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# HARMONIC FUNCTIONS ON SEMIDIRECT EXTENSIONS OF TYPE $H$ NILPOTENT GROUPS

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ABSTRACT. Let $S = NA$ be a semidirect extension of a Heisenberg type nilpotent group $N$ by the one-parameter group of dilations, equipped with the Riemannian structure, which generalizes this of the symmetric space. Let $\{P_a(y)\}_{a>0}$ be a Poisson kernel on $N$ with respect to the Laplace-Beltrami operator. Then every bounded harmonic function $F$ on $S$ is a Poisson integral $F(yb) = f * P_b(y)$ of a function $f \in L^\infty(N)$. Moreover the harmonic measures $\mu_a^b$ defined by $P_b = P_a * \mu_a^b$, $b > a$, are radial and have smooth densities. This seems to be of interest also in the case of a symmetric space of rank 1.

**Introduction.** We continue to investigate the harmonic functions on the Riemannian spaces $S$, studied in [**2**, **3**], with respect to the Laplace-Beltrami operator $\Delta$. $S$ is a semidirect product of a type $H$ nilpotent group $N$ and the one-parameter group of dilations $A$, equipped with the Riemannian structure modeled on one of the symmetric spaces of rank 1. $S$ includes those spaces as well as many more nonsymmetric ones [**3**].

The formula for what should be the Poisson kernel $P_a$, $a \in A$, has been written down by J. Cygan. In [**2**] the author has proved that the function $P(ya) = P_a(y)$, $y \in N$, is harmonic, as is the function $f * P_a(y)$ for every $f \in L^p(N)$, $1 \leq p \leq \infty$. Also it has been shown that $\lim_{a \to 0} f * P_a(y) = f(y)$ a.e.

The aim of this paper is to show that every bounded harmonic function on $S$ is a Poisson integral of a $L^\infty$ function on $N$. For the symmetric space it is, of course, well known [**6**, **9**]. However all the proofs we know are based on the fact that $S$ admits a large group $K$ of isometries, which leaves a point in $S$ invariant. By [**3**, **12**] we know that such a group is in fact quite small for general $S$.

Thus our proof is based on a different idea, which seems to be new in the classical case also. The idea is based on a maximum principle and certain properties or reproducing measures $\mu_a^b$ on $N$, which are uniquely defined as solutions of the equation $P_b = P_a * \mu_a^b$, $b > a$. We show that these measures are radial and have smooth densities, though the explicit formulas for them, even in the case of the Siegel domain, seems to be hopeless. To investigate $\mu_a^b$ we apply a method, which began with [**7**] and was further developed in [**1**, **11**].

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**1. Preliminaries.** Let $\mathcal{N} = V \oplus Z$ be a nonabelian Lie algebra of type $H$ [**8**] with centre $Z$. $V$ is the orthogonal complement to $Z$. Let $N = \exp \mathcal{N}$. We denote the element $\exp_N(v, z)$ by $(v, z)$ so that

$$(v, z)(v', z') = (v + v', z + z' + \tfrac{1}{2}[v, v']).$$

As in [**2**] let $S = NA$, where $A$ is the multiplicative group of $R^+$, be a semidirect product of $N$ and $A$, $A$ acting on $N$ as dilations $\delta_a(v, z) = (av, a^2 z)$. We identify $S$ with $V \times Z \times R^+$ so that $(v, z, a)(v', z', a') = (v + av', z + a^2 z' + \frac{1}{2}a[v, v'], aa')$ and in the Lie algebra $\mathcal{S} = \mathcal{N} \oplus R$ of $S$ we define the inner product

$$\langle (v, z, \log a), (v', z', \log a') \rangle_S = \langle v, v' \rangle + \langle z, z' \rangle + 4(\log a)(\log a')$$

[**2, 3**]. Let $\{e_i\}_{i=1,\ldots,2m}$, $\{e_r\}_{r=2m+1,\ldots,2m+l}$, $\{e_0\}$, where $2m = \dim V$, $l = \dim Z$, be an orthonormal basis of $\mathcal{S}$ corresponding to the decomposition $\mathcal{S} = V \oplus Z \oplus R$. Denote by $E_\beta$ the left-invariant vector field on $S$ determined by $e_\beta$, $\beta = 0, 1, \ldots, 2m + l$. It has been shown in [**2**] that the Laplace-Beltrami operator associated to the left-invariant metric $\langle\ ,\ \rangle_S$ has the form

$$\Delta = \sum_\beta E_\beta^2 - \frac{Q}{2} E_0,$$

where $Q = 2l + 2m$.

Let $\partial_i, \partial_r, \partial_0$ be the partial derivatives for the system of coordinates $(v^i, z^r, a)$ corresponding to $\{e_i, e_r, e_0\}$. Since

$$E_0 = \tfrac{1}{2} a \partial_0, \qquad E_r = a^2 \partial_r,$$

$$E_i = a\partial_i + \tfrac{1}{2} a \sum_{r=2m+1}^{2m+l} \langle [v, e_i], e_r \rangle \partial_r,$$

a straightforward calculation yields

$$\begin{aligned} \Delta = {} & \tfrac{1}{4}(1 - Q) a \partial_0 + \tfrac{1}{4} a^2 \partial_0^2 + a^2 \left(a^2 + \tfrac{1}{4}|v|^2\right) \sum_{r=2m+1}^{2m+l} \partial_r^2 \\ & + a^2 \sum_{i=1}^{2m} \partial_i^2 + a^2 \sum_{r=2m+1}^{2m+l} \sum_{i=1}^{2m} \langle [v, e_i], e_r \rangle \partial_r \partial_i. \end{aligned} \tag{1.1}$$

If $f$ is a function on $S$ depending only on $|v|, z$ and $a$, then

$$\sum_{r=2m+1}^{2m+l} \sum_{i=1}^{2m} \langle [v, e_i], e_r \rangle \partial_r \partial_i f = 0. \tag{1.2}$$

Finally the formula for the Poisson kernel (cf. [**2**]) is

$$P_a(y) = \frac{c a^Q}{((a^2 + |v|^2/4)^2 + |z|^2)^{Q/2}}, \tag{1.3}$$

where $y = (v, z)$ and $c$ is such that $\int_N P_a(y)\, dy = 1$.

**2. A maximum principle.** Let

$$N_a = \{ya \colon y \in N\}, \qquad S_a = \{yb \colon y \in N, b > a\}.$$

We shall prove the following form of the maximum principle.

THEOREM 2.1. *For every $\varepsilon > 0$, $M > 0$ and $x = y_1 b \in S_{a_0}$ there is a ball $B \subset N_{a_0}$ with centre in $y_1 a_0$ such that, if $F$ is harmonic in $S_{a_0}$ (i.e. $\Delta F = 0$ on $S_{a_0}$), continuous in $\bar{S}_{a_0}$ and $|F| \le M$, then*

$$F(x) \le \sup_{y \in B} F(y) + \varepsilon.$$

PROOF. Since $\Delta$ commutes with left translations, it is sufficient to prove the theorem for $y_1 = e = (0,0)$. Obviously, we can assume that $F \le 0$ on $B$ and $M/\varepsilon > 1$. We consider the function $G_\varepsilon = G^1_\varepsilon + \eta G^2_\varepsilon$, where

$$G^1_\varepsilon(v,z,a) = -\varepsilon \frac{a - a_0}{b - a_0},$$

and

$$G^2_\varepsilon(v,z,a) = -\frac{\varepsilon}{M} \log(1 + |v|^2 + |z|^2)$$

in the domain

$$D = \left\{ (v,z,a) \colon a_0 < a < \frac{b - a_0}{\varepsilon} M + a_0, |v|^2 + |z|^2 < R^2 \right\},$$

where $\eta$ is sufficiently small (see below) and $R$ is such that $\varepsilon\eta \log(1 + R^2) \ge M^2$. We put

$$B = \{(v,z,a_0) \colon |v|^2 + |z|^2 \le R^2\}.$$

Since by (1.1), (1.2)

$$\Delta G^1_\varepsilon \ge \frac{1}{4}(Q-1)\frac{\varepsilon a_0}{b - a_0} > 0$$

and $\Delta G^2_\varepsilon$ is bounded in $\{(v,z,a) \colon a_0 \le a \le (b - a_0)M/\varepsilon + a_0\}$, taking $\eta$ sufficiently small we obtain $\Delta G_\varepsilon \ge 0$. Moreover $G_\varepsilon + F \le 0$ on $\partial D$, because $F \le 0$ on $B$ and $F \le M$ on the rest of the boundary. Applying the classical maximum principle for elliptic operators to $\Delta$ on $D$ we obtain $F(b) \le \varepsilon$.

As an immediate consequence of Theorem 2.1 we have

COROLLARY 2.2. *For every $\varepsilon > 0, M > 0$ and $x = y_1 b \in S_{a_0}$ there is a ball $B \subset N_{a_0}$ with centre in $y_1 a_0$ such that if $F$ is harmonic in $S_{a_0}$, continuous in $\bar{S}_{a_0}$ and $|F| \le M$, then*

$$|F(x)| \le \sup_{y \in B} |F(y)| + \varepsilon.$$

**3. The representation theorem.** First we are going to prove some properties of radial functions and measures on $N$. By $O(N)$ we denote the set of transformations $B \colon N \to N$ such that $B|_V$ is orthogonal and $B|_Z$ is the identity.

DEFINITION 3.1. We say that a Borel measure $\mu$ is radial if for every $B \in O(N)$ and every Borel set $X \subset N, \mu(BX) = \mu(X)$, and similarly a function $f$ is radial if $f(By) = f(y)$, $y \in N$ [**11**].

By $M_v(N)$ and $L^1_v(N)$ we denote the set of radial measures and the set of radial integrable functions, respectively. In [**11**] F. Ricci proved that $L^1_v(N)$ is a commutative algebra and consequently since $L^1_v(N)$ is *-weakly dense in $M_v(N)$, so is $M_v(N)$.

Let $C_{0,v}(N)$ ($C_{c,v}(N)$) denote the set of radial continuous functions vanishing at infinity (with compact support). Since

$$Ef(y) = \int_{O(N)} f(By)\, dB$$

is a projection on the radial functions and $\langle E\mu, f\rangle = \langle \mu, Ef\rangle$ is a projection on radial measures, the dual of $C_{0,v}(N)$ is $M_v(N)$. Of course, by (1.3), $P_a$, $a > 0$, are radial.

LEMMA 3.2. *The set* $R^a = \{f * P_a : f \in C_{c,v}(N)\}$ *is dense in* $C_{0,v}(N)$.

PROOF. Our proof is a modification of that of Przebinda [**10**]. Let $\mu \in M_v(N)$ be a functional vanishing on $R^a$, i.e. $0 = \langle f * P_a, \mu\rangle = \langle f, \mu * P_a\rangle$. Thus

$$\mu * P_a = 0. \tag{3.1}$$

Now we consider the harmonic function $F(yb) = \mu * P_b(y)$, $b > 0$, on $S$. By (3.1) and the maximum principle $F(yb) = 0$ for $b > a$. Since $F$ is harmonic, it is real analytic and so $F(yb) = 0$ for all $b > 0$. Thus $\mu = 0$, because $P_a$ is an approximate identity.

Similarly we have

LEMMA 3.3. *The set* $T^a = \{f * P_a : f \in C_c(N)\}$ *is dense in* $C_0(N)$.

Now we return to our main theme. Let

$$H^a = \{F \in C^2(S_a) \cap C(\bar{S}_a) \cap L^\infty(\bar{S}_a) : \Delta F = 0\}, \quad U^a = \{F|_{N_a} : F \in H^a\}.$$

In view of the maximum principle the mapping $F \to F|_{N_a}$ is injective. Since $T^a$ is dense in $C_0(N)$, the maximum principle implies that for every $f$ in $C_0(N)$ there is an $F$ in $H^a$ such that $f = F|_{N_a}$. For every $x \in S$ we can define the functional

$$\langle \varphi_a^x, f\rangle = F(x), \qquad f \in U^a \text{ and } f = F|_{N_a}.$$

By Theorem 2.1 $\|\varphi_a^x\| \le 1$. $\varphi_a^x$ defines a bounded measure $\mu_a^x$ such that

$$F(x) = f * \mu_a^x(e), \qquad f \in C_0(N) \text{ and } f = F|_{N_a}. \tag{3.2}$$

Let us list a few elementary properties of $\mu_a^x$. We have

$$\mu_a^{yx} = \mu_a^x * \delta_{y^{-1}} \tag{3.3}$$

because for $g$ in the dense set $T^a$

$$g * \mu_a^{yx}(e) = g * \mu_a^x * \delta_{y^{-1}}(e). \tag{3.4}$$

Putting $x = b$ and $g = F_a = F|_{N_a}$ in (3.4) we obtain

$$F(yb) = F_a * \mu_a^b(y) \quad \text{for } F \in H^a \cap C_0(\bar{S}_a). \tag{3.5}$$

In particular

$$P_b = P_a * \mu_a^b. \tag{3.6}$$

(3.6) immediately gives

$$\|\mu_a^b\| = 1 \quad \text{and} \quad \mu_a^b \ge 0, \tag{3.7}$$

and (3.6) combined with Lemma 3.3 yields

$$\mu_c^a * \mu_a^b = \mu_c^b. \tag{3.8}$$

LEMMA 3.4. *The measures* $\mu_a^b$ *are uniquely defined by* (3.6).

PROOF. If $P_a * \nu = 0$, then by Lemma 3.3 $\nu$ defines the zero functional on $C_0$; hence $\nu = 0$.

LEMMA 3.5. *If $f \in C_{0,v}(N)$, then $f * \mu_a^b \in C_{0,v}(N)$.*

PROOF. If $f \in R^a$, that is $f = g * P_a$ for a $g \in C_{c,v}(N)$, then by (3.6) $f * \mu_a^b = g * P_b$, which is radial. The assertion follows now by Lemma 3.2.

LEMMA 3.6. *The measures $\mu_a^b$ are radial.*

PROOF. Let $f \in C_0(N)$ and $B \in O(N)$. By the previous lemma $P_\varepsilon * \mu_a^b \in L_v^1(N)$ for $\varepsilon > 0$. Then we have

$$\langle P_\varepsilon * f, \mu_a^b\rangle = \langle f, P_\varepsilon * \mu_a^b\rangle = \langle f \circ B, P_\varepsilon * \mu_a^b\rangle = \langle P_\varepsilon * (f \circ B), \mu_a^b\rangle.$$

Hence, if $\varepsilon \to 0$, we obtain $\langle f, \mu_a^b\rangle = \langle f \circ B, \mu_a^b\rangle$, which completes the proof.

Now we look at the relation of these measures to bounded harmonic functions. First of all we notice that (3.2) is true for all $f \in U^a$, because the functional $\varphi_a^x$ is defined on the space $U^a \subset C(N) \cap L^\infty(N)$ and in view of (3.7) attains its norm on $C_0(N)$. As before, (3.2) combined with (3.3) gives

$$F(yb) = F_a * \mu_a^b(y) \quad \text{for } F \in H^a. \tag{3.9}$$

Now we are in a position to prove the main theorem of this section.

THEOREM 3.7. *If $F$ is a bounded harmonic function on $S$, then there is an $f \in L^\infty(N)$ such that $F(yb) = f * P_b(y)$.*

PROOF. Since the family $\{F_a\}_{a>0}$ is uniformly bounded there is a sequence $F_{a_n}$ which is convergent *-weakly to a function $f \in L^\infty(N)$ when $a_n \to 0$. In particular, $F_{a_n} * P_b \to f * P_b$. On the other hand, Lemma 3.6 and (3.9) imply

$$F_{a_n} * P_b = F_{a_n} * (P_{a_n} * \mu_{a_n}^b) = (F_{a_n} * \mu_{a_n}^b) * P_{a_n} = F_b * P_{a_n}.$$

Since $P_{a_n}$ is an approximate identity, $F_b * P_{a_n} \to F_b$ a.e. when $a_n \to 0$ [2] and the theorem follows.

**4. Smoothness of $\mu_a^b$.** In this section we investigate more precisely the measures $\mu_a^b$ by means of the Gelfand transform of $L_v^2(N)$. The Gelfand transform of $L_v^1(N)$ is described in [**11**]. There are two families of multiplicative functionals. For a real nonnegative $\rho$ we write

$$\hat{f}(\rho) = \int_N f(v,z) e^{i\rho\langle v, v_0\rangle}\, dv\, dz,$$

where $v_0$ is a fixed unit vector in $V$. For $w \in Z \backslash \{0\}$ and a nonnegative integer $n$ we have

$$\hat{f}(w,n) = \binom{n+m-1}{n}^{-1} \int_N f(v,z) e^{i\langle w,z\rangle} \mathcal{L}_n^{m-1}\left(\tfrac{1}{2}|w|\,|v|^2\right) dz\, dv, \tag{4.1}$$

where $\mathcal{L}_n^{m-1}(r) = e^{-r/2} L_n^{m-1}(r)$ and $L_n^{m-1}(r)$ is the Laguerre polynomial of degree $n$ and order $m-1$ [**4**]. Formula (4.1) defines a unitary operator

$$\mathcal{F}\colon L_v^2(N) \to \mathcal{H} = \left\{\chi\colon \|\chi\|_{\mathcal{H}}^2 = \int_Z \sum_{n=0}^\infty |\chi(w,n)|^2 \binom{n+m-1}{n} |w|^m\, dw < \infty\right\};$$

that is a Plancherel theorem holds [**11**]:

$$\|f\|_{L^2}^2 = (2\pi)^{-m-l} \|\hat{f}\|_{\mathcal{H}}^2.$$

Thus, of course, we have

$$\mathcal{F}(P_a * f) = \mathcal{F}P_a \cdot \mathcal{F}f. \tag{4.2}$$

Let

$$\mathcal{F}^{-1}(\chi) = (2\pi)^{-m-l} \sum_{n=0}^{\infty} \int_Z \chi(w,n) e^{-i\langle w,z\rangle} \mathcal{L}_n^{m-1}\left(\tfrac{1}{2}|w|\,|v|^2\right) |w|^m \, dw. \tag{4.3}$$

Applying the ordinary inversion formula for the Fourier transform and the orthogonality relation

$$\frac{n!}{(n+m-1)!} \int_0^\infty L_n^{m-1}(r) L_{n_1}^{m-1}(r) r^{m-1} e^{-r}\, dr = \delta_{n,n_1}$$

we obtain by routine calculation that

$$\mathcal{F}^{-1} \circ \mathcal{F} = I \quad \text{and} \quad \mathcal{F} \circ \mathcal{F}^{-1} = I.$$

Using (4.3) we can write the formula for what should be the density of $\mu_a^b$, $b > a$. First we show that $\hat{P}_b(w,n)/\hat{P}_a(w,n) \in \mathcal{H}$, and second that $\psi_a^b = \mathcal{F}^{-1}(\hat{P}_b/\hat{P}_a)$ satisfies the equation $P_b = P_a * \psi_a^b$. If we also prove that $\psi_a^b \in L_v^1(N)$, then we shall have $\mu_a^b = \psi_a^b$.

LEMMA 4.1.

$$\begin{aligned} \hat{P}_a(w,n) &= c_1 |w|^{-m} e^{-|w|a^2} \\ &\quad \times \int_0^\infty \left(\frac{t}{1+t}\right)^n e^{-2|w|a^2 t} t^{(2m+l-1)/2} (1+t)^{(l-1)/2}\, dt, \end{aligned} \tag{4.4}$$

*where*

$$c_1 = 2^{m+1} \pi^{(2m+l+1)/2} c a^Q \left(\Gamma\left(\frac{Q}{2}\right)\right)^{-1} \left(\Gamma\left(\frac{2m+l+1}{2}\right)\right)^{-1}$$

*and $c$ is the constant in* (1.3).

PROOF. By (5.1) of [1] and (1.3) we have

$$\begin{aligned} \int_Z P_a(v,z) e^{i\langle w,z\rangle}\, dz &= c_2 e^{-|w|(a^2+|v|^2/4)} \\ &\quad \times \int_0^\infty e^{-|w|(a^2+|v|^2/4)t} (t^2+2t)^{(2m+l-1)/2}\, dt, \end{aligned} \tag{4.5}$$

where

$$c_2 = c a^Q 2^{1-2m-l} \pi^{(l+1)/2} \left(\Gamma\left(\frac{Q}{2}\right) \cdot \Gamma\left(\frac{2m+l+1}{2}\right)\right)^{-1}.$$

Hence

$$\begin{aligned} \hat{P}_a(w,n) = \binom{n+m-1}{n}^{-1} c_2 e^{-|w|a^2} \int_V \int_0^\infty e^{-|w|(a^2+|v|^2/4)t} \\ \times (t^2+2t)^{(2m+l-1)/2} e^{-|w|\,|v|^2/2} L_n^{m-1}\left(\frac{1}{2}|w|\,|v|^2\right) dt\, dv. \end{aligned} \tag{4.6}$$

Integrating first over $v$, by

$$L_n^{m-1}(r) = \frac{1}{n!} r^{-m+1} e^r D^n (r^{n+m-1} e^{-r})$$

we obtain

$$\int_V e^{-|w|\,|v|^2 t/4} e^{-|w|\,|v|^2/2} L_n^{m-1}\left(\tfrac{1}{2}|w|\,|v|^2\right) dv$$
$$= \left(\frac{2}{|w|}\right)^m \frac{\pi^m}{n!(m-1)!} \int_0^\infty e^{-rt/2} D^n (r^{n+m-1} e^{-r})\, dr,$$

which by parts is equal to

$$\left(\frac{2}{|w|}\right)^m \frac{\pi^m}{n!(m-1)!} \left(\frac{t}{2}\right)^n \frac{(n+m-1)!}{(1+t/2)^{n+m}}. \tag{4.7}$$

Finally putting (4.7) into (4.6) we get (4.4).

LEMMA 4.2.

$$\frac{\hat{P}_b(w,n)}{\hat{P}_a(w,n)} \le \begin{cases} c_3 e^{-c_4\sqrt{|w|n}} e^{-|w|(b^2-a^2)} & \text{if } 0 < |w| \le 1, \\ c_3 e^{-c_4\sqrt{n}} e^{-|w|(b^2-a^2)} |w|^{(l-1)/2} & \text{if } |w| > 1, \end{cases} \tag{4.8}$$

*where* $c_3 = (b/a)^{l-1} + (b^2+1)^{(l-1)/2} c_5$, $c_4 = (b^2-a^2)/(c_6+b^2)$, $c_6 = \frac{3}{2}a + 1, c_5 > 0$ *depends only on* $m + l - 1$ *and* $a$.

PROOF. By (4.4)

$$\frac{\hat{P}_b(w,n)}{\hat{P}_a(w,n)} = e^{-|w|(b^2-a^2)} \frac{\int_0^\infty e^{-2t} (t/(t+|w|b^2))^n t^{(2m+l-1)/2} (|w|b^2+t)^{(l-1)/2}\, dt}{\int_0^\infty e^{-2t} (t/(t+|w|a^2))^n t^{(2m+l-1)/2} (|w|a^2+t)^{(l-1)/2}\, dt}.$$

Let

$$I_n(a,A) = \int_0^A e^{-2t} \left(\frac{t}{t+|w|a^2}\right)^n t^{(2m+l-1)/2} (|w|a^2+t)^{(l-1)/2}\, dt$$

and

$$J_n(a,A) = \int_A^\infty e^{-2t} \left(\frac{t}{t+|w|a^2}\right)^n t^{(2m+l-1)/2} (|w|a^2+t)^{(l-1)/2}\, dt.$$

Analogously we define $I_n(b,A)$ and $J_n(b,A)$. We estimate separately

$$\text{(i)} \quad \frac{I_n(b, c_6\sqrt{n|w|})}{I_n(a,\infty)} \quad \text{and} \quad \text{(ii)} \quad \frac{J_n(b, c_6\sqrt{n|w|})}{I_n(a,\infty)}.$$

Since $(t+|w|a^2)/(t+|w|b^2)$ increases,

$$I_n(b, c_6\sqrt{n|w|}) = \int_0^{c_6\sqrt{n|w|}} e^{-2t} \left(\frac{t}{t+|w|a^2}\right)^n \left(\frac{t+|w|a^2}{t+|w|b^2}\right)^n t^{(2m+l-1)/2}$$
$$\times (|w|a^2+t)^{(l-1)/2} \left(\frac{|w|b^2+t}{|w|a^2+t}\right)^{(l-1)/2} dt$$
$$d \le \left(\frac{b}{a}\right)^{l-1} \left(\frac{c_6\sqrt{n} + \sqrt{|w|}a^2}{c_6\sqrt{n} + \sqrt{|w|}b^2}\right)^n I_n(a, c_6\sqrt{n|w|}).$$

Hence for (i) we have

$$\frac{I_n(b, c_6\sqrt{n|w|})}{I_n(a,\infty)} \le \left(\frac{b}{a}\right)^{l-1} \exp\left(-\frac{b^2-a^2}{c_6+\sqrt{|w|}b^2}\sqrt{|w|n}\right). \tag{4.9}$$

To estimate (ii), we notice that if $t \ge c_6\sqrt{n|w|}$, then

$$(|w|b^2+t) \le \begin{cases} (b^2+1)t & \text{if } |w| \le 1, \\ (b^2+1)|w|t & \text{if } |w| > 1 \end{cases}$$

and

$$J_n(b, c_6\sqrt{n|w|}) \le c_7 \int_{c_6\sqrt{n|w|}}^{\infty} e^{-2t} t^{m+l-1}\, dt, \tag{4.10}$$

where $c_7 = (b^2+1)^{(l-1)/2}\max(1, |w|^{(l-1)/2})$. Since

$$\left(\frac{t}{t+|w|a^2}\right)^n \ge \left(\frac{t}{t+|w|a^2}\right)^{t^2/a^2|w|} \ge e^{-t}$$

for $t \ge a\sqrt{n|w|}$, we have

$$I_n(a,\infty) \ge \int_{a\sqrt{n|w|}}^{\infty} e^{-3t} t^{m+l-1}\, dt. \tag{4.11}$$

There is a constant $c_5$ depending only on $m+l-1$, $a$ and $c_6$ such that

$$\begin{aligned} \frac{\int_{c_6\sqrt{n|w|}}^{\infty} e^{-2t} t^{m+l-1}\, dt}{\int_{a\sqrt{n|w|}}^{\infty} e^{-3t} t^{m+l-1}\, dt} &\le c_5 \exp((-2c_6+3a)\sqrt{n|w|}) \\ &= c_5 e^{-2\sqrt{n|w|}}. \end{aligned} \tag{4.12}$$

Finally putting (4.10)–(4.12) together we obtain

$$\frac{J_n(b, c_6\sqrt{n|w|})}{I_n(a,\infty)} \le c_7 c_5 e^{-2\sqrt{n|w|}}$$

which with (4.9) implies (4.8).

LEMMA 4.3. *For every nonnegative integer p*

$$\sum_{n=0}^{\infty} \int_Z \left|\frac{\hat{P}_b(w,n)}{\hat{P}_a(w,n)}\right|^2 (2n+m)^p n^{m-1} |w|^{m+p}\, dw < \infty.$$

PROOF. Let $h_n(w) = |\hat{P}_b(w,n)/\hat{P}_a(w,n)|^2\, |w|^{m+p}$. Since

$$\int_{|w|\ge 1} h_n(w)\, dw \le c_3^2 e^{-2c_4\sqrt{n}} \int_{|w|\ge 1} \exp(-2|w|(b^2-a^2))|w|^{m+p+l-1}\, dw$$

and

$$\sum_{n=0}^{\infty} e^{-2c_4\sqrt{n}} (2n+m)^p n^{m-1} < \infty$$

by (4.8), we have

$$\sum_{n=0}^{\infty} (2n+m)^p n^{m-1} \int_{|w|\ge 1} h_n(w)\, dw < \infty.$$

On the other hand,

$$\begin{aligned}\int_{|w|\le 1} h_n(w)\,dw &\le c_3^2 \int_{|w|\le 1} \exp(-2c_4\sqrt{n|w|})\exp(-2|w|(b^2-a^2))|w|^{m+p}\,dw\\ &\le c_3^2\int_Z e^{-2c_4\sqrt{n|w|}}|w|^{m+p}\,dw\\ &= c_3^2\frac{2l\pi^{1/2}}{\Gamma(l/2+1)}\frac{(2(m+p+l)-1)!}{(2c_4)^{2(m+p+l)}}\frac{1}{n^{m+p+l}}\end{aligned}$$

and $\sum_{n=0}^{\infty}(2n+m)^p/n^{1+p+l} < \infty$ for $l > 0$. Hence also

$$\sum_{n=0}^{\infty}(2n+m)^p n^{m-1}\int_{|w|\le 1} h_n(w)\,dw < \infty$$

which concludes the proof.

THEOREM 4.4. *The measures $\mu_a^b$ are absolutely continuous and their densities $\psi_a^b$ are given by the formula*

$$\psi_a^b(v,z) = (2\pi)^{-l-m}\sum_{n=0}^{\infty}\int_Z \frac{\hat{P}_b(w,n)}{\hat{P}_a(w,n)}\mathcal{L}_n^{m-1}(\tfrac{1}{2}|w|\,|v|^2)e^{-i\langle w,z\rangle}|w|^m\,dw. \tag{4.13}$$

PROOF. The previous lemma and the inequality

$$\binom{n+m-1}{n} \le \frac{\text{const}}{(m-1)!}n^{m-1}$$

yield $\hat{P}_b/\hat{P}_a \in \mathcal{H}$. Hence $\psi_a^b \in L_v^2(N)$, $\mathcal{F}(\psi_a^b) = \hat{P}_b/\hat{P}_a$ and by (4.2) $\mathcal{F}(P_b) = \mathcal{F}(P_a * \psi_a^b)$. Thus we have

$$P_b = P_a * \psi_a^b. \tag{4.14}$$

Let $f$ be a continuous function with compact support. Then by (4.14) $(f * P_a) * \psi_a^b(e) = (f * P_a) * \mu_a^b(e)$, which shows that $\psi_a^b$ defines a continuous functional on $C_0(N)$. Hence $\psi_a^b \in L^1(N)$ and in view of the uniqueness of $\mu_a^b$ as the solution of (3.6) $\psi_a^b$ is the density of $\mu_a^b$.

THEOREM 4.5. *The functions $\psi_a^b$ are smooth and, for every left-invariant operator $\partial$ on $N$, $\partial(\psi_a^b) \in L^2(N)$.*

PROOF. Let $L = -\sum_{i=1}^{2m} E_i^2$. Here $E_i$ is the left-invariant field on $N$ corresponding to $e_i$. Analogously to the case of $\Delta$ we can easily check that

$$L = \sum_{i=1}^{2m}\partial_i^2 + \sum_{s=2m+1}^{2m+l}\sum_{i=1}^{2m}\langle [v,e_i],e_s\rangle\partial_s\partial_i + \frac{1}{4}|v|^2\sum_{s=2m+1}^{2m+l}\partial_s^2.$$

If $f$ is a radial function, then by (1.2)

$$Lf = \left(\frac{\partial^2}{\partial r^2} + \frac{2m-1}{r}\frac{\partial}{\partial r} + \frac{1}{4}r^2\sum_{s=2m+1}^{2m+l}\partial_s^2\right)f, \tag{4.15}$$

where $r = |v|$. Applying (4.15) and the equality

$$r(L_n^{m-1})''(r) + (m-r)(L_n^{m-1})'(r) + nL_n^{m-1}(r) = 0$$

[**4**, vol. 2, p. 188] we obtain that $\mathcal{L}_n^{m-1}(\frac{1}{2}|w|\,|v|^2)e^{-i\langle w,z\rangle}$ are eigenfunctions of $L$ with eigenvalues $(2n+m)|w|$. Let

$$g_n(v,z) = (2\pi)^{-l-m} \int_Z \frac{\hat{P}_b(w,n)}{\hat{P}_a(w,n)} \mathcal{L}_n^{m-1}\left(\frac{1}{2}|w|\,|v|^2\right) e^{-i\langle w,z\rangle} |w|^m \, dw.$$

$g_n \in C^\infty(N)$ and by (4.13) $\sum_{n=0}^\infty g_n$ is convergent in $L^2(N)$ to $\psi_a^b$. We have

$$L^k g_n(v,z) = (2\pi)^{-l-m} \int_Z \frac{\hat{P}_b(w,n)}{\hat{P}_a(w,n)} (2n+m)^k \mathcal{L}_n^{m-1}\left(\frac{1}{2}|w|\,|v|^2\right) e^{-i\langle w,z\rangle} |w|^{m+k} \, dw$$

and

$$\|L^k g_n\|_{L^2}^2 = (2\pi)^{-l-m} \binom{n+m-1}{n} (2n+m)^{2k} \int_Z \left| \frac{\hat{P}_b(w,n)}{\hat{P}_a(w,n)} \right|^2 |w|^{m+2k} \, dw.$$

Hence by Lemma 4.3

$$\sum_{n=0}^\infty \|L^k g_n\|_{L^2}^2 < \infty.$$

This shows that $\psi_a^b$ belongs to the domain of the closure of $L^k$ and so $\psi_a^b$ is smooth. Since, by [**5**], for every left-invariant differential operator $\partial$ on $N$ there is a $k$ and $d$ such that

$$\|\partial f\|_{L^2} \le d(\|L^k f\|_{L^2} + \|f\|_{L^2}),$$

the theorem follows.

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