TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 290, Number 1, July 1985

ON THE DECOMPOSITION NUMBERS OF THE FINITE GENERAL LINEAR GROUPS[1]

BY

RICHARD DIPPER

ABSTRACT. Let $G = \mathrm{GL}_n(q)$, q a prime power, and let r be an odd prime not dividing q. Let s be a semisimple element of G of order prime to r and assume that r divides $q^{\deg(\Lambda)} - 1$ for all elementary divisors Λ of s. Relating representations of certain Hecke algebras over symmetric groups with those of G, we derive a full classification of all modular irreducible modules in the r-block B_s of G with semisimple part s. The decomposition matrix D of B_s may be partly described in terms of the decomposition matrices of the symmetric groups corresponding to the Hecke algebras above. Moreover D is lower unitriangular. This applies in particular to all r-blocks of G if r divides $q - 1$. Thus, in this case, the r-decomposition matrix of G is lower unitriangular.

Introduction. The modular representation theory of finite groups of Lie type defined over a field of characteristic p is naturally divided into the cases of equal characteristic $r = p$ and unequal characteristic $r \neq p$. This paper begins the study of decomposition numbers in the unequal characteristic case of the finite general linear groups when $r > 2$.

Let G be the full linear group of degree n over $\mathrm{GF}(q)$ and $2 \neq r$ be a prime not dividing q. Let $s \in G$ be semisimple and consider the geometric conjugacy class $(s)^G$ of irreducible characters corresponding to the G-conjugacy class of s. Using basic properties of the Deligne-Lusztig operators introduced in **[7]** we will determine a parabolic subgroup $P = P_s$ of G and a cuspidal irreducible character of χ of P such that $(s)^G$ is just the set of constituents of χ^G, the induced character. Let $\{\overline{R}, R, K\}$ be an r-modular system for G and M a KP-module affording χ. Then $\mathrm{End}_{KG}(M^G)$ is called the Hecke algebra of M (compare **[5]**) and has been calculated in a more general context by R. B. Howlett and G. I. Lehrer in **[12]**. It turns out that $\mathrm{End}_{KG}(M^G) \cong K[W]$, the Hecke algebra of W, where $W = W_s$ denotes the Weyl group of $C_G(s)$ (the centralizer of s in G) (compare **[1, 2, 4, 13]**). In particular, $K[W]$ has a basis $\{T_w | w \in W\}$ such that the structure constants of the multiplication are all contained in $R \subseteq K$. So let $R[W]$ be the R-order in $K[W]$ generated by $\{T_w | w \in W\}$ and $\overline{R}[W] = \overline{R} \otimes_R R[W]$. Choosing a special RP-lattice S in M, and using the classification of the r-blocks of RG given by P. Fong and B. Srinivasan in

Received by the editors July 25, 1983 and, in revised form, October 18, 1983, May 23, 1984 and September 1, 1984.

1980 *Mathematics Subject Classification*. Primary 20C20; Secondary 20G40.

[1] This work was supported by the Deutsche Forschungsgemeinschaft.

their fundamental paper **[9]**, we will show that $\text{End}_{\tilde{R}G}(\tilde{R}S^G) \cong \tilde{R}[W]$ for every choice of $\tilde{R} \in \{\bar{R}, R, K\}$ if $s \in G$ satisfies certain conditions (Hypothesis 2.4), which are fulfilled, in particular, if the order of s is prime to r. This generalizes parts of **[12]**. We define a certain subalgebra A of RG, which does not arise from a subgroup of G, and which will play the role of an inertia algebra of S in G. Using this algebra we will prove a homological lemma of Clifford type (Lemma 3.7), which enables us to connect the submodule structure of S^G and $R[W]$. It is a well-known theorem that $K[W] \cong KW$, the usual group algebra of W over K. This does not remain true, in general, if we replace K by R or $\bar{R}$. But there is an important special case where $\bar{R}[W] \cong \bar{R}W$ canonically, namely if r divides $q^{\deg \Lambda}-1$ for all elementary divisors of s, and we may apply then the known facts of the representation theory of symmetric groups.

Thus let s be a semisimple r'-element of G and assume that r divides $q^{\deg \Lambda}-1$ for all elementary divisors Λ of s. By **[9]** there is just one r-block B of RG with semisimple part s and the union of the geometric conjugacy classes $(sy)^G$, where y runs through a Sylow r-subgroup D of $C_G(s)$, is just the set of irreducible characters in B. Note that r divides $q^{\deg \Gamma} - 1$ for all elementary divisors Γ of sy, too. We define a sequence $1 = y_1, \ldots, y_k$ in D such that $t_i = sy_i$ satisfies Hypothesis 2.4, so we may apply the results of §§3 and 4 to the geometric classes $(t_i)^G$. This leads to a full classification of the irreducible $\bar{R}G$-modules contained in B. We obtain parts of the decomposition matrix of B in terms of the decomposition matrices of W_{t_i} $(1 \le i \le k)$, the Weyl group of $C_G(t_i)$. In particular, we show that the irreducible characters and irreducible Brauer characters in B may be ordered such that the decomposition matrix of B has generalized lower triangular form with one's in the main diagonal, a fact known for the decomposition matrices of symmetric groups. As a corollary we get that a cuspidal irreducible character is an irreducible Brauer character if it is restricted to the set of r'-elements of G for all primes r which divide $q - 1$. In fact, it seems to be likely that this holds in general.

Throughout this paper we will use the standard notation **Z**, **R**, etc. for the integers, real numbers respectively. If not stated otherwise, all occurring modules are finitely generated right modules. All rings are R-free, finite dimensional R-algebras for some domain R. For an R-algebra A and an A-module M, we denote the Jacobson radical of M by $J(M)$. The factor module $M/J(M)$ is called the head of M.

The author is very indebted to P. Fong and B. Srinivasan for many helpful suggestions and discussions.

1. First let us fix some notation. Throughout this paper let q be a power of some fixed prime p. Let $F = \text{GF}(q)$ be the finite field with q elements and $G = \text{GL}_n(q)$ the full linear group of degree n over F for a fixed natural number n. Let R be a discrete rank one valuation ring with maximal ideal $J(R)$ (the Jacobson radical of R), and assume that the quotient field K of R has characteristic zero, and $\bar{R} = R/J(R)$ is a field of characteristic $r > 0$ for some odd prime $r \neq p$. For an R-module V we will write $\bar{R}V$ or $\bar{V}$ for $\bar{R} \otimes_R V \cong V/J(R)V$, and KV for $K \otimes_R V$. Furthermore, we will choose R such that K and $\bar{R}$ are both splitting fields for all subgroups of G.

Next we will describe briefly the semisimple conjugacy classes and the irreducible characters of G, where we mostly adopt the notation of [**9**].

So let X be an indeterminant and let $\mathscr{F}$ be the set of monic irreducible polynomials in X over F, different from the polynomial $p(X) = X$. For $\Lambda \in \mathscr{F}$, $s \in G$ and $k \in \mathbf{N}$ let deg Λ be the degree of Λ, $k(\Lambda) \in \mathrm{Gl}_{k \cdot \deg \Lambda}(q)$ the matrix direct sum of k copies of the companion matrix of Λ and $m_\Lambda(s)$ the multiplicity of Λ as elementary divisor of s. If $s \in G$ is semisimple, all elementary divisors of s are contained in $\mathscr{F}$, hence s is conjugate in G to $\prod_{\Lambda \in \mathscr{F}} m_\Lambda(s)(\Lambda)$, where $\prod$ stands for matrix direct sum, and factors of multiplicity zero have to be dropped. This is just the rational canonical form of s and is uniquely determined up to the order of the factors. So let us make the following convention: First we order $\mathscr{F}$ totally such that $\deg \Lambda < \deg \Gamma$ ($\Lambda, \Gamma \in \mathscr{F}$) implies $\Lambda < \Gamma$. All occurring products will be taken in this fixed order, and, in the following, $\prod$, Σ, $\bigcup$, etc. always mean $\prod_{\Lambda \in \mathscr{F}}$, $\Sigma_{\Lambda \in \mathscr{F}}$, $\bigcup_{\Lambda \in \mathscr{F}}$, etc. With this convention

$$\left\{ \prod k_\Lambda(\Lambda) | 0 \leq k_\Lambda \in \mathbf{N}, \Sigma k_\Lambda \cdot \deg \Lambda = n \right\}$$

is a full set of representatives of semisimple conjugacy classes of G. For semisimple $s = \prod m_\Lambda(s)(\Lambda)$ in G, the centralizer $C_G(s)$ of s in G is given by

$$C_G(s) = \prod \mathrm{GL}_{m_\Lambda(s)}(q^{\deg \Lambda}),$$

where we identify $\mathrm{GF}(q^{\deg \Lambda})$ with the subring of the ($\deg \Lambda \times \deg \Lambda$)-matrix ring over F generated by F and (Λ).

For the notation of maximal tori, regular, parabolic and Levi subgroups we refer to [**15**]. In particular, recall that the set of all diagonal matrices in G is a maximal torus of G, the standard split torus T_0. The Weyl group $W_G = W_0$ of G is defined to be $N_G(T_0)/T_0$, where $N_G(T_0)$ denotes the normalizer of T_0 in G. In the following, we identify always W_0 with the subgroup of all permutation matrices in G, which is canonically isomorphic to S_n, the symmetric group on n letters (permuting the natural basis of the natural module for G). The G-conjugacy classes of maximal tori are in bijection with the conjugacy classes of W_0, hence with partitions of n. More precisely, to the partition $\lambda = (1^{r_1}, \ldots, n^{r_n})$ of n we may construct a representative T_λ of the corresponding G-conjugacy class of maximal tori in the following way: First, for $1 \leq i \leq n$ choose an irreducible polynomial $\Lambda \in \mathscr{F}$ whose roots generate $\mathrm{GF}(q^i)$ over $F = \mathrm{GF}(q)$. Then, as above, F and (Λ) generate a subalgebra F_i of the $i \times i$-matrix over $\mathscr{F}$ isomorphic to $\mathrm{GF}(q^i)$. Now define $T_\lambda = (F_n^*)^{r_n} \times \cdots \times (F_1^*)^{r_1} \leq G$ (matrix direct sum), where F_i^* denotes the multiplicative group of F_i ($1 \leq i \leq n$). In particular $T_0 = T_\lambda$ for $\lambda = (1^n)$. A maximal torus corresponding to $\lambda = (n)$ is called a Coxeter torus. If T is a maximal torus of G, the Weyl group $W(T) = W_G(T)$ of T (in G) is defined to be $N_G(T)/T$. If T corresponds to $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$, then $W(T)$ is isomophic to $C_{W_0}(w_\lambda)$, the centralizer of $w_\lambda \in W_0 = W_G(T_0)$ in W_0, $w_\lambda = w_1 \cdots w_k$, where $w_i \in W_0$ is defined to be the cycle $(j_i + 1, \cdots, j_{i+1})$ in W with $j_i = \Sigma_{m=1}^{i-1} \lambda_m$. Of course $C_{W_0}(w_\lambda)$ is the semidirect product of $\prod_{j=1}^{k} \langle w_i \rangle$ and $W_\lambda \cong S_{r_i} \times \cdots \times S_{r_n}$ ($\lambda = (\lambda_1 \geq \cdots \geq \lambda_k) = (1^{r_i}, \ldots, n^{r_n})$), generated by the involutions

$(j_i + 1, j_{m+1} + 1) \cdots (j_{i+1}, j_{m+1})$, where $1 \leq i, m \leq k$ are such that $\lambda_i = \lambda_m$. If $T = T_\lambda$ is constructed as above, then it can easily be seen that T has a complement in $N_G(T) \leq G$ which is the semidirect product of a subgroup of $\prod_{i=1}^n \mathrm{GL}_i(q)^{r_i} \leq G$ isomorphic to $\prod_{i=1}^k \langle w_i \rangle$ by W_j (in fact conjugate to $\prod_{i=1}^k \langle w_i \rangle$ in $\mathrm{GL}_n(q^m) \geq G$ for a suitable $m \in \mathbf{N}$). Of course $N_G(T)/T$ is isomorphic to this complement. Abusing notation, we call this complement Weyl group of T, too. In addition, throughout this paper, $W_G(T)$ always denotes the complement of T in $N_G(T)$, whereas we shall denote the Weyl group of T in G in the original sense by $N_G(T)/T$. So $W_G(T) \leq G$. Note in particular that $W_0 = W_G(T_0)$ is the set of permutation matrices in G.

Now let L be a regular subgroup of G. Let R_L^G be the additive operator from $X(L)$ to $X(G)$ defined in the Deligne-Lusztig theory, where $X(L)$ and $X(G)$ are the character rings of L and G respectively over K, following the notation used in **[9]** (there is no harm in replacing the algebraically closed field in **[9]** by K). For the definition and basic properties of these operators we refer to **[15]**.

In **[10]** J. A. Green has classified the irreducible characters of G. In the Deligne-Lusztig theory these are parametrized by G-conjugacy classes of pairs (s, μ) where $s \in G$ is semisimple and $\mu = \prod \mu_\Lambda$ (cartesian product), with μ_Λ a partition of $m_\Lambda(s)$. So for semisimple s in G, let M_s be the set of all such products of partitions of $m_\Lambda(s)$. The corresponding character of G will be denoted by $\chi_{s,\mu}$.

In **[9]** P. Fong and B. Srinivasan have classified the blocks of RG with their defect groups. Furthermore, they determined for each r-block the characters contained in it. Following the notation introduced in **[9]** we have a bijecton $B \to B_{s,\lambda}$ between the blocks B of RG and G-conjugacy classes (s^G, λ) of pairs (s, λ), where s is a semisimple r'-element of G and λ the unipotent factor of B as defined in **[9]**. λ can be derived from elements of M_s in terms of hooks and cores. We will need the following special case. Let $\Lambda \in \mathscr{F}$ be such that $\deg \Lambda = n$ and all roots of Λ have order prime to r. Then $s = (\Lambda) \in G$ is a semisimple r'-element. It can easily be seen that there is just one possible unipotent factor associated with s, say λ, and we may write $B_s = B_{s,\lambda}$. Furthermore, it follows from **[9]** that B_s is a block with cyclic defect group and inertial index one. So there is just one irreducible RG-module M in B_s, and every irreducible KG-module in B_s, when reduced modulo r, is isomorphic to M. Note that all definitions and formulas above (and the following as well) may easily be generalized to finite direct products of finite full linear groups.

Now let $s \in G$ be semisimple, and consider $L_0 = C_G(s)$. The center of L_0 is isomorphic to the multiplicative group of linear characters of L_0, and this isomorphism may be chosen such that $\widehat{s^g} = \hat{s}g$ for all $g \in N_G(L_0)$, where the linear character corresponding to s will be denoted by $\hat{s}$.

L_0 is conjugate in G to $\prod \mathrm{GL}_{m_\Lambda(s)}(q^{\deg \Lambda})$. In particular, the Weyl group $W_{L_0} = W_s = W$ of L_0 is isomorphic to $\prod S_{m_\Lambda(s)}$. Consequently, the L_0-conjugacy classes of maximal tori of L_0 correspond bijectively to M_s. For $\mu = \prod \mu_\Lambda \in M_s$ define the partition $\tilde{\mu}$ of n in the following way: First, for $\Lambda \in \mathscr{F}$, $\mu_\lambda = (1^{k_1}, 2^{k_2}, \ldots)$, define the partition $\deg \Lambda \cdot \mu_\Lambda$ of $\deg \Lambda \cdot m_\Lambda(s)$ setting

$$\deg \Lambda \cdot \mu_\Lambda = \left((\deg \Lambda)^{k_1}, (2 \deg \Lambda)^{k_2}, \ldots \right).$$

Let $r_{i,\Lambda}$ be the exponent of i in $\deg\Lambda \cdot \mu_\Lambda$ for $1 \le i \le \deg\Lambda \cdot m_\Lambda(s)$, and $r_{i,\Lambda} = 0$ for $\deg\Lambda \cdot m_\Lambda(s) + 1 \le i \le n$. Set $r_i = \Sigma r_{i,\Lambda}$, and define $\tilde{\mu}$ setting $\tilde{\mu} := (1^{r_1}, \ldots, n^{r_n})$. It can easily be seen that a maximal torus of L_0 corresponding to $\mu \in M_s$ is also a maximal torus of G, contained in the G-conjugacy class of maximal tori of G corresponding to $\tilde{\mu}$. The map $\sim : \mu \to \tilde{\mu}$ is not injective in general (i.e. there are maximal tori in L_0, which are not conjugate in L_0 but in G), but if $\mu = \prod(1^{m_\Lambda(s)}) \in M_s, \nu \in M_s$ and $\mu \neq \nu$, then $\tilde{\mu} \neq \tilde{\nu}$. In other words, the maximal split torus $T = T_s$ $(= T_\mu)$ of L_0 is not conjugate in G to $T_{\tilde{\nu}}$.

It is well known (see e.g. [**15**]) that there is a bijection between the set of all irreducible characters of $W = W_s$ and M_s, and between the set of conjugacy classes of W and M_s as well. So, for $\mu, \lambda \in M_s$, let $\phi^\mu(\lambda)$ be the value of the irreducible character of W corresponding to μ on the conjugacy class of W corresponding to λ. Furthermore, let $\hat{\lambda}$ be the number of elements in this class. Finally denote the restriction of $\hat{s}$ to T_μ, $\mu \in M_s$, again by $\hat{s}$. Now define the generalized character $\chi^{s,\mu}$ of G by

$$\chi^{s,\mu} = \frac{1}{|W|} \sum_{\lambda \in M_s} \hat{\lambda}\phi^\mu(\lambda) R^G_{T_{\tilde{\lambda}}}(\hat{s}).$$

Then $\chi^{s,\mu}$ is up to sign the irreducible character $\chi_{s,\mu}$ of G (see e.g. [**9**, 1.18]).

Note that we may replace s by $s^g = g^{-1}sg$ for $g \in G$. So we may assume in the following that s is given in the rational canonical form. In particular, $s \in T$, $L_0 = C_G(s) = \prod \mathrm{GL}_{m_\Lambda(s)}(q^{\deg\Lambda}) \le G$ canonically, and $W = W_s \le W_0$.

1.1. LEMMA. $R^G_T(\hat{s}) = \Sigma_{\mu \in M_s} \phi^\mu(1)\chi^{s,\mu}$.

PROOF. Let $(\cdot, \cdot) = (\cdot, \cdot)_G$ be the usual scalar product in $X(G)$. By the orthogonality relations for the Deligne-Lusztig operators (see e.g. [**9**, 1.6]) the following holds:

$$\left(R^G_{T_{\tilde{\lambda}}}(\hat{s}), R^G_T(\hat{s})\right) = 0 \quad \text{for } \prod(1^{m_\Lambda(s)}) \neq \lambda \in M_s,$$

and

$$\left(R^G_T(\hat{s}), R^G_T(\hat{s})\right) = \left|C_{W_G(T)}(\hat{s})\right| = \left|C_{W_G(T)}(s)\right| = |W|;$$

for, since T is abelian and $s \in T$,

$$C_{N_G(T)/T}(s) \cong C_{N_G(T)}(s)/T = \left(N_G(T) \cap C_G(s)\right)/T = N_{L_0}(T)/T \cong W.$$

Consequently

$$\left(\chi^{s,\mu}, R^G_T(\hat{s})\right) = |W|^{-1} \sum_{\lambda \in M_s} \hat{\lambda}\phi^\mu(\lambda)\left(R^G_{T_{\tilde{\lambda}}}(\hat{s}), R^G_T(\hat{s})\right)$$

$$= |W|^{-1}\phi^\mu(1)\left(R^G_T(\hat{s}), R^G_T(\hat{s})\right) = \phi^\mu(1).$$

Now $\Sigma_{\mu \in M_s}(\phi^\mu(1))^2 = |W|$. This proves the lemma. □

Now let $L = L_s = \prod (\mathrm{GL}_{\deg\Lambda}(q))^{m_\Lambda(s)}$, embedded in G in the obvious way. Then $s \in L$, in fact the standard split torus $T = T_s$ of L_0 (which contains s) is a Coxeter torus of L and $C_L(s) = L \cap L_0 = T$. Of course, L is a Levi complement of some parabolic subgroup $P = LU$ with Levi kernel $U = U_s$, and we can choose $P = LU_0$, where U_0 denotes the subgroup of all unipotent upper triangular matrices in G. Every L-module M (over R, $\bar{R}$ or K) may be considered as P-module via the epimorphism $P \to L$. The corresponding P-module will be denoted by $\tilde{M}$. Similarly, for a character χ of L let $\tilde{\chi}$ be the pull back of χ to P. Furthermore, if $H \leq G$ and if M is an H-module, N a G-module, let M^G be the induced and N_H the restricted module.

1.2 LEMMA. $R_T^G(\hat{s}) = (\widetilde{R_T^L(s)})^G$.

PROOF. [**15**, 6.24].

1.3. LEMMA. $R_T^L(\hat{s})$ *is up to sign an irreducible cuspidal character of* L.

PROOF. $C_G(s) \cap W_L(T) \leq L \cap L_0 = T$, hence the centralizer of s (therefore of $\hat{s}$ too) in $N_L(T)/T$ is trivial, i.e. $\hat{s}$ is a regular character of $T \leq L$. So $\pm R_T^L(\hat{s})$ is irreducible by [**15**, 6.19]. Of course $(R_T^L(\hat{s}), R_{\tilde{T}}^L(\theta))_L = 0$ for every maximal torus $\tilde{T}$ of L not conjugate to $T = T_s$ in L [**15**, 6.14], and for any character θ of $\tilde{T}$. So $\pm R_T^L(\hat{s})$ is cuspidal by [**15**, 6.25]. □

Next let $L(\Lambda) = \mathrm{GL}_{\deg \Lambda}(q)$ $(\Lambda \in \mathscr{F})$; thus $L = \prod L(\Lambda)^{m_\Lambda(s)}$. Let $s = \prod m_\Lambda(s)(\Lambda)$, $\hat{s} = \prod \widehat{(\Lambda)}^{m_\Lambda(s)}$ (tensor product), and $T = \prod T(\Lambda)^{m_\Lambda(s)}$ be the corresponding decompositions of s, $\hat{s}$ and $T_s = T$ respectively. Then $T(\Lambda)$ is a Coxeter torus of $L(\Lambda)$ and $C_{L(\Lambda)}((\Lambda)) = T(\Lambda)$. Applying 1.3 to $L(\Lambda)$ we conclude that $R_{T(\Lambda)}^{L(\Lambda)}(\widehat{(\Lambda)})$ is up to sign an irreducible character of $L(\Lambda)$. Let $S(\Lambda)$ be an $RL(\Lambda)$-lattice in the irreducible $KL(\Lambda)$-module affording this irreducible character, and set $S_0 = \prod S(\Lambda)^{m_\Lambda(s)}$ (tensor product).

1.4. LEMMA. KS_0 *is an irreducible* KL-*module. The character afforded by* KS_0 *is up to sign* $R_T^L(\hat{s})$.

PROOF. Note first that by the choice of the isomorphism between the center of $L_0 = C_G(s)$ and the multiplicative group of linear characters of L_0, in fact $\hat{s} = \prod \widehat{(\Lambda)}^{m_\Lambda(s)}$. As an easy consequence from the Künneth formula for 1-adic cohomology (see e.g. [**15**, 5.9]) we now get that

$$R_T^L(\hat{s}) = \prod \left(R_{T(\Lambda)}^{L(\Lambda)}\left(\widehat{(\Lambda)}\right)\right)^{m_\Lambda(s)}.$$

Now the lemma follows easily from 1.3. □

Let $S = \tilde{S}_0$, i.e. the pull back of S_0 to P. Then 1.1, 1.2 and 1.4 imply

1.5. LEMMA. $KS^G = \sum_{\mu \in M_s} \phi^\mu(1) N_\mu$, *where* N_μ, $\mu \in M_s$, *is an irreducible* KG-*module affording* $\chi_{s,\mu}$.

In particular, counting multiplicities, this implies the following well-known corollary.

1.6. COROLLARY. *The endomorphism ring* $\mathrm{End}_{KG}(KS^G)$ *of* KS^G *is isomorphic to the group algebra* KW_s.

Throughout this paper we will keep the notation introduced above. In particular, $s \in G$ will always denote a semisimple element of G, given in the rational canonical form, and M_s, $W = W_s$, $T = T_s$, $L = L_s$, $P = P_s = LU$, $U = U_s$ and S will be defined as above. If necessary, we will use the index s. Furthermore let $b = b_s$ be the block of RP containing S and $e = e_s$ the block idempotent of b. By general arguments (see e.g. [**8**, 4.3]), the blocks of RL are just those blocks of RP which contain U in their kernel, because U is a normal r'-subgroup of P. In particular, U is in the kernel of b and all irreducible KP-modules in b contain U in their kernel. So we may consider b as the block of RL and decompose it in $b = \prod b_\Lambda^{m_\Lambda(s)}$ corresponding to the decomposition of L. If e_0 is the block idempotent of b considered as the block of RL, then $e = e_0 f$ with $f = f_s = |U|^{-1}\Sigma_{u \in U} u$. Note that $f \in \tilde{R}G$ for every choice of $\tilde{R} \in \{\overline{R}, R, K\}$.

2. For the moment we will assume in addition that the order of $s \in G$ is prime to r. For $\Lambda \in \mathscr{F}$ consider the block b_Λ of $L(\Lambda)$. As remarked in the first section there is just one block, namely b_Λ, of $RL(\Lambda)$ with semisimple part $(\Lambda) \in L(\Lambda)$, because there is only one possible unipotent factor associated with (Λ). Furthermore, b_Λ has cyclic defect group and inertial index 1, so $\overline{S(\Lambda)} = \overline{R}S(\Lambda)$ is the unique irreducible $\overline{R}L(\Lambda)$-module in b_Λ. This implies immediately

2.1. Lemma. *Let N be an irreducible KP-module in b, and S_1 and RP-lattice in N. Then $\overline{S}_1 = \overline{R}S_1 \cong \overline{R}S$, and $\overline{R}S$ is the unique irreducible $\overline{R}P$-module in b.*

Recall that the set of characters $\chi_{t,\mu}$, where t is a fixed semisimple element of G and μ runs through M_t, is called the geometric conjugacy class corresponding to t and is denoted by $(t)^G$. Assume that $\chi \in (t)^L$ and $\chi \in b$. Then [**9**, 7A] implies in particular that $t = sy$, for some r-element $y \in T_s$. So we conclude

2.2. Lemma. *Let ψ be an irreducible character of P contained in b. Then $\psi = \chi$ for some $\chi \in (sy)^L$ and some r-element $y \in T$.*

2.3. Lemma. *Let V be an irreducible KP-module contained in b. Assume that V occurs as the constituent of KS_P^G. Then $V \cong KS$.*

Proof. Let ψ be the character afforded by V. Then, by 2.2, $\psi = \chi$ and $\chi \in (sy)^L$ for some $y \in T$. Let $t = sy$. Then $C_L(t) = C_L(s) = T$. In particular, $C_L(t)$ is a direct product of multiplicative groups of fields, so, considered as a product of linear groups, its Weyl group is trivial, and $(t)^L$ contains only one element, namely χ. Therefore $\chi = \pm R_T^L(\hat{t})$, and by [**15**, 6.24], $\psi^G = \pm R_T^G(\hat{t})$. Obviously the multiplicity of V in KS_P^G is just $|(\psi, R_T^G(\hat{s}))_P|$. By Frobenius reciprocity and [**15**, 6.24],

$$\left(\psi, R_T^G(\hat{s})\right)_P = \left(\tilde{\chi}^G, R_T^G(\hat{s})\right)_G = \pm\left(R_T^G(\hat{t}), R_T^G(\hat{s})\right) = \left|\left\{ g \in N_G(T)/T \mid t^g = s \right\}\right|.$$

Now, if $y \neq 1$, the orders of s and t are different and they cannot be conjugate by an element of $N_G(T)/T$. Thus $y = 1$, that is $t = s$ and $\chi \in (s)^L$. Because $(s)^L$ contains exactly one element, namely the character afforded by KS, we conclude $V \cong KS$ as desired. □

Now let $s \in G$ be again an arbitrary semisimple element.

2.4. Hypothesis. *Assume that s satisfies the following conditions*:

(i) *$\overline{R}S$ is an irreducible $\overline{R}P$-module.*

(ii) *Let $V \in b$ be an irreducible constituent of KS_P^G. Then $V \cong KS$.*

We have seen in 2.1 and 2.3 that $s \in G$ satisfies 2.4 as long as its order is prime to r. So the results of the following sections are always true in this case, but we will need it also for some special elements of the form sy, where s is a semisimple r'-element of G and y is an r-element of $C_G(s)$. We have to verify 2.4 in these cases. We will prove later that (i) is always true if r divides $q-1$; in fact, it seems to be true in general, whereas (ii) is not satisfied by all semisimple elements of G.

3. Let $s \in G$ be semisimple, given in the rational canonical form, and assume 2.4. Now W, permuting the equal factors of $s = \prod m_\Lambda(s)(\Lambda)$, normalizes L. On the other hand $L \cap W \leq L \cap C_G(s) = T$, $W \cap T = (1)$ implies that $H = LW \leq G$ is a semidirect product.

Let $\tilde{R} \in \{\overline{R}, R, K\}$. Recall that $\tilde{R}S_0 = \prod \tilde{R}S(\Lambda)^{m_\Lambda(s)}$. Now $w \in W$ acts as permutation on the factors of $\tilde{R}S(\Lambda)^{m_\Lambda(s)}$ for each $\Lambda \in \mathscr{F}$. So $\tilde{R}S_0^w \cong \tilde{R}S_0$ for all $w \in W$, and obviously the $\tilde{R}L$-isomorphisms t_w: $\tilde{R}S_0^{w^{-1}} \to \tilde{R}S_0$ (considered as elements of $\mathrm{End}_{\tilde{R}}(\tilde{R}S_0)$) may be chosen such that $t_w t_{w'} = t_{ww'}$ for all w, $w' \in W$. In other words, in the language of stable Clifford theory (see e.g. [**6**]), $\mathrm{End}_{\tilde{R}H}(\tilde{R}S_0^H) \cong \tilde{R}W\,(\cong \tilde{R}(H/L))$ canonically.

In this section we will give a presentation of $\mathrm{End}_{\tilde{R}_G}(\tilde{R}S^G)$. For $\tilde{R} = K$ this has been done by R. B. Howlett and G. I. Lehrer in [**12**] in a more general situation. They proved in particular that $\mathrm{End}_{KG}(KS^G)$ is the Hecke algebra $K[W_s]$ (later we will give a concrete description of this algebra). This will be generalized to arbitrary $\tilde{R} \in \{\overline{R}, R, K\}$. But first we have to describe S_P^G.

Consider W_L, the Weyl group of L. Note that $W_L \leq W_0$. By [**2**, Chapter IV, 2.5] there exists a bijection between the double cosets of W_L in W_0 and the double cosets of P in G given by $W_L w W_L \to PwP$ $(w \in W_0)$.

3.1. Lemma. *Let w, $w' \in W$. Then $w' \in PwP$ if and only if $w = w'$.*

Proof. By the above it is enough to prove $w' \in W_L w W_L$, if and only if $w = w'$. Considering W as a symmetric group, W_L becomes a Young subgroup of W and the lemma follows from the well-known description of the double cosets of Young subgroups of symmetric groups. □

Next we want to decompose the double coset PwP $(w \in W)$ into right cosets of P in G. So let $V_0 \leq G$ be the subgroup of G of all unipotent, lower triangular matrices, and set, for $w \in W_0$, $U_w^- = V_0^{w^{-1}} \cap U_0$ and $U_w^+ = U_0 \cap U_0^w$. It is well known that $U_0 = U_w^+ U_w^-$ and $U_w^+ \cap U_w^- = (1)$. Let $B \leq G$ be the group of all upper triangular matrices in G (Borel subgroup of G). Then every element of $BwB \subset G$ may be uniquely written as gwu with $g \in B$ and $u \in U_w^-$ (the Bruhat decomposition). So BwB is the disjoint union $\dot{\bigcup}_{u \in U_w^-} Bwu$. Now let in particular $w \in W_s$. Then $U_L = L \cap U_0$ is contained in U_w^+, because w permutes the factors of U_L. Of course $U_0 = U_L \cdot U$, hence $U_w^- \subset U$. Therefore

$$PwP = PwLU = PwU = PwU_w^- = \bigcup_{u \in U_w^-} Pwu.$$

Let u_1, $u_2 \in U_w^-$ and assume $wu_1 \in Pwu_2$. Then $wu_1 = u_3 x w u_2$ for some $x \in L$ and $u_3 \in U$. By the Bruhat decomposition for L there exists $g \in B \cap L$, $w_1 \in W_L$ and

$u_4 \in U_L \cap U_{w_1}^-$ such that $x = gw_1u_4$. Now $u_4 \in U_L \subset U_w^+$, so $wu_1 = u_3gw_1wu_5u_2$ with $u_5 = u_4^w \in B$, hence $wu_1 \in Bw_1wB$ forcing $w_1 = 1$. Thus $x \in B$, and again by the Bruhat decomposition $u_3x = 1$ and $u_1 = u_2$. We have shown

3.2. LEMMA. $PwP = \dot{\bigcup}_{u \in U_w^-} Pwu$ (*disjoint union*) ($w \in W_s$).

We still need

3.3. LEMMA. *Let V be a free finitely generated R-module with basis $\{v_i | i \in I\}$ for some index set I. Let J_t ($t \in \mathscr{T}$) be a family of pairwise disjoint subsets of I, $x_t = \sum_{i \in J_t} v_i$. Then the sum $\sum_{t \in \mathscr{T}} x_t R$ is direct and has an R-complement in V.*

PROOF. Obvious. □

Remember that $f = |U|^{-1}\sum_{u \in U} u$. Let $\tilde{R} \in \{\overline{R}, R, K\}$. Then

3.4. LEMMA. $\tilde{R}S^G{}_P = \sum_{w \in W}^{\oplus} (\tilde{R}S \otimes 1)fwf \oplus \tilde{R}S^G(1 - e)$.

PROOF. $\tilde{R}S^G = \sum \tilde{R}S \otimes g$ as a $\tilde{R}$-space, where g runs through a system of right coset representatives of P in G, which we may choose such that it contains $\{wu | w \in W, u \in U_w^-\}$ by 3.2. Therefore $\sum_{w \in W}(\tilde{R}S \otimes 1)fwf + \tilde{R}S^G(1 - e) \leq \tilde{R}S^G$. Note that $f = f_w^+ f_w^-$ ($w \in W$) with $f_w^+ = |U_w^+ \cap U|^{-1}\sum_{u \in U \cap U_w^+} u$ and $f_w^- = |U_w^-|^{-1}\sum_{u \in U_w^-} u$, because $U = (U_w^+ \cap U)U_w^-$. Furthermore $u^{w^{-1}} \in U \cap U_w^+$ for $u \in U \cap U_w^+$, hence

$$(S \otimes 1)fwf = S \otimes wf = S \otimes wf_w^+ f_w^- = S \otimes wf_w^- = S \otimes \widehat{wU_w^-},$$

where $\widehat{wU_w^-} = \sum_{u \in U_w^-} wu$. Choosing an R-basis of S and applying 3.3 we conclude that the sum $\sum_{w \in W}(S \otimes 1)fwf$ is direct and has an R-module complement in S^G. Furthermore $(S \otimes 1)fwf \cong S^{w^{-1}} \cong S$ as an L-module, because $g \in L$ commutes with f, and S is trivial as a U-module, hence $(S \otimes 1)fwf \cong S$ as a P-module. Of course $\sum_{w \in W}(S \otimes 1)fwf \leq S^G \cdot e$, thus the sum $X = \sum_{w \in W}(S \otimes 1)fwf + S^G(1 - e) \leq S^G$ is direct, and, by the above, has an R-complement in S^G. Of courses this is a decomposition of X as a P-module. From 2.4 and the Frobenius reciprocity it follows that $KS^G{}_P = |W| \cdot KS \oplus V$, where the constituents of V are not contained in b, i.e. $V = KS^G(1 - e)$. Thus X is an RG-lattice in KS^G, hence, counting R-rank, $X = S^G$. Tensoring by $\tilde{R}$ we get the desired result. □

Recall the definition of $t_w \in \mathrm{End}_R(S)$ ($w \in W$). Define $A_w \in \mathrm{Hom}_{RP}(S, S^G)$ by $A_w(s) = t_w(s) \otimes fwf$. Then, by 3.4, $\{A_w | w \in W\}$ is an R-basis of $\mathrm{Hom}_{RP}(S, S^G)$. By the Nakayama relations every A_w, $w \in W$, induces an element $B_w \in \mathrm{End}_{RG}(S^G)$ defined by $B_w(s \otimes g) = A_w(s)g$ for all $g \in G$. Note that for $w = 1 \in W$, $B_w = B_1 = 1 \in \mathrm{End}_{RG}(S^G)$. Consider KS^G and extend B_w to an endomorphism of KS^G also denoted by B_w ($= 1 \otimes B_w$). It can easily be seen that our definition of B_w coincides with the one given in [**12**, 3.8].

Remember that $C_G(s) = \prod \mathrm{GL}_{m_\Lambda(s)}(q^{\deg \Lambda}) \leq G$, $W = W_{C_G(s)}$. Let $W = \prod W(\Lambda)$ be the corresponding decomposition of W, where we may consider $W(\Lambda)$ as the symmetric group $S_{m_\Lambda(s)}$ permuting the natural basis of the vector space $(\mathrm{GF}(q^{\deg \Lambda}))^{m_\Lambda(s)}$ over $\mathrm{GF}(q^{\deg \Lambda})$. Let $\mathscr{T}(\Lambda)$ be the set of basic transpositions in $W(\Lambda)$ (that is the permutation matrices corresponding to the transpositions of the form $(i, i+1)$ in $S_{m_\Lambda(s)}$, $1 \leq i \leq m_\Lambda(s) - 1$). Then $\mathscr{T} = \bigcup \mathscr{T}(\Lambda)$ generates W, where the union has to be taken over all $\Lambda \in \mathscr{F}$. Furthermore, for $w \in W$, let $\mathrm{lt}(w)$ denote as usual the number of factors in a reduced (i.e. minimal) expression of w as product of basic transpositions, and note that $\mathrm{lt}(w)$ is uniquely determined.

The following lemma has been proved by R. B. Howlett and G. I. Lehrer in **[12]** considering B_w ($w \in W$) as element of $\mathrm{End}_{KG}(KS^G)$ in a more general context. We will give here a more simple proof of it (corresponding to the more simple situation), since we will need parts of the proof later on.

3.5. LEMMA. *Let $\Lambda \in \mathscr{F}$, $\deg \Lambda = d$, $v \in \mathscr{T}(\Lambda)$ and $w \in W$. Then*

$$B_v \cdot B_w = \begin{cases} B_{vw} & \text{if } \mathrm{lt}(vw) > \mathrm{lt}(w), \\ q^{-d^2} B_{vw} + \beta B_w & \text{otherwise}, \end{cases}$$

with $\beta = \varepsilon q^{d(d-1)/2 - d^2}(q^d - 1)$, where $\varepsilon = 1$ if the order of $(\Lambda) \in L(\Lambda)$ is odd and $\varepsilon = -1$ otherwise.

PROOF. It is enough to calculate $B_v B_w(m \otimes 1) \in S^G$ for $m \in S$. Now

$$B_v B_w(m \otimes 1) = t_v t_w(m) \otimes fvffwf = t_{vw}(m) \otimes fvfwf.$$

Recall from 3.4 the definition of f_v^-, f_v^+ and note that $f = f_v^+ f_v^-$. Thus

$$fvfwf = fvf_v^- wf = |U_v^-|^{-1} \sum_{u \in U_v^-} fvuwf.$$

As an easy consequence of [3, 2.5.8] we get $\mathrm{lt}_0(vw) = \mathrm{lt}_0(v) + \mathrm{lt}_0(w)$ if $\mathrm{lt}(vw) > \mathrm{lt}(w)$, i.e. $\mathrm{lt}(vw) = \mathrm{lt}(w) + 1$, where $\mathrm{lt}_0\colon W_0 \to \mathbf{N}$ denotes the length function on W_0, the Weyl group of G. Thus [3, 7.2.1] implies $U_v^- \leq U_w^+$, hence $fvf_v^- wf = fvwf$ and $B_v B_w(m \otimes 1) = t_{vw}(m) \otimes fvwf = B_{vw}(m \otimes 1)$ if $\mathrm{lt}(vw) > \mathrm{lt}(w)$, as desired. So we may assume that $\mathrm{lt}(vw) \leq \mathrm{lt}(w)$, i.e. $\mathrm{lt}(vw) = \mathrm{lt}(w) - 1$. By the above, using induction on the length of w, we may assume that $w = v$.

Now let $D \subset W_0$ be a set of representatives of double cosets of P containing W (3.1). Then $S^G{}_P = \sum_{x \in D}^{\oplus} S_x$, with $S_x = (S \otimes x)_{x^{-1}Px \cap P}{}^P$ by Mackey's decomposition theorem. For $x \in D$ set $a_x = \sum_{u \in U_v^-,\, vuv \in PxP} fvuvf$. Then $m \otimes a_x \in S_x$, in particular, $\sum_{x \in D, x \notin W} m \otimes a_x \in S^G(1 - e)$, since $S_x \subset S^G(1 - e)$ for $x \in D$, $x \notin W$ by 3.4. Furthermore, $\sum_{x \in W} m \otimes a_x \in S^G \cdot e$, again by 3.4. Now

$$B_v^2(m \otimes 1) = |U_v^-|^{-1} \sum_{x \in D} m \otimes a_x = |U_v^-|^{-1} \sum_{x \in W} m \otimes a_x + |U_v^-|^{-1} \sum_{x \in D,\, x \in W} m \otimes a_x.$$

Since $B_v^2(S^G e) = (B_v^2(S^G))e \leq S^G e$ we conclude that

$$\sum_{x \in D, x \notin W} m \otimes a_x \in S^G e \cap S^G(1 - e) = (0).$$

In particular, $a = \sum_{x \in D,\, x \notin W} a_x$ annihilates the P-submodule $S^G e$ of S^G and

$$B_v^2(m \otimes 1) = |U_v^-|^{-1} \sum_{x \in W} m \otimes a_x.$$

Thus assume that $vuv \in PxP$ with $x \in W$. Let $v = v_1 \cdots v_k$ be a reduced expression for v as element of W_0. By [3, 8.1.5] $vuv \in ByB \subseteq PyP$, $y \in W$, where a reduced expression of y as element of W_0 involves only basic transpositions $v_i \in W_0$,

$1 \leq i \leq k$. By [2, Chapter IV, 2.5] $W_L x W_L = W_L y W_L$, and we conclude $x = v$ or $x = 1$. Now

$$vuv = \begin{pmatrix} E & & & & & \\ & \ddots & & & & \\ & & E & 0 & & \\ & & A & E & & \\ & & & & \ddots & \\ & & & & & E \end{pmatrix}$$

for a certain $(d \times d)$-matrix A, where E denotes the $(d \times d)$-identity matrix. A quick calculation shows that $vuv \in PvP$ if and only if A is invertible, and then $vuv = u's_A vu'$ with

$$u' = \begin{pmatrix} E & & & & & \\ & \ddots & & & & \\ & & E & A^{-1} & & \\ & & 0 & E & & \\ & & & & \ddots & \\ & & & & & E \end{pmatrix}$$

and

$$s_A = \begin{pmatrix} E & & & & & & & \\ & \ddots & & & & & & \\ & & E & & & & & \\ & & & -A^{-1} & & & & \\ & & & & A & & & \\ & & & & & E & & \\ & & & & & & \ddots & \\ & & & & & & & E \end{pmatrix}.$$

Of course $vuv \in P \cdot 1 \cdot P = P$ if and only if $A = 0$, i.e. $u = 1$.

Summarizing we get

$$B_v^2(m \otimes 1) = t_{v^2}(m) \otimes fvfvf = |U_v^-|^{-1}(m \otimes 1) + |U_v^-|^{-1} t \otimes fvf$$

with $t = \sum_{A \in \mathrm{GL}_d(q)} ms_A$.

For an arbitrary finite group G and an irreducible representation $T\colon G \to \mathrm{GL}(V)$ over $\mathbf{C}$, the well-known formula

$$\sum_{g \in G} t_{ij}(g^{-1}) t_{kl}(g) = \delta_{jk}\delta_{il}\frac{|G|}{\dim V},$$

where $\mathrm{GL}(V) \ni (t_{ij}(g)) = T(g)$ for $g \in G$, implies immediately that

$$\sum_{g \in G} e_i T(g^{-1}) \otimes_{\mathbf{C}} e_k T(g) = \sum_{j,s} e_j \otimes_{\mathbf{C}} e_s \Big(\sum_{g \in G} t_{ij}(g^{-1}) t_{ks}(g) \Big)$$

$$= \frac{|G|}{\dim V} \sum_{j,s} e_j \otimes_{\mathbf{C}} e_s \delta_{jk} \delta_{is} = \frac{|G|}{\dim V} e_k \otimes_{\mathbf{C}} e_i$$

for all elements e_i, e_k of the natural basis of V, hence

$$\sum_{g \in G} v_1 T(g^{-1}) \otimes_{\mathbf{C}} v_2 T(g) = \frac{|G|}{\dim V} v_2 \otimes_{\mathbf{C}} v_1$$

for all $v_1, v_2 \in V$.[2] Applying this to our situation, replacing **C** by K and V by the irreducible $RL(\Lambda)$-lattice $S(\Lambda)$, we easily get $t = \varepsilon|L(\Lambda)|t_v(m)/\dim S(\Lambda)$, observing that $m(\Lambda)\cdot A = \varepsilon m(\Lambda)$ for $A = -E$ by [**15**, 6.8]. By [**15**, 6.21] $\dim S(\Lambda) = |L(\Lambda)|/|U_{L(\Lambda)}||T(\Lambda)|$, where $U_{L(\Lambda)}$ is the subgroup of all upper triangular unipotent matrices of $L(\Lambda)$. Now $|U_v^-| = q^{d^2}$, $|U_{L(\Lambda)}| = q^{d(d-1)/2}$ and $|T(\Lambda)| = q^d - 1$. Thus

$$B_v^2(m\otimes 1) = q^{-d^2}(m\otimes 1) + \varepsilon q^{d(d-1)/2-d^2}(q^d-1)t_v(m)\otimes fvf$$
$$= q^{-d^2}(m\otimes 1) + \beta B_v(m\otimes 1)$$

for all $m \in S$, hence $B_v^2 = q^{-d^2} + \beta B_v$ as desired. □

As an immediate consequence from the proof above we get

COROLLARY. *Let* $w_1, w_2 \in W$. *Then* $fw_1fw_2f = \sum_{w\in W} r_w fwf + a$ *for certain* $r_w \in RL$, *where* $a \in RG$ *annhilates* $S^G \cdot e \subseteq S^G$.

Now, for $\Lambda \in \mathscr{F}$, $v \in \mathscr{T}(\Lambda)$, $d = \deg\Lambda$ and $w \in W$ define $T_v = \varepsilon q^{d(d+1)/2}B_v \in \operatorname{End}_{RG}(S^G)$ and $T_w = T_{v_1}\cdots T_{v_k}$ if $w = v_1\cdots v_k$ is a reduced expression for w. By the above $T_w = q^m B_w$ for a certain $m \in \mathbf{N}$. Of course, by 3.4, $\{T_w | w \in W\}$ is a basis of $\operatorname{End}_{\tilde{R}G}(\tilde{R}S^G)$ for $\tilde{R} \in \{\overline{R}, R, K\}$, denoting $1\otimes T_w \in \tilde{R}\otimes_R \operatorname{End}_{RG}(S^G)$ again by T_w. Furthermore, by 3.5, $T_v\cdot T_w = T_{vw}$, if $\mathrm{lt}(vw) > \mathrm{lt}(w)$ and $T_vT_w = q^dT_{vw} + (q^d-1)T_w$ otherwise, where $v, w \in W$ as in 3.5. For an arbitrary commutative ring $\tilde{R}$ (with 1) let $\tilde{R}[W]$ be the $\tilde{R}$-free $\tilde{R}$-algebra with basis $\{T_w | w\in W\}$, where the multiplication is induced by the relations above. We will see later that $\tilde{R}[W]$ is an associative $\tilde{R}$-algebra for every choice of $\tilde{R}$. Of course $\overline{R}[W] \cong \overline{R}\otimes_R R[W]$ and $K[W] \cong K\otimes_R R[W]$ canonically, thus we have shown, generalizing [**12**] partially:

3.6. THEOREM. *Let* $s \in G$ *be semisimple satisfying* 2.4. *Then* $\operatorname{End}_{\tilde{R}G}(\tilde{R}S^G) = \tilde{R}[W] = \tilde{R}\cdot\operatorname{End}_{RG}(S^G)$ *for every choice of* $\tilde{R} \in \{\overline{R}, R, K\}$.

For the rest of this paper let A be the subalgebra of RG generated by $\{fRPf, fwf | w \in W\}$. Note that f is the identity of A. Furthermore, $eA = Ae$, thus $M\cdot e \le M_A$ for an arbitrary RG-module M, where M_A denotes the restriction of M to the subalgebra A of RG. In particular,

$$S^G\cdot e = \sum_{w\in W} S\otimes fwf = (S\otimes 1)A \le S_A^G.$$

Let I be the annihilator ideal of $S^G\cdot e$ in A, and $\pi\colon A \to A/I$ the natural epimorphism. Then $\mathscr{A} = A/I$ is R-free, being isomorphic to a subalgebra of the R-free R-algebra $\operatorname{End}_R(S^G\cdot e)$. From 3.4 and the corollary to 3.5 we conclude immediately that $\mathscr{A} = \sum_{w\in W}^{\oplus}\pi(fRPf)\pi(fwf) = \sum_{w\in W}^{\oplus}\pi(fwf)\pi(fRPf)$, i.e. $\mathscr{A}$ is free as a $\pi(fRPf)$-module with basis $\{\pi(fwf) | w \in W\}$ (on both sides). Note that $\pi(fRPf) \cong RL/I_0$ canonically, where I_0 denotes the annihilator of S in RL. Of course, KI is the annihilator of KS^Ge in KA, and $K\mathscr{A} \cong KA/KI$ canonically. Let I_1 be the annihilator of $\overline{R}S^Ge$ in $\overline{R}A$, and $\pi_1\colon \overline{R}A \to \overline{R}A/I_1 = \mathscr{A}_1$ the natural epimorphism. Because A, I and $A/I = \mathscr{A}$ are R-free, we may consider $\overline{R}I$ as an ideal of $\overline{R}A$ canonically, and of course $\overline{R}I \subset I_1$. Thus there exists an epimorphism from $\overline{R}A/\overline{R}I$

[2] I am indebted to R. Knörr who showed me the proof of this general result.

onto $\mathscr{A}_1$. Now 3.5 holds analogously if we replace S by $\bar{R}S$ and RG by $\bar{R}G$, by 3.4. Consequently $\mathscr{A}_1 = \sum^{\oplus}_{w \in W} \pi_1(f\bar{R}Pf)\pi_1(fwf)$, and $\pi_1(f\bar{R}Pf) \cong \bar{R}L/I_2$, where I_2 denotes the annihilator of $\bar{R}S$ in $\bar{R}L$. We again get an epimorphism from $\bar{R}L/\bar{R}I_0$ onto $\bar{R}L/I_2$. Since $\bar{R}S$ and KS are irreducible, we get, counting dimensions, $\bar{R}L/\bar{R}I_0 \cong \bar{R}L/I_2$, i.e. $\bar{R}I_0 = I_2$. This implies $\bar{R}A/\bar{R}I \cong \mathscr{A}_1$. Of course $\bar{R}A/\bar{R}I \cong \bar{R}\mathscr{A}$. Thus $\tilde{R}\mathscr{A}$ is isomorphic to $\tilde{R}A$ modulo the annihilator of $\tilde{R}S^Ge$ in $\tilde{R}A$ for every choice of $\tilde{R} \in \{\bar{R}, R, K\}$ canonically.

3.4 and 3.5 immediately imply that $\tilde{R}S^Ge = (S \otimes 1)\tilde{R}\mathscr{A} \cong \tilde{R}S \otimes_{\pi(f\tilde{R}Pf)} \tilde{R}\mathscr{A}$, the induced module. Let $\tau \in \operatorname{End}_{\tilde{R}G}(\tilde{R}S^G)$. Since $\tau(\tilde{R}S^Ge) \leq \tilde{R}S^Ge$, the restriction $\tilde{\tau}$ of τ to $\tilde{R}S^Ge$ defines an endomorphism of $\tilde{R}S^Ge$ as an $\tilde{R}\mathscr{A}$-module. By adjointness of induction and restriction (Frobenius reciprocity), $\operatorname{End}_{\tilde{R}\mathscr{A}}(\tilde{R}S^Ge) \cong \operatorname{Hom}_{\pi(f\tilde{R}Pf)}(\tilde{R}S, \tilde{R}S^Ge)$. So 3.4 implies immediately that $\{\tilde{T}_w | w \in W\}$ is a basis of $\operatorname{End}_{\tilde{R}\mathscr{A}}(\tilde{R}S^Ge)$, and we may identify $\operatorname{End}_{\tilde{R}G}(\tilde{R}S^G)$ and $\operatorname{End}_{\tilde{R}\mathscr{A}}(\tilde{R}S^Ge)$. Furthermore, $\tilde{R}[W] \cong \tilde{R}\mathscr{A}/\tilde{R}\Omega_p$, where Ω_p is the ideal of $\mathscr{A}$ generated by $\{\pi(f - fgf) | g \in P\}$. The next lemma tells us that we may argue as in stable Clifford theory, regarding $\tilde{R}\mathscr{A}$ as a kind of semidirect product of $\pi(f\tilde{R}Pf)$ with $\tilde{R}[W]$, and as an "intertial algebra" of $\tilde{R}S$ in $\tilde{R}G$.

Let $\operatorname{mod}(\tilde{R}[W]/\tilde{R})$ be the category of all finitely generated $\tilde{R}$-free $\tilde{R}[W]$-modules and $\operatorname{mod}(\tilde{R}\mathscr{A}/\tilde{R}S)$ the category of all finitely generated $\tilde{R}S$-homogeneous $\tilde{R}\mathscr{A}$-modules, where we call an $\tilde{R}\mathscr{A}$-module $\tilde{R}S$-homogeneous if its restriction to $\pi(f\tilde{R}Pf)$ is isomorphic to a direct sum of copies of $\tilde{R}S$.

3.7. LEMMA. *The functors* $? \otimes_{\tilde{R}[W]} (S \otimes 1)\tilde{R}\mathscr{A}$ *and* $\operatorname{Hom}_{\tilde{R}\mathscr{A}}((S \otimes 1)\tilde{R}\mathscr{A}, ?)$ *form an equivalence between* $\operatorname{mod}(\tilde{R}[W]/\tilde{R})$ *and* $\operatorname{mod}(\tilde{R}\mathscr{A}/\tilde{R}S)$.

PROOF. Obviously $? \otimes_{\tilde{R}[W]} (S \otimes 1)\tilde{R}\mathscr{A}$ is a functor from $\operatorname{mod}(\tilde{R}[W]/\tilde{R})$ to $\operatorname{mod}(\tilde{R}\mathscr{A}/\tilde{R}S)$ and $\operatorname{Hom}_{\tilde{R}\mathscr{A}}((S \otimes 1)\tilde{R}\mathscr{A}, ?)$ from $\operatorname{mod}(\tilde{R}\mathscr{A}/\tilde{R}S)$ to $\operatorname{mod}(\tilde{R}[W]/\tilde{R})$. The Nakayama relations show that $\operatorname{Hom}_{\tilde{R}\mathscr{A}}((S \otimes 1)\tilde{R}\mathscr{A}, X) \cong \operatorname{Hom}_{f\tilde{R}Pf}(\tilde{R}S, X)$ as $\tilde{R}$-modules by restriction of a: $(S \otimes 1)\tilde{R}\mathscr{A} \to X$ to $\tilde{R}S = \tilde{R}S \otimes 1$ for $X \in \operatorname{mod}(\tilde{R}\mathscr{A}/\tilde{R}S)$. Note also that

$$B \otimes_{\tilde{R}[W]} ((S \otimes 1)\tilde{R}\mathscr{A}) \cong B \otimes_{\tilde{R}[W]} (\tilde{R}[W] \otimes_{\tilde{R}} \tilde{R}S) \cong B \otimes_{\tilde{R}} \tilde{R}S$$

as $f\tilde{R}Pf$-modules for $B \in \operatorname{mod}(\tilde{R}[W]/\tilde{R})$.

Now define

$$\varepsilon\colon \operatorname{Hom}_{\tilde{R}\mathscr{A}}((S \otimes 1)\tilde{R}\mathscr{A}, X) \otimes_{\tilde{R}[W]} (S \otimes 1)\tilde{R}\mathscr{A} \to X$$

as the homomorphism induced by

$$\varepsilon(a \otimes m) = a(m)$$

for $a \in \operatorname{Hom}_{\tilde{R}\mathscr{A}}((S \otimes 1)\tilde{R}\mathscr{A}, X)$ and $m \in (S \otimes 1)\tilde{R}\mathscr{A}$, and

$$\gamma\colon B \to \operatorname{Hom}_{\tilde{R}\mathscr{A}}((S \otimes 1)\tilde{R}\mathscr{A}, B \otimes_{\tilde{R}[W]} (S \otimes 1)\tilde{R}\mathscr{A})$$

by $\gamma(b) = a_b \in \operatorname{Hom}_{\tilde{R}\mathscr{A}}((S \otimes 1)\tilde{R}\mathscr{A}, B \otimes_{\tilde{R}[W]} (S \otimes 1)\tilde{R}\mathscr{A})$ for $b \in B$, where $a_b(x) = b \otimes x$ for $x \in \tilde{R}S \otimes 1$.

Obviously ε is natural in X and γ in B. Now as in stable Clifford theory (see e.g. [6]) it can easily be seen, by the remarks made above, that ε and γ are isomorphisms. This proves the lemma. □

In the following we identify $\mathrm{mod}(\tilde{R}\mathscr{A}/\tilde{R}S)$ with the full subcategory of $\mathrm{mod}(\tilde{R}A/\tilde{R}S)$ of all finitely generated $\tilde{R}S$-homogeneous $\tilde{R}A$-modules, which are annihilated by $\tilde{R}I$, where $\mathrm{mod}(\tilde{R}A/\tilde{R}S)$ denotes the category of all finitely generated $\tilde{R}S$-homogeneous $\tilde{R}A$-modules. Note that simple objects of $\mathrm{mod}(\tilde{R}\mathscr{A}/RS)$ are irreducible $\tilde{R}A$-modules.

3.8. COROLLARY. *$I \to I(S \otimes 1)\tilde{R}A$ defines an isomorphism between the lattice of all $\tilde{R}$-free right ideals I of $\tilde{R}[W]$ and the lattice of all $\tilde{R}S$-homogeneous submodules of $(S \otimes 1)\tilde{R}A$.*

4. Theorem 3.6 and Corollary 1.6 particularly imply that $KW \cong K[W]$. In fact this is a well-known theorem, not only for Weyl groups of type A_n, but also for arbitrary Coxeter groups, if K contains $\mathbf{Q}[q^{1/2}, q^{-1/2}]$. In our case ($W = W_s$) it is true for $K = \mathbf{Q}$. The proof of the general theorem uses the fact that separable algebras with the same numerical invariants (the degrees of the irreducible characters) have to be isomorphic by Wedderburn's theorem.

In the following we will identify $\mathbf{Q}[W]$ and $\mathbf{Q}W$ by a certain isomorphism, which we will extend to an isomorphism between $K[W]$ and KW. So we will find in $K[W]$ two R-orders, one generated by the basis $\{T_w | w \in W\}$, the other by the basis $\{w | w \in W\} \subset KW = K[W]$, namely $R[W]$ and RW respectively. These are not isomorphic in general, in fact, choosing $s = 1$ in $G = \mathrm{GL}_2(5)$ and $r = 3$, then $W = S_2$ and $\overline{R}S_2 = \overline{R} \otimes_R RS_2$ is semisimple, whereas $\overline{R}[S_2] = \overline{R} \otimes R[S_2]$ has a one-dimensional nilpotent ideal, namely $\{a + aT_w | a \in \overline{R}\}$, where w is the unique element of order 2 in S_2. In particular, RS_2 and $R[S_2]$ cannot be isomorphic in this case.

The proof of the theorem mentioned above uses the generic algebra. We will introduce it here only for Weyl groups of type A_n (i.e. symmetric groups) using K instead of $\mathbf{Q}[q^{1/2}, q^{1/2}]$. For the general case, more details and proofs of the following claims, the reader is referred to [1, 2, 4, 13].

So let $\{W_i | i \in I\}$ be a family of symmetric groups W_i over a finite index set I, and let $\mathbf{d} = \{d_i | i \in I\}$ be a family of positive integers. Let $K[u, u^{-1}]$ be the ring of Laurent polynomials over K in the indeterminant u, $K(u)$ the quotient field of $K[u, u^{-1}]$. Let $W = \prod_{i \in I} W_i$. Then the generic K-algebra $\mathscr{G}[W, \mathbf{d}]$ of W over K with respect to $\mathbf{d}$ is defined as follows: $\mathscr{G}[W, \mathbf{d}]$ is a free $K[u, u^{-1}]$-module with basis $\{T_w | w \in W\}$, where the multiplication in $\mathscr{G}[W, \mathbf{d}]$ is given by

$$T_v T_w = \begin{cases} T_{vw} & \text{if } \mathrm{lt}(vw) = \mathrm{lt}(w) + 1, \\ u^{d_i} T_{vw} + (u^{d_i} - 1) T_w & \text{otherwise} \end{cases}$$

for $w \in W$ and $v \in W_i \subseteq W$, where v is a basic transposition. Let h_∞: $K[u, u^{-1}] \to K(u)$ be the natural embedding, and, for $0 \neq k \in K$, let h_k: $K[u, u^{-1}] \to K$ be the homomorphism induced by $h_k(u) = k$. Consider K and $K(u)$ as a $K[u, u^{-1}]$-module via h_k and h_∞ respectively, and set

$$K[W, \mathbf{d}, k] = \mathscr{G}[W, \mathbf{d}] \otimes_{K[u, u^{-1}]} K$$

and

$$K[W, \mathbf{d}, \infty] = \mathscr{G}(W, \mathbf{d}) = \mathscr{G}[W, \mathbf{d}] \otimes_{K[u, u^{-1}]} K(u),$$

respectively.

$T_w \to T_w \otimes 1$ induces a K-algebra homomorphism from $\mathscr{G}[W, \mathbf{d}]$ into $K[W, \mathbf{d}, k]$ $(0 \neq k \in K \cup \{\infty\})$ which is also denoted by h_k and is surjective for $k \in K$ and injective for $k = \infty$. For simplicity we will denote $T_w \otimes 1 = h_k(T_w)$ by T_w again.

We call $h_k(K[W, \mathbf{d}, k])$ the specialization of $\mathscr{G}[W, \mathbf{d}]$ with respect to k. Since $K[W, \mathbf{d}, q]$ (q an arbitrary prime power) is the endomorphism ring of an induced module (choosing $s = 1$ in $G = \prod_{i \in I} \mathrm{GL}_{k_i}(q^{d_i})$, $k_i \in \mathbf{N}$, such that $W_i \cong S_{k_i}$ in 3.6), hence is in particular an associative algebra, it can easily be seen that $\mathscr{G}[\mathrm{W}, \mathbf{d}]$ is associative. This also remains true if we replace K by an arbitrary commutative ring with 1, because the multiplication of the T_w's ($w \in W$) only involves Laurent polynomials in u with integer coefficients.

Note that $K[W, \mathbf{d}, 1] = KW$, the usual group algebra of W over K. Moreover, T_w ($w \in W$) is invertible in $\mathscr{G}[W, \mathbf{d}]$. In fact, if $v \in W_i$, lt $v = 1$, then $T_v^{-1} = u^{-d_i} T_v + (u^{-d_i} - 1)$, and if $w = v_1 \cdots v_k$ is a reduced representation of $w \in W$ as a product of basic transpositions, then $T_w^{-1} = T_{v_k}^{-1} \cdots T_{v_1}^{-1}$.

For $W = W_s$ we choose $I = \{\Lambda \in \mathscr{F} | m_\Lambda(s) \neq 0\}$ and $\mathbf{d} = \{d_\Lambda = \deg \Lambda | \Lambda \in I\}$. In particular, $K[W_s, \mathbf{d}, q] = K[W_s]$ and $K[W_s, \mathbf{d}, 1] = KW_s$, where q is defined as in §1, and $K[W_s]$ as in §3. For simplicity we will write $\mathscr{G}[W_s] = \mathscr{G}[W]$ and $\mathscr{G}(W_s) = \mathscr{G}(W)$ instead of $\mathscr{G}[W_s, \mathbf{d}]$ and $\mathscr{G}(W_s, \mathbf{d})$ respectively, and we will formulate the following for $\mathscr{G}[W]$, although it is true in general, as we can see if we choose $s = 1$ in G as above. It can be shown that $\mathscr{G}(W)$ is semisimple. If χ is a character of $\mathscr{G}(W)$, then $\chi(T_w)$ is a polynomial in u for all $w \in W$. Given a specialization h_k: $u \to k$ for some prime power k, then χ_k: $K[W, \mathbf{d}, k] \to K$ defined by $\chi_k(T_w \otimes 1) = h_k(\chi(T_w))$ is a character of $K[W, \mathbf{d}, k]$. It turns out that χ_k is irreducible if and only if χ is irreducible. Of course $\chi_k(1)$ is independent of the choice of k. So we may identify $K[W] = K[W_s, \mathbf{d}, q]$ and $KW = K[W_s, \mathbf{d}, 1]$ by an isomorphism, say τ, such that χ_q and χ_1 define the same character of $K[W] = KW$ for every irreducible character χ of $\mathscr{G}(W)$. Moreover, $\mathscr{G}(W)$ and $K[W, \mathbf{d}, k]$ have the same numerical invariants, and $\chi = \phi$ if and only if $\chi_k = \phi_k$ for two irreducible characters χ and ϕ of $\mathscr{G}(W)$. In particular, we get all irreducible characters of $K[W] = KW$ as specialized characters $\chi_q = \chi_1$, χ an irreducible character of $\mathscr{G}(W)$. Obviously $\chi_q = \chi_1$ for all characters χ of $\mathscr{G}(W)$.

Let $\mathscr{T}$ again be the set of basic transpositions of W, and $J \subset \mathscr{T}$. Then the subalgebra of $\mathscr{G}[W]$ generated by $\{T_w | w \in J\}$ is isomorphic to $\mathscr{G}[W_J] = \mathscr{G}[W_J, \mathbf{d}']$, where W_J is the subgroup of W generated by J and $\mathbf{d}'$ denotes the restriction of $\mathbf{d}$ to $\{\Lambda \in I | W_J \cap W(\Lambda) \neq (1)\}$. Similarly we define $\mathscr{G}(W_J)$, and in the following we always consider $\mathscr{G}[W_J]$ and $\mathscr{G}(W_J)$ as subalgebras of $\mathscr{G}[W]$ and $\mathscr{G}(W)$ respectively.

If $0 \neq k \in K \cup \{\infty\}$, $h_k(K[W_J]) = K[W_J, \mathbf{d}', k] \subset K[W]$, i.e. h_k restricted to $\mathscr{G}[W_J]$ is the specialization of $\mathscr{G}[W_J]$ with respect to k. In particular, $K[W_J] = K[W_J, \mathbf{d}, q]$ and $KW_J = K[W_J, \mathbf{d}', 1]$ may be considered as subalgebras of $K[W] = KW$. But we have to be careful here; an isomorphism between $K[W_J]$ and KW_J cannot be extended to an isomorphism between $K[W]$ and KW in general. In other words, $K[W_J]$ and KW_J are different subalgebras of $K[W] = KW$ in general.

As above let W_J be the subgroup of W generated by J, and $R[W_J] \subset R[W]$ the R-order in $K[W_J]$ generated by $\{T_w | w \in W_J\}$. Note that $\bar{R}[W_J] = \bar{R} \otimes_R R[W_J]$. Let D_J be a system of right coset representatives of W_J in W. We may choose D_J such that $\mathrm{lt}(wv) = \mathrm{lt}(w) + \mathrm{lt}(v)$ for all $w \in W_J$, $v \in D_J$, a so-called distinguished set of coset representatives (see e.g. [**3**, 2.5.8]).

4.1. Lemma. (i) *$\mathscr{G}[W]$ and $\mathscr{G}(W)$ are free as $\mathscr{G}[W_J]$- and $\mathscr{G}(W_J)$-left modules respectively with basis $\{T_v | v \in D_J\}$.*

(ii) *$\bar{R}[W_J] \subset \bar{R}[W]$, and $\bar{R}[W]$ is free as a $\tilde{R}[W_J]$-left module with basis $\{T_v | v \in D_J\}$ for every choice of $\tilde{R} \in \{\bar{R}, R, K\}$.*

Proof. Choosing D_J as a distinguished set of coset representatives, $\mathscr{G}[W] = \sum_{v \in D_J} \mathscr{G}[W_J] T_v$ as a $\mathscr{G}[W_J]$-left module. Counting $K[u, u^{-1}]$-rank, the sum must be direct. Now the lemma follows immediately. □

Consider the trivial and the alternating character of KW_J. By the above these must be specializations of certain linear characters of $\mathscr{G}(W_J)$, which we will also call the trivial respectively alternating character of $\mathscr{G}(W_J)$. Being linear characters, the corresponding idempotents in $\mathscr{G}(W_J)$ are central and uniquely determined.

For $\Lambda \in \mathscr{F}$ and $J \subset \mathscr{T}$ let $W_J(\Lambda) = W_{J \cap \mathscr{T}(\Lambda)} = W_J \cap W(\Lambda)$, where $\mathscr{T}(\Lambda) = W(\Lambda) \cap \mathscr{T}$. Let $y_J(\Lambda) = \sum_{w \in W_J(\Lambda)} (-1)^{\mathrm{lt}\, w} u^{-\deg \Lambda\, \mathrm{lt}\, w} T_w$. Define $y_J = \prod y_J(\Lambda)$ and $x_J = \sum_{w \in W_J} T_w$.

4.2. Lemma. *Let $J \subset \mathscr{T}$. Then there exist $\alpha, \beta \in K(u)$ such that αx_J and βy_J are the unique idempotents in $\mathscr{G}(W_J)$ corresponding to the trivial and the alternating character of $\mathscr{G}(W_J)$ respectively.*

Proof. Of course $x_J = \prod x_{J \cap \mathscr{T}(\Lambda)}$, hence we may assume that $W_J = W_J(\Lambda)$ for some $\Lambda \in \mathscr{F}$, i.e. W_J has one nontrivial component, and the trivial and the alternating character are the only linear characters of KW_J. Obviously we may assume, too, that $\deg \Lambda = 1$. So $\mathscr{G}(W_J)$ has just two different one-dimensional right ideals. Let $x = \sum_{w \in W_J} a_w T_w \in \mathscr{G}(W_J)$ such that $x\mathscr{G}(W_J)$ is one-dimensional over $K(u)$. This is equivalent to $xT_v = x\beta_v$ for all $v \in J$, where $\beta_v \in K(u)$, because $\mathscr{G}(W_J)$ is generated by $\{T_v | v \in J\}$ as algebra. Note that in particular $x \neq 0$. For $v \in J$ let $W_v = \{w \in W_J | \mathrm{lt}(vw) = \mathrm{lt}(w) + 1\}$; then W_J is the disjoint union of W_v and vW_v, and $x = \sum_{w \in W_v} (a_w T_w + a_{vw} T_{vw})$. Now x is in the center of $\mathscr{G}(W_J)$, because it is a scalar multiple of some central idempotent. Consequently

$$xT_v = T_v x = \sum_{w \in W_v} a_{vw} u T_w + (a_w + a_{vw}(u-1)) T_{vw}.$$

So $xT_v = x\beta_v$ forces, for $w \in W_v$,

(i) $a_{vw}u = a_w\beta_v$ and,

(ii) $a_{vw}\beta_v = a_w + a_{vw}(u-1)$.

Hence,

(iii) $a_w(\beta_v^2 - \beta_v(u-1) - u) = 0$.

(ii) implies $a_w = 0$ if $a_{vw} = 0$, hence, repeating the argument using a reduced expression for w as the product of basic transpositions, $a_1 = 0$. Using (i) we conclude $a_{w'} = 0$ for all $w' \in W_J$, forcing $x = 0$, a contradiction. So $a_{w'} \neq 0$ for all $w' \in W_J$, and (iii) implies $\beta_v = u$ or $\beta_v = -1$.

Obviously the character χ afforded by $x\mathscr{G}(W_J)$ satisfies $\chi(T_v) = \beta_v$ for all $v \in J$. Therefore, using the specialization $u \to 1$, either $\beta_v = u$ for all $v \in \mathscr{T}$(specializing to the trivial character of W_J) or $\beta_v = -1$ for all $v \in \mathscr{T}$(specializing to the alternating character of W_J). Normalizing $a_1 = 1$ we get x_J in the first and y_J in the second case. □

Identify $K[W]$ and KW as above, and consider the subalgebras KW_J and $K[W_J]$ ($J \subset \mathscr{T}$). Let $X_J^{(k)} = h_k(x_J)KW$, $Y_J^{(k)} = h_k(y_J)KW$ for $k = 1$ or q.

4.3. LEMMA. *$X_J^{(1)} \cong X_J^{(q)}$ and $Y_J^{(1)} \cong Y_J^{(q)}$ as KW-modules.*

PROOF. Let D_J be a distinguished system of coset representatives of W_J in W. Then $x_J\mathscr{G}[W] \cong x_J\mathscr{G}[W_J] \otimes_{\mathscr{G}[W_J]} \mathscr{G}[W]$ is a free $K[u, u^{-1}]$-module with basis $\{x_J \otimes T_w | w \in D_J\}$ by 4.1. Furthermore $h_k(x_J) \neq 0$ and

$$h_1(x_J)KW \cong h_1(x_J)KW_J \otimes_{KW_J} KW,$$

$$h_q(x_J)K[W] \cong h_q(x_J)K[W_J] \otimes_{K[W_J]} K[W]$$

with basis $\{h_1(x_J) \otimes w | w \in D_J\}$, $\{h_q(x_J) \otimes T_w | w \in D_J\}$ respectively. Let χ be the character afforded by $X_J = x_J\mathscr{G}(W)$. Then, by the above, χ_k is the character afforded by $X_J^{(k)}$, and $X_J^{(1)} \cong X_J^{(q)}$ by construction of the identifying isomorphism τ: $K[W] \to KW$. Similarly $Y_J^{(1)} \cong Y_J^{(q)}$. □

We need some results from the representation theory of symmetric groups. For details the reader is referred to [**14**]. Remember that there is a bijection between M_s and the set of all inequivalent irreducible characters of W, denoted by $\lambda \leftrightarrow \phi^\lambda$ ($\lambda \in M_s$). If S^λ denotes the Specht module over R corresponding to $\lambda \in M_s$, then KS^λ affords ϕ^λ. It may be constructed in the following way:

Let $\lambda_\Lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. Define $J = J_\lambda$ to be the set of basic transpositions $(i, i+1)$, where $(\sum_{j=1}^{m}\lambda_j) + 1 \leq i \leq (\sum_{j=1}^{m+1}\lambda_j) - 1$ for some $\Lambda \in \mathscr{F}$, $0 \leq m \leq k - 1$. Then $W_\lambda = W_J$ is just the standard Young subgroup of W corresponding to $\lambda \in M_s$. Let $J(\Lambda) = J \cap \mathscr{T}(\Lambda)$ ($\Lambda \in \mathscr{F}$). Let t be the λ_Λ-tableau in which the numbers $\{1, 2, \ldots, m_\Lambda(s)\}$ increase along the rows. Then $W_\lambda(\Lambda) = W_{J(\Lambda)}$ is just the row stabilizer of t. Let $C_J(\Lambda)$ be the column stabilizer of t and set $C_J = \prod C_J(\Lambda)$. Note that $x = h_1(x_J)$ is just the sum over the elements of $W_J = W_\lambda$. Let y be the alternating sum over the elements of C_J, and set $xyRW = S^\lambda$. Then KS^λ affords ϕ^λ. Moreover, xKW and yKW have just one composition factor in common, namely KS^λ.

This procedure cannot be transposed directly to Hecke (respectively generic) algebras, because the $K(u)$-subspace (K-subspace) of $\mathcal{G}(W)$ ($K[W]$) generated by $\{T_w | w \in C_J\}$ is not a subalgebra in general. But C_J is conjugate in W to $W_{\lambda'}$, where $\lambda' = \prod \lambda'_\Lambda$ and λ'_Λ is the dual partition of λ_Λ ($\lambda = \prod \lambda_\Lambda$). So let us fix $w \in W$ such that $w^{-1} W_{\lambda'} w = C_J$. Set $J' = J_{\lambda'}$ and define $\tilde{z}_\lambda = x_J T_{w}^{-1} y_{J'} T_w \in \mathcal{G}(W)$, where $x_{J'}, y_{J'} \in \mathcal{G}[W]$ are defined as in 4.2. Finally, set $z_\lambda = h_q(\tilde{z}_\lambda) \in K[W] = KW$. Note that $T_{w}^{-1} \in \mathcal{G}[W]$. Inspecting the coefficients of T_v^{-1} ($v \in \mathcal{T}$) we see that $T_w \otimes 1$ is invertible in $R[W] \subset K[W]$. So $z_\lambda \in R[W]$.

4.4. LEMMA. *Assume $q^{d_\Lambda} > 2$ for all $\Lambda \in I$. Then $z_\lambda K[W] \cong KS^\lambda$.*

PROOF. First note that $h_1(\tilde{z}_\lambda) = xw^{-1}h_1(y_{J'})w = xy \neq 0$, where $x, y \in RW$ are defined as above, hence $\tilde{z}_\lambda \neq 0$. Now xKW and yKW have just one composition factor in common, namely KS^λ. Obviously $yKW \cong wyw^{-1}KW = Y_{J'}^{(1)} \cong Y_{J'}^{(q)}$ by 4.3, so the same is true for $xKW = X_J^{(1)} \cong X_J^q = h_q(x_J)K[W]$ and $Y_{J'}^{(q)} = h_q(y_{J'})K[W]$. Consequently either $z_\lambda = h_q(x_J)h_q(y_{J'}) = 0$ or $z_\lambda K[W] \cong KS^\lambda$. Now $\tilde{z}_\lambda \in x_J \mathcal{G}[W] = x_J \mathcal{G}[W_J] \otimes \mathcal{G}[W]$, and a basis of $x_J \mathcal{G}[W_J] \otimes \mathcal{G}[W]$ is given by $\{x_J \otimes T_W | w \in D_J\}$, where D_J is defined as in 4.3. Furthermore, it can easily be seen that the coefficients of $\tilde{z}_\lambda = \sum_{w \in D_J} x_J \otimes T_w r_w$ ($r_w \in K(u)$) are Laurent polynomials in u with integer coefficients. In particular, $r_1 = f(u)/u^k$ for some $k \in \mathbf{N}$ and a polynomial $f(u) \in K[u]$ with integer coefficients. Using the specialization $u \to 1$ we see that $f(1) = 1$. Hence $f(q) = 1 \bmod (q-1)$ (note that $q > 2$ by assumption), in particular $f(q) \neq 0$. This implies $z_\lambda \neq 0$ as desired. □

We will see later that the assumption in 4.4 is no real restriction for our purpose. So let us assume in the following that $q^{\deg \Lambda} > 2$ for all $\Lambda \in \mathcal{F}$ with $m_\Lambda(s) \neq 0$.

Let us return now to the KG-module KS^G defined in §2. Recall that $\operatorname{End}_{RG}(S^G) = R[W]$ by 3.6. For $\lambda \in M_s$ define the A-module S_λ to be $z_\lambda R[W](S \otimes 1) = z_\lambda R[W](S \otimes 1)A \leq S^G \cdot e$, where $A \subseteq RG$ is defined as in §3. Then $S_\lambda RG = z_\lambda \cdot S^G$ is an RG-submodule of S^G.

4.5. LEMMA. *$S_\lambda KG$ is an irreducible KG-module affording $\chi_{s,\lambda}$.*

PROOF. $S_\lambda KG = z_\lambda K[W](S \otimes 1)KG = e_\lambda \cdot KS^G$, where $e_\lambda \in K[W]$ is the unique primitive idempotent contained in the irreducible $K[W]$-module $z_\lambda K[W] \cong KS^\lambda$ by 4.4. So $S_\lambda KG$ is irreducible.

By [**15**, 6.24, 8.6], $R_T^G(s) = R_{L_0}^G(\theta)$, where $L_0 = C_G(s)$ and $\theta = \pi^{L_0}$ setting $\pi = \tilde{\hat{s}}$, the lifting of $\hat{s}$ to the standard Borel subgroup of L_0. Now $R_{L_0}^G(\chi)$ is (up to sign) irreducible for every irreducible constituent χ of θ. So we may assume $G = L_0$, that is $s \in Z(G)$, the center of G. Furthermore, $\theta = \hat{s} \otimes 1_{B_0}^G$, where 1_{B_0} denotes the trivial character of B_0, $\otimes$ the usual product of characters, and s on the right-hand side is considered as the linear character of $G = L_0$. So we may assume that $s = 1 \in G$. But this is just the classical case, where it is well known that $e_\lambda KS^G$ affords $\chi_{s,\lambda}$ (see e.g. [**4**, 13]). □

We are now prepared to prove the main technical lemma. So define $D^\lambda = \overline{R}S^\lambda / J(\overline{R}S^\lambda)$, $D_\lambda = \overline{R}S_\lambda / J(\overline{R}S_\lambda)$ and $E_\lambda = \overline{R}(z_\lambda S^G)/J(\overline{R}(z_\lambda S^G))$. We have to be careful here. In fact, if $\bar{z}_\lambda$ denotes the image of $z_\lambda \in R[W]$ in $\overline{R}[W]$, the RG-modules

$\overline{R}(z_\lambda S^G)$ and $\bar{z}_\lambda(\overline{R}S^G)$ are not isomorphic in general, although there always exists an epimorphism from $\overline{R}(z_\lambda S^G)$ onto $\bar{z}_\lambda(\overline{R}S^G)$. In particular, $\overline{R}(z_\lambda S^G)$ may not necessarily be a submodule of $\overline{R}S^G$ in general.

4.6. LEMMA. *Let $\lambda \in M_s$ and assume that D^λ is a simple $\overline{R}[W]$-module. Then D_λ is a simple $\overline{R}S$-homogeneous $\overline{R}A$-module occurring in $E_\lambda e$ as a composition factor. Finally E_λ is simple.*

PROOF. By 3.7, $S_\lambda = z_\lambda R[W](S \otimes 1)$ is S-homogeneous. Hence $\overline{R}S_\lambda$ and its epimorphic image D_λ are $\overline{R}S$-homogeneous, too. Again by 3.7 D_λ is irreducible. Note that $Me \leq M_{\tilde{R}A}$ for $\tilde{R} \in \{\overline{R}, R, K\}$ and an arbitrary $\tilde{R}G$-module M. Furthermore, $\overline{R}(z_\lambda S^G e) = (\overline{R}(z_\lambda S^G))e$, so $E_\lambda e$ is the $\overline{R}S$-homogeneous component of E_λ. Let $X \leq \overline{R}(z_\lambda S^G)$. Then $Xe \leq \overline{R}(z_\lambda S^G e) = \overline{R}(z_\lambda(S \otimes 1)A) = \overline{R}S_\lambda$. By the above, $J(\overline{R}S_\lambda)$ is the unique maximal $\overline{R}A$-submodule of $\overline{R}S_\lambda$, hence $Xe \leq J(\overline{R}S_\lambda)$ or $Xe = \overline{R}S_\lambda$. But $\overline{R}S_\lambda \subset Xe \subset X$ implies $\overline{R}(S_\lambda RG) = \overline{R}(z_\lambda S^G) \leq X$, because $\overline{R}(S_\lambda RG) = (\overline{R}S_\lambda)RG$. Therefore every proper submodule of $\overline{R}(z_\lambda S^G)$ is contained in the proper $\overline{R}A$-submodule

$$J\left(\overline{R}\left(z_\lambda S^G\right)e\right) \oplus \overline{R}\left(z_\lambda S^G\right)(1-e) = J\left(\overline{R}\left(z_\lambda S^G e\right)\right) + \overline{R}\left(z_\lambda S^G\right)(1-e),$$

hence the sum of all proper submodules of $\overline{R}(z_\lambda S^G)$, too. Thus E_λ is irreducible, and D_λ occurs as a composition factor in $E_\lambda e$ (in fact $D_\lambda \leq E_\lambda \cdot e/J(E_\lambda \cdot e)$).

REMARK. Note that $\overline{R}(z_\lambda S^G e)$, hence D_λ and $E_\lambda \cdot e$, are contained in $\operatorname{mod}_{\overline{R}\mathscr{A}}(\overline{R}\mathscr{A}/\overline{R}S) \subset \operatorname{mod}_{\overline{R}\mathscr{A}}(\overline{R}A/\overline{R}S)$.

5. In the previous section, we have worked out the way to use the Specht modules of $K[W]$ for defining irreducible RG-modules. However, this only works after having achieved sufficient knowledge on the representation theory of $R[W]$, which at the moment is in the state of development. So we have to restrict ourselves to special cases.

5.1. HYPOTHESIS. *Let $s \in G$ be semisimple satisfying* 2.4 *and assume r divides $q^{\deg \Lambda} - 1$ for all $\Lambda \in \mathscr{F}$ with $m_\Lambda(s) \neq 0$.*

5.2. REMARK. Let $s \in G$ be semisimple, and assume that the order of s is prime to r. Then s satisfies 2.4 by 2.1 and 2.3. Let B be an r-block of G with semisimple part $s \in G$. Then, by [**9**, 5.6], s satisfies 5.1 if B has inertia index one. However, if r divides $q-1$, that is, if r divides the order of $Z(G)$, then 5.1 is satisfied for all semisimple $s \in G$, and the following will hold for all r-blocks of G. Note, if $s \in G$ satisfies 5.1 and has order prime to r, then there exists only one r-block $B = B_s$ of G with semisimple part s. Furthermore, a Sylow r-subgroup of $C_G(s)$ is a defect group $\delta(B_s)$ of B_s and all geometric conjugacy classes $(t)^G$ with $t = sy$, $y \in \delta(B_s)$, are contained in B_s, by [**9**]. These are all of the irreducible characters contained in B_s and the set of characters in $(s)^G$ restricted to the set of r'-elements of G form a basis of the space of Brauer characters of G contained in B_s. In particular, B_s just contains $|M_s|$ many inequivalent irreducible Brauer characters.

Recall the definition of $z_\lambda \in R[W]$, and let $\bar{z}_\lambda$ again be the image of z_λ in $\overline{R}[W]$. Note that the assumption in 4.4 is satisfied if $s \in G$ satisfies 5.1, because $r \geq 2$, so $z_\lambda \neq 0$ by 4.4.

5.3. LEMMA. *Assume* 5.1. *Then* $\bar{R}[W]$ *and* $\bar{R}W$ *may be identified canonically such that* $\bar{z}_\lambda \in \bar{R}[W]$ *generates the usual Specht module* $\bar{R}S^\lambda$ *of* $\bar{R}[W] = \bar{R}W$ $(\lambda \in M_s)$.

PROOF. Let $\bar{T}_w$ denote the image of $T_w \in R[W]$ in $\bar{R}[W]$. Now 5.1 implies $q^{\deg \Lambda} \equiv 1 \bmod r$ and $q^{\deg \Lambda} - 1 \equiv 0 \bmod r$ for $\Lambda \in \mathscr{F}$ with $m_\Lambda(s) \neq 0$. The defining relations for $\bar{R}[W]$ show immediately that $\bar{T}_w \to w \in \bar{R}W$ induces an isomorphism between $\bar{R}[W]$ and $\bar{R}W$. Identifying $\bar{R}[W]$ and $\bar{R}W$ by this isomorphism,

$$\bar{z}_\lambda = \overline{h_q(x_J)h_q(T_w^{-1}y_{J'}T_w)} = \overline{h_1(x_J)h_1(w^{-1}y_{J'}w)},$$

where $J, J' \subset \mathscr{T}$, $T_w, x_J, y_{J'} \in \mathscr{G}[W]$ are defined as in 4.4 and $-$: $R[W] \to \bar{R}[W]$, respectively $-$: $RW \to \bar{R}W$ denotes the canonical epimorphism. Now, for $\bar{R}W$, it is well known (see e.g. [**14**]) that

$$\bar{R}\big(h_1(x_J)h_1(w^{-1}y_{J'}w)RW\big) \cong \overline{h_1(x_J)h_1(w^{-1}y_{J'}w)}\,\bar{R}W.$$

Thus $\bar{R}(z_\lambda R[W]) = \bar{z}_\lambda \bar{R}W$. □

Summarizing we get by 4.4 and 5.3

5.4. THEOREM. *Assume* 5.1. *Then there exist isomorphisms* τ: $K[W] \to KW$ *and* $\bar{R}[W] \to \bar{R}W$ *such that the following diagram commutes, where* $\rightsquigarrow$ *denotes the decomposition map via* $R[W]$ *and* RW *respectively*:

$$\begin{array}{ccc} K[W] & \stackrel{\tau}{\to} & KW \\ \wr\downarrow & & \wr\downarrow \\ \bar{R}[W] & \stackrel{\sim}{\to} & \bar{R}W \end{array}$$

In particular, the decomposition matrices of $R[W]$ *and* RW *may be identified.*

REMARK. We do not know if $RW \cong R[W]$ inducing the above isomorphisms. However, the theorem is trivially true in this case.

Theorem 5.4 says that we may apply the results from the representation theory of symmetric groups (extended to direct products of symmetric groups) to $R[W]$, as they are presented in the standard literature (see e.g. [**14**]). Remember that a partition $\lambda = (1^{r_1}, \dots, k^{r_k})$ of k is called r-regular if $r_i < r$ for all $1 \leq i \leq k$. So call $\lambda = \prod \lambda_\Lambda \in M_s$ r-regular if λ_Λ is r-regular for all $\Lambda \in \mathscr{F}$, otherwise r-singular. Let $M_{s,r'} = \{\lambda \in M_s | \lambda \ r\text{-regular}\}$ and $M_{s,r} = M_s \setminus M_{s,r'}$. Denote by $\leq$ the dominance order on the set of partitions of $k \in \mathbf{N}$ as defined in [**14**, 3.2], and extend this to M_s setting $\lambda \leq \mu$ $(\lambda = \prod \lambda_\Lambda,\ \mu = \prod \mu_\Lambda \in M_s)$ if $\lambda_\Lambda \leq \mu_\Lambda$ for all $\Lambda \in \mathscr{F}$. Now, if $\lambda \in M_{s,r'}$, the Specht module $\bar{R}S^\lambda$ (with respect to $\bar{R}W$) has a unique maximal submodule by [**14**, 4.9], hence the same is true if 5.1 holds for $\bar{R}S^\lambda$ as a $\bar{R}[W]$-module, by 5.4, that is D^λ is irreducible. Furthermore, $D^\lambda \not\cong D^\mu$ $(\lambda, \mu \in M_{s,r'})$ if $\lambda \neq \mu$ and $\{D^\lambda | \lambda \in M_{s,r'}\}$ is a full set of nonisomorphic irreducible $\bar{R}[W]$-modules, again by 5.4. Finally, if $(a_{\lambda\rho})$ $(\lambda \in M_s,\ \rho \in M_{s,r'})$ denotes the decomposition matrix of RW $(R[W])$, then $a_{\rho\rho} = 1$ and $a_{\lambda\rho} \neq 0$ implies $\rho \geq \lambda$.

Recall the definition of the $\bar{R}G$-module E_λ $(\lambda \in M_s)$. By the above and 4.6, E_ρ $(\rho \in M_{s,r'})$ is irreducible if 5.1 holds.

5.5. LEMMA. *Assume* 5.1. *Let* ρ, $\rho' \in M_{s,r'}$ *and* $\rho \neq \rho'$. *Then* E_ρ *and* $E_{\rho'}$ *are two nonisomorphic irreducible* $\overline{R}G$*-modules.*

PROOF. By the above, E_ρ and $E_{\rho'}$ are irreducible, and, by 4.6, the $\overline{R}A$-modules D_ρ, $D_{\rho'}$ are irreducible, too. Furthermore, they are composition factors of $E_\rho e$ and $E_{\rho'}e$ respectively. By 3.7, $D_\rho \not\cong D_{\rho'}$. If D_μ ($\mu \in M_{s,r'}$) occurs as a composition factor of $E_\rho e$, then it occurs as a composition factor of $\overline{R}(z_\rho S^G e) = RS_\rho$, because $E_\rho = \overline{R}(z_\rho S^G)/J(\overline{R}(z_\rho S^G))$. Thus, by 3.7, D^μ occurs as a composition factor of $\overline{R}S^\rho$, forcing $\mu \geq \rho$. If $E_\rho \cong E_{\rho'}$, then $E_\rho e \cong E_{\rho'}e$, hence D_ρ is a composition factor of $E_{\rho'}e$ and $D_{\rho'}$ of $E_\rho e$ by 4.6. So, by the above, $\rho \leq \rho'$ and $\rho' \leq \rho$ forcing $\rho = \rho'$, a contradiction. □

Assume again 5.1 and let $b_{\lambda\rho}$ be the multiplicity of E_ρ as a composition factor of $\overline{R}(z_\lambda S^G)$. Inspecting the $\overline{R}A$-module $\overline{R}(z_\lambda S^G e)$ and applying 3.7, 4.6 we conclude

5.6. COROLLARY. *Let* $\rho \in M_{s,r'}$, $\lambda \in M_s$. *Then* $b_{\lambda\rho} \leq a_{\lambda\rho}$. *In particular*, $b_{\lambda\rho} \neq 0$ *forces* $\lambda \leq \rho$. *Moreover*, $b_{\rho\rho} = 1$.

Assume 5.1. Let $i \in \overline{R}[W] = \overline{R}W$ be a primitive idempotent, and let $i\overline{R}W/J(i\overline{R}W) \cong D^\rho$, $\rho \in M_{s,r'}$. Thus $Q^\rho = i\overline{R}W$ is a projective indecomposable $\overline{R}W$-module with head D^ρ. Let $Q_\rho = i(\overline{R}S^G e) = Q^\rho(\overline{R}S^G e)$. Then Q_ρ is an indecomposable direct summand of $\overline{R}S^G e = (S \otimes 1)\overline{R}A$. Similarly $P_\rho = i(\overline{R}S^G) = Q^\rho(\overline{R}S^G)$ is an indecomposable direct summand of $\overline{R}S^G$ by 3.6. Note that $P_\rho e = Q_\rho$ and $P_\rho e\overline{R}G = Q_\rho \overline{R}G = P_\rho$.

5.7. LEMMA. *Assume* 5.1, *and let* $\rho \in M_{s,r'}$. *Then* $P_\rho/J(P_\rho)$ *is simple and the head of* $P_\rho/J(P_\rho) \cdot e$ *is* D_ρ.

PROOF. Let $X \leq P_\rho$. Then $Xe \leq P_\rho e = Q_\rho$, hence $X = P_\rho$ or $Xe \leq J(Q_\rho)$. So all proper submodules of P_ρ are contained in the $\overline{R}A$-submodule $J(Q_\rho) + P_\rho \cdot (1 - e)$, hence $P_\rho/J(P_\rho)$ is irreducible. The canonical epimorphism $P_\rho \to P_\rho/J(P_\rho)$ induces an epimorphism $Q_\rho = P_\rho e \to P_\rho/J(P_\rho) \cdot e$, whose kernel is contained in $J(Q_\rho)$. Thus D_ρ, being the head of Q_ρ, is the head of $P_\rho/J(P_\rho)e$, too. □

5.8. COROLLARY. $P_\rho/J(P_\rho) \cdot e \cong D_\rho$.

PROOF. Let $1 = i_1 + \cdots + i_k$ be a decomposition of $1 \in \overline{R}W$ into a sum of primitive orthogonal idempotents and set $P_j = P_{i_j}$ $(1 \leq j \leq k)$. Because $\overline{R}[W] = \overline{R}W = \mathrm{End}_{\overline{R}G}(\overline{R}S^G)$, $P_j \cong P_m$ if and only if $\rho_j = \rho_m$, $1 \leq j, m \leq k$, where $i_j\overline{R}W/J(i_j\overline{R}W) = D^{\rho_j}$, $1 \leq j \leq m$, $\rho_j \in M_{s,r'}$ suitable. 5.7 implies that an epimorphism $P_j \to P_m$ has to be bijective for $1 \leq i, j \leq m$, in particular $\rho_j = \rho_m$. Now it follows easily from Fitting's theorem (see e.g. [**11**, 5.2]) that $J(i_j\overline{R}W) \cdot (\overline{R}S^G)$ is a proper $\overline{R}G$-submodule of P_j for $1 \leq j \leq k$. By 5.7 this must be $J(P_j)$, and the corollary follows immediately. □

6. Now we fix a semisimple element $s \in G$ of order prime to r and assume that r divides $q^{\deg \Lambda} - 1$ for all $\Lambda \in \mathscr{F}$ with $m_\Lambda(s) \neq 0$. Then s satisfies 5.1 and, by 5.2, there exists only one r-block B_s of G with semisimple part s. Furthermore, $\delta(B_s)$ may be chosen as a Sylow r-subgroup of $C_G(s)$ and there are just $|M_s|$ nonisomorphic irreducible $\overline{R}G$-modules in B_s by [**9**]. So far, we have determined $|M_{s,r'}|$ many of them, namely $\{E_\rho | \rho \in M_{s,r'}\}$, by 5.5.

Let $\delta(B_s)$ be a Sylow r-subgroup of $C_G(s)$, $y \in \delta(B_s)$ and $t = sy$. We will vary t, thus all objects have to have the index t in the following.

Note that b_t may be considered as an r-block of RL_t with semisimple part $s \in L_t$. Let the RP_t-module S_t be defined as in 1.4 with t replacing s and S_t replacing S. Our aim now is to find a sequence $1 = y_1, y_2, \ldots, y_k$ of r-elements in $\delta(B_s)$ such that $t_i = sy_i$ satisfies 5.1, and the resulting irreducible $\overline{R}G$-modules build a full set of nonisomorphic irreducible $\overline{R}G$-modules in B_s. But first we still need an auxiliary result. So let $t_i = sy_i$ $(i = 1,2)$ with $y_i \in \delta(B_s)$. Assume that $t_i \in G$ satisfies 2.4. Because r divides $q^{\deg \Lambda} - 1$ for all $\Lambda \in \mathscr{F}$ with $m_\Lambda(s) \neq 0$ by assumption, the same is true for t_i instead of s, hence t_i satisfies 5.1 $(i = 1,2)$. Set $S_i = S_{t_i}$, $L_i = L_{t_i}$ and $P_i = P_{t_i} = L_i U_i$ with $U_i = U_{t_i}$, the Levi kernel of P_i. Let $\rho \in M_{t_1,r'}$, $E = E_\rho$ and $\lambda \in M_{t_2}$, $M = \overline{R}(z_\lambda S_2^G)$. Note that E is simple by 5.5. For an arbitrary group H with subgroups H_1 and H_2 we write $H_1 \leq_H H_2$ if $H_1^h \leq H_2$ for some $h \in H$.

6.1. LEMMA. (i) *If E occurs as a composition factor of $\overline{R}S_2^G$ or M, then $L_2 \leq_G L_1$.*
(ii) *If $Ee_{t_2} \neq (0)$, then $L_1 \leq_G L_2$.*

PROOF. First note that a Coxeter torus T_2 of L_2 is conjugate (in G) to a maximal torus of L_1 if and only if $L_1 \leq_G L_1$, because $L_i = \prod_{j=1}^k \mathrm{GL}_{\mu_j}(q)$ for a suitable partition $\mu = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$ of n. Assume that E occurs as a composition factor of M, then it occurs as a composition of $\overline{R}S_2^G$, because the composition factors of $\overline{R}S^{(1)}$ are independent of the choice of an RG-lattice $S^{(1)}$ in KS^G. By 4.6, $E \cdot e_{t_1} \neq (0)$, hence $\overline{R}S_2^G e_{t_1} \neq (0)$, and therefore $KS_2^G e_{t_1} \neq (0)$, that is, the trivial KU_1-module I_1 occurs as a composition factor of KS_2^G. If $E \cdot e_{t_2} \neq (0)$, then $\overline{R}(z_\rho \cdot S_1^G)e_{t_2} \neq (0)$, hence $z_\rho S_1^G e_{t_2} \neq (0)$. Thus the trivial KU_2-module I_2 occurs as a composition factor of $z_\rho KS_1^G \leq KS_1^G$ in this case, and it is enough to show (i). Taking characters and using Frobenius reciprocity we conclude in the first case:

$$0 \neq \left(1_{U_1}, R_{T_2}^G(\hat{t}_2)\right)_{U_1} = \left(\tilde{\rho}^G, R_{T_2}^G(\hat{t}_2)\right)_G$$

$$= \sum_T \sum_{\theta \in \hat{T}} \varepsilon \left(\widetilde{R_T^{L_1}(\theta)}^G, R_{T_2}^G(\hat{t}_2)\right)_G$$

$$= \sum_T \sum_{\theta \in \hat{T}} \varepsilon \left(R_T^G(\theta), R_{T_2}^G(\hat{t}_2)\right)_G,$$

where ρ denotes the regular representation of L_1, T runs through all maximal tori of L_1, $\hat{T}$ denotes the set of irreducible characters of T and ε is a sign depending on T and L_1 [**15**, 6.23]. By [**15**, 6.14] T_2 has to be conjugate in G to a maximal torus T of L_1, forcing $L_2 \leq_G L_1$. □

Next let $y \in \delta(B_s)$ be arbitrary, $t = sy$. Then $C_G(t) \leq C_G(s) = \prod \mathrm{GL}_{m_\Lambda(s)}(q^{\deg \Lambda})$. Let $\prod C_G(t)_\Lambda$, $\prod y_\Lambda$ and $\prod t_\Lambda$ be the corresponding decompositions of $C_G(t)$, y and t respectively. Let $\Lambda \in \mathscr{F}$ with $m_\Lambda(s) \neq 0$ and let $\tilde{\Gamma}$ be an elementary divisor of y_Λ. So $\tilde{\Gamma}$ is a monic irreducible polynomial with coefficients in $\mathrm{GF}(q^{\deg \Lambda})$. Note that the order of $(\tilde{\Gamma}) \in \mathrm{GL}_{\deg \tilde{\Gamma}}(q^{\deg \Lambda})$ is a power of r, so the order of each root ω of $\tilde{\Gamma}$ is a power of r, say r^k. Of course $\mathrm{GF}(q^{\deg \Lambda})[\omega] = \mathrm{GF}(q^{\deg \Lambda \cdot \deg \Gamma}) = \mathrm{GF}(q)[\sigma \cdot \omega]$, where σ is a root of Λ, because the order of σ is prime to r, hence prime to the order of ω. Let ν_r: $\mathbf{Q} \to \mathbf{Z}$ denote the r-adic valuation and $\nu_r(q^{\deg \Lambda} - 1) = j$. Set $i = k - j$ if $k \geq j$ and $i = 0$ otherwise. Note that $k \geq 0$. It can easily be seen that $\deg \tilde{\Gamma} = r^i$ (compare e.g. [**9**, 3A]), hence the minimum polynomial Γ of $\sigma\omega$ over F has degree $\deg \Lambda \cdot r^i$, and over $\mathrm{GF}(q^{\deg \Lambda})$ degree r^i. Replacing G by $\mathrm{GL}_{m_\Lambda(s)}(q^{\deg \Lambda}) \leq \mathrm{GL}_{m_\Lambda(s) \cdot \deg \Lambda}(q)$, we conclude that $m_\Gamma(t_\Lambda) \neq 0$ implies $\deg \Gamma = \deg \Lambda \cdot r^i$ for some $0 \leq i \in \mathbf{Z}$. Let $N_i = \{\Gamma \in \mathscr{F} | \deg \Gamma = \deg \Lambda \cdot r^i\}$ and $d_i = \sum_{\Gamma \in N_i} m_\Gamma(t_\Lambda)$. Then $m_\Lambda(s) = \sum_{i=0}^{\infty} d_i \cdot r^i$. Let d_Λ be the sequence $(d_i)_{0 \leq i \in \mathbf{Z}}$ and D_Λ be the set of sequences $(d_i)_{0 \leq i \in \mathbf{Z}}$ of natural numbers d_i with $\sum_{i=0}^{\infty} d_i r^i = m_\Lambda(s)$. For the moment let λ, μ be partitions of some natural number k, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j)$, $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m)$. Define the Levi subgroups L_λ, L_μ of $\mathrm{GL}_k(q)$ setting $L_\lambda = \prod_{i=1}^{j} \mathrm{GL}_{\lambda_i}(q)$, $L_\mu = \prod_{i=1}^{m} \mathrm{GL}_{\mu_i}(q)$. On the set of partitions of k define a partial order $\prec$ setting $\lambda \prec \mu$ if there exists a set partition $\{\Omega_1, \ldots, \Omega_m\}$ of $\{1, \ldots, j\}$ with m parts (i.e. $\Omega_i \subseteq \{1, \ldots, j\}$, the Ω_i $(1 \leq i \leq m)$ are pairwise disjoint and $\{1, \ldots, j\} = \bigcup_{i=1}^{m} \Omega_i$), such that $\mu_i = \sum_{h \in \Omega_i} \lambda_h$. Note that this is not the dominance order in general.

It is easy to see that $L_\lambda \leq_{\mathrm{GL}_k(q)} L_\mu$ if $\lambda \prec \mu$. So assume that $L_\lambda^g \leq L_\mu$ for some $g \in \mathrm{GL}_k(q)$. Let T_λ be a Coxeter torus of L_λ. Then T_λ^g is isomorphic to $\prod_{i=1}^{j} \mathrm{GF}(q^{\lambda_i})$ and is a maximal torus of L_μ, corresponding to a product $\prod_{h=1}^{m} \nu^{(h)}$ of partitions $\nu^{(h)}$ of μ_h $(1 \leq h \leq m)$. So, for each $h \in \{1, \ldots, m\}$, there exists a subset Ω_h of $\{1, \ldots, j\}$ such that $\sum_{i \in \Omega_h} \lambda_i = \nu^{(h)}$. Of course $\{\Omega_1, \ldots, \Omega_m\}$ is a set partition of $\{1, \ldots, j\}$. Thus we have shown that $L_\lambda \leq_{\mathrm{GL}_k(q)} L_\mu$ if and only if $\lambda \prec \mu$.

Now let D_Λ be as above. With $d_\Lambda \in D_\Lambda$ we may associate the proper partition $(\deg \Lambda^{d_0}, (\deg \Lambda \cdot r)^{d_1}, \ldots, (\deg \Lambda \cdot r^k)^{d_k})$ of $k = m_\Lambda(s) \cdot \deg \Lambda$. Denote this by λ_{d_Λ} and notice that $d_\Lambda \mapsto \lambda_{d_\Lambda}$ is an injective map. Define a partial order $\prec$ on D_λ setting $d_\Lambda \prec d'_\Lambda$ $(d_\Lambda, d'_\Lambda \in D_\Lambda)$ if $\lambda_{d_\Lambda} \prec \lambda_{d'_\Lambda}$.

Let $D_s = \prod D_\Lambda$ (cartesian product) and define a partial order $\prec$ on D_s setting $\prod d_\Lambda \prec \prod d'_\Lambda$ $(d_\Lambda, d'_\Lambda \in D_\lambda$ for $\Lambda \in \mathscr{F})$ if $d_\Lambda \prec d'_\Lambda$ for all $\Lambda \in \mathscr{F}$. Finally, fix a linear order $\leq$ on D_s compatible with $\prec$. As we have seen above, we have a map from $s \cdot \delta(B_s)$ into D_s. For $t = sy \in s \cdot \delta(B_s)$ let $\mathbf{d}_t$ be the image of t under this map. Of course $D_s = \mathbf{d}_1 \lneqq \mathbf{d}_2 \lneqq \cdots \lneqq \mathbf{d}_k$ is finite, where $k = |D_s|$. If $\mathbf{d}_t = \mathbf{d}_j$, $t = sy \in s \cdot \delta(B_s)$, call $j = \mathrm{ht}_s(y)$ the s-height of y. Later we will construct for each $j \in \{1, \ldots, k\}$ a $y_j \in \delta(B_s)$ of height j, such that in fact the map defined above is surjective.

6.2. LEMMA. *Assume that $s = m_\Lambda(s)(\Lambda)$, i.e. s has only one elementary divisor, and let $y, y' \in \delta(B_s)$, $t = sy$, $t' = sy'$. Then $L_t \leq_G L_{t'}$ if and only if $\mathbf{d}_t \prec \mathbf{d}_{t'}$. In particular $L_t = L_{t'}$ if and only if $\mathrm{ht}_s(y) = \mathrm{ht}_s(y')$.*

PROOF. Obvious by the construction of $\mathbf{d}_t$ and $\mathbf{d}_{t'}$ and the definition of L_t and $L_{t'}$. □

Obviously the construction above may be extended to all r-elements y of $C_G(s)$. If $y, y' \in C_G(s)$ are $C_G(s)$-conjugate elements of $C_G(s)$, $t = sy$, $t' = sy'$, then $\mathbf{d}_t = \mathbf{d}_{t'}$ and $\mathrm{ht}_s(y) = \mathrm{ht}_s(y')$. Note that t and t' are G-conjugate if and only if y and y' are $C_G(s)$-conjugate.

Fix again $\Lambda \in \mathscr{F}$ and let $1 \le i \in \mathbf{Z}$. Let σ be a root of Λ. Assume that the order of σ is prime to r, and that r divides $q^{\deg \Lambda} - 1$. Let $m = \nu_r(q^{\deg \Lambda} - 1)$, $j = \deg \Lambda \cdot r^i$. Note that $\nu_r(q^j - 1) = m + i$, because r is odd. So fix an element $\omega = \omega_i$ of $\mathrm{GF}(q^j)$ of order r^{m+i}. Then $\mathrm{GF}(q^{\deg \Lambda})[\omega] = \mathrm{GF}(q)[\sigma \cdot \omega] = \mathrm{GF}(q^j)$. Consequently the minimal polynomial Λ_i of ω over $\mathrm{GF}(q^{\deg \Lambda})$ has degree r^i, and the minimal polynomial Λ'_i of $\sigma\omega$ over $\mathrm{GF}(q)$ has degree j. Set $\Lambda_0 = 1$, $\Lambda'_0 = \Lambda$. Let $d_\Lambda = (d_i)_{0 \le i \in \mathbf{Z}} \in D_\Lambda$, where $s \in G$ is chosen as above. Then define $y_\Lambda \in C_G(s)_\Lambda$ setting $y_\Lambda = \prod_{i=0}^{\infty} d_i(\Lambda_i)$ (matrix direct sum).

For $\mathbf{d} = \prod d_\Lambda \in D_s$ set $y_{\mathbf{d}} = \prod y_\Lambda \in C_G(s)$. Note that y is an r-element. Now, for $j \in \{1, \dots, k\}$, $\mathbf{d} = \mathbf{d}_j$, choose $y_j = \delta(B_s)$ $C_G(s)$-conjugate to $y_{\mathbf{d}}$, which is possible since we have chosen $\delta(B_s)$ to be a Sylow r-subgroup of $C_G(s)$. Set $t_j = sy_j$. In particular, $t_1 = s$. Note that all elementary divisors of t_j are of the form Λ'_i for some $0 \le i \in \mathbf{Z}$, where Λ runs through the set of elementary divisors of $s \in G$. The next lemma follows immediately from the definitions.

6.3. LEMMA. *Let $j, j' \in \{1, 2, \dots, k\}$. Then $\mathbf{d}_{t_j} = \mathbf{d}_j$ and $\mathrm{ht}_s(y_j) = j$. In particular, the map $t \to \mathbf{d}_t$ from $s \cdot \delta(B_s)$ into D_s is surjective. If $y_j^g = y_{j'}$ for some $g \in C_G(s)$, then $j = j'$. So this map induces a bijection between $\{t_j | 1 \le j \le k\}$ and D_s. Finally, if $\mathbf{d}_j = \prod d_\Lambda$, $d_\Lambda = (d_i)_{0 \le i \in \mathbf{Z}}$, then $m_{\Lambda'_i}(t_j) = d_i$ $(0 \le i \in \mathbf{Z})$.*

In the following we will replace all occurring indices of the form t_j $(1 \le j \le k)$ by j itself, e.g. $W_j = W_{t_j}$, $S_j = S_{t_j}$, etc. Note in particular that $M_{1,r'} = M_{s,r'}$, $M_1 = M_s$ and $W_1 = W_s$. Let $\lambda = \prod \lambda_\Lambda \in M_1$, i.e. λ_Λ is a partition of $m_\Lambda(s)$ $(\Lambda \in \mathscr{F})$. Let $\lambda_\Lambda = (1^{k_1}, \dots, m^{k_m})$, where $m = m_\Lambda(s)$, i.e. $\sum_{\nu=1}^{m} k_\nu \cdot \nu = m$. Let $k_\nu = \sum_{i=0}^{\infty} d_i^{(\nu)} r^i$ be the r-adic expansion of k_ν for $1 \le \nu \le m$ and set $d_i = \sum_{\nu=1}^{m} d_i^{(\nu)} \cdot \nu$ $(0 \le i \in \mathbf{Z})$. Then $\sum_{i=0}^{\infty} d_i r^i = m = m_\Lambda(s)$, hence $d_\Lambda = (d_i)_{0 \le i \in \mathbf{Z}} \in D_\Lambda$ and $\mathbf{d} = \prod d_\Lambda \in D_s$, i.e. $\mathbf{d} = \mathbf{d}_j$ for some $1 \le j \le k$. By 6.3, $m_{\Lambda'_i}(t_j) = d_i$, and obviously $d_i = \sum_{\nu=1}^{m} d_i^{(\nu)} \nu = \sum_{\nu=1}^{d_i} d_i^{(\nu)} \nu$, thus $(1^{d_i^{(1)}}, \dots, \nu^{d_i^{(\nu)}}, \dots) = \lambda_{\Lambda'_i}$ is an r-regular partition of $m_{\Lambda'_i}(t_j)$. Taking the (cartesian) product over all $\Lambda \in \mathscr{F}$ and all $0 \le i \le \mathbf{Z}$ we have defined an element μ_λ of $M_{j,r'}$. This defines a map $\lambda \to \mu_\lambda$ from M_1 into $\bigcup_{j=1}^{k} M_{j,r'}$. Note that this union is disjoint by 6.3.

On the other hand, let $\mu \in M_{j,r'}$, $1 \le j \le k$ and let Γ be an elementary divisor of t_j. Then $\Gamma = \Lambda'_i$ for some $0 \le i \in \mathbf{Z}$ and some elementary divisor Λ of s. Let $\mathbf{d}_j = \prod d_\Lambda$, $d_\Lambda = (d_i)_{0 \le i \in \mathbf{Z}}$, and let $\mu_\Gamma = (1^{d_i^{(1)}}, \dots, \nu^{d_i^{(\nu)}}, \dots)$, where $\mu = \prod \mu_\Gamma$. By 6.3, $m_\Gamma(t_j) = d_i$, thus $\sum_{\nu=1}^{\infty} d_i^{(\nu)} \nu = d_i$. Define $k_\nu = \sum_{i=0}^{\infty} d_i^{(\nu)} r^i$ and note that $\sum_{\nu=1}^{\infty} \sum_{i=0}^{\infty} d_i^{(\nu)} r^i \nu = m_\Lambda(s)$, hence $\sum_{\nu=1}^{\infty} k_\nu \cdot \nu = m_\Lambda(s)$, i.e. $\lambda_\mu(\Lambda) = (1^{k_1}, \dots, \nu^{k_\nu}, \dots)$ is a partition of $m_\Lambda(s)$. Taking $\lambda_\mu = \prod \lambda_\mu(\Lambda)$ we have defined a map $\mu \to \lambda_\mu$ from $\bigcup_{j=1}^{k} M_{j,r'}$ into M_1, which is obviously the inverse of the map $\lambda \to \mu_\lambda$ defined above. So we have shown

6.4. LEMMA. *$\lambda \to \mu_\lambda$ defines a bijection between M_1 and the disjoint union $\bigcup_{j=1}^{k} M_{j,r'}$.*

In the following we will identify M_1 with $\bigcup_{j=1}^{k} M_{j,r'}$ by this bijection.

We still need the following results:

6.5. LEMMA. *Let $T \le G$ be a maximal torus of G and θ a linear character of T. Let $u \in G$ such that the minimum polynomial of u is $(X - 1)^n$. Then $(R_T^G(\theta))(u) = 1$.*

PROOF. u is a unipotent element of G, in fact u is conjugate to the matrix

$$\begin{pmatrix} 1 & 1 & & & 0 \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & 0 & & \ddots & 1 \\ & & & & 1 \end{pmatrix} \quad \text{(the Jordan Normalform of } u\text{)},$$

and the canonical rational form of u is the companion matrix of the polynomial $(X-1)^n$.

So $(R_T^G(\theta))(u) = Q_T^G(u)$, where Q_T^G denotes the Green function of G with respect to T and q. If T is not a Coxeter torus, the lemma follows easily from [**15**, 6.24]. If T is a Coxeter torus, it follows from [**15**, 8.2]. Compare as well [**10**, Appendix]. □

Let U_0 be again the subgroup of all unipotent upper triangular matrices of G, and consider the restriction $R_T^G(\theta)_{U_0}$ of $R_T^G(\theta)$ to U_0. Because $R_T^G(\theta)$ is a virtual character of G, this restriction is a virtual character of U_0, i.e. a linear combination of irreducible characters of U_0 with integer coefficients. Let m be the greatest common divisor of these coefficients. Because $(R_T^G(\theta))(u)$ is an algebraic integer for all $u \in U_0$ we conclude

6.6. COROLLARY. $m = 1$.

6.7. LEMMA. *Let $j \in \{1,\ldots,k\}$ and $y \in \delta(B_s) \cap L_j$, $t = sy$. Assume that $t^g = t_j$ for some $g \in G$. Then $t^h = t_j$ for some $h \in L_j$.*

PROOF. The elementary divisors of t_j are of the form Λ'_i, where $0 \leq i \in \mathbf{Z}$ and Λ is an elementary divisor of s. Because t and t_j are conjugate in G, they have the same elementary divisors. Obviously we may assume $s = 1$ (hence $r|q-1$). So $n = \sum_{i=0}^{\infty} d_i r^i$ with $\mathbf{d}_\Lambda = (d_i)_{0 \leq i \in \mathbf{Z}}$, $\Lambda = X - 1 \in \mathscr{F}$, hence $L_j = \prod_{i=0}^{\infty} (\mathrm{GL}_{r^i}(q))^{d_i}$, and $\deg \Lambda'_i = r^i$. Now the lemma follows immediately, because $y \in L_j$. □

6.8. REMARK. If we replace y_j by an arbitrary $y' \in \delta(B_s)$, the lemma remains no longer true. The point is that we have fixed the polynomials Λ'_i. For example, if we take $n = 2$, $r|q-1$, $s = 1 \in G$, and w and w' as two different elements of order r in $\mathrm{GF}(q)$, $y = \left(\begin{smallmatrix} w & 0 \\ 0 & w' \end{smallmatrix}\right)$, $y' = \left(\begin{smallmatrix} w' & 0 \\ 0 & w \end{smallmatrix}\right)$, then $y \in L_{y'} = \mathrm{GF}(q)^* \times \mathrm{GF}(q)^*$, $y \in \delta(B_1)$, y and y' are conjugate in G, but cannot be conjugate in the abelian group $L_{y'}$.

Now we are prepared to prove the main technical lemma of this section.

6.9. LEMMA. *Let $j \in \{1,\ldots,k\}$. Then $t_j \in G$ satisfies* 5.1.

PROOF. As we have seen, all elementary divisors Γ of t_j are of the form Λ'_i, $0 \leq i \in \mathbf{Z}$, where Λ is an elementary divisor of s. In particular, $\deg \Gamma = \deg \Lambda \cdot r^i$, hence r divides $q^{\deg \Gamma} - 1$ by [**9**, 3A], because r is odd dividing $q^{\deg \Lambda} - 1$.

So we have to show that t_j satisfies 2.4. For $j = 1$, i.e. $t_j = s$, this is true by 2.1 and 2.3. So let $j > 1$ and assume that $t_{j'}$ satisfies 2.4 hence 5.1 for all $1 \leq j' < j$. In order to prove 2.4(i) we have to consider S_j as an RL_j-module contained in the block b_j whose semisimple part is $s \in L_j \leq G$. We may restrict ourselves to a component $S_j(\Gamma)$ for some elementary divisor Γ of t_j, so, using induction over n, we may assume that $j = k$, $n = \deg \Lambda \cdot r^h$ and $s = r^h(\Lambda)$ for some $h \in \mathbf{N}$. In particular, $t_k = (\Lambda'_h)$, and $M_{k,r'} = M_k$ contains one element $\lambda_k = \prod \lambda_\Gamma$, with $\lambda_\Gamma = (1)$ for $\Gamma = \Lambda'_h$ and $\lambda_\Gamma = (-)$ otherwise. By 5.5 and 6.4, for each $\mu \in M_s = M_1 = \bigcup_{j=1}^k M_{j,r'}$ with $\mu \neq \lambda_k$, we have defined an irreducible $\overline{R}G$-module E_μ. Let $\lambda_k \neq \mu, \mu' \in M_s$ and assume $E_\mu \cong E_{\mu'}$. Let $\mu \in M_{j,r'}$, $\mu' \in M_{j',r'}$. Then 6.1 implies immediately $L_j = L_{j'}$, hence $j = j'$ by 6.2 and $\mu = \mu'$ by 5.5. So we have found all but one irreducible $\overline{R}G$-module contained in $b_k = B_s$. Let E be the remaining irreducible $\overline{R}G$-module contained in B_s. By 6.1, E_μ does not occur as a composition factor of $\overline{R}S_k$, where $\lambda_k \neq \mu \in M_1$. So all composition factors of $\overline{R}S_k$ are isomorphic to E. Let m be the multiplicity of E as a composition factor of $\overline{R}S_k$. Then $\overline{R}S_k$ restricted to U_0 is isomorphic to a direct sum of m copies of E_{U_0}, because U_0 is an r'-group. Consequently $KS_k \cong m \cdot KE$ as U_0-modules, where E is an RU_0-module such that $\overline{R}E \cong E_{U_0}$. Let χ be the character afforded by KE. Then $\pm R^G_{T_k}(\hat{t}_k)_{U_0} = m\chi$, forcing $m = 1$ by 6.6. Hence $\overline{R}S_k = E = E_{\lambda_k}$ is irreducible, proving 2.4(i).

Returning to the general case we now have proved that t_j satisfies 2.4(i) if $t_{j'}$ satisfies 5.1 for all $1 \leq j' < j$. So let $V \in b_j$ be an irreducible constituent of $KS^G_{jP_j}$, where S_j is considered as a P_j-module. Let ψ be the character afforded by V. Then, because the semisimple part of b_j is s, $\psi = \tilde{\chi}$ for some $\chi \in (sy)^{L_j}$ and some r-element $y \in C_{L_j}(s)$, i.e. $\chi = \chi_{sy,\lambda}$ for some $\lambda \in M_{sy}$, where we take M_{sy} with respect to L_j. Now $\chi_{sy,\lambda}$ is a constituent of $\pm R^{L_j}_{T_{sy}}(\widehat{sy})$, and $R^{L_j}_{T_{sy}}(\widehat{sy}) = (\widetilde{R^{L_{sy}}_{T_{sy}}(\widehat{sy})})^{L_j}$ by [**15**, 6.24]. In particular, $R^{L_j}_{T_{sy}}(\widehat{sy})$ is up to sign a proper character of L_j. The character afforded by $KS_j^G{}_{P_j}$ is up to sign $R^G_{T_j}(\hat{t}_j)_{P_j}$. So V is an irreducible constituent of $KS_j^G{}_{P_j}$ if and only if

$$\left(\widetilde{R^{L_j}_{T_{sy}}(\widehat{sy})},\, R^G_{T_j}(\hat{t}_j)_{P_j}\right) = \left(R^G_{T_{sy}}(\widehat{sy}),\, R^G_{T_j}(\hat{t}_j)\right) \neq 0$$

using Frobenius reciprocity and [**15**, 6.24]. So sy has to be conjugate to t_j in G by [**15**, 6.14], hence in L_j by 6.7, forcing $R^{L_j}_{T_{sy}}(\widehat{sy}) = R^{L_j}_{T_j}(\hat{t}_j)$, i.e. $V = KS_j$, as desired. So the lemma is proved by induction over j. □

Let $\rho \in M_1 = \bigcup_{j=1}^k M_{j,r'}$. Then, by 6.9 and 5.5, E_ρ is an irreducible $\overline{R}G$-module in B_s. Suppose that $s = m_\Lambda(s)(\Lambda)$ for some $\Lambda \in \mathscr{F}$. Then we conclude as in the proof of 6.9 from 6.1, 6.2 and 5.5 that $E_\rho \cong E_{\rho'}$ ($\rho, \rho' \in M_1$) implies $\rho = \rho'$. So we get immediately from [**9**, 8A] the following

6.10. Lemma. *Assume that s has only one elementary divisor. Then $\{E_\rho | \rho \in M_1\}$ is a full set of nonisomorphic irreducible $\overline{R}G$-modules in B_s.*

For $\rho \in M_{j,r'}$ let $P_\rho \leq RS_j^G$ be defined as in 5.7. Recall the definitions of the subalgebra $A_j = A_{t_j}$ of RG, and the irreducible $\overline{R}A_j$-module D_ρ.

6.11. Lemma. *Suppose that s has only one elementary divisor $\Lambda \in \mathscr{F}$, and let $1 \leq j \leq k$, $\rho \in M_{j,r'}$. Then $P_\rho/J(P_\rho) \cong E_\rho$. In particular, $E_\rho \cdot e_j = D_\rho$ as $\overline{R}A_j$-modules.*

PROOF. Let $E = P_\rho/J(P_\rho)$. Then, by 6.10 and 5.7, $E = E_{\rho'}$ for some $\rho' \in M_1$, say $\rho' \in M_{j',r'}$. Now $E_{\rho'}$ is a composition factor of $\overline{R}S_j^G$, so $L_j \leq_G L_{j'}$ by 6.1. On the other hand, $E_{\rho'} \cdot e_j = D_\rho \neq (0)$ by 5.8, hence $L_{j'} \leq_G L_j$ again by 6.1. Therefore $j = j'$ by 6.2. By 4.6 $D_{\rho'}$ is a composition factor of $E_{\rho'} \cdot e_j = (P_\rho/J(P_\rho)) \cdot e_j$, so $D_\rho = D_{\rho'}$ by 5.8. Therefore $\rho = \rho'$ as desired. □

Let s again be an arbitrary semisimple r'-element of G satisfying 5.1. Let $\rho \in M_{j,r'}$, $\lambda \in M_{j'}$, $1 \leq j, j' \leq k$, and denote the multiplicity of E_ρ as a composition factor of $\overline{R}(z_\lambda S_{j'}^G)$ by $d_{\lambda\rho}$, the $(|M_{j'}| \times |M_{j,r'}|)$-matrix $(d_{\lambda\rho})$ by $D_{j'j}$ and the decomposition matrix of RW_j by D^j.

6.12. THEOREM. *Let s be a semisimple r'-element of G such that r divides $q^{\deg\Lambda} - 1$ for all $\Lambda \in \mathscr{F}$ with $m_\Lambda(s) \neq 0$. Then the following holds*:

(i) $\{E_\rho | \rho \in M_1\}$ *is a full set of nonisomorphic irreducible $\overline{R}G$-modules in B_s.*

(ii) *Let $1 \leq j, j' \leq k$, $\rho \in M_{j,r'}$, $\lambda \in M_{j'}$. Then $d_{\lambda\rho} \neq 0$ implies that $\mathbf{d}_{j'} \prec \mathbf{d}_j$. In particular, if $j \lneq j'$, then $D_{j'j} = 0$.*

(iii) $D_{jj} = D^j$ *for $j \in \{1, \ldots, k\}$.*

PROOF. Assume first that $s = m_\Lambda(s)(\Lambda)$ for some $\Lambda \in \mathscr{F}$. Then (i) is 6.10 and (ii) follows immediately from 6.1 and 6.2(i). So let $j = j'$, and let $\overline{R}(z_\lambda S_j^G) = C_1 \gneq C_2 \gneq \cdots \gneq C_m \gneq (0)$ be composition series of $\overline{R}(z_\lambda S_j^G)$, $C_i/C_{i+1} = E_i$ for $1 \leq i \leq m$ setting $C_{m+1} = (0)$. Then, by 6.10, $E_i = E_{\rho_i}$ for some $\rho_i \in M_1$. Obviously $C_i \cdot e_j/C_{i+1}e_j = (C_i/C_{i+1}) \cdot e_j = E_j \cdot e_j$. Let $i \in \{1, \ldots, m\}$ and assume that $E_ie_j \neq (0)$. Then 6.1 implies immediately that $\rho_i \in M_{j,r'}$, hence $C_ie_j/C_{i+1}e_j = E_ie_j = D_{\rho_i}$ is simple by 6.11. This shows that $C_1e_j \geq C_2e_j \geq \cdots \geq C_me_j \geq (0)$ is a composition series of $\overline{R}(z_\lambda S_j^G)e_j$. The standard basis theorem for Specht modules [**14**, 8.5] implies immediately that $\overline{R}(z_\lambda S_j^G)e_j = z_\lambda(\overline{R}S_j^Ge_j) \leq \overline{R}S_j^Ge_j$ is a reduction mod r of the irreducible KA_j-module $z_\lambda(KS_j^Ge_j)$. Counting multiplicities and applying 3.7, (iii) follows.

Now let $s = \prod m_\Lambda(s)(\Lambda) \in G$ again be arbitrary. Define the Levi subgroup $\tilde{L}$ of G setting $\tilde{L} = \prod \mathrm{GL}_{m_\Lambda(s)\cdot\deg\Lambda}(q)$, the parabolic subgroup $\tilde{P}$ of G to be $\tilde{L}U_0$, and let $\tilde{U}$ be the unipotent radical of $\tilde{P}$. Then $\tilde{L}$ contains $C_G(s)$ and all L_j $(1 \leq j \leq k)$, thus M_i with respect to $\tilde{L}$ and M_i with resepect to G coincide. Let $\tilde{b}$ be the block of $R\tilde{L}$ with semisimple part s lifted to $\tilde{P}$. Note that we may extend all results developed so far to parabolic subgroups of general linear groups, as long as we restrict ourselves to blocks containing the unipotent radical of the parabolic subgroup in the kernel. In particular, we may apply it to $\tilde{b}$. Note as well that the theorem holds for $\tilde{b}$ by the construction of $\tilde{L}$ and $\tilde{P}$, and by the above. So, for $j, j' \in \{1, \ldots, k\}$, $\rho \in M_{j,r'}$, $\lambda \in M_{j'}$, we denote by $\tilde{S}_j$, $\tilde{E}_\rho$, $\tilde{P}_\rho$, $\tilde{D}_{j'j}$ the corresponding objects taken with respect to $\tilde{P}$.

Let $1 \leq i \leq k$. Then $\tilde{R}S_i^G \cong \Sigma_g \tilde{R}S_i^{\tilde{P}} \otimes g$ as $\tilde{R}$-space ($\tilde{R} \in \{\overline{R}, R, K\}$), where g runs through a set of coset representatives of $\tilde{P}$ in G. Note that $\tilde{R}[W_i](S_i \otimes 1) \leq \tilde{R}S_i^{\tilde{P}}$, and that $\tilde{R}[W_i] = \mathrm{End}_{\tilde{R}\tilde{P}}(\tilde{R}S_i^{\tilde{P}})$. So, if V is an arbitrary $\tilde{R}\tilde{P}$-submodule of $\tilde{R}S_j^{\tilde{P}} = (\tilde{R}S_j \otimes 1)\tilde{R}P$, then $V\tilde{R}G \cong V^G$ canonically. Consider

$$\tilde{P}_\rho = Q_\rho \overline{R}\tilde{P} = Q^\rho\left(\overline{R}S_j^{\tilde{P}}e_j\right)\overline{R}\tilde{P} = Q^\rho\overline{R}S_j^{\tilde{P}},$$

where Q_ρ and Q^ρ are defined as in 5.7. Then $\tilde{P}_\rho^G = (Q^\rho \overline{R}S_j^{\tilde{P}})^G = Q^\rho(\overline{R}S_j^G) = P_\rho$ and similarly $J(\tilde{P}_\rho)^G = J(P_\rho)$. Thus $\tilde{P}_\rho^G/J(\tilde{P}_\rho)^G = P_\rho/J(P_\rho)$. By 6.11, $\tilde{P}_\rho/J(\tilde{P}_\rho) = \tilde{E}_\rho$, and obviously $\tilde{P}_\rho^G/J(\tilde{P}_\rho)^G = \tilde{E}_\rho^G$. Now $P_\rho/J(P_\rho) = \tilde{E}_\rho^G$ is simple by 5.7. Since all composition factors of $\overline{R}S_i^{\tilde{P}}$ are of the form $\tilde{E}_\mu$ for some $\mu \in M_1$, $V \mapsto V^G$ defines an isomorphism from the submodule lattice of $\overline{R}S_i^{\tilde{P}}$ onto the submodule lattice of $\overline{R}S_i^G$. So all composition factors of $\overline{R}S_i^G$ are of the form $\tilde{E}_\mu^G$. From [9, 8A] we conclude that $\{\tilde{E}_\mu^G | \mu \in M_1\}$ is a complete set of nonisomorphic irreducible $\overline{R}G$-modules in B_s. So, for an arbitrary $\overline{R}\tilde{P}$-module V_0 in $\tilde{b}$, $V \mapsto V^G$ defines an isomorphism from the submodule lattice of V_0 onto the submodule lattice of $V_0^G \in B_s$.

In particular, $E_\rho \cong \tilde{E}_\mu^G$ for some $\mu \in M_1$, say $\mu \in M_{i,r'}$, $1 \le i \le k$. Thus $\tilde{E}_\mu^G$ is a composition factor of $\overline{R}S_j^G$, hence $\tilde{E}_\mu$ of $\overline{R}S_j^{\tilde{P}}$ by the above, forcing $\mathbf{d}_j \prec \mathbf{d}_i$, since the theorem holds for $\tilde{b}$. On the other hand L_j and L_i must be conjugate in G by 6.1. From the definition of the partial order $\prec$ we get immediately $i = j$. As in the proof of 6.11 we conclude $\rho = \mu$, i.e. $\tilde{E}_\rho^G = E_\rho$. Now

$$z_\lambda S_{j'}^G = \left(z_\lambda S_{j'}^{\tilde{P}}\right)^G \quad \text{and} \quad \overline{R}\left(z_\lambda S_{j'}^G\right) = \overline{R}\left(z_\lambda S_{j'}^{\tilde{P}}\right)^G.$$

Thus the multiplicity of $\tilde{E}_\rho$ in $\overline{R}(z_\lambda S_{j'}^{\tilde{P}})$ equals the multiplicity of E_ρ in $\overline{R}(z_\lambda S_{j'}^G)$. This proves the theorem. □

We have also shown

6.13. Corollary. *Let s and $\tilde{L}$ be as above, and b be the block of $\tilde{L}$ with semisimple part s. Then the decomposition matrices of B_s and b coincide.*

Proof. Since $C_G(s)$ is contained in $\tilde{L}$ and $\delta(B_s)$ may be chosen in $C_G(s)$, $\delta(B_s)$ is a defect group of b as well. Moreover the geometric conjugacy classes in B_s and b are the same, because $t^g = t'$ ($t = sy$, $t' = sy'$, $y, y' \in \delta(B_s)$), for some $g \in G$ implies $g \in C_G(s) \le \tilde{L}$. So the sizes of the decomposition matrices are the same. If $y \in \delta(B_s)$, $t = sy$, $\lambda \in M_t$ and $\rho \in M_1$, we conclude as in 6.12 that the multiplicity of $\tilde{E}_\rho$ in $\overline{R}(z_\lambda S_t^{\tilde{P}})$ equals the multiplicity of E_ρ in $\overline{R}(z_\lambda S_t^G) = (\overline{R}(z_\lambda S_t^{\tilde{P}}))^G$, where $\tilde{P}$ and $\tilde{b}$ are defined as in 6.12. Thus the decomposition matrices of B_s and $\tilde{b}$ coincide. However the same is true with b instead of $\tilde{b}$. □

Next let y be an arbitrary r-element of $C_G(s)$, $t = sy$. Arguing as in the proof of 6.9 with y instead y_j ($1 \le j \le k$), we see that $\overline{R}S_t$ is irreducible, in fact $\overline{R}S_t = \overline{R}S_j$ with $j = \mathrm{ht}_s(y)$.

6.14. Lemma. *t satisfies* 2.4(i) *and* $\overline{R}S_t = \overline{R}S_j$, *where* $j = \mathrm{ht}_s(y)$.

Remark. It is not true in general that t satisfies 2.4(ii) too, as the example in 6.8 shows. In fact, here $\mathrm{End}_{KG}(KS_y^G) = K$, whereas $\mathrm{End}_{\overline{R}G}(\overline{R}S_y^G) = \overline{R}[W_1]$, and W_1 is isomorphic to the symmetric group on two letters.

As an easy consequence we get

6.15. Theorem. *Assume that r divides $q - 1 = |Z(G)|$ and let χ be an irreducible cuspidal character of G. Then χ, restricted to the r'-elements of G, is an irreducible Brauer character.*

Summarizing we get the following theorem.

6.16. THEOREM. *Let $s \in G$ be semisimple and assume that r divides $q^{\deg \Lambda} - 1$ for all elementary divisors of s. Then there exist $1 = y_1, y_2, \ldots, y_k \in \delta(B_s)$ and a bijection between M_s and the disjoint union $\bigcup_{j=1}^{k} M_{j,r'}$ such that $\{E_\rho \mid \rho \in M_{j,r'}, 1 \le j \le k\}$ is a full set of nonisomorphic irreducible $\overline{R}G$-modules in B_s. Ordering $\{1, \ldots, k\}$ downwards, the decomposition matrix D of B_s has the form*

$$D = \begin{pmatrix} D^k & & \\ & D^{k-1} & 0 \\ * & & \\ & & D^1 \\ * & * & * \end{pmatrix},$$

where D^j $(1 \le j \le k)$ denotes the decomposition matrix of RW_j.

Furthermore, the irreducible characters and Brauer characters in B_s may be ordered such that D has the (lower triangular) form

$$D = \begin{pmatrix} 1 & & \\ & \ddots & 0 \\ * & & \\ & & 1 \\ * & * & * \end{pmatrix}.$$

PROOF. For both presentations of D order the set of irreducible Brauer characters by ordering $\{1, \ldots, k\}$ downwards and $M_{j,r'}$ lexicographically downwards. In the first case take the set $\{\chi_{t_j,\lambda} \mid 1 \le j \le k, \lambda \in M_j\}$ ordered analogously first, then the other irreducible characters in B_s. In the second case take $\{\chi_{t_j,\lambda} \mid 1 \le j \le k, \lambda \in M_{j,r'}\}$ ordered in the same way first, then the remaining irreducible characters in B_s. Now the theorem follows from 6.16 and **[14]**.

6.17. COROLLARY. *Let $G = \mathrm{GL}_n(q)$ for some natural number n and some prime power q, and let r be an odd prime not dividing $q - 1$. Then the r-decomposition matrix of G is lower unitriangular.*

REFERENCES

1. C. T. Benson and C. W. Curtis, *On the degrees and rationality of certain characters of finite Chevalley groups*, Trans. Amer. Math. Soc. **165** (1972), 251–273; **202** (1975), 405–406.
2. N. Bourbaki, *Groupes et algèbres de Lie*, Chapitres 4–6, Hermann, Paris, 1968.
3. R. W. Carter, *Simple groups of Lie type*, Wiley, New York, 1972.
4. C. W. Curtis, N. Iwahori and R. Kilmoyer, *Hecke algebras and characters of parabolic type of finite groups with (B, N) pairs*, Inst. Hautes Etudes Sci. Publ. Math. **40** (1972), 81–116.
5. C. W. Curtis and I. Reiner, *Methods of representation theory with applications to finite groups and orders*, Vol. 1, Wiley, New York, 1981.
6. E. C. Dade, *Group-graded rings and modules*, Math. Z. **174** (1980), 241–262.
7. P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (12) **103** (1976), 103–161.
8. W. Feit, *Representations of finite groups*, Part I, Yale Univ., 1969.
9. P. Fong and B. Srinivasan, *The blocks of finite general and unitary groups*, Invent. Math. **69** (1982), 109–153.

10. J. A. Green, *The characters of the finite general linear groups*, Trans. Amer. Math. Soc. **80** (1955), 402–447.

11. ______, *Vorlesungen über Modulare Darstellungstheorie Endlicher Gruppen*, Vorlesungen aus dem Mathematischen Institut Giessen, Heft 2, 1974.

12. R. B. Howlett and G. I. Lehrer, *Induced cuspidal representations and generalized Hecke rings*, Invent. Math. **58** (1980), 37–64.

13. N. Iwahori, *On the structure of the Hecke ring of a Chevalley group over a finite field*, J. Fac. Sci. Univ. Tokyo Sect. 1A Math. **10** (1964), 215–236.

14. G. D. James, *The representation theory of the symmetric groups*, Lecture Notes in Math., vol. 682, Springer, Berlin and New York, 1978.

15. B. Srinivasan, *Representations of finite Chevalley groups*, Lecture Notes in Math., vol. 764, Springer, Berlin and New York, 1979.

MATHEMATISCHES INSTITUT DER UNIVERSITÄT ESSEN, 4300 ESSEN, WEST GERMANY

Current address: Department of Mathematics/Computer Science, Clark University, Worcester, Massachusetts 01610