & & -\bar{z}_2 \\ & & I_n & & & \vdots \\ & & & & & -\bar{z}_n \\ 0 & \cdots & \cdots & & 0 & 1 \end{bmatrix}, \qquad t \in \mathbf{R}, z_i \in \mathbf{C}.$$

The subgroup V is clearly isomorphic to H_n. It is known by the general theory of semisimple Lie groups that the map $B \times V \to G$ given by $(b, v) \to bv$ is a diffeomorphism onto an open subset of G whose complement has Haar measure zero. By the Bruhat lemma we have $G = (BpB) \cup B$. Applying p yields $G = (BV) \cup Bp$ so that each element $x \notin Bp$ may be written as the unique product of elements from M, A, N and V.

For $s \in \mathbf{R}$, define the character χ_s on the subgroup B by $\chi_s(ma(r)n) = r^{-s}$. We let $T(\cdot, s)$ denote the representation of G induced by χ_s. The family $T(\cdot, s)$ is called the spherical principal series of G and may be realized on $L^2(H_n)$ as follows:

(6.1) $$T(a(r), s)f(t, z) = r^{-Q/2-s}f(r^{-2}t, r^{-1}z), \qquad r > 0;$$

(6.2) $$T(m(w, u), s)f(t, z) = f(t, \overline{w}zu), \qquad u \in U(n), \det u = w^{-2};$$

(6.3) $$T((t', z'), s)f(t, z) = f((t, z)(t', z')), \qquad (t', z') \in H_n;$$

(6.4) $$T(p, s)f(t, z) = \left|it + |z|^2/2\right|^{-Q/2-s} f\left(-\left|it + |z|^2/2\right|^{-2} t, -z'/\left(it + |z|^2/2\right)\right);$$

where $Q = 2n + 2$ and $z' = (-z_1, z_2, \ldots, z_n)$;

(6.5) $$T(g, s)f(t, z) = 0 \quad \text{if } g \in Bp.$$

7. Analytic continuation of operators. In this section we will analytically continue various operators into strips in the complex plane. We shall state some results without giving detailed proofs since complete proofs of analogous results may be found elsewhere in the literature and arguments presented there carry over almost verbatim to our situation.

We recall the normalizing constant $g(s) = 2\pi^{n+1-s/2}\Gamma(s/2)/\Gamma(\frac{1}{4}(Q - s))^2$ given in (5.2). The basic properties of $\Gamma(z)$ along with Stirling's formula **[10]** establish the next result.

LEMMA 7.1. *The function $s \to g(s)^{-1}$ is analytic in the strip $|\mathrm{Re}(s)| < Q/2$ and*

$$|g(s)^{-1}| \leqslant K(1 + |\tau|)^{n+1/2-\sigma}, \qquad s = \sigma + i\tau, |\sigma| < Q/2,$$

where K is a number depending only on n.

If $\mathrm{Re}(s) = 0$, we define the operator $M(s)$ on $L^2(H_n)$ by

$$M(s)f(h) = \|h\|_0^{-s} f(h), \qquad f \in L^2(H_n).$$

Clearly $M(s)$ is unitary with inverse $M(-s)$.

LEMMA 7.2. *The composition* $A(s)M(s)$, *initially defined on* $L^2(H_n)$ *for* $\mathrm{Re}(s) = 0$, *can be continued into the strip* $0 \leqslant \mathrm{Re}(s) < 1$ *so that*

(1) *for each* f, k *in* $L^2(H_n)$ *the function* $s \to (A(s)M(s)f|k)$ *is continuous in* $0 \leqslant \mathrm{Re}(s) < 1$ *and analytic in* $0 < \mathrm{Re}(s) < 1$;

(2) $\|A(s)M(s)\|_\infty \leqslant K_d(1 + |\tau|)^{n+1/2-\sigma}$, *where* $s = \sigma + i\tau$, $0 \leqslant \sigma \leqslant d < 1$.

PROOF. Complete details may be found in [**8** and **11**]. We shall give a brief outline. Let f, k be continuous functions compactly supported away from the origin. Now set $\phi(f, k; s) = (M(s)f|A(\bar{s})k)$. One then uses the properties of $M(s)$ and $A(\bar{s})$ to conclude $\phi(f, k; s)$ is continuous in $0 \leqslant \mathrm{Re}(s) < 1$ and uniformly bounded in $0 \leqslant \mathrm{Re}(s) \leqslant d < 1$. One then uses Theorem 4.3 to show that $g(s)\phi(f, k; s)$ is analytic in $0 < \mathrm{Re}(s) < 1$ and $|g(s)\phi(f, k; s)| \leqslant K(\sigma)\|f\|_2\|k\|_2$, where $0 < \sigma < 1$, $\sigma = \mathrm{Re}(s)$. The sesquilinear form $(f, k) \to \phi(f, k; s)$ then extends to all of $L^2(H_n)$ but remains continuous in $0 \leqslant \mathrm{Re}(s) < 1$ and analytic in $0 < \mathrm{Re}(s) < 1$. The estimate for the operator norm of $A(s)M(s)$ then follows from Lemma 7.1. □

For $\mathrm{Re}(s) = 0$, define the operator $C(s)$ on $L^2(H_n)$ by $C(s) = A(s)M(s) - M(s)A(s)$.

LEMMA 7.3. *The operators* $C(s)$, *initially defined on* $L^2(H_n)$ *for* $\mathrm{Re}(s) = 0$, *can be continued into the strip* $-1 < \mathrm{Re}(s) < Q/2$ *so that*

(1) *for each* $f, k \in L^2(H_n)$, *the function* $s \to (C(s)f, k)$ *is analytic in* $-1 < \mathrm{Re}(s) < Q/2$;

(2) $\|C(s)\|_\infty \leqslant K(\sigma)(1 + |\tau|)^{n-\sigma+3/2}$, $s = \sigma + i\tau$, $-1 < \sigma < Q/2$ *and* $K(\sigma)$ *is uniformly bounded on closed subintervals of* $(-1, Q/2)$.

PROOF. Let f, k be continuous and compactly supported away from the origin. For $0 \leqslant \sigma < 1$, let $\phi(f, k; s) = (M(s)f, A(\bar{s})k) - (A(s)f, M(\bar{s})k)$. We then have $\phi(f, k; s) = (C(s)f|k)$ when $\mathrm{Re}(s) = 0$. For $0 < \mathrm{Re}(s) < 1$, we have the estimate

$$(7.1) \qquad |g(s)\phi(f, k; s)| \leqslant \int_{H_n \times H_n} \frac{|f(v)k(h)|}{\|hv^{-1}\|_0^{Q-\sigma}} \left| \|v\|_0^{-s} - \|h\|_0^{-s} \right| dv\, dh.$$

Now the proof of Lemma 7.4 of [**11**] shows the integral in (7.1) converges in the wider strip $-1 < \mathrm{Re}(s) < Q/2$. We shall briefly outline the steps. Let S_i $(i = 1, 2, 3)$ denote the subsets of $H_n \times H_n$ as defined in Remark (3) of §4. Let I_i denote the integral in (7.1) taken over S_i. Now let $s = \sigma + i\tau$. Using Remark (3) of §4, we find that for $i = 1, 2$ the integral I_i is bounded by $K_i(\sigma)\|f\|_2\|k\|_2$ whenever $\sigma < Q/2$, while the integral I_3 is bounded by $K_3(\sigma)(1 + |\tau|)\|f\|_2\|k\|_2$ if $-1 < \sigma < Q - 1$. By Remark (2) of §4, each of the functions $\sigma \to K_i(\sigma)$ is uniformly bounded on closed subintervals of $(-1, Q/2)$. Thus the integral defining $g(s)\phi(f, k; s)$ converges absolutely in the strip $-1 < \mathrm{Re}(s) < Q/2$. Furthermore, we may use Lemma 7.1 to obtain

$$|\phi(f, k; s)| \leqslant K(\sigma)(1 + |\tau|)^{n-\sigma+3/2}\|f\|_2\|k\|_2, \qquad -1 < \mathrm{Re}(s) < Q/2.$$

The same arguments proving Lemma 7.2 now establish the analyticity of $\phi(f, k; s)$. □

For $\mathrm{Re}(s) = 0$, define the operator $E(s)$ on $L^2(H_n)$ by

$$E(s) = A(s)M(s)A(-s)M(-s).$$

LEMMA 7.4. *The operators $E(s)$, initially defined on $L^2(H_n)$ for $\mathrm{Re}(s) = 0$, can be continued into the strip $-1 < \mathrm{Re}(s) < 1$ so that*

(1) *for each $f, k \in L^2(H_n)$, the function $s \to (E(s)f|k)$ is analytic in $-1 < \mathrm{Re}(s) < 1$;*

(2) $\|E(s)\|_\infty \leqslant K(\sigma)(1 + |\tau|)^Q$, $s = \sigma + i\tau$, $|\sigma| < 1$, *where $K(\sigma)$ is uniformly bounded on closed subintervals of $(-1, 1)$.*

PROOF. The complete proof of (1) may be found in the proof of Lemma 20 of **[8]**. Briefly, the proof proceeds by using the equation

$$E(s) = C(s)A(-s)M(-s) + \text{Identity}, \qquad \mathrm{Re}(s) = 0,$$

to extend the definition of $E(s)$ into the strip $-1 < \mathrm{Re}(s) < 0$. One extends $E(s)$ into the strip $0 < \mathrm{Re}(s) < 1$ by observing

$$E(s) = A(s)M(s)C(-s) + \text{Identity}, \qquad \mathrm{Re}(s) = 0.$$

We observe that the veracity of (2) is established by Lemmas 7.2 and 7.3. □

8. Analytic uniformly bounded representations. The major portion of this section will be spent showing that the normalized principal series representations $R(\cdot, s) = A(s)T(\cdot, s)A(-s)$, $\mathrm{Re}(s) = 0$, have an analytic continuation in the strip $|\mathrm{Re}(s)| < 1$ which is uniformly bounded. In doing this, we shall subsequently prove that $A(2s)$ intertwines $T(\cdot, s)$ and $T(\cdot, -s)$. In the last part of this section we shall construct analytic uniformly bounded representations of $SU(1, n+1)$ which are symmetric in s and unitary when $s \in (-1, 1)$.

Let H_n' denote all nonzero elements of H_n. For each $(t, z) \in H_n'$, let $j(t, z) = it + |z|^2/2$ and $J(t, z) = (-t/|j(t, z)|^2, -z'/j(t, z))$, where $z' = (-z_1, z_2, \ldots, z_n)$ if $z = (z_1, \ldots, z_n)$.

LEMMA 8.1. *The function $J\colon H_n' \to H_n'$ has the following properties:*

(1) *J^2 is the identity on H_n';*

(2) $\|J(h)\|_0 = \|h\|_0^{-1}$, $h \in H_n'$;

(3) $\|h\|_0 \|vJ(h)^{-1}\|_0 = \|v\|_0 \|hJ(v)^{-1}\|_0$ *for all $h, v \in H_n'$;*

(4) *if $J'(h)$ denotes the determinant of the Jacobian matrix of J at $h \in H_n'$, then* $|J'(h)| = \|h\|_0^{-2Q}$.

PROOF. (1) follows immediately once we observe that $j(J(t, z)) = j(t, z)^{-1}$ for all $(t, z) \in H_n'$. In order to show (2), we write

$$\|J(t, z)\|_0^4 = |j(t, z)|^{-4}\left(t^2 + |z|^4/4\right)^4 = |j(t, z)|^{-2} = \|(t, z)\|_0^{-4}.$$

We shall prove (3) by first showing

$$\|h\|_0^{-1}\|v\|_0^{-1}\|hv^{-1}\|_0 = \left\|J(h)J(v)^{-1}\right\|_0 \tag{8.1}$$

for all $h, v \in H_n'$. We observe that

$$\left\|(t_1, z_1)(t, z)^{-1}\right\|_0^4 = \left|j(t_1, z_1) + \overline{j(t, z)} - (z_1|z)\right|^2.$$

Consequently, we have

$$\left\|J(t_1, z_1)J(t, z)^{-1}\right\|_0^4 = \left|j(t_1, z_1)^{-1} + \overline{j(t, z)}^{-1} - j(t_1, z_1)^{-1}\overline{j(t, z)}^{-1}(z_1|z)\right|^2.$$

On the other hand,

$$\begin{aligned}\|(t_1, z_1)\|_0^{-4}\|(t, z)\|_0^{-4}&\left\|(t_1, z_1)(t, z)^{-1}\right\|_0^4\\ &= \left|j(t_1, z_1)\right|^{-2}\left|\overline{j(t, z)}\right|^{-2}\left|j(t_1, z_1) + \overline{j(t, z)} - (z_1, z)\right|^2\end{aligned}$$

so that (8.1) is now verified. We see (3) is proved by replacing h in (8.1) with $J(h)$ and using (1) and (2) of the lemma.

We shall prove (4) by switching to homogeneous polar coordinates. If $(t, z) \in H_n$, let $(t, z)'$ denote (t, z'). For $\eta \in \Sigma_n$ and $r > 0$ we have $J(\delta_r\eta) = \delta_{r^{-1}}\eta'^{-1}$. Now let f be continuous and compactly supported on H_n. From (4.2) we then have

$$\int_{H_n} f(J(t, z))\|(t, z)\|_0^{-2Q}\,dt\,dz = \int_0^\infty \int_{\Sigma_n} f\left(\delta_{r^{-1}}\eta'^{-1}\right)r^{-Q}\,d\sigma(\eta)\,\frac{dr}{r}.$$

If we apply the transformations $r \to r^{-1}$ and $\eta \to \eta'^{-1}$ we see this last integral is simply the integral of f over H_n. This establishes (4). □

For $\operatorname{Re}(s) = 0$ and $g \in G$, let $R(g, s) = A(s)T(g, s)A(-s)$. We also let $\overline{P} = MAV$.

LEMMA 8.2. *If* $g \in \overline{P}$ *and* $\operatorname{Re}(s) = 0$, *then*
(1) $R(g, s) = T(g, 0)$,
(2) $A(2s)T(g, s) = T(g, -s)A(2s)$.

PROOF. By using the integral formula (5.2), one may easily verify that $T(g, 0)A(s)f = A(s)T(g, s)f$ and $T(g, -s)A(2s)f = A(2s)T(g, s)f$ for $f \in D$ and $0 < \operatorname{Re}(s) < 1$. Continuity then insures the validity of these equations when $\operatorname{Re}(s) = 0$. We also refer to this result found in **[6]**. □

LEMMA 8.3. *If* $\operatorname{Re}(s) = 0$, *then*
(1) $T(p, 0)A(s)T(p, 0) = M(s)A(s)M(s)$,
(2) $A(2s)T(p, s) = T(p, -s)A(2s)$.

PROOF. Let D_0 denote the linear space of bounded, compactly supported Baire functions on H_n which vanish in a neighborhood of the origin. If $0 < \operatorname{Re}(s) < 1$, then both $M(s)$ and $T(p, 0)$ leave D_0 invariant. If $f \in D_0$ and $h \in H_n'$, then we have

$$T(p, 0)A(s)T(p, 0)f(h) = \frac{\|h\|_0^{-Q}}{g(s)}\int_{H_n} \|v\|_0^{s-Q}\|v^{-1}J(h)\|_0^{-Q} f\left(J\left(v^{-1}J(h)\right)\right)dv.$$

By applying the transformation $v \to J(h)v^{-1}$ and using (3) of Lemma 8.1, we see this last integral becomes

$$\frac{\|h\|_0^{-s}}{g(s)}\int_{H_n} \|v\|_0^{s-2Q}\left\|hJ(h)^{-1}\right\|_0^{s-Q} f(J(v))\,dv.$$

We now apply the transformation $v \to J(v)$ and use (1), (2) and (4) of Lemma 8.1 to obtain

$$\frac{\|h\|_0^{-s}}{g(s)}\int_{H_n} \|v\|_0^{s-Q}\|v^{-1}h\|_0^{-s} f(v^{-1}h)\,dh.$$

We recognize this integral as $M(s)A(s)M(s)f(h)$. Just as in the proof of Lemma 8.2, we conclude $T(p,0)A(s)T(p,0) = M(s)A(s)M(s)$, $\operatorname{Re}(s) = 0$, by continuity.

In order to prove (2), we begin by noting that $T(p, s) = M(2s)T(p,0)$. From (1), we have for $\operatorname{Re}(s) = 0$

$$T(p,0)A(2s) = M(2s)A(2s)M(2s)T(p,0) = M(2s)A(2s)T(p,0).$$

Since $M(2s)^{-1}T(p,0) = T(p,-s)$, we conclude

$$T(p,-s)A(2s) = A(2s)T(p,s), \qquad \operatorname{Re}(s) = 0. \quad \square$$

REMARK. From the Bruhat lemma we know $G = \overline{P} \cup \overline{P}p\overline{P}$ so that Lemmas 8.2 and 8.3 prove

$$A(2s)T(g,s) = T(g,-s)A(2s), \qquad g \in G, \operatorname{Re}(s) = 0. \tag{8.2}$$

LEMMA 8.4. *The operator $R(p,s) = A(s)T(p,s)A(-s)$, initially defined for $\operatorname{Re}(s) = 0$, can be continued into the strip $-1 < \operatorname{Re}(s) < 1$ so that*

(1) *for each $f, k \in L^2(H_n)$, the function $s \to (R(p,s)f|k)$ is analytic in $|\operatorname{Re}(s)| < 1$;*

(2) *$\|R(p,s)\|_\infty \leqslant K(\sigma)(1 + |\tau|)^Q$, $s = \sigma + i\tau$, $|\sigma| < 1$, where $K(\sigma)$ is uniformly bounded on closed subintervals of $(-1,1)$.*

PROOF. From Lemma 8.3 we have

$$\begin{aligned} R(p,s) &= A(s)M(2s)T(p,0)A(-s) \\ &= A(s)M(2s)M(-s)A(-s)M(-s)T(p,0) = E(s)T(p,0). \end{aligned}$$

Both (1) and (2) now follow from Lemma 7.4. $\square$

THEOREM 8.5. *For each s satisfying $|\operatorname{Re}(s)| < 1$, there exists a continuous representation $g \to R(g,s)$ of G on $L^2(H_n)$ such that*

(1) *if $\operatorname{Re}(s) = 0$, then $R(g,s)$ is unitarily equivalent to the principal series representation $g \to T(g,s)$;*

(2) *for each fixed $g \in G$ and $f, k \in L^2(H_n)$, the function $s \to (R(g,s)f|k)$ is analytic in the strip $|\operatorname{Re}(s)| < 1$;*

(3) *$\sup\{\|R(g,s)\|_\infty : g \in G\} \leqslant K(\sigma)(1 + |\tau|)^Q$, $s = \sigma + i\tau$, $|\sigma| < 1$ and $K(\sigma)$ is uniformly bounded on closed subintervals of $(-1,1)$.*

PROOF. A complete argument may be found in [**8**]. Briefly, one sets $R(g,s) = T(g,s)$ if $g \in \overline{P}$ and $R(g_1 p g_2, s) = R(g_1, s)R(p,s)R(g_2,s)$ if $g_1 p g_2 \in \overline{P}p\overline{P}$. Analyticity then insures this defines a representation of G. Finally, (2) and (3) follow from Lemma 8.4. $\square$

The representation $R(\cdot, s)$ and $R(\cdot, -s)$, $|\operatorname{Re}(s)| < 1$, given by Theorem 8.5 are not equal. We shall now normalize $R(\cdot, s)$ by an analytic operator $D(s)$ in order to obtain a symmetric representation $S(\cdot, s)$ of G which will be both analytic and uniformly bounded in $|\operatorname{Re}(s)| < 1$ and unitary when $s \in (-1,1)$.

LEMMA 8.6. *If $s, s' \in \mathbf{R}$, then the unitary operators $A(s)$ and $A(s')$ commute.*

PROOF. This follows immediately from the Plancherel theorem and the fact that $\hat{A}(s)$ and $\hat{A}(s')$ commute. $\square$

LEMMA 8.7. *For each complex number s satisfying* $|\mathrm{Re}(s)| < 1$, *we have*

$$R(g^{-1}, s)^* = R(g, -\bar{s}), \qquad g \in G.$$

PROOF. Let $f, k \in L^2(H_n)$ and $g \in G$. By Theorem 8.5 the functions $\phi_1(s) = (R(g^{-1}, s)f|k)$ and $\phi_2(s) = (f|R(g_1 - \bar{s})k)$ are analytic in the strip $|\mathrm{Re}(s)| < 1$. However if $\mathrm{Re}(s) = 0$, then $R(g, s)$ is unitary so that $\phi_1(s) = \phi_1(s)$, $\mathrm{Re}(s) = 0$. Consequently, $\phi_1(s) = \phi_2(s)$ for $|\mathrm{Re}(s)| < 1$. □

By taking the principal value for the log, we set

$$d^\gamma(s) = \frac{\Gamma((Q+s)/4 + |\gamma|)}{\Gamma((Q-s)/4 + |\gamma|)}\left(\frac{\Gamma((Q-2s)/4 + |\gamma|)}{\Gamma((Q+2s)/4 + |\gamma|)}\right)^{1/2}$$

for $\gamma \in \mathbf{N}^n$ and $|\mathrm{Re}(s)| < \frac{1}{2}Q$. We define the operator $D'(s)$ on $C(\alpha)$ ($\alpha \neq 0$) by

$$D'(s)f_\gamma^\alpha = d^\gamma(s)f_\gamma^\alpha, \qquad f_\gamma^\alpha \in B(\alpha),$$

and then extend $D'(s)$ linearly to all of $C(\alpha)$. Suppose ε is chosen such that $0 < \varepsilon < \frac{1}{2}Q$. We know from Lemma 2 of **[16]** that there exists a constant C such that if $s = \sigma + i\tau$ with $|\sigma| < \frac{1}{2}Q - \varepsilon$, then

$$|d^\gamma(s)| \leqslant Ce^{3|\sigma|}(1 + |\sigma|\varepsilon^{-1})(1 + 2|\sigma|\varepsilon^{-1})^{1/2} \times \frac{|n + 3 + 2|\gamma| + i\tau/2|}{|n + 3 + 2|\gamma| + i\tau|}^{\mathrm{Re}(s)/2}.$$

However, $\frac{1}{2} \leqslant |n + 3 + 2|\gamma| + i\tau/2|/|n + 3 + 2|\gamma| + i\tau| \leqslant 1$ for all $\gamma \in \mathbf{N}^n$ and $\tau \in \mathbf{R}$. It now follows that if $0 < \varepsilon < \frac{1}{2}Q$, then there exists a constant K_ε such that $|d^\gamma(s)| \leqslant K_\varepsilon$ for all s satisfying $|\mathrm{Re}(s)| < \frac{1}{2}Q - \varepsilon$ and all $\gamma \in \mathbf{N}^n$. Furthermore, for each $\gamma \in \mathbf{N}^n$, the function $s \to d^\gamma(s)$ is analytic in the strip $|\mathrm{Re}(s)| < \frac{1}{2}Q$. By Lemma 3.2, $D'(s)$ extends to an operator on $L^2(\mathbf{R}'; \mathrm{HS})$, again denoted by $D'(s)$, such that $s \to D'(s)$ is analytic and $\|D'(s)\|_\infty \leqslant K_\varepsilon$ whenever $|\mathrm{Re}(s)| < \frac{1}{2}Q - \varepsilon$. By the Plancherel theorem, we know $D'(s)$ defines an analytic operator $D(s)$ on $L^2(H_n)$ satisfying $\|D(s)\|_\infty \leqslant K_\varepsilon$ whenever $|\mathrm{Re}(s)| \leqslant \frac{1}{2}Q - \varepsilon$, $0 < \varepsilon < \frac{1}{2}Q$. Furthermore, it follows from the definition of $d^\gamma(s)$ that $D(s)^* = D(\bar{s})$ and if $\mathrm{Re}(s) = 0$, then $D(s)$ is unitary with inverse $D(-s)$.

LEMMA 8.8. *If* $\mathrm{Re}(s) = 0$, *then* $D(s)^2 = A(-s)^2A(2s)$.

PROOF. We have $\hat{A}(-s)^2\hat{A}(2s) = G(-s)^2N(-s)^2G(2s)N(2s) = G(-s)^2G(2s)$. On the other hand,

$$D'(s)^2 f_\gamma^\alpha = q^\gamma(-s)^2 q^\gamma(2s) f_\gamma^\alpha = G(-s)^2 G(2s) f_\gamma^\alpha.$$

The lemma now follows from the Plancherel theorem. □

For $\mathrm{Re}(s) = 0$, let $S(g, s) = D(s)R(g, s)D(-s)$.

THEOREM 8.9. *For each s such that* $|\mathrm{Re}(s)| < 1$ *there exists a continuous representation* $g \to S(g, s)$ *of G on* $L^2(H_n)$ *such that*

(1) *if* $\mathrm{Re}(s) = 0$, *then* $S(g, s)$ *is unitarily equivalent to the principal series representation* $g \to T(g, s)$;

(2) *for each fixed* $g \in G$ *and* $f, k \in L^2(H_n)$, *the function* $s \to (S(g, s)f|k)$ *is analytic in the strip* $|\mathrm{Re}(s)| < 1$;

(3) $\sup\{\|S(g, s)\|_\infty : g \in G\} \leqslant K(\sigma)(1 + |\tau|)^Q$;

(4) *for all* $g \in G$ *and* s *in the strip* $|\mathrm{Re}(s)| < 1$, $S(g, s) = S(g, -s)$;

(5) *for all* $g \in G$ *and* s *in the strip* $|\mathrm{Re}(s)| < 1$, $S(g^{-1}, s)^* = S(g, -\bar{s})$;

(6) *if* $-1 < s < 1$, *then* $S(\cdot, s)$ *is a unitary representation of* G.

PROOF. Since $D(s)$ is unitary when $\mathrm{Re}(s) = 0$, (1) follows from Theorem 8.5. By a theorem of Hille [5, p. 53], the product of analytic operators is analytic. Since $D(s)$ and $R(g, s)$ are both analytic in $|\mathrm{Re}(s)| < 1$, so must be $S(g, s)$. Thus (2) is established. In addition, (3) follows from Theorem 8.5 and the fact that $\|D(s)\|_\infty \leqslant K$ for all s in the strip $|\mathrm{Re}(s)| < 1$.

In order to prove (4), we let $f, k \in L^2(H_n)$ and set $\phi_1 = (S(g, s)f|k)$ and $\phi_2(s) = (S(g, -s)f|k)$ for $|\mathrm{Re}(s)| < 1$. From Lemma 8.7, we have for $\mathrm{Re}(s) = 0$

$$\begin{aligned}\phi_1(s) &= \left(D(-s)D(s)^2R(g, s)D(-s)^2D(s)f|k\right)\\ &= \left(D(-s)A(-s)^2A(2s)R(g, s)A(s)^2A(-2s)D(s)f|k\right).\end{aligned}$$

By using Lemma 8.6 and the intertwining property (8.2), we see $A(2s)R(g, s) = A(s)T(g_1 - s)A(2s)A(-s)$ so that for $\mathrm{Re}(s) = 0$ we obtain

$$\begin{aligned}\phi_1(s) &= (D(-s)A(-s)T(g, -s)A(s)D(s)f|k)\\ &= (S(g, -s)f|k) = \phi_2(s).\end{aligned}$$

By analyticity, it now follows that $\phi_1(s) = \phi_2(s)$ on the entire strip $|\mathrm{Re}(s)| < 1$.

We now prove (5) by using Lemma 8.8 and the fact that $D(s)^* = D(\bar{s})$. We have

$$S(g^{-1}, s)^* = D(-s)^*R(g^{-1}, s)^*D(s)^* = D(-\bar{s})R(g, -\bar{s})D(\bar{s}) = S(g, -\bar{s}).$$

Finally, if $\mathrm{Im}(s) = 0$, then we conclude from (4) and (5) that $S(g^{-1}, s)^* = S(g, s)$. This proves (6) of the theorem. □

REMARK. A two-stage normalization of $T(\cdot, s)$ was required in order to obtain an analytic uniformly bounded representation $S(\cdot, s)$ which enjoys symmetry in s. In the case $G = SO(1, n+1)$, Wilson [16] shows the normalization $A(s)T(\cdot, s)A(-s)$ automatically produces symmetry in s. For the Lorentz groups this simplification occurs because $A(s)^2 = A(2s)$.

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