

THE DETERMINANT OF THE EISENSTEIN MATRIX AND HILBERT CLASS FIELDS

BY

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ABSTRACT. We compute the determinant of the Eisenstein matrix associated to the Hilbert-Blumenthal modular group $\mathrm{PSL}_2(\mathcal{O}_K)$, and express it in terms of the zeta function of the Hilbert class field of K .

0. Introduction. The Eisenstein series for the modular group $\Gamma = \mathrm{PSL}_2(\mathbf{Z})$ is given by

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y(\gamma z)^s,$$

where $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbf{Z} \right\}$, z is in the upper half-plane and $y(z) = \mathrm{Im}(z)$. The Fourier expansion of $E(z, s)$ is well known (see Hejhal [4, p. 76]):

$$E(z, s) = y^s + \phi(s)y^{1-s} + \text{nonzero coefficients},$$

where

$$\phi(s) = \frac{\pi^{-(s-1/2)}\Gamma(s-1/2)\zeta(2s-1)}{\pi^{-s}\Gamma(s)\zeta(2s)},$$

$\Gamma(s)$ is the gamma function and $\zeta(s)$ is the Riemann zeta function.

In the general case of a group with h cusps one has an Eisenstein series at each cusp. The Fourier expansions of these series in the various cusps lead to an $h \times h$ matrix $\Phi(s)$ (see (1.11) below). This matrix is the constant term of the Eisenstein series, and is sometimes referred to as the scattering matrix [7], or simply the Φ -matrix.

The determinant $\phi(s) = \det \Phi(s)$ plays an important role in the theory. In the first place it controls the Eisenstein series in the sense that if $\phi(s)$ is analytic at some point, then so are all the Eisenstein series. More importantly, $\phi(s)$ (or rather its logarithmic derivative) accounts for the contribution of the continuous spectrum to the Selberg Trace Formula. A classical situation in which multiplicity of cusps occurs is that of congruence subgroups of the modular group, and the computation of $\phi(s)$ for these is quite involved (see Hejhal [4, Chapter 12] and Huxley [5]).

Our aim in this paper is to calculate $\phi(s)$ for the Hilbert-Blumenthal modular groups $\mathrm{PSL}_2(\mathcal{O}_K)$, \mathcal{O}_K being the ring of integers of a number field K . In this case the number of cusps is the class number of K . What we show is that $\phi(s)$ is elegantly expressed in terms of the zeta function of the Hilbert class field of K . An interesting feature of our approach is that we do not use any Fourier developments

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in cusps. Instead we obtain our result purely from considerations of functional equations. To keep notations to a minimum we shall deal only with imaginary quadratic fields. The general case may be dealt with in a similar way (see Efrat [2] for the set up for totally real fields).

1. General set up and results. Let \mathfrak{h}^3 be the hyperbolic three space $\{w = (y, x_1, x_2) = (y, z) | y > 0\}$ with its Lobachevskii metric

$$ds^2 = \frac{dy^2 + dx_1^2 + dx_2^2}{y^2}.$$

If we think of \mathfrak{h}^3 as the set of all quaternions $x_1 + ix_2 + jy + kt$ for which $t = 0$, then $G = \mathrm{PSL}_2(\mathbf{C})$ is the group of all orientations preserving isometries of \mathfrak{h}^3 . It acts via linear fractional transformations, i.e., for $\tau = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$,

$$\tau(w) = (\alpha w + \beta)(\gamma w + \delta)^{-1}.$$

Consider next a discrete subgroup $\Gamma \subset G$ for which $\mathcal{F} = \mathfrak{h}/\Gamma$ is not compact but is of finite volume. Let $\kappa_1 = \infty, \kappa_2, \dots, \kappa_n$ be a complete set of inequivalent cusps, and let Γ_i be the subgroup of Γ that fixes κ_i . We choose $\rho_i \in G$ such that $\rho_i(\kappa_i) = \infty$ and such that

$$\rho_i \Gamma_i \rho_i^{-1} = \left\{ \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \middle| l \in L_i \right\},$$

where L_i is a lattice in \mathbf{C} whose fundamental domain has volume 1. We let

$$w^{(i)} = \rho_i w = (y(\rho_i w), z(\rho_i w)).$$

DEFINITION 1.1. For each cusp κ_i , define its Eisenstein series to be

$$E_i(w, s) = \sum_{\gamma \in \Gamma_i \backslash \Gamma} y^{(i)}(\gamma w)^s, \quad \mathrm{Re}(s) > 2.$$

Then Selberg's theory shows that $E_i(w, s)$ can be meromorphically continued in s to all of \mathbf{C} , and gives a Γ -automorphic eigenfunction of the Laplacian Δ of \mathfrak{h}^3 , with

$$\Delta E_i(w, s) + s(2-s)E_i(w, s) = 0.$$

Since $E_i(w, s)$ is Γ -automorphic, it is invariant under the lattice at κ_j . It thus admits a Fourier expansion there, which is of the form

$$(1.1.1) \quad E_i(w, s) = \delta_{ij} y^{(j)s} + \phi_{ij}(s) y^{(j)2-s} \\ + \text{nonzero coefficients}, \quad \text{rapidly decaying as } y^{(j)} \rightarrow \infty.$$

If we let $\Phi(s) = (\phi_{ij}(s))_{i,j=1,\dots,h}$ and

$$\vec{E}(w, s) = \begin{bmatrix} E_1(w, s) \\ \vdots \\ E_h(w, s) \end{bmatrix},$$

then we have the functional equation

$$(1.1.2) \quad \vec{E}(w, s) = \Phi(s) \vec{E}(w, 2-s).$$

(For these facts see Cohen and Sarnak [1], Hejhal [4] or Sarnak [8].) The following lemma shows that $\Phi(s)$ is characterized by this equation.

LEMMA 1.2. If $\vec{E}(w, s) = \Psi(s)\vec{E}(w, 2-s)$ for every $w \in \mathfrak{h}^3$, then $\Psi(s) = \Phi(s)$.

PROOF. We have

$$\begin{aligned} E_i(w, s) &= \sum_{k=1}^h \psi_{ik}(s) E_k(w, 2-s) \\ (1.2.1) \quad &= \sum_{k=1}^h \psi_{ik}(s) (\delta_{kj} y^{(j)2-s} + \phi_{kj}(2-s) y^{(j)s} + \dots). \end{aligned}$$

Now choose s with $\operatorname{Re}(s) < 0$. Then as $y^{(j)} \rightarrow \infty$ only the terms involving $y^{(j)2-s}$ will increase. Comparing (1.2.1) with (1.1.1) gives

$$\phi_{ij}(s) = \sum_{k=1}^h \psi_{ik} \delta_{kj} = \psi_{ij}(s). \quad \square$$

Let $\Gamma = \Gamma_D$ be the Hilbert modular group associated to the imaginary quadratic number field $K = \mathbb{Q}(\sqrt{-D})$, i.e.,

$$\Gamma_D = \operatorname{PSL}_2(\mathcal{O}_K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathcal{O}_K, ad - bc = 1 \right\} / \{\pm 1\},$$

where \mathcal{O}_K is the ring of integers of K . We also assume $D \neq 1, 3$.

Then the cusps of Γ are the numbers $\alpha/\beta \in \mathbb{C}$ with $\alpha, \beta \in \mathcal{O}_K$ and ∞ . Furthermore, α/β and α'/β' are equivalent under Γ as cusps if and only if (α, β) and (α', β') are equivalent ideals. So if $\mathcal{A}_1, \dots, \mathcal{A}_h$, $\mathcal{A}_i = (\gamma_i, \delta_i)$, are a complete set of representatives of the ideal classes of K , then $\kappa_i = -\delta_i/\gamma_i$, $i = 1, \dots, h$, form a complete set of inequivalent cusps.

To define the ρ_i 's we first note that we can choose $\alpha_i, \beta_i \in \mathcal{A}_i^{-1}$ such that

$$\hat{\rho}_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \in G.$$

Note that $\hat{\rho}_i(\kappa_i) = \infty$.

To guarantee the volume 1 condition, we scale $\hat{\rho}_i$ and define

$$\rho_i = \begin{pmatrix} \lambda_i^{1/2} & 0 \\ 0 & \lambda_i^{-1/2} \end{pmatrix} \hat{\rho}_i,$$

where $\lambda_i = \omega_K N \mathcal{A}_i$, $\omega_K = \sqrt{2}/d_K^{1/4}$ and d_K is the absolute value of the discriminant of K .

By the discussion above, we now have our Eisenstein series defined for Γ_D . Our main theorem is

THEOREM 1. For Γ_D ,

$$\phi(s) = (-1)^{(h-2t-1)/2} \omega_K^{2s-2} \frac{\xi_H(s-1)}{\xi_H(s)},$$

where $\xi_H(s) = (d_H^{1/2}/(2\pi)^h)^s \Gamma(s)^h \zeta_H(s)$, H is the Hilbert class field of K and t is the number of prime divisors of d_K .

REMARKS. (i) Since H is unramified everywhere it follows that $\xi_H(s)$ is the usual ξ function for the Hilbert class field H , so that $\xi_H(1-s) = \xi_H(s)$.

(ii) The exponent $(h - 2^{t-1})/2$ of -1 in the formula is always an integer by genus theory (see, for example, Hecke [3, p. 160]).

Besides the contribution $\phi(s)$ to the trace formula there is also one other term that appears and which comes from the Eisenstein series. It is the term $\text{tr}(\Phi(1))$ (see [1 or 4]).

Since $\Phi(s)$ is real symmetric for real s and is unitary for s of the form $1 + it$, it follows that at this special point (the middle of the critical strip) $\Phi(1)$ is both unitary and real symmetric. Its eigenvalues are therefore ± 1 , so that the trace is an integer which counts the excess of the eigenvalue $+1$ over -1 (or to put it another way we want the signature of $\Phi(1)$). We have

THEOREM 2. $\text{tr}(\Phi(1)) = 2^{t-1} - 2$, where, as above, t is the number of prime divisors of d_K .

One final comment before turning to the proofs of these theorems: In the general number field case the formulas of Theorems 1 and 2 are similar, however the number 2^{t-1} is replaced by the number of elements of order two in the class group. In the quadratic case this number may be determined in terms of the divisors of d as we have done.

2. Functional equations and the matrix $\Phi(s)$. We begin this section by deriving the functional equation for the Eisenstein series using theta functions. A nice discussion of Eisenstein series and Epstein zeta functions appears in Terras [9, especially Chapter 5]. Since these functional equations are central to the approach in this paper we derive them directly using the notation we have introduced.

For a quaternion $w = x_1 + ix_2 + jy + kt$ we denote by $N(w)$ its norm: $N(w) = x_1^2 + x_2^2 + y^2 + t^2$. Its fundamental property is $N(w_1 w_2) = N(w_1)N(w_2)$. If $w \in \mathfrak{h}^3$ and $y(w)$ is its y -coordinate, then for $\tau \in \text{PSL}_2(\mathbb{C})$, $\tau = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, we have $y(\tau w) = y/N(\gamma w + \delta)$.

DEFINITION 2.1. Define $\tilde{E}_i(w, s)$ by

$$\tilde{E}_i(w, s) = \lambda_i^s \sum_{(c,d) \in \mathcal{A}_i} \frac{y^s}{N(cw + d)^s}.$$

We will need the following, by now standard, lemma due to Selberg.

LEMMA 2.2. Let $f(w)$ be a γ -automorphic function on \mathfrak{h}^3 satisfying

$$\Delta f + s(2-s)f = 0$$

for some s with $\text{Re}(s) > 2$ and such that

$$f(w) = \delta_{ij}(y^{(j)})^s + O(1) \quad \text{as } y^{(j)} \rightarrow \infty.$$

Then $f(w) = E_i(w, s)$.

PROOF. $H_i(w) = E_i(w, s) - f(w)$ is a Γ -automorphic eigenfunction of Δ with eigenvalue $s(2-s)$. In view of the behaviour of $f(w)$ in the cusps, H_i is in $L^2(\Gamma \backslash \mathfrak{h}^3)$. However, from the selfadjointness and nonnegativity of Δ it follows that its eigenvalue must be real and nonnegative. For $s(2-s)$ with $\text{Re}(s) > 2$ this is not the case, so $H_i \equiv 0$.

PROPOSITION 2.3. $\tilde{E}_i(w, s) = E_i(w, s)$.

PROOF. We verify the conditions of Lemma 2.2 for \tilde{E}_i . First, $\tilde{E}_i(w, s)$ is Γ -automorphic, because if $\gamma = \begin{pmatrix} x & y \\ u & v \end{pmatrix} \in \Gamma$, then

$$\begin{aligned} \tilde{E}_i(\gamma w, s) &= \lambda_i^s \sum_{(c,d) \in \mathcal{A}_i} \frac{y^s / N(uw + v)^s}{N(c(xw + y)(uw + v)^{-1} + d)^s} \\ &= \lambda_i^s \sum_{(c,d) \in \mathcal{A}_i} \frac{y^s / N(uw + v)^s}{N(c(xw + y) + d(uw + v))^s N(uw + v)^{-s}} \\ &= \lambda_i^s \sum_{(c,d) \in \mathcal{A}_i} \frac{y^s}{N((cx + du)w + (cy + dv))^s} \end{aligned}$$

and $cx + du$ and $cy + dv$ run over the basis of \mathcal{A}_i as c and d do.

To check the behaviour of \tilde{E}_i at the j th cusp, recall that

$$\rho_j^{-1} = \begin{pmatrix} \delta_j & -\beta_j \\ -\gamma_j & \alpha_j \end{pmatrix} \begin{pmatrix} \lambda_j^{-1/2} & 0 \\ 0 & \lambda_j^{1/2} \end{pmatrix}$$

so that

$$\begin{aligned} \tilde{E}_i(w, s) &= \tilde{E}_i(\rho_j^{-1} w^{(j)}, s) = \lambda_i^s \sum_{(c,d) \in \mathcal{A}_i} \frac{y(\rho_j^{-1} w^{(j)})^s}{N(c\rho_j^{-1} w^{(j)} + d)^s} \\ &= \lambda_i^s \sum_{(c,d) \in \mathcal{A}_i} \frac{y^{(j)s} / N(-\gamma_j \lambda_j^{-1/2} w^{(j)} + \alpha_j \lambda_j^{1/2})^s}{N(c(\delta_j \lambda_j^{-1/2} w^{(j)} - \beta_j \lambda_j^{1/2})(-\gamma_j \lambda_j^{-1/2} w^{(j)} + \alpha_j \lambda_j^{1/2})^{-1} + d)^s} \\ &= \lambda_i^s \sum_{(c,d) \in \mathcal{A}_i} \frac{y^{(j)s}}{N((c\delta_j \lambda_j^{-1/2} - d\gamma_j \lambda_j^{-1/2})w^{(j)} + (-c\beta_j \lambda_j^{1/2} + d\alpha_j \lambda_j^{1/2}))^s} \\ &= \frac{\lambda_i^s}{\lambda_j^s} \sum_{(c,d) \in \mathcal{A}_i} \frac{y^{(j)s}}{N(\lambda_j^{-1}(c\delta_j - d\gamma_j)w^{(j)} + (-c\beta_j + d\alpha_j))^s}. \end{aligned}$$

The only term in this sum which grows with $y^{(j)}$ is the one for which $c\delta_j - d\gamma_j = 0$, $-c\beta_j + d\alpha_j = 1$, which occurs if and only if $i = j$, so that the above is $\delta_{ij}[y^{(j)s} + O(1)]$. Since \tilde{E}_i is clearly an eigenfunction with eigenvalue $s(2 - s)$, Lemma 2.2 implies our claim. \square

DEFINITION 2.4.

$$F_i(w, s) = \lambda_i^s \sum'_{\substack{c,d \in \mathcal{A}_i \\ (\text{mod } \pm 1)}} \frac{y^s}{N(cw + d)^s}$$

(so that we run over half the lattice $\mathcal{A}_i \times \mathcal{A}_i$).

PROPOSITION 2.5.

$$\vec{F}(w, s) = \begin{bmatrix} F_1(w, s) \\ \vdots \\ F_h(w, s) \end{bmatrix} = [\zeta_{\mathcal{A}_i^{-1} \mathcal{A}_j}(s)] \begin{bmatrix} E_1(w, s) \\ \vdots \\ E_h(w, s) \end{bmatrix},$$

where $\zeta_{\mathcal{A}}(s)$ is the Dedekind zeta function of the ideal class of \mathcal{A} .

PROOF (SEE TERRAS [9]).

$$\begin{aligned} F_i(w, s) &= \lambda_i^s \sum_{\mathcal{A}_i | \mathcal{B}} \sum_{(c, d) \in \mathcal{B}} \frac{y^s}{N(cw + d)^s} \\ &= \lambda_i^s \sum_{j=1}^h \sum_{\substack{\mathcal{A}_i | \mathcal{B} \\ \mathcal{B} \sim \mathcal{A}_j}} \sum_{(c, d) \in \mathcal{B}} \frac{y^s}{N(cw + d)^s}. \end{aligned}$$

Write $\mathcal{B} = (\theta_{\mathcal{B}})\mathcal{A}_j$ for some $\theta_{\mathcal{B}} \in K$. Then

$$\begin{aligned} &\lambda_i^s \sum_{j=1}^h \sum_{\substack{\mathcal{A}_i | \mathcal{B} \\ \mathcal{B} \sim \mathcal{A}_j}} \frac{1}{N\theta_{\mathcal{B}}^s} \sum_{(c/\theta_{\mathcal{B}}, d/\theta_{\mathcal{B}}) \in \mathcal{A}_j} \frac{y^s}{N(cw/\theta_{\mathcal{B}} + d/\theta_{\mathcal{B}})^s} \\ &= \lambda_i^s \sum_{j=1}^h \left(\sum_{\substack{\mathcal{A}_i | \mathcal{B} \\ \mathcal{B} \sim \mathcal{A}_j}} \frac{1}{N\mathcal{B}^s} \right) N\mathcal{A}_j^s \sum_{(c, d) \in \mathcal{A}_j} \frac{y^s}{N(cw + d)^s}. \end{aligned}$$

Now let $\mathcal{C} = \mathcal{A}_i^{-1}\mathcal{B}$ so that $\mathcal{C} \sim \mathcal{A}_i^{-1}\mathcal{A}_j$, $N\mathcal{B} = N\mathcal{C} \cdot N\mathcal{A}_i$; then

$$= \sum_{j=1}^h \left(\sum_{\mathcal{C} \sim \mathcal{A}_i^{-1}\mathcal{A}_j} \frac{1}{N\mathcal{C}^s} \right) \lambda_j^s \sum_{(c, d) \in \mathcal{A}_j} \frac{y^s}{N(cw + d)^s}. \quad \square$$

PROPOSITION 2.6. *The function $\omega_K^{-s} (d_K^{1/2}/2\pi)^s \Gamma(s) F_i(w, s)$ is invariant under $s \rightarrow 2 - s$ and $\mathcal{A}_i \rightarrow (\partial_K \mathcal{A}_i)^{-1}$, where ∂_K is the different of K .*

PROOF. Using the formula

$$\frac{\Gamma(s)}{a^s} = \int_0^\infty e^{-at} t^s \frac{dt}{t}$$

with $a = 2\pi/d_K^{1/2} N\mathcal{A}_i \cdot N(cw + d)/y$, we get

$$\begin{aligned} &\omega_K^{-s} \cdot \left(\frac{d_K^{1/2}}{2\pi} \right)^s \Gamma(s) (\omega_K N\mathcal{A}_i)^s \sum'_{c, d \in \mathcal{A}_i} \frac{y^s}{N(cw + d)^s} \\ (2.6.1) \quad &= \int_0^\infty \sum'_{c, d \in \mathcal{A}_i} \exp \left(-\frac{2\pi}{d_K^{1/2} N\mathcal{A}_i} \frac{N(cw + d)}{y} t \right) t^s \frac{dt}{t} \\ &= \int_0^\infty \sum'_{l \in \mathcal{A}_i \times \mathcal{A}_i} \exp \left(-\frac{2\pi}{d_K^{1/2} N\mathcal{A}_i} \langle l, Al \rangle t \right) t^s \frac{dt}{t}, \end{aligned}$$

where A is the Hermitian matrix

$$A = \begin{pmatrix} Nw/y & z/y \\ \bar{z}/y & 1/y \end{pmatrix}.$$

We wish to apply the Poisson summation formula to this last expression. To this end, let

$$f(z_1, z_2) = e^{-r\langle Z, AZ \rangle}, \quad Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

be thought of as a function of four real variables. Then for $(\beta, \gamma) \in \mathbf{C}^2$,

$$\hat{f}(\beta, \gamma) = \iint_{\mathbf{C}^2} f(z_1, z_2) e^{-2\pi i \operatorname{Re}(\bar{\beta} z_1 + \bar{\gamma} z_2)} dZ,$$

where $dZ = -\frac{1}{4} \cdot dz_1 \wedge \overline{dz_1} \cdot dz_2 \wedge \overline{dz_2}$,

$$= \iint_{\mathbf{C}^2} e^{-r\langle Z, AZ \rangle} e^{-2\pi i \operatorname{Re}(\langle \beta, \gamma \rangle, Z)} dZ.$$

Since A is positive definite we can change variables $W = A^{1/2}Z$. Since $|A| = 1$ this leads to

$$\iint_{\mathbf{C}^2} e^{-r\langle W, W \rangle} e^{-2\pi i \operatorname{Re}(A^{-1/2} \begin{pmatrix} \beta \\ \gamma \end{pmatrix}, W)} dW.$$

Writing

$$W = \begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad A^{-1/2} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \nu_1 \\ \nu_2 \end{bmatrix}$$

we have

$$\int \dots \int_{\mathbf{R}^4} e^{-r(u_1^2 + u_2^2 + v_1^2 + v_2^2)} e^{-2\pi i(\mu_1 v_1 + \mu_2 u_2 + \nu_1 v_1 + \nu_2 v_2)} du_1 du_2 dv_1 dv_2.$$

Recalling that

$$\int_{-\infty}^{\infty} e^{-rx^2} e^{-2\pi ixy} dx = \sqrt{\frac{\pi}{r}} e^{-\pi^2 y^2/r},$$

we obtain

$$\begin{aligned} & \frac{\pi^2}{r^2} e^{-\pi^2(\mu_1^2 + \mu_2^2 + \nu_1^2 + \nu_2^2)/r} \\ &= \frac{\pi^2}{r^2} e^{-\pi^2 \langle A^{-1/2} \begin{pmatrix} \beta \\ \gamma \end{pmatrix}, A^{-1/2} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \rangle / r} \\ &= \frac{\pi^2}{r^2} e^{-\pi^2 \langle \begin{pmatrix} \beta \\ \gamma \end{pmatrix}, A^{-1} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \rangle / r}. \end{aligned}$$

Going back to (2.6.1), we separate the integral into two parts to obtain

$$\begin{aligned} & \int_1^\infty \sum'_{l \in \mathcal{A}_i \times \mathcal{A}_i} \exp \left(-\frac{2\pi}{d_K^{1/2} N \mathcal{A}_i} \langle l, Al \rangle t \right) t^s \frac{dt}{t} \\ &+ \int_0^1 \sum_{l \in \mathcal{A}_i \times \mathcal{A}_i} \exp \left(-\frac{2\pi}{d_K^{1/2} N \mathcal{A}_i} \langle l, Al \rangle t \right) t^s \frac{dt}{t} - \frac{1}{s} \\ &= \text{I} + \text{II}. \end{aligned}$$

Applying the Poisson summation formula to Π we get from our computation above

$$\Pi = \frac{1}{v} \int_0^1 \frac{\pi^2}{r^2} \sum_{l \in \mathcal{A}'_i \times \mathcal{A}'_i} \exp\left(-\frac{\pi^2}{r} \langle l, A^{-1}l \rangle\right) t^s \frac{dt}{t} - \frac{1}{s},$$

where $r = 2\pi t/d_K^{1/2} N\mathcal{A}_i$ and \mathcal{A}'_i is the dual lattice of \mathcal{A}_i , so that $\mathcal{A}'_i = 2(\partial_K \mathcal{A}_i)^{-1}$. Here $v = \text{vol}(\mathbf{C}^2/\mathcal{A}_i \times \mathcal{A}_i) = d_K/4 \cdot N\mathcal{A}_i^2$. Thus

$$\Pi = \int_0^1 \sum_{l \in (\partial_K \mathcal{A}_i)^{-1} \times (\partial_K \mathcal{A}_i)^{-1}} \exp\left(-\frac{\pi d_K^{1/2} N\mathcal{A}_i}{2} \langle 2l, 2A^{-1}l \rangle t^{-1}\right) t^{s-2} \frac{dt}{t} - \frac{1}{s}.$$

Now change variables $t \rightarrow t^{-1}$

$$\begin{aligned} &= \int_1^\infty \sum_{l \in (\partial_K \mathcal{A}_i)^{-1} \times (\partial_K \mathcal{A}_i)^{-1}} \exp(-2\pi d_K^{1/2} N\mathcal{A}_i \langle l, A^{-1}l \rangle t) t^{2-s} \frac{dt}{t} - \frac{1}{s} \\ &= \int_1^\infty \sum'_{c,d \in (\partial_K \mathcal{A}_i)^{-1}} \exp\left(-2\pi d_K^{1/2} N\mathcal{A}_i \frac{N(cw+d)}{y} t\right) t^{2-s} \frac{dt}{t} - \frac{1}{s} - \frac{1}{2-s}. \end{aligned}$$

Finally we note that

$$d_K^{1/2} N\mathcal{A}_i = \frac{1}{d_K^{1/2} N(\mathcal{A}_i \partial_K)^{-1}}$$

and comparing our last expression with (2.6.1) completes the proof. \square

COROLLARY 2.7. *Let P be the permutation matrix of the permutation of the ideal class group given by $\mathcal{A}_i \rightarrow (\partial_K \mathcal{A}_i)^{-1}$. Then*

$$\vec{F}(w, s) = \omega_K^{2s-2} \frac{(d_K^{1/2}/2\pi)^{2-s} \Gamma(2-s)}{(d_K^{1/2}/2\pi)^s \Gamma(s)} \cdot P \cdot \vec{F}(w, 2-s).$$

Putting Proposition 2.5 and Corollary 2.7 together we obtain

$$[\zeta_{\mathcal{A}_i^{-1} \mathcal{A}_j}(s)] \vec{E}(w, s) = \omega_K^{2s-2} \frac{(d_K^{1/2}/2\pi)^{2-s} \Gamma(2-s)}{(d_K^{1/2}/2\pi)^s \Gamma(s)} \cdot P \cdot [\zeta_{\mathcal{A}_i^{-1} \mathcal{A}_j}(2-s)] \vec{E}(w, 2-s).$$

But then Lemma 1.2 implies

COROLLARY 2.8.

$$\Phi(s) = \omega_K^{2s-2} \frac{(d_K^{1/2}/2\pi)^{2-s} \Gamma(2-s)}{(d_K^{1/2}/2\pi)^s \Gamma(s)} [\zeta_{\mathcal{A}_i^{-1} \mathcal{A}_j}(s)]^{-1} \cdot P \cdot [\zeta_{\mathcal{A}_i^{-1} \mathcal{A}_j}(2-s)].$$

3. Computations of $\phi(s)$ and $\text{tr } \Phi(1)$. We can finally turn to the computation of $\phi(s)$. We first find $\det[\zeta_{\mathcal{A}_i^{-1} \mathcal{A}_j}(s)]$.

PROPOSITION 3.1. *Let χ be a character of the ideal class group and let $L(s, \chi)$ be its L -function. Then*

$$[\zeta_{\mathcal{A}_i^{-1} \mathcal{A}_j}(s)] \begin{bmatrix} \chi(\mathcal{A}_1) \\ \vdots \\ \chi(\mathcal{A}_h) \end{bmatrix} = L(s, \chi) \begin{bmatrix} \chi(\mathcal{A}_1) \\ \vdots \\ \chi(\mathcal{A}_h) \end{bmatrix}.$$

PROOF.

$$\begin{aligned} \sum_{j=1}^h \zeta_{\mathcal{A}_i^{-1} \mathcal{A}_j}(s) \cdot \chi(\mathcal{A}_j) &= \sum_{j=1}^h \sum_{\mathcal{B} \sim \mathcal{A}_i^{-1} \mathcal{A}_j} \frac{\chi(\mathcal{A}_j)}{N\mathcal{B}^s} \\ &= \chi(\mathcal{A}_i) \sum_{j=1}^h \sum_{\mathcal{B} \sim \mathcal{A}_i^{-1} \mathcal{A}_j} \frac{\chi(\mathcal{B})}{N\mathcal{B}^s} = \chi(\mathcal{A}_i) \cdot L(s, \chi). \quad \square \end{aligned}$$

Since the determinant is the product of the eigenvalues, we infer using class field theory (see Lang [6, XII, §1])

COROLLARY 3.2. $\det[\zeta_{\mathcal{A}_i^{-1} \mathcal{A}_j}(s)] = \prod_{\chi} L(s, \chi) = \zeta_H(s)$, where H is the Hilbert class field of K .

It remains to find $\det(P)$. First, for an imaginary quadratic field the different is a principal ideal, so that the permutation is actually $\mathcal{A}_i \rightarrow \mathcal{A}_i^{-1}$, and therefore $\det(P) = (-1)^{(h-m)/2}$, where m is the number of elements in the ideal class group of order 2. This, however, can be made more explicit using genus theory (see Hecke [3, p. 176]), which asserts that $m = 2^{t-1}$, where t is the number of prime divisors of d_K .

Combining Corollaries 2.8 and 3.2 and the above remark concludes the proof of Theorem 1.

We now prove Theorem 2. Let $M = (\bar{\chi}_i(\mathcal{A}_j))_{i,j=1,\dots,h}$ so that by Proposition 3.1

$$[\zeta_{\mathcal{A}_i^{-1} \mathcal{A}_j}(s)] = M^{-1} \begin{bmatrix} L(s, \chi_1) & & \\ & \ddots & \\ & & L(s, \chi_h) \end{bmatrix} M.$$

Then

$$\begin{aligned} \text{Tr}(\Phi(s)) &= \text{Tr} \left[\omega_K^{2s-2} \frac{(d_K^{1/2}/2\pi)^{2-s} \Gamma(2-s)}{(d_K^{1/2}/2\pi)^s \Gamma(s)} \cdot P \cdot [\zeta_{\mathcal{A}_i^{-1} \mathcal{A}_j}(2-s)] [\zeta_{\mathcal{A}_i^{-1} \mathcal{A}_j}(s)]^{-1} \right] \\ &= \text{Tr} \left[\omega_K^{2s-2} \frac{(d_K^{1/2}/2\pi)^{2-s} \Gamma(2-s)}{(d_K^{1/2}/2\pi)^s \Gamma(s)} \cdot P \cdot M^{-1} \begin{bmatrix} L(2-s, \chi_1) & & \\ & \ddots & \\ & & L(2-s, \chi_h) \end{bmatrix} \right. \\ &\quad \left. \times \begin{bmatrix} L(s, \chi_1)^{-1} & & \\ & \ddots & \\ & & L(s, \chi_h)^{-1} \end{bmatrix} M \right]. \end{aligned}$$

If we assume that $\chi_1 = 1$, then

$$\lim_{s \rightarrow 1} \frac{L(2-s, \chi_1)}{L(s, \chi_1)} = -1$$

so that

$$\begin{aligned} \text{Tr}(\Phi(1)) &= \lim_{s \rightarrow 1} \text{Tr}(\Phi(s)) \\ &= \text{Tr} \left(P M^{-1} \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} M \right) = \text{Tr} \left(P \frac{-2}{h} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} + P \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \right). \end{aligned}$$

Recalling the definition of P we have $\text{Tr}(\Phi(1)) = m - 2 = 2^{t-1} - 2$, which is Theorem 2.

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