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REGULAR LINEAR ALGEBRAIC MONOIDS

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ABSTRACT. In this paper we study connected regular linear algebraic monoids. If ϕ: $G_0 \to \mathrm{GL}(n, K)$ is a representation of a reductive group G_0, then the Zariski closure of $K\phi(G_0)$ in $\mathcal{M}_n(K)$ is a connected regular linear algebraic monoid with zero. In §2 we study abstract semigroup theoretic properties of a connected regular linear algebraic monoid with zero. We show that the principal right ideals form a relatively complemented lattice, that the idempotents satisfy a certain connectedness condition, and that these monoids are V-regular. In §3 we show that when the ideals are linearly ordered, the group of units is nearly simple of type A_l, B_l, C_l, F_4 or G_2. In §4, conjugacy classes are studied by first reducing the problem to nilpotent elements. It is shown that the number of conjugacy classes of minimal nilpotent elements is always finite.

1. Preliminaries. Throughout this paper Z^+ will denote the set of all positive integers. If X is a set, then $|X|$ denotes the cardinality of X. Let S be a semigroup. We let $E(S)$ denote the set of all idempotents of S. If $a, b \in S$, then $a|b$ (*a divides b*) if $xay = b$ for some $x, y \in S^1$, $a \mathcal{J} b$ if $a|b|a$, $a \mathcal{R} b$ if $aS^1 = bS^1$, $a \mathcal{L} b$ if $S^1a = S^1b$, $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$, $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$ (see **[1, 7]**). The semigroups we consider turn out to have the property that a power of each element lies in a subgroup and hence by **[10]**, $\mathcal{J} = \mathcal{D}$. A $\mathcal{J}$-class with an idempotent is called *regular*. We let $\mathcal{U}(S)$ denote the partially ordered set of all regular $\mathcal{J}$-classes of S. If $a \in S$, then J_a, H_a will denote the $\mathcal{J}$-class and $\mathcal{H}$-class of a, respectively. If $X \subseteq S$, then $E(X) = E(S) \cap X$ and $\langle X \rangle$ is the subsemigroup of S generated by X. S is *regular* if $a \in aSa$ for all $a \in S$. A monoid S with group of units G is *unit regular* if for all $a \in S$ there exist $e \in E(S)$ and $x \in G$ such that $a = ex$. Clearly any submonoid of a unit regular monoid, containing the group of units, is also unit regular.

Let K be an algebraically closed field, $K^* = K \setminus \{0\}$. We let $\mathcal{M}_n(K)$ denote the monoid of all $n \times n$ matrices over K. We consider $\mathcal{M}_n(K)$ with the Zariski topology on it (see **[8]**). By an *algebraic monoid S* we mean a closed submonoid of $\mathcal{M}_n(K)$. The irreducible component of 1 in the Zariski topology of S will be denoted by S^c. If $S = S^c$, then we say that S is a *connected monoid*. Let S be a connected monoid with zero and group of units G. Let T be a maximal torus of G. Thus $E(\overline{T})$ consists of the 'diagonal' idempotents in S. Then $E(\overline{T})$ is a finite relatively complemented lattice

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and $\mathscr{U}(S)$ is a finite lattice. Moreover $\dim T$ = the length of any maximal chain in $E(\overline{T})$ = the length of any maximal chain in $\mathscr{U}(S)$ = the length of any maximal chain in $E(S)$. We call this number the *rank* of S. Let $X, Y \subseteq S$. Then the *centralizer* in Y of X is $C_Y(X) = \{a \in Y | ax = xa \text{ for all } x \in X\}$ and the *normalizer* in G of X is $N_G(X) = \{a \in G | a^{-1}Xa = X\}$. Let $\Gamma \subseteq E(\overline{T})$. Then the *right centralizer* of Γ in G is $C_G^r(\Gamma) = \{a \in G | ae = eae \text{ for all } e \in \Gamma\}$. The *left centralizer* of Γ in G is $C_G^l(\Gamma) = \{a \in G | ea = eae \text{ for all } e \in \Gamma\}$. The groups $C_G^l(\Gamma)$, $C_G^r(\Gamma)$, $C_G(\Gamma)$ are all connected by [**18**, Corollary 4.2]. Let $e \in E(S)$. Then $S_e = \{a \in S | ea = ae = e\}^c$, $G_e = G \cap S_e$ (see [**17**, **18**] for further details). We refer to [**8**, **28**] for the basic notions in the theory of algebraic groups. In particular, $W = N_G(T)/C_G(T)$ is the *Weyl group* of G.

Let S be a connected algebraic monoid with zero and group of units G. If $\operatorname{char} K = 0$, then the author [**21**] showed that G is a reductive group if and only if S is a regular semigroup. This result was extended to arbitrary characteristic by Renner [**24**]. Renner uses some results in algebraic geometry such as Zariski's main theorem to prove this result. Because of the central importance of this result in the theory of linear algebraic monoids, we present here an alternate proof.

THEOREM 1.1 [**21**, **24**]. *Let S be a connected algebraic monoid with zero and group of units G. Then G is a reductive group if and only if S is a* (*unit*) *regular monoid.*

PROOF. If S is a regular semigroup, then by [**18**, Theorem 2.11], G is a reductive group. So assume that G is a reductive group. We prove by induction on $\dim S$ that S is regular. If $e \in E(S)$, then G_e and H_e are reductive groups by [**21**, Theorem 2.3]. Thus $e \neq 0$ implies that S_e is regular, and $e \neq 1$ implies that eSe is regular. Let $S \subseteq \mathscr{M}_n(K)$, $\dim S = p$. Consider det: $S \to K$ where 'det' denotes the matrix determinant. Let $Y = S \setminus G = \det^{-1}(0)$. Then $Y = SYS$. So by [**8**, Theorem 4.1], every irreducible component of Y is an ideal of S by dimension $p - 1$. Let Y_0 be an irreducible component of Y. Then $\dim Y_0 = p - 1$. We first show that Y_0 is not nil. For suppose otherwise. Let B, B^- be opposite Borel subgroups of G with respect to some maximal torus T. By [**8**, Proposition 27.2] we can choose an appropriate change of basis such that every element of B is upper triangular and every element of B^- is lower triangular. Thus the same is true for $\overline{B}$ and $\overline{B}^-$. Thus every element of $\overline{B} \cap \overline{B}^-$ is a diagonal matrix. Let W denote the Weyl group of G. So $B^- = \sigma^{-1}B\sigma$ for some $\sigma \in W$. Now B^-B and hence $B\sigma B$ is an open subset of G by [**8**, Proposition 28.5]. So $X = G \setminus B\sigma B$ is a closed subset of G. Thus $\dim X \leqslant p - 1$. We claim that $Y_0 \subseteq \overline{X}$. So let $a \in Y_0$, $a \notin \overline{X}$. Since $0 \in \overline{B} \subseteq \overline{X}$, $a \neq 0$. By the Bruhat decomposition [**8**, Theorem 28.3], G is the disjoint union of $B\theta B$ ($\theta \in W$). Thus $a \notin \overline{B\theta B}$ for any $\theta \in W$, $\theta \neq \sigma$. So $a \notin \overline{B}B\theta B \cup B\theta B\overline{B}$ for $\theta \in W$, $\theta \neq \sigma$. Also by [**29**, p. 68 or **23**, Proposition 3.4.1] $S = G\overline{B} = \overline{B}G$. Hence $a \in \overline{B}B\sigma B \cap B\sigma B\overline{B} \subseteq \overline{B}\sigma B \cap B\sigma\overline{B}$. Let $\sigma = gT$, $g \in N_G(T)$. Then there exist $b_1, b_2 \in B$, $u_1, u_2 \in \overline{B}$ such that $a = u_1 g b_1 = b_2 g u_2$. So $u_2 b_1^{-1} = g^{-1} b_2^{-1} u_1 g \in \overline{B} \cap \overline{B}^-$. Thus $u_2 b_1^{-1}$ is a diagonal matrix. But $u_2 b_1^{-1} = g^{-1} b_2^{-1} a b_1^{-1} \in Y_0 \setminus \{0\}$. This contradicts the assumption that Y_0 is nil. Thus $Y_0 \subseteq \overline{X}$. Let $X_1, \ldots, X_r$ be the irreducible components of X_i. Then $Y_0 \subseteq \overline{X}_i$ for some i. Since $\dim Y_0 = p - 1$ and $\dim X_i \leqslant p - 1$ we see that $Y_0 = \overline{X}_i$. This is a contradiction since $Y_0 \subseteq S \setminus G$, $X_i \subseteq G$. Therefore Y_0 is not nil. So there exists $e \in E(Y_0)$,

$e \neq 0$. Choose e such that J_e is maximal in $\mathscr{U}(Y_0)$. Let $Y_1 = \{y \in Y_0 | y \nmid e\}$. Then Y_1 is closed by [**13**, Lemma 2.1]. Let $u \in Y \setminus Y_1$. Then $u | e$. By [**21**, Theorem 1.4], $u \in \overline{GS_eG}$. Hence u is regular. By the maximality of J_e, $u \mathscr{J} e$. Thus $u \in SeS$. Hence $Y_0 = \overline{SeS} \cup Y_1$. Since $e \notin Y_1$, and since Y_0 is irreducible, we see that $Y_0 = \overline{SeS}$. Now $e \in \Gamma$ for some maximal chain in $E(S)$. Let $B_1 = C_G^r(\Gamma)$, $B_2 = C_G^l(\Gamma)$. Then B_1, B_2 are Borel subgroups of G by [**18**, Theorem 4.5]. Clearly, $B_1 e = eB_1 e$, $eB_2 = eB_2 e$. Thus $\underline{B_1 eSe B_2 = eSe.}$ So by [**29**, p. 68], $GeSeG$ is closed in S. Since $GeG \subseteq GeSeG$, we see that $Y_0 = \overline{SeS} = \overline{GeG} \subseteq GeSeG$. Since eSe is regular, so is Y_0. Thus $Y = S \setminus G$ is regular. This proves the theorem.

Let S be a connected regular monoid with zero and group of units G. Let T be a maximal torus of G. If Γ is a maximal chain in $E(\overline{T})$, then by [**18**] $C_G^r(\Gamma)$ is a Borel subgroup of G. However, the map: $\Gamma \to C_G^r(\Gamma)$ is not one-to-one. To obtain a better correspondence, the author [**18–20**] considers cross-section lattices. A subset Λ of $E(\overline{T})$ is a *cross-section lattice* if: (1) $|\Lambda \cap J| = 1$ for all $J \in \mathscr{U}(S)$, and (2) for all $e, f \in \Lambda$, $J_e \geqslant J_f$ implies $e \geqslant f$. Thus a cross-section lattice is a diagonal idempotent cross-section of $\mathscr{J}$-classes preserving the $\mathscr{J}$-class ordering. The author [**18–20**] has shown that cross-section lattices correspond one-to-one with Borel subgroups of G containing T. The correspondence is as follows:

$$\Lambda \to B = C_G^r(\Lambda),$$
$$B \to \Lambda = \{e \in E(\overline{T}) | ae = eae \text{ for all } a \in B\}$$
$$= \{e \in E(\overline{T}) | \text{for all } f \in E(S),\ e \mathscr{R} f \text{ implies } f \in \overline{B}\}.$$

Let C denote the center of the reductive group G. Then by [**8**] $G = CG_0$, where G_0 is a semisimple group (the commutator subgroup of G). We will say that G is *nearly semisimple* if $\dim C = 1$. In such a case Renner [**25**, **26**] calls the monoid S semisimple. We avoid the terminology in this paper because the term 'semisimple' has a completely different meaning in abstract semigroup theory. With the additional assumption that S is normal, Renner classifies these monoids geometrically. Let $X(T)$, $X(\overline{T})$ denote the character group and character monoid of T and $\overline{T}$, respectively. Let Φ denote the root system of G. Then Renner [**26**] shows that the map $S \to (X(T), \Phi, X(\overline{T}))$ classifies normal 'semisimple' monoids up to isomorphism. Since $X(T), \Phi, X(\overline{T})$ are all geometrical objects, this classification may be thought of as being geometrical. This theorem of Renner is quite deep, having many important consequences. For example, Renner uses his classification to show that any such monoid S has an involution fixing $\overline{T}$. A key step in the proof of Renner's classification theorem is the generalization of Chevalley's big cell from groups to monoids. Renner uses cross-section lattices in the construction of his big cell.

Let S be a connected regular monoid with zero and let $\hat{S}$ be its normalization (see [**23**]). Then there exists a finite homomorphism ϕ: $\hat{S} \twoheadrightarrow S$. The term 'finite' is being used in the sense of algebraic geometry [**8**, §4.2]. Finite homomorphisms are of abstract semigroup theoretic interest also, because of the following result of Renner [**23**, Propositions 3.4.13, 4.1.6].

THEOREM 1.2 (RENNER). *Let* $\phi\colon S \to S'$ *be a homomorphism between connected regular monoids with zero such that* $\overline{\phi(S)} = S'$. *Then* ϕ *is finite if and only if it is idempotent separating.*

Following Renner [**26**] we say that a connected regular monoid S with zero is $\mathcal{J}$*-irreducible* if $\mathcal{U}(S)\setminus\{0\}$ has a minimum element. The following result is due to Renner [**26**, Corollary 8.3.3].

THEOREM 1.3 (RENNER). *Let S be a connected regular monoid with zero. Then S is* $\mathcal{J}$*-irreducible if and only if S has a finite (i.e. idempotent separating) irreducible matrix representation. In particular the group of units of such a monoid is nearly semisimple.*

We will say that $G = CG_0$ is *nearly simple* if $\dim C = 1$ and G_0 is *almost simple*, i.e. G_0 has no nontrivial normal closed connected subgroups. We denote by $\mathrm{Ren}(S)$ the finite inverse monoid $\overline{N_G(T)}/T$. Renner [**27**] has used this monoid to obtain a Bruhat decomposition for S. $\mathrm{Ren}(S)$ is a fundamental (i.e. has no nontrivial idempotent separating congruences) unit regular monoid with group of units $W = N_G(T)/T$ and idempotent set $E(\overline{T})$.

2. Abstract properties. Let S be a regular semigroup. We let $S/\mathcal{R}$ denote the partially ordered set of all principal right ideals of S under inclusion. Then $S/\mathcal{R}$ can be identified with the partially ordered set of all $\mathcal{R}$-classes of S.

THEOREM 2.1. *Let S be a connected regular monoid with zero and group of units G. Then* $S/\mathcal{R}$ *is a relatively complemented, complete lattice with all maximal chains having the same finite length equal to* rank S.

PROOF. By [**14**] all maximal chains in $E(S)$ have the same finite length. Thus the same is true for $S/\mathcal{R}$. Let $eS, fS \in S/\mathcal{R}$ where $e, f \in E(S)$. By [**18**, Theorem 4.6] $C_G^r(e)$ and $C_G^r(f)$ are parabolic subgroups of G. So by [**8**, Corollary 28.3] $C_G^r(e) \cap C_G^r(f)$ contains a maximal torus T of G. By [**14**] there exist $a \in C_G^r(e)$, $b \in C_G^r(f)$ such that $e' = a^{-1}ea$, $f' = b^{-1}fb \in \overline{T}$. Then $eS = e'S$, $fS = f'S$, $e'f' = f'e'$. So $eS \cap fS = e'f'S$. Since $S/\mathcal{R}$ has a maximum element, and since all maximal chains in $S/\mathcal{R}$ have the same finite length, it follows that $S/\mathcal{R}$ is a complete lattice.

Let $e, f, h \in E(S)$ such that $eS \supseteq fS \supseteq hS$. Without loss of generality we can assume that $e \geqslant f \geqslant h$. By [**14**] there exists a maximal torus T of G such that $e, f, h \in E(\overline{T})$. Since $E(\overline{T})$ is relatively complemented [**13**], there exists $f' \in E(\overline{T})$ such that $e \geqslant f' \geqslant h$ and $ff' = h$. Then $eS \supseteq f'S \supseteq hS$ and $fS \cap f'S = hS$. It follows that $S/\mathcal{R}$ is relatively complemented.

Let S be a regular semigroup, $a \in S$. Then $V(a) = \{x \in S \mid axa = a,\ xax = x\}$ denote the set of *inverses* of a in S. Let $e, f \in E(S)$. Of crucial importance in the theory of regular semigroups is the *sandwich set* $\mathcal{S}(e, f) = V(ef) \cap fSe$ which is a rectangular band (see [**11**]).

THEOREM 2.2. *Let S be a connected regular monoid with zero and group of units G. Let* $e, f \in E(S)$. *Then*

(i) $C_G^r(f) \cap C_G^l(e)$ *is a connected group and* $fSe \subseteq \overline{C_G^r(f) \cap C_G^l(e)}$.

(ii) *If* $h \in \mathcal{S}(e, f)$, *then* $\mathcal{S}(e, f) = \{x^{-1}hx \mid x \in C_G^r(f) \cap C_G^l(e)\}$.

(iii) $\mathscr{S}(e, f) = \{e'f' | e', f' \in E(\bar{T})$ *for some maximal torus* T *of* G *such that* $e \mathscr{L} e', f \mathscr{R} f'\} = \{e'f' | e', f' \in E(S), e \mathscr{L} e', f \mathscr{R} f', e'f' = f'e'\}$.

PROOF. By [**18**, Theorem 4.6], $C_G^r(f)$, $C_G^l(e)$ are parabolic subgroups of G. By [**8**, Corollary 28.3], there exists a maximal torus T of G contained in $C_G^r(f) \cap C_G^l(e)$. By [**14**], there exist $a \in C_G^l(e)$, $b \in C_G^r(f)$ such that $e' = a^{-1}ea$, $f' = b^{-1}fb \in E(\bar{T})$. Then $f \mathscr{R} f'$, $e \mathscr{L} e'$. Hence by [**15**, Theorems 1, 9], $C_G^l(e) = C_G^l(e')$, $C_G^r(f) = C_G^r(f')$. Since $f' \in \bar{T} \subseteq \overline{C_G^l(e')}$, it follows from [**18**, Theorem 4.1] that $G' = C_G^l(e) \cap C_G^r(f)$ is a connected group. Let $S' = \bar{G'}$, $S_1 = \overline{C_G^l(e')}$. By [**15**, Theorem 1] $Se' \subseteq S_1$. By the same theorem, $f'S_1 \subseteq S'$. Clearly $f'Se' \subseteq f'S_1$ and $fSe = f'Se'$. Thus $fSe \subseteq S'$. Since $e'f' = f'e'$, we see that $e'f' \in \mathscr{S}(e, f)$. Let $x \in G'$. Then $x^{-1}e'f'x = e''f''$ where $e'' = x^{-1}e'x$, $f'' = x^{-1}f'x \in x^{-1}Tx$, $e \mathscr{L} e''$, $f \mathscr{R} f''$. Hence $e''f'' \in \mathscr{S}(e, f)$. Finally let $h \in \mathscr{S}(e, f)$. Since $\mathscr{S}(e, f)$ is a rectangular band, and since $\mathscr{S}(e, f) \subseteq fSe \subseteq S'$, we see that $h \mathscr{J} e'f'$ in S'. By [**15**, Theorem 9], there exists $x \in G'$ such that $h = x^{-1}e'f'x$. This proves the theorem.

COROLLARY 2.3. *Let* S *be a connected regular monoid with zero and group of units* G. *Let* $a \in S$. *Then there exists a maximal torus* T *of* G, $e, f \in E(\bar{T})$ *such that* $e \mathscr{R} a \mathscr{L} f$. *In particular the number of conjugacy classes of the* $\mathscr{H}$-*classes of* S *is finite.*

Let S be a regular semigroup. Then S is *V-regular* if $V(ab) \subseteq V(b)V(a)$ for all $a, b \in S$. S is *strongly V-regular* if for all $e, f \in E(S)$, $h \in \mathscr{S}(e, f)$, there exist $e', f' \in E(S)$ such that $e \mathscr{L} e'$, $f \mathscr{R} f'$, $h \in \mathscr{S}(f', e')$. Nambooripad and Pastijn [**12**] show that every strongly V-regular semigroup is V-regular. They also show that many classes of naturally arising regular semigroups are strongly V-regular. We add to this list. The following theorem is immediate from Theorem 2.2.

THEOREM 2.4. *Every connected regular monoid with zero is strongly V-regular.*

Let S be a regular semigroup. Then Fitz-Gerald [**5**] showed that $\langle E(S) \rangle$ is a regular semigroup. The determination of $\langle E(S) \rangle$ represents an important step in the study of S (see [**6**, **11**]).

Problem 2.5. Let S be a connected regular monoid with zero. Determine $\langle E(S) \rangle$.

We solve the above problem in the special case when $S \setminus G$ has a maximum $\mathscr{J}$-class. Erdos [**4**] (see also Dawlings [**2**, **3**]) showed that the singular matrices in $\mathscr{M}_n(K)$ can always be written as products of idempotent matrices.

LEMMA 2.6. *Let* S *be a connected regular monoid with zero*, S_1 *a closed connected submonoid of* S. *If* $E(S) = E(S_1)$, *then* $S = S_1$.

PROOF. We prove the lemma by induction on dim S. Let $S \subseteq \mathscr{M}_n(K)$. Consider det: $S \to K$ where 'det' denotes the determinant. Let V be an irreducible component of $\det^{-1}(0)$. Then V is clearly an ideal of S and by [**8**, Theorem 4.1], $\dim V = \dim S - 1$. Since S is regular, $V = SeS$ for some $e \in E(S)$, $e \neq 1$. Now $eS_1e \subseteq eSe$, $E(eS_1e) = E(eSe)$, and eSe is regular. By the induction hypothesis, $eSe = \overline{eS_1e}$. Let $X = \{f \in E(S) | e \mathscr{R} f\}$, $Y = \{f \in E(S) | e \mathscr{L} f\}$. By [**15**, Theorem 14], $\dim \overline{S_1eS_1} = \dim X + \dim Y + \dim eS_1e$, $\dim SeS = \dim X + \dim Y + \dim eSe$. Thus $\dim SeS = \dim S_1eS_1 < \dim S_1$. So $\dim S \leqslant \dim S_1$ and $S = S_1$.

THEOREM 2.7. *Let S be a connected regular monoid with zero and group of units G. Suppose $S \setminus G$ has a maximum $\mathscr{J}$-class. Then $S \setminus G \subseteq E(S)^m = \langle E(S) \rangle$ for some $m \in Z^+$.*

PROOF. Let J denote the maximum $\mathscr{J}$-class of $S \setminus G$, $E_0 = E(S) \cap J$. Then E_0 is closed and irreducible by [**15**, Theorem 8]. We have the product maps $E_0 \times \cdots \times E_0 \to \overline{E_0^k}$. Thus each $\overline{E_0^k}$ is also irreducible. Clearly

$$E_0 \subseteq \overline{E_0^2} \subseteq \overline{E_0^3} \subseteq \cdots.$$

So there exists $p \in Z^+$, $p \leqslant \dim S$ such that $\overline{E_0^p} = \overline{E_0^q}$ for all $q \in Z^+$, $q \geqslant p$. Let $S_1 = \overline{E_0^p}$. Since $E(\overline{T})$ is relatively complemented for any maximal torus T of G, it follows that $E(S) \setminus \{1\} \subseteq \langle E_0 \rangle$. Thus $E(S_1) = E(S) \setminus \{1\}$. Let $e \in E_0$. Then $E(eS_1e) = E(eSe)$. By Lemma 2.6, $eSe = eS_1e$. Let $X = E_0 \times \cdots \times E_0$, the p-fold direct product. Define ϕ: $X \to eSe$ as $\phi(e_1, \ldots, e_p) = ee_1 \cdots e_pe$. Thus $\phi(X) = eE_0^pe$. So

$$\overline{\phi(X)} = \overline{eE_0^pe} = eS_1e = eSe.$$

By [**8**, Theorem 4.4], $\phi(X)$ is constructible and hence contains a nonempty open subset U of eSe. Let H denote the group of units of eSe. Then H is an open subset of eSe. Thus $U_1 = U \cap H$ is a nonempty open subset of H. By [**8**, Lemma 7.4], $U_1^2 = H$. Thus $H \subseteq E_0^{2p+3}$.

Now let $a \in J$. Then there exist $e, f \in E_0$ such that $e \mathscr{R} a \mathscr{L} f$. By [**16**, Lemma 1.12] there exist $e_1, f_1 \in E_0$ such that $e \mathscr{R} e_1 \mathscr{L} f_1 \mathscr{R} f$. Then $e_1 f \mathscr{H} a$. So $a \in H_a = H_e e_1 f \subseteq E_0^{2p+5}$. Finally, let $a \in S \setminus G$. Then $a \mathscr{R} f$ for some $f \in E(S)$, $f \neq 1$. Now $f' \geqslant f$ for some $f' \in E_0$ (see [**15**]). By [**15**], $a = fg$ for some $g \in G$. So $a = f(f'g)$, $f'g \in J \subseteq E_0^{2p+5}$. Thus $a \in E_0^{2p+6}$. This proves the theorem.

EXAMPLE 2.8. Let $S = \{A \otimes B | A, B \in \mathscr{M}_2(K)\}$. Then for $B \in \mathrm{GL}(2, K)$, $\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right) \otimes B$ cannot be written as a product of idempotents in S unless B is a scalar.

EXAMPLE 2.9. Let $S = \overline{KG_0}$ where $G_0 = \{A \otimes (A^{-1})^t | A \in \mathrm{SL}(3, K)\}$. Then $\mathscr{U}(S)$ is the same as in Example 2.8. However $S \setminus G$ is generated by idempotents where $G = K^*G_0$. Note that $S \setminus G$ has two maximal $\mathscr{J}$-classes.

Problem 2.10. Let S be a connected regular monoid with zero and group of units G. Find necessary and sufficient conditions for $S \setminus G$ to be idempotent generated.

Let S be a regular semigroup. The *fundamental congruence* μ on S is the largest congruence on S contained in $\mathscr{H}$. If μ equals the equality, then S is said to be *fundamental* (see [**6**, **7**, **11**]). We now show that when S is a connected regular monoid with zero, then the extension S of S/μ is a special 'central' extension and hence susceptible to a cohomological approach [**9**].

THEOREM 2.11. *Let S be a connected regular monoid with zero and group of units G. Consider the natural (semigroup) homomorphism ϕ: $S \to S/\mu$. Then for all $e \in E(S)$, $\phi^{-1}\phi(e)$ is the center of H_e.*

PROOF. Let $e \in E(S)$. Then by Hall [**6**, Corollary 6], μ restricted to eSe is the fundamental congruence on eSe. Thus without loss of generality we can assume that

$e = 1$. Let C denote the center of G. Let $x \in G$ such that $x \mu 1$. Then for all $f \in E(S)$, $fx \mathcal{H} f \mathcal{H} xf$. So $fx = xf$ for all $f \in E(S)$. By [**18**, Theorem 4.5], $x \in B$ for all Borel subgroups B of G. It follows [**8**, p. 162, Exercise 2] that $x \in C$. Now the congruence θ defined by $a \theta b$ if $a \in Cb$, is idempotent separating and hence $\theta \subseteq \mu$. Thus $c \mu 1$ for all $c \in C$.

Since all maximal chains in $E(S)$ have the same finite length, we can define an obvious height function ht on $E(S)$ such that $\mathrm{ht}(1) = \operatorname{rank} S$ and $\mathrm{ht}(0) = 0$.

THEOREM 2.12. *Let S be a connected regular monoid with zero and group of units G. Let $e, e' \in E(S)$ with $\mathrm{ht}(e) = \mathrm{ht}(e') = p > 0$. Then there exist $e = e_0, e_1, \ldots, e_k = e'$, $f_1, \ldots, f_k \in E(S)$ such that $\mathrm{ht}(e_i) = p$, $\mathrm{ht}(f_i) = p - 1$, $e_i > f_i$, $e_{i-1} > f_i$, $i = 1, \ldots, k$.*

PROOF. By [**16**, Lemma 1.15], the theorem is true when G is a torus. Since any idempotent in S is in the closure of a maximal torus and since any two maximal tori are conjugate, we are reduced to the case when $e \mathcal{J} e'$. By [**16**, Lemma 1.12] we are then further reduced to the case when $e \mathcal{R} e'$. There exists a cross-section lattice Λ of S such that $e \in \Lambda$. Let $B = C_G^r(\Lambda)$, $T = C_G(\Lambda)$. Then B is a Borel subgroup of G, T is a maximal torus of G, $T \subseteq B$, $e \in E(\overline{T})$, $e' \in \overline{B}$. By [**20**, Theorem 1.2], there exists a cross-section lattice $\Lambda' \subseteq E(\overline{T})$ such that $B = C_G^l(\Lambda')$. There exists $u \in \Lambda'$ such that $e \mathcal{J} u$ in S. Then $\mathrm{ht}(e) = \mathrm{ht}(u)$. So there exist $e = u_0, u_1, \ldots, u_k = u$, $v_1, \ldots, v_k \in E(\overline{T})$ such that $\mathrm{ht}(u_i) = p$, $\mathrm{ht}(v_i) = p - 1$, $u_i > v_i$, $u_{i-1} > v_i$, $i = 1, \ldots, k$. By [**18**, Proposition 4.3], $e, e', v_1 \in \overline{C_B^r(e, v_1)}$. There exists $x \in C_B^r(e, v_1)$ such that $e' = x^{-1}ex$. Let $v_1' = x^{-1}v_1x \in E(\overline{B})$. Then $v_1 \mathcal{R} v_1'$, $e' \geqslant v_1'$. Now $v_1, v_1', u_2 \in \overline{C_B^r(u_2, v_1)}$. There exists $y \in C_B^r(u_2, v_1)$ such that $v_1' = y^{-1}v_1y$. Let $u_2' = y^{-1}u_2y \in E(\overline{B})$. Then $u_2' \geqslant v_1'$, $u_2 \mathcal{R} u_2'$. Continuing, we find $e' = u_0'$, $u_1', \ldots, u_k' = u'$, $v_1', \ldots, v_k' \in E(\overline{B})$ such that $\mathrm{ht}(u_i') = p$, $\mathrm{ht}(v_i') = p - 1$, $u_i' > v_i'$, $u_{i-1}' > v_i'$, $u_i \mathcal{R} u_i'$, $v_i \mathcal{R} v_i'$, $i = 1, \ldots, k$. In particular, $u \mathcal{R} u'$ and $u' = zuz^{-1}$ for some $z \in C_B^r(u)$. Since $B \subseteq C_G^l(u)$, we see that $u = u'$. This proves the theorem.

3. $\mathcal{J}$-linear monoids. The monoid $\mathcal{M}_n(K)$ has the very nice property that the $\mathcal{J}$-classes are linearly ordered. We will call such a monoid *$\mathcal{J}$-linear*. We start by obtaining a useful test for $\mathcal{J}$-linearity.

THEOREM 3.1. *Let S be a connected regular monoid with zero and group of units G. Let $\operatorname{rank} S = m$. Then the following conditions are equivalent.*

(i) *S if $\mathcal{J}$-linear.*

(ii) *There exists $a \in S$ such that $a^m = 0$, $a^{m-1} \neq 0$.*

(iii) *There exists $a \in \operatorname{Ren}(S)$ such that $a^m = 0$, $a^{m-1} \neq 0$.*

PROOF. We first show that (ii) $\Rightarrow$ (i). We proceed by induction on $\dim S$. If $\dim S = 1$, then $|\mathcal{U}(S)| = 2$ and hence S is $\mathcal{J}$-linear. So let $\dim S > 1$. Let $a \in S$ such that $a^m = 0$, $a^{m-1} \neq 0$. Let J_i denote the $\mathcal{J}$-class of a^i. Then $J_i \neq J_k$ for $1 \leqslant i < k \leqslant m$ (recall that S is a matrix semigroup). Since S is regular, each J_i is regular. Thus $\{G > J_1 > J_2 > \cdots > J_m = 0\}$ is a maximal chain in $\mathcal{U}(S)$. By Corollary 2.3, there exists a maximal torus T of G, $e, f \in E(\overline{T})$ such that $e \mathcal{R} a \mathcal{L} f$. So there exist $x, y \in G$ such that $a = ex = yf$. Now $\operatorname{rank} eSe = m - 1$, $(exe)^{m-1} = 0$, $(exe)^{m-2} \neq 0$. By the induction hypothesis, eSe is $\mathcal{J}$-linear. Hence by [**17**, §4], the interval $[0, J_1] = \{0, J_{m-1}, \dots, J_1\}$. Now $a^2 = yfex \mathcal{J} ef$. So $ef = fe \in J_2$, $e \neq f$. By [**16**, Proposition 1.14(6)], the interval $[J_2, G] = \{J_2, J_1, G\}$. It then follows from [**16**, Lemma 1.15] that $\mathcal{U}(S) = \{0, J_{m-1}, \dots, J_1, G\}$. Hence S is $\mathcal{J}$-linear.

That (iii) $\Rightarrow$ (ii) is obvious. We are therefore left with showing (i) $\Rightarrow$ (iii). So assume that S is $\mathcal{J}$-linear. Let T be a maximal torus of G. If $e \in E(\overline{T})$, $\sigma \in W$, then let $e^\sigma = \sigma^{-1} e \sigma$. Let $\{1 = e_0 > e_1 > \cdots > e_m = 0\}$ be a maximal chain in $E(\overline{T})$. Since $E(\overline{T})$ is relatively complemented, there exist $1 = f_0, f_1, \dots, f_m = 0 \in E(\overline{T})$ such that $e_{i+1} \leqslant f_i \leqslant f_{i-1}$, $e_i f_i = e_{i+1}$, $i = 1, \dots, m-1$. Since S is $\mathcal{J}$-linear, $e_i \mathcal{J} f_i$, $i = 1, \dots, m-1$. Thus by [**17**, §4], there exist $\sigma_i \in W$, $i = 1, \dots, m-1$, such that $e_i^{\sigma_i} = f_i$, $f_j^{\sigma_i} = f_j$ for $j < i$, $e_k^{\sigma_i} = e_k$ for $k > i$. Let $\sigma = \sigma_1 \cdots \sigma_{m-1}$. Then

$$e_i^\sigma = \left(e_i^{\sigma_1 \cdots \sigma_{i-1}}\right)^{\sigma_i \cdots \sigma_{m-1}} = e_i^{\sigma_i \cdots \sigma_{m-1}} = f_i^{\sigma_{i+1} \cdots \sigma_{m-1}} = f_i, \qquad i = 1, \dots, m-1.$$

So $e_i e_i^\sigma = e_{i+1}$, $i = 1, \dots, m-1$. Thus

$$e_1 e_1^\sigma = e_2, e_1 e_1^\sigma e_1^{\sigma^2} = \left(e_1 e_1^\sigma\right)\left(e_1 e_1^\sigma\right)^\sigma = e_2 e_2^\sigma = e_3, \dots, e_1 e_1^\sigma e_1^{\sigma^2} \cdots e_1^{\sigma^k} = e_{k+1}, \dots.$$

Now $e_1 e_1^\sigma \cdots e_1^{\sigma^k} = (e_1 \sigma^{-1})^{k+1} \sigma^{k+1}$. Thus $(e_1 \sigma^{-1})^m = 0$, $(e_1 \sigma^{-1})^{m-1} \neq 0$. This proves the theorem.

REMARK 3.2. Let G_0 be an almost simple group of type A_l, B_l or C_l with the representation given in Humphreys [**8**, §7.2]. Let $S = \overline{KG_0}$. Then Theorem 3.1 can be used to show that S is $\mathcal{J}$-linear. The $\mathcal{J}$-linearity of S does indeed very much depend on the particular representation of G_0. For example, let $G_0 = \{A \otimes (A^{-1})^t | A \in \mathrm{SL}(3, K)\}$. Then $S = \overline{KG_0}$ is certainly not $\mathcal{J}$-linear.

THEOREM 3.3. *Let S be a connected regular $\mathcal{J}$-linear monoid with zero and group of units G. Then G is a nearly simple group of type A_l, B_l, C_l, F_4 or G_2.*

PROOF. Let R denote the radical of G. Then R lies in the center of G and $G = RG_0$ where G_0 is a semisimple group. By Theorem 1.3, $\dim R = 1$. Suppose that G_0 is not almost simple. Then by [**8**, Theorem 27.5], $G_0 = G_1 G_2$ where G_1, G_2 are nontrivial semisimple subgroups of G_0 and G_1 centralizes G_2. Let T be a maximal torus of G. Then $T_1 = T \cap G_1$, $T_2 = T \cap G_2$ are maximal tori of G_1 and G_2, respectively. Let e_i be a nonzero minimal idempotent of $\overline{RT_i}$, $i = 1, 2$. Let J_i denote the $\mathcal{J}$-class of e_i in S, $i = 1, 2$. By symmetry suppose that $J_1 \geqslant J_2$. Then $e_1 \geqslant e_2'$ for some $e_2' \in E(J_2)$. There exists $x \in G$ such that $x^{-1} e_2 x = e_2'$. Now $x = x_1 x_2$ for some $x_1 \in G_1$, $x_2 \in RG_2$. Since $e_2 \in S_2 = \overline{RG_2}$ and G_1 centralizes S_2, we see that $e_2' = x_2^{-1} e_2 x \in S_2$. Now since $\overline{RG_1}$ is nearly semisimple, we see by [**18**, Theorems 2.3, 2.13] that there exists $e_1' \in E(\overline{RT_1})$, $e_1 \neq e_1'$ such that $e_1 \mathcal{J} e_1'$ in $S_1 = \overline{RG_1}$. Thus $e_1' = y^{-1} e_1 y$ for some $y \in G_1$. Then $e_1' \geqslant y^{-1} e_2' y = e_2'$. So $0 = e_1 e_1' \geqslant e_2'$, a contradiction. This shows that G is nearly simple.

Let T be a maximal torus of G and let $\Lambda = \{0 < e_1 < \cdots < e_k < 1\}$ be a maximal chain in $E(\overline{T})$. Then Λ is a cross-section lattice. Let $B = C_G^r(\Lambda)$. For $i = 1, \dots, k$, there exists $\sigma_i \in W$, $\sigma_i^2 = 1$ such that $e_i^{\sigma_i} \neq e_i$, $e_j^{\sigma_i} = e_j$ for $j \neq i$. By [**20**, Corollary 2.8], $\mathcal{F} = \{\sigma_i | i = 1, \dots, k\}$ is the set of simple reflections with respect to B. Let $\mathcal{F}_i = \mathcal{F} \setminus \{\sigma_i\}$, $P_i = C_G^r(e_i) \supseteq B$. Then $W_i = \langle \mathcal{F}_i \rangle$ is the Weyl group of P_i and hence $P_i = BW_iB$. Clearly $C_G(e_i)$ is a Levi factor of P_i. Let H_i denote the $\mathcal{H}$-class of

e_i, $G_i = G_{e_i}$. Then H_i, G_i are the groups of units of e_iSe_i and S_{e_i}, respectively. By [**17**, Theorem 4.6], e_iSe_i, S_{e_i} are $\mathscr{J}$-linear monoids. Thus H_i, G_i are nearly simple groups. By [**15**, Theorem 4], the homomorphism ψ: $C_G(e_i) \to H_i$ given by $\psi(a) = e_ia$ is surjective. Clearly G_i is the identity component of the kernel of ψ. It follows that for $i \neq 1, k$, W_i is reducible. Hence G is not of type D_l, E_6, E_7 or E_8. This proves the theorem.

THEOREM 3.4. *Let S be a connected regular $\mathscr{J}$-linear monoid with zero and group of units G. Let T be a maximal torus of G. Then*:

(i) *Cross-section lattices are just the maximal chains in $E(\overline{T})$.*

(ii) *The Weyl group of G is isomorphic to the abstract automorphism group of $E(\overline{T})$.*

(iii) Ren(S) *is isomorphic to the Munn semigroup of $E(\overline{T})$.*

PROOF. (i) is clear and (ii) follows from [**18**, Theorem 3.9]. We now prove (iii). The fundamental congruence μ on the inverse monoid $\overline{N_G(T)}$ is given by $a\,\mu\, b$ if and only if $b \in Ta$. Thus Ren(S) is isomorphic to the submonoid of the Munn semigroup of $E(\overline{T})$ generated by W and $E(\overline{T})$. It therefore suffices to show that any isomorphism between two principal ideals of $E(\overline{T})$ extends to an automorphism of $E(\overline{T})$. Let ϕ: $eE(\overline{T}) \cong fE(\overline{T})$ where $e, f \in E(\overline{T})$. Then $e \mathscr{J} f$ and so $\sigma^{-1}e\sigma = f$ for some $\sigma \in W$. Thus we are reduced to the case when $e = f$. Then ϕ is an automorphism of $eE(\overline{T}) = E(\overline{eT})$ and eSe is $\mathscr{J}$-linear. So by (ii) ϕ belongs to the Weyl group eW of eSe. Hence ϕ extends to an automorphism of $E(\overline{T})$. This proves the theorem.

4. Conjugacy classes. We start with the following preliminary result.

THEOREM 4.1. *Let S be a connected regular monoid with zero and group of units G,* rank $S = m$. *Let $e \in E(S)$. Let N_e denote the set of nilpotent elements of S_e and let $G^e = C_G(S_e)^c$. Let $a, b \in S$ such that $a^m \mathscr{H} e \mathscr{H} b^m$. Then there exist $\eta \in N_e$, $g \in G^e$ such that $a = \eta g$. Moreover a is conjugate to b if and only if $b = \eta' g'$ for some $\eta' \in N_e$, $g' \in G^e$ such that η is conjugate to η' in S_e and g is conjugate to g' in G^e.*

PROOF. Now $G_e \lhd C_G(e)$ and $C_G(e)$ is a reductive group. Thus by [**8**, Theorem 27.5], $C_G(e) = G_eG^e$. Now $ea \in Sa^{m+1} \subseteq Se$, $ae \in a^{m+1}S \subseteq eS$. So $ea = ae \mathscr{H} e$. By [**15**, Theorem 4] there exists $x \in G^e$ such that $ea = ex$. So $eax^{-1} = ax^{-1}e = e$. Thus $ax^{-1} \in S_e' = \{z \in S \mid ez = ze = e\}$. Let $G_e' = G \cap S_e'$. Then by [**21**, Theorem 1.3], $\underline{S_e'} = \overline{G_e'}$. By [**21**, Lemma 1.2], $S_e' = S_eG_e' \subseteq S_eC_G(e) = S_eG_eG^e = S_eG^e$. So there exists $\eta \in S_e$, $g_1 \in G^e$ such that $ax^{-1} = \eta g_1$. Let $g = g_1x \in G^e$. Then $a = \eta g$. Now $\eta^m \mathscr{H} f$ for some $f \in E(S_e)$. So $f \geqslant e$. Now $e \mathscr{H} a^m = (\eta g)^m = \eta^m g^m \mathscr{J} \eta^m \mathscr{H} f$. Thus $e \mathscr{J} f$, $f \geqslant e$. Hence $e = f$. Since $\eta \in S_e$, $\eta^m = e$. Thus $\eta \in N_e$. Now suppose a is conjugate to b. Then $y^{-1}ay = b$ for some $y \in G$. Then $e \mathscr{H} b^m = y^{-1}a^m y \mathscr{H} y^{-1}ey$. So $y^{-1}ey = e$ and $y \in C_G(e)$. Hence $y = y_1y_2$ for some $y_1 \in G_e$ and $y_2 \in G^e$. Then $b = \eta' g'$ where $\eta' = y^{-1}\eta y = y_1^{-1}\eta y_1$, $g' = y^{-1}gy = y_2^{-1}g_2y_2$. Conversely let $b = \eta' g'$, $\eta' \in N_e$, $g' \in G^e$, such that $\eta' = y_1^{-1}\eta y_1$ for some $y_1 \in G_e$ and $g' = y_2^{-1}gy_2$ for some $y_2 \in G^e$. Then $\eta' = y^{-1}\eta y$, $g' = y^{-1}gy$ where $y = y_1y_2$. So $y^{-1}ay = b$. This proves the theorem.

Let S be a connected regular monoid with zero and group of units G, rank $S = m$. Let $a, b \in S$. Then for some $e, f \in E(S)$, $a^m \mathscr{H} e$, $b^m \mathscr{H} f$. If a is conjugate to b

then e is conjugate to f. If $e = f$, then by Theorem 4.1, the conjugacy problem for a and b reduces to that for nilpotent elements in S_e and that for elements in the reductive group G^e. Thus the conjugacy problem in S reduces to the following three problems:

(A) Conjugacy problem for idempotents.

(B) Conjugacy problem within a reductive group.

(C) Conjugacy problem for nilpotent elements.

Much is known about (A) and (B). So we are left with problem (C). An obvious question is whether the number of conjugacy classes of nilpotent elements in S is always finite. That this is not always the case was pointed out to the author by L. Renner. His example is: $S = \{(\alpha A, \beta A) | \alpha, \beta \in K, A \in \mathcal{M}_2(K)\}$. We now prove the following general result.

THEOREM 4.2. *Let S be a connected regular monoid with zero and group of units G. Suppose that S is $\mathcal{J}$-irreducible. If the number of conjugacy classes of nilpotent elements in S is finite, then G is nearly simple.*

PROOF. Let J denote the nonzero minimum $\mathcal{J}$-class of S. Let R denote the radical of G. Then $G = RG_0$ where G_0 is a semisimple group. By Theorem 1.3, $\dim R = 1$. Also R is contained in the center of G. Suppose G_0 is not almost simple. Then by [**8**, Theorem 27.5], $G_0 = G_1 G_2$ where G_1 is a nontrivial semisimple subgroup of G_0 and G_2 is a nontrivial almost simple subgroup of G_0 and G_1 centralizes G_2. Let $S_1 = \overline{RG_1}$ and let J_1 be a nonzero minimal $\mathcal{J}$-class of S_1. Since RG_1 is nearly semisimple, we see by [**18**, Theorem 2.3] that $J_1^2 \not\subseteq J_1$. So there exists $a \in J_1$ such that $a^2 = 0$. Suppose $aG_2 \subseteq S_1 a S_1$. Then $GaG \subseteq S_1 a S_1$ whereby $SaS = S_1 a S_1$. So $J \subseteq SaS = J_1 \cup \{0\}$. Thus $J = J_1$. Let $S_2 = \overline{RG_2}$. Since RG_2 is nearly simple we see by [**18**, Theorem 2.3] that there exist nonzero minimal idempotents e_1, e_1', $e_1 \neq e_1'$ in the closure of a maximal torus of RG_2 such that $e_1 \mathcal{J} e_1'$ in S_2. Then $e_1 e_1' = 0 = e_1' e_1$. Since J is the minimum nonzero $\mathcal{J}$-class of S, there exists $e_2 \in E(J)$ such that $e_1 \geqslant e_2$. Now $e_1' = x^{-1} e_1 x$ for some $x \in RG_2$. Then since $e_2 \in S_1$, $e_1' \geqslant x^{-1} e_2 x = e_2$. So $0 = e_1 e_1' \geqslant e_2$, a contradiction. Hence $aG_2 \not\subseteq S_1 a S_1$. Let

$$V = \left\{ g \in G_2 | ag \in S_1 a S_1 \right\}.$$

Clearly V is a closed subset of G_2. Let $g, g' \in V$. Then $ag, ag' \in S_1 a S_1$. So $agg' \in S_1 a S_1 g' = S_1 a g' S_1 \subseteq S_1 a S_1$. Now let $x \in G_2$, $g \in V$. Then $ag \in S_1 a S_1$ and $ax^{-1} g x = x^{-1} a g x \in x^{-1} S_1 a S_1 x = S_1 a S_1$. Thus V is a closed normal subgroup of G_2. By the above, $V \neq G_2$. Since G_2 is almost simple, V is a finite subgroup of G_2 lying in the center of G_2. Let $|V| = m$. We can find a sequence of positive integers $n_1 < n_2 < \cdots$, $\alpha_1, \alpha_2, \ldots \in K^*$ such that the order of α_i is n_i and $n_{i+1} > m n_i$, $i = 1, 2, \ldots$. Thus we can find $u_1, u_2, \ldots$ in a maximal torus of G_2 such that the order of u_i is n_i, $i = 1, 2, \ldots$. Now au_i is a nilpotent element of S for any $i \in Z^+$. Thus it suffices to show that au_i is not conjugate to au_j for $i < j$. So suppose to the contrary. Then there exists $x \in G$ such that $x^{-1} a u_i x = a u_j$. Now $x = yz$ for some $y \in RG_1$, $z \in G_2$. Then $a u_i y z = y z a u_j$. Thus $a y u_i z = y a z u_j$. So $a(z u_j z^{-1} u_i^{-1}) = y^{-1} a y \in S_1 a S_1$ and $z u_j z^{-1} u_i^{-1} \in G_2$. Hence $z u_j z^{-1} u_i^{-1} \in V$ and $z u_j z^{-1} \in V u_i$. Since $u_i^{n_i} = 1$, $|V| = m$, we see that $(zu_jz^{-1})^{n_im} = 1$. Hence $u_j^{n_im} = 1$ and $n_im \leqslant n_j$, a contradiction. This proves the theorem.

EXAMPLE 4.3. Theorem 4.2 shows that the $\mathscr{J}$-irreducible monoid $S = \{A \otimes B | A, B \in \mathscr{M}_2(K)\}$ has infinitely many conjugacy classes of nilpotent elements.

EXAMPLE 4.4. Let $G_0 = \{A \otimes (A^{-1})^t | A \in \mathrm{SL}(3, K)\}$, $S = \overline{KG_0}$, $G = K^*G_0$. Then G is nearly simple. Also S is $\mathscr{J}$-irreducible. However the number of conjugacy classes of nilpotent elements in S can be shown to be infinite.

CONJECTURE 4.5. Let S be a $\mathscr{J}$-irreducible connected regular monoid with zero. Then the number of conjugacy classes of nilpotent elements is finite if and only if S is $\mathscr{J}$-linear.

CONJECTURE 4.6. Let S be a connected $\mathscr{J}$-linear regular monoid with zero. Then the number of conjugacy classes of nilpotent elements in S is equal to the number of conjugacy classes of nilpotent elements in the finite inverse monoid $\mathrm{Ren}(S)$ ($\cong$ Munn semigroup of $E(\overline{T})$).

REMARK 4.7. The Jordan canonical form shows the above conjecture to be true for $\mathscr{M}_n(K)$.

If S is a regular semigroup with zero and if a is a nilpotent element in S, then we say that a is *minimal* if a lies in a nonzero minimal $\mathscr{J}$-class of S.

THEOREM 4.8. *Let S be a connected regular monoid with zero. Then the number of conjugacy classes of minimal nilpotent elements in S equals the number of conjugacy classes of minimal nilpotent elements in* $\mathrm{Ren}(S)$.

PROOF. Let T be a maximal torus of G. Let $X = \{(e, f) | e, f$ are nonzero minimal idempotents of $\overline{T}$, $e \neq f$, $e \mathscr{J} f\}$. Define an equivalence relation $\equiv$ on X as follows: $(e, f) \equiv (e', f')$ if there exists $\theta \in W$ such that $e^\theta = e'$, $f^\theta = f'$. If a is a minimal nilpotent element of $\mathrm{Ren}(S)$, then there exists a unique $(e, f) \in X$ such that $e \mathscr{R} a \mathscr{L} f$. It follows that the number of $\equiv$-classes of X equals the number of conjugacy classes of minimal nilpotent elements in $\mathrm{Ren}(S)$. Now by Corollary 2.3, any minimal nilpotent element in S is conjugate to an element of eSf for some $(e, f) \in X$. Now let $(e, f) \in X$, $U = U(e, f) = eSf \setminus \{0\}$. Then every element of U is a minimal nilpotent element and U is an $\mathscr{H}$-class of S. Since eSe is regular and $E(eSe) = \{0, e\}$ we see that $\dim eSe = 1$. There exists $y \in G$ such that $f = y^{-1}ey$. Then $eSf = eSey$. It follows that $\dim U = 1$ and that U is affine and irreducible. Now T acts on U by conjugation. By [**8**, Proposition 8.3] some orbit A is closed in U. Suppose $\dim A = 0$. Then $A = \{a\}$, $a \in C_S(T) = \overline{T}$ by [**23**, Theorem 4.4.4]. This is a contradiction since $a^2 = 0$, $a \neq 0$. Hence $\dim A = 1$ and $A = U$. Thus any two elements in U are conjugate. Now let $(e, f), (e', f') \in X$, $a \in U(e, f)$, $b \in U(e', f')$ such that a is conjugate to b. We must show that $(e, f) \equiv (e', f')$. Now $e \mathscr{R} a \mathscr{J} b \mathscr{R} e'$. Hence by [**16**, Lemma 1.7], there exists $\sigma \in W$ such that $e^\sigma = e'$. Thus without loss of generality we can assume that $e = e'$. Now $b = x^{-1}ax$ for some $x \in G$. Then $x^{-1}ex \mathscr{R} x^{-1}ax = b \mathscr{R} e$. So $e \mathscr{R} x^{-1}ex$ and $x \in C_G^r(e)$. By [**15**, Theorem 1], $a, b \in eS \subseteq S' = \overline{C_G^r(e)}$. Now in S, $f' \mathscr{L} b = x^{-1}ax \mathscr{L} x^{-1}fx$. Hence $f' \mathscr{L} x^{-1}fx$ in S and hence in S'. Thus $f' \mathscr{J} f$ in S'. So by [**16**, Lemma 1.7] there exists

$u \in N_G(T) \cap C_G^r(e) = N_G(T) \cap C_G(e)$ such that $f' = u^{-1}fu$. Thus there exists $\theta \in W$ such that $e^\theta = e, f^\theta = f'$. This proves the theorem.

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