

SUBELLIPTICITY OF THE $\bar{\partial}$ -NEUMANN PROBLEM ON NONPSEUDOCONVEX DOMAINS

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ABSTRACT. Following the work of Kohn, we give a sufficient condition for subellipticity of the $\bar{\partial}$ -Neumann problem for not necessarily pseudoconvex domains. We define a sequence of ideals of germs and show that if 1 is in any of them, then there is a subelliptic estimate. In particular, we prove subellipticity under some specific conditions for $n - 1$ forms and for the case when the Levi-form is diagonalizable. For the necessary conditions, we use another method to prove that there is no subelliptic estimate for q forms if the Levi-form has $n - q - 1$ positive eigenvalues and q negative eigenvalues. This was proved by Derridj. Using similar techniques, we prove a necessary condition for subellipticity for some special domains. Finally, we give a remark to Catlin's theorem concerning the hypoellipticity of the $\bar{\partial}$ -Neumann problem in the case of nonpseudoconvex domains.

1. Introduction. The solution of the $\bar{\partial}$ -Neumann problem has many important applications in the theory of several complex variables, particularly in the study of boundary regularity of the $\bar{\partial}$ -problem. (For a detailed discussion of the $\bar{\partial}$ -Neumann problem, we refer the reader to [4].) The $\bar{\partial}$ -Neumann problem in strongly pseudoconvex domains was solved by Kohn [6], and the technique there showed that the existence of a subelliptic estimate will give the solution. In the past twenty years a great deal of work has been done on subelliptic estimates in weakly pseudoconvex domains. Not much is known in the case when the domain is not necessarily pseudoconvex except the results of Derridj [3] and Hormander [5]. We study here subellipticity on domains which are not necessarily pseudoconvex where the Levi-form degenerates.

We first deal with the sufficient condition. Hormander [5] proved the necessary and sufficient conditions for $\frac{1}{2}$ estimate on nonpseudoconvex domains. The condition is simply counting the number of positive or negative eigenvalues of the Levi-form. On weakly pseudoconvex domains Kohn [7] gave a sufficient condition for subellipticity by introducing a sequence of ideals of subelliptic multipliers. We use Kohn's idea to define a sequence of ideals of subelliptic multipliers when the domain is nonpseudoconvex. The Levi-form of the domain must satisfy some conditions in order that we have a basic estimate. This has to do with the positiveness of the eigenvalue of the Levi-form, which corresponds to the result of Hormander. We use this result to prove that in the case of $n - 1$ forms if there is a vector field whose Levi-form is nonnegative and of finite type, then there is a subelliptic estimate. We also deal with the case when the Levi-form is diagonalisable, giving conditions for subellipticity which requires some 'finite type' condition.

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For necessary conditions we give another proof to a theorem of Derridj [3], that if the Levi-form is nondegenerate, then there is a subelliptic estimate if and only if the $\frac{1}{2}$ estimate holds. Catlin [1] proved, in weakly pseudoconvex domains, that if there is an $(n-1)$ -dimensional variety with order of contact η with the boundary, and a subelliptic estimate of order ε holds, then $\varepsilon \leq 1/\eta$. In nonpseudoconvex domains we prove that in some special domains it turns out that the negative direction of the Levi-form has no contribution to the order of subellipticity for $n-1$ forms, as shown by Proposition 3.2. We also get a condition $\varepsilon \leq 1/\eta$ in the other direction.

Terminology and notation. Let Ω be a domain in \mathbf{C}^n and let $b\Omega$ denote the boundary of Ω . We call r a defining function for Ω if r is a C^∞ real-valued function such that $dr \neq 0$ on $b\Omega$, $r < 0$ in Ω , and $r > 0$ outside $\bar{\Omega}$.

Let $T_x^{1,0}$ and $T_x^{0,1}$ denote the holomorphic and antiholomorphic vectors at x , respectively.

We denote by $A_x^{p,q}$ the space of (p, q) forms on Ω which are smooth up to and including the boundary. In terms of z_1, \dots, z_n we can write $\phi \in A^{p,q}$ as

$$\phi = \sum'_{I,J} \phi_{IJ} dz_I \wedge d\bar{z}_J,$$

where $\phi_{IJ} \in C^\infty(\bar{\Omega})$, $I = (i_1, \dots, i_p)$ with $1 \leq i_1 < \dots < i_p \leq n$, $J = (j_1, \dots, j_q)$ with $1 \leq j_1 < \dots < j_q \leq n$, $dz_I = dz_{i_1} \wedge \dots \wedge dz_{i_p}$, and $d\bar{z}_J = d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$. The symbol $'$ means that the indexes I and J are increasing.

Let $L_2^{p,q}(\Omega)$ denote the space of square-integrable (p, q) forms on Ω . The inner product and norm are defined by

$$\langle \alpha, \beta \rangle = \sum'_{I,J} \int \alpha_{IJ} \bar{\beta}_{IJ} dV, \quad \|\alpha\|^2 = \langle \alpha, \alpha \rangle,$$

where $\alpha = \sum' \alpha_{IJ} dz_I \wedge d\bar{z}_J$ and $\beta = \sum' \beta_{IJ} dz_I \wedge d\bar{z}_J$. Then we have

$$L_2^{p,q-1}(\Omega) \xrightarrow[\bar{\partial}^*]{\bar{\partial}} L_2^{p,q}(\Omega) \xrightarrow[\bar{\partial}^*]{\bar{\partial}} L_2^{p,q+1}(\Omega),$$

where $\bar{\partial}$ is the closed operator which is the maximal extension of the differential operator $\bar{\partial}$ on $A^{p,q}$, where

$$\bar{\partial}\phi = \sum'_{kIJ} \frac{\partial \phi_{IJ}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J \quad \text{if } \phi = \sum'_{IJ} \phi_{IJ} dz_I \wedge d\bar{z}_J.$$

$\bar{\partial}^*$ means the L_2 -adjoint of $\bar{\partial}$.

Define $D^{p,q} = A^{p,q} \cap \text{dom}(\bar{\partial}^*)$. Then it can be shown that

$$D^{p,q} = \{\phi \in A^{p,q}: (\text{int}(\bar{\partial}r)\phi)_x = 0 \text{ for } x \in b\Omega\},$$

where $\text{int } \theta: A_x^{p,q} \rightarrow A_x^{p,q+1}$ is defined by

$$\langle \text{int}(\theta)\phi, \omega \rangle_x = \langle \phi, \theta \wedge \omega \rangle_x,$$

where $\phi \in A_x^{p,q}$ and $\omega \in A_x^{p,q-1}$.

We later give a simpler expression for $D^{p,q}$ for some special coordinates.

The quadratic form Q on $D^{p,q}$ is defined by

$$Q(\phi, \psi) = \langle \bar{\partial}\phi, \bar{\partial}\psi \rangle + \langle \bar{\partial}^*\phi, \bar{\partial}^*\psi \rangle \quad \text{for } \phi, \psi \in D^{p,q}.$$

Let $x \in \text{b}\Omega$. Denote by $\text{CT}_x(\text{b}\Omega)$ the space of complex-valued vectors tangential to $\text{b}\Omega$, i.e., $T \in \text{CT}_x(\text{b}\Omega)$ if $T(r) = 0$.

DEFINITION 1.1. The Levi-form at x is the Hermitian form on the $(n-1)$ -dimensional space $\text{CT}_x(\text{b}\Omega) \cap T_x^{1,0}$.

$$L_x(L_1, L_2) = \langle \partial \bar{\partial} r, L_1 \wedge \bar{L}_2 \rangle_x.$$

We call Ω pseudoconvex if L_x is nonnegative for all $x \in \text{b}\Omega$.

For all $x_0 \in \text{b}\Omega$ there exists a neighbourhood U of x_0 such that we can choose C^∞ vector fields L_1, \dots, L_n , with values in $T^{1,0}$, which form an orthonormal basis for $T^{1,0}$ at each $x \in U \cap \bar{\Omega}$.

Let $\omega_1, \dots, \omega_n$ be the dual basis to L_1, \dots, L_n of $(1,0)$ forms on $U \cap \bar{\Omega}$. Moreover, we may fix r so that $|\partial r|_x = 1$ in a neighbourhood of $\text{b}\Omega$, and we may choose $\omega_n = \partial r$.

Denote the conjugates of L_1, \dots, L_n by $\bar{L}_1, \dots, \bar{L}_n$, i.e., $\bar{L}_i(f) = \overline{L_i(\bar{f})}$, and the conjugates of $\omega_1, \dots, \omega_n$ by $\bar{\omega}_1, \dots, \bar{\omega}_n$.

On $U \cap \text{b}\Omega$ we have

$$L_j(r) = \bar{L}_j(r) = \delta_{jn}.$$

Therefore, L_1, \dots, L_{n-1} and $\bar{L}_1, \dots, \bar{L}_{n-1}$ are local bases of $T^{1,0}(U \cap \text{b}\Omega)$ and $T^{0,1}(U \cap \text{b}\Omega)$, respectively. Define a vector field T on $U \cap \text{b}\Omega$ by $T = L_n - \bar{L}_n$. Then $L_1, \dots, L_{n-1}, \bar{L}_1, \dots, \bar{L}_{n-1}, T$ is a basis for $\text{CT}_x(\text{b}\Omega)$ at each point $x \in U \cap \text{b}\Omega$. We denote the Levi-form in terms of this basis by

$$c_{ij}(x) = \langle \partial \bar{\partial} r, L_i \wedge \bar{L}_j \rangle_x$$

for $i, j = 1, \dots, n-1$ and $x \in U \cap \text{b}\Omega$. We can write

$$(1.1) \quad [L_i, \bar{L}_j] = c_{ij}T + \sum_1^{n-1} a_{ij}^k L_k + \sum_1^{n-1} b_{ij}^k \bar{L}_k,$$

where $a_{ij}^k, b_{ij}^k \in C^\infty(U \cap \bar{\Omega})$.

Let $\varphi \in A_x^{p,q}$. Then on $U \cap \bar{\Omega}$ we can write φ as

$$\varphi = \sum_{I,J} \varphi_{IJ} \omega_I \wedge \bar{\omega}_J,$$

where I, J, ω_I, ω_J are the same as before. In this coordinate $\varphi \in D^{p,q}$ is equivalent to the condition

$$(1.2) \quad \varphi_{IJ}(x) = 0 \quad \text{for } x \in \text{b}\Omega \text{ whenever } n \in J.$$

Define $D_U^{p,q} = \{\varphi \in D^{p,q}: \text{supp}(\varphi) \subset U \cap \bar{\Omega}\}$.

We can also express $\bar{\partial}\varphi$ and $\bar{\partial}^* \varphi$ in terms of these bases.

$$(1.3) \quad \bar{\partial}\varphi = (-1)^p \sum_{IJ} \sum_j \bar{L}_j(\varphi_{IJ}) \omega_I \wedge \bar{\omega}_j \wedge \bar{\omega}_J + \sum_{\substack{I,J|H|=p \\ |L|=q+1}} f_{HL}^{IJ} \varphi_{IJ} \omega_H \wedge \bar{\omega}_L,$$

where $||$ means the number of elements in H . The second term consists of those terms in which φ_{IJ} is not differentiated and $f_{HL}^{IJ} \in C^\infty(U \cap \bar{\Omega})$.

For $\varphi \in D_U^{p,q}$ we can write $\bar{\partial}^*$ as follows:

$$(1.4) \quad \bar{\partial}^* \varphi = (-1)^{p+1} \sum'_{IJ} \left(\varepsilon_{jH}^J L_j(\varphi_{IJ}) \omega_I \wedge \bar{\omega}_H + \sum_{\substack{|H|=p \\ |K|=q-1}} g_{HK}^{IJ} \varphi_{IJ} \omega_H \wedge \bar{\omega}_K \right),$$

where

$$\varepsilon_{jH}^J = \begin{cases} 0 & \text{if } \{j\} \cup H \neq K, \\ \text{sign of permutation taking } jH \text{ to } J & \text{if } \{j\} \cup H = J, \end{cases}$$

and $g_{HK}^{IJ} \in C^\infty(U \cap \bar{\Omega})$.

We call a system t_1, \dots, t_{2n-1} of real C^∞ coordinates r boundary coordinates if r is the defining function for Ω .

For $U \in C_0^\infty(U \cap \bar{\Omega})$ we define the tangential fourier transform \hat{u} of u as follows:

$$\hat{u}(\tau, r) = \int_{\mathbf{R}^{2n-1}} e^{-it \cdot \tau} u(t, r) dt,$$

where $\tau = (\tau_1, \dots, \tau_{2n-1})$, $t = (t_1, \dots, t_{2n-1})$.

$$t \cdot \tau = \sum_1^{2n-1} t_j \tau_j \quad \text{and} \quad dt = dt_1 \cdots dt_{2n-1}.$$

We define $\hat{\Lambda}^s u$ and $|||u|||_s$ by

$$\Lambda^s u(\tau, r) = (1 + |\tau|^2)^{s/2} u(\tau, r) \quad \text{for } s \in \mathbf{R}$$

and

$$|||u|||_s^2 = \int_{-\infty}^0 \int_{\mathbf{R}^{2n-1}} |\hat{\Lambda}^s u(t, r)|^2 dt dr,$$

and we call $||| \cdot |||_s$ the tangential Sobolev-norm of order s .

DEFINITION 1.2. If $x_0 \in \bar{\Omega}$, we say that a subelliptic estimate holds at x_0 for (p, q) forms for the $\bar{\partial}$ -Neumann problem if there exists a neighbourhood U of x_0 and constants $\varepsilon, C > 0$ such that

$$|||\varphi|||_\varepsilon^2 \leq C(Q(\varphi, \varphi) + \|\varphi\|^2)$$

for all $\varphi \in D_U^{p,q}$. The norm $|||\varphi|||_\varepsilon^2 = \sum'_{IJ} |||\varphi_{IJ}|||_\varepsilon^2$.

DEFINITION 1.3. If $x_0 \in \bar{\Omega}$, $f \in C^\infty(U \cap \bar{\Omega})$ for some neighbourhood U of x_0 . We say that f is a subelliptic multiplier at x_0 if for some $\varepsilon, C > 0$ we have

$$|||f\varphi|||_\varepsilon^2 \leq C(Q(\varphi, \varphi) + \|\varphi\|^2)$$

for all $\varphi \in D_U^{p,q}$.

The set of all subelliptic multipliers of (p, q) forms is denoted by $S(x_0)$, where it is understood that we are working on (p, q) forms.

REMARK. It is clear that a subelliptic estimate holds at x_0 for (p, q) forms if and only if a subelliptic estimate holds at x_0 for $(0, q)$ forms.

2. Sufficient conditions for subellipticity. In this section we follow Kohn's idea to define a sequence of ideals of subelliptic multipliers in nonpseudoconvex domains. Kohn's proof [7], to a large extent, only uses the property of some $(n - q) \times (n - q)$ minor of the Levi-form in the case of (p, q) forms. Hormander's result says that $n - q$ positive eigenvalues at a point imply a $\frac{1}{2}$ estimate. In view of these two facts it is natural to guess that, without the assumption of pseudoconvexity, if some $(n - q) \times (n - q)$ minor of the Levi-form is positive semidefinite, we can define ideals similar to that of Kohn, and if 1 is in one of these ideals, then there is a subelliptic estimate for (p, q) forms. It turns out to be true for $(p, n - 1)$ forms. When $q \leq n - 2$ we prove a similar result under more complicated assumptions. Much of the technique in the proof here follows Kohn [7].

First we introduce the following notation.

Let C be an $(n - 1) \times (n - 1)$ matrix defined in a neighbourhood U of x_0 , and let k be an integer. We associate two matrices $A^{(k)}$ and $B^{(k)}$ to this matrix, each of which is a $(n - 1)!/q!(n - 1 - q)! \times (n - 1)!/q!(n - 1 - q)!$ matrix defined in the same neighbourhood U of x_0 . We denote entries of the matrices $A^{(k)}$ and $B^{(k)}$ by $A_{IJ}^{(k)}$ and $B_{IJ}^{(k)}$, respectively, where I, J are strictly increasing sequences of positive integers of length q taking values from 1 to $n - 1$.

Define $A^{(k)}$ and $B^{(k)}$ as follows:

$$A_{IJ}^{(k)} = \begin{cases} \varepsilon_{iK}^I \varepsilon_{jK}^J c_{ij} & \text{if } I \text{ and } J \text{ have exactly } q - 1 \\ & \text{elements } K \text{ in common,} \\ \sum_{\substack{i \leq k \\ i \in I}} c_{ii} & \text{if } I = J, \\ 0 & \text{otherwise,} \end{cases}$$

$$B_{IJ}^{(k)} = \begin{cases} \varepsilon_{iK}^I \varepsilon_{jK}^J c_{ij} & \text{if } I \text{ and } J \text{ have exactly } q - 1 \\ & \text{elements } K \text{ in common,} \\ - \sum_{\substack{i \leq k \\ i \in I}} c_{ii} & \text{if } I = J, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose L_1, \dots, L_n is a C^∞ basis for $T^{1,0}$ on $U \cap \bar{\Omega}$. Consider the functions defined in a neighbourhood of x_0 as germs at x_0 . Let

$$\sqrt[n]{I} = \{g \in C^\infty: |g|^n \leq |f| \text{ for some } f \text{ in the ideal } I\}.$$

Define ideals of germs of functions at x_0 by

$$I_1(x_0) = \sqrt[n]{r, \det A^{(k)}}, \quad I_{m+1}(x_0) = \sqrt[n]{I_m, M_m},$$

where

$$M_m(x_0) = \left\{ \det M: \begin{array}{l} \text{rows of } M \text{ are either (i) a row of } A^{(k)}, \\ \text{(ii) for some fixed } H \text{ and } f \in I_m(x_0), \\ M_{IJ} = \varepsilon_{jH}^J L_j f \quad \text{if } \langle jH \rangle = J \text{ and } j \leq k \\ \quad \quad \quad = 0 \quad \quad \quad \text{otherwise} \end{array} \right\}.$$

Similarly, we define

$$J_1(x_0) = \sqrt[n]{r, \det B^{(k)}}, \quad J_{m+1}(x_0) = \sqrt[n]{J_m, N_m}$$

where

$$N_m(x_0) = \left\{ \begin{array}{l} \det N: \text{ rows of } N \text{ are either (i) a row of } B^{(k)}, \\ \text{(ii) for some fixed } K \text{ and } f \in J_m(x_0), \\ N_{IJ} = \varepsilon_{jK}^J \bar{L}_j f \quad \text{if } \langle jJ \rangle = K \text{ and } j \leq k \\ \quad = 0 \quad \text{otherwise} \end{array} \right\}.$$

We can now state the main theorem in this section.

THEOREM 2.1. *Let Ω be a domain in \mathbb{C}^n with C^∞ boundary, $x_0 \in \text{b}\Omega$, and L_1, \dots, L_n a C^∞ basis for $T^{1,0}$ so that L_1, \dots, L_{n-1} are tangential on $\text{b}\Omega$. Assume there exists a neighbourhood U of x_0 such that:*

(1) *for some $k \geq n - q$ the matrix $A^{(k)}$ associated to the matrix of the Levi-form is positive semidefinite in U ;*

(2) *for all $\varepsilon > 0$ there exists $C > 0$ (C depends on S^0 but not on ϕ) such that*

$$\langle D\phi_I, S^0\phi_J \rangle \leq \varepsilon Q(\phi, \phi) + C\|\phi\|^2 \quad \text{for all } \phi \in D_U^{p,q},$$

where I, J are q -tuples and each contains at least $q - 2$ elements in common, and $D \in \{\bar{L}_{k+1}, \dots, \bar{L}_{n-1}\}$, S^0 is a tangential pseudodifferential operator of order zero.

If $1 \in I_m(x_0)$ for some m , there is a subelliptic estimate for (p, q) forms at x_0 .

The above theorem corresponds to the case in which the Levi-form has $n - q$ positive eigenvalues in the $\varepsilon - \frac{1}{2}$ estimate. Corresponding to the $q + 1$ negative eigenvalues we have the following theorem dual to Theorem 2.1.

THEOREM 2.2. *Let Ω be a domain in \mathbb{C}^n with C^∞ boundary, $x_0 \in \text{b}\Omega$, and L_1, \dots, L_n a C^∞ basis for $T^{1,0}$ so that L_1, \dots, L_{n-1} are tangential on $\text{b}\Omega$. Assume there exists a neighbourhood U of x_0 such that:*

(1) *for some $k \geq q + 1$ the matrix $B^{(k)}$ associated to the matrix of the Levi-form is positive semidefinite in U ;*

(2) *same as Theorem 2.1(2).*

If $1 \leq J_m(x_0)$ for some m , there is a subelliptic estimate for (p, q) forms at x_0 .

Before proving Theorems 2.1 and 2.2 we give a few lemmas. We refer the reader to [7] for the proofs of these lemmas.

LEMMA 2.3. *If $f \in S(x_0)$ and $g \in C^\infty(x_0)$ with the property $|g|^n \leq |f|$ for some integer n in a neighbourhood of x_0 , then $g \in S(x_0)$.*

LEMMA 2.4 (INTEGRATION BY PARTS). *Let L_1, \dots, L_n be C^∞ vector fields defined in a neighbourhood U of $x_0 \in \text{b}\Omega$ as in the theorems and let $u, v \in C_0^\infty(U \cap \bar{\Omega})$. Then for $i = 1, \dots, n$ we have*

$$\langle L_i u, v \rangle = -\langle u, \bar{L}_i v \rangle + \delta_{in} \int_{\text{b}\Omega} u \bar{v} dS + \langle u, g_i v \rangle,$$

where $g_i \in C^\infty(U \cap \bar{\Omega})$.

LEMMA 2.5. *Let L_1, \dots, L_n, U, x_0 be as in Lemma 2.4. Then for $i \leq n - 1$,*

$$\|\bar{L}_i u\|^2 = \|L_i u\|^2 - \int_{\text{b}\Omega} c_{ii} |u|^2 dS + O(\|u\| \|E(u)\| + \|u\|^2)$$

for all $u \in C^\infty(U \cap \bar{\Omega})$, where $E(u) = \sum_{i=1}^n \|\bar{L}_i u\|^2$.

The notation $O(u)$ represents a quantity such that $|O(u)| \leq C|u|$, where C is independent of u .

PROOF OF THEOREM 2.1. In view of Proposition 1.4 it suffices to prove the theorem for $(0, q)$ forms. Let L_1, \dots, L_n be as in the theorem. Then $\phi \in D_U^{0,q}$ can be written as $\phi = \sum_J \phi_J \bar{\omega}_J$. By standard integration by parts (Folland–Kohn, Hormander, and others) one has

$$Q(\phi, \phi) \geq C \left(\sum_{j \notin J} \|\bar{L}_j \phi_J\|^2 + \sum_{j \in J} \|L_j \phi_J\|^2 + \sum_{\substack{j, h, H, S \\ j \neq h}} \int_{b\Omega} \varepsilon_{hS}^J \varepsilon_{jS}^H c_{hj} \phi_J \bar{\phi}_H dS + R^{(1)}(\phi) \right),$$

with $0 < C < 1$; $R^{(1)}(\phi) = O(\|\bar{L}\phi\| \|\phi\| + \|\phi\|^2)$.

Integrate by parts for the terms $\|L_j \phi_J\|^2$ where $j \leq k$ or $j = n$ in the above expression. For $j \leq k$ we use Lemma 2.5.

$$\|L_j \phi_J\|^2 = \|\bar{L}_j \phi_J\|^2 + \sum_{b\Omega} c_{jj} |\phi_J|^2 dS + R_{jJ}(\phi),$$

where

$$R_{jJ}(\phi) = \sum_{j=1}^{n-1} \langle \bar{L}_j \phi_J, h_j \phi_J \rangle + O(\|\bar{L}_n \phi\| \|\phi\| + \|\phi\|^2),$$

$h_j \in C^\infty(U \cap \bar{\Omega})$.

For $j = n$, $\phi_J = 0$ on $b\Omega$. Integrating by parts gives no boundary term and, hence, gives the same form as above, except that the boundary term is zero. Hence,

$$Q(\phi, \phi) \geq C \left(\sum_{\substack{j \notin J \text{ or} \\ j \in \{1, \dots, k, n\}}} \|\bar{L}_j \phi_J\|^2 + \sum_{\substack{j \in J \text{ and} \\ k < j < n}} \|L_j \phi_J\|^2 + \sum_{IJ} \int_{b\Omega} A_{IJ}^{(k)} \phi_I \bar{\phi}_J dS \right) + R(\phi),$$

where $R(\phi) = R^{(1)}(\phi) + \sum_{jJ} R_{jJ}(\phi)$.

Using assumption (2) in the theorem, given any $\varepsilon > 0$ there exists $C > 0$ such that

$$|R(\phi)| \leq (Q(\phi, \phi) + \|\bar{L}_n \phi\|^2) + C\|\phi\|^2.$$

Finally, we get

$$(2.3) \quad C(Q(\phi, \phi) + \|\phi\|^2) \geq \sum_{\substack{j \notin J \text{ or} \\ j \in \{1, \dots, k, n\}}} \|\bar{L}_j \phi_J\|^2 + \sum_{\substack{j \in J \text{ and} \\ k < j < n}} \|L_j \phi_J\|^2 + \sum_{IJ} \int_{b\Omega} A_{IJ}^{(k)} \phi_I \bar{\phi}_J dS$$

for all $\phi \in D_U^{0,q}$.

Before proceeding, we make an observation. For all those J containing n we have

$$\|\phi_J\|_1^2 \leq C(Q(\phi, \phi) + \|\phi\|^2).$$

To complete the proof of the theorem we need the following four facts.

$$(2.4) \quad |||r\phi|||_1^2 \leq C(Q(\phi, \phi) + \|\phi\|^2) \quad \text{for all } \phi \in D_U^{0,q}.$$

$$(2.5) \quad ||| \sum_I A_{IJ}^{(k)} \phi_I |||_{1/2}^2 \leq C(Q(\phi, \phi) + \|\phi\|^2) \quad \text{for all } \phi \in D^{0,q} \text{ and } J.$$

(2.6) Let

$$m_I = \begin{cases} L_j(f) \varepsilon_{jH}^I & \text{if } \langle jH \rangle = I \text{ and } j \leq k, \\ 0 & \text{otherwise,} \end{cases}$$

where H is fixed. If f satisfies $|||f\phi|||_\varepsilon^2 \leq C(Q(\phi, \phi) + \|\phi\|^2)$, then there exists $C' > 0$ such that

$$||| \sum_I m_I \phi_I |||_{\varepsilon/2}^2 \leq C'(Q(\phi, \phi) + \|\phi\|^2).$$

(2.7) If (d_{IJ}) is a matrix and for each J there exists $C > 0$ such that

$$||| \sum_I d_{IJ} \phi_I |||_\varepsilon^2 \leq C(Q(\phi, \phi) + \|\phi\|^2),$$

then there exists $C' > 0$ such that

$$||| \det(d_{IJ}) \phi_K |||_\varepsilon^2 \leq C'(Q(\phi, \phi) + \|\phi\|^2) \quad \text{for all } K.$$

In view of the definition of I_m , $1 \in I_m$ means there exists some (d_{IJ}) such that $\det(d_{IJ})(x_0) \neq 0$, where (d_{IJ}) consists of rows of $A^{(k)}$ or rows (m_I) where m_I is of the form (2.6). Therefore (2.7) proves the theorem. It remains to prove (2.4)–(2.7). For the proof of (2.4) we refer to [7].

PROOF OF (2.5). We show that for some $C > 0$,

$$\sum_J ||| \sum_I A_{IJ}^{(k)} \phi_I |||_{1/2}^2 \leq C(Q(\phi, \phi) + \|\phi\|^2) \quad \text{for all } \phi \in D_U^{0,q}.$$

We have

$$\begin{aligned} \sum_J ||| \sum_I A_{IJ}^{(k)} \phi_I |||_{1/2}^2 &= \left| \sum_J \left\langle \sum_I A_{IJ}^{(k)} \phi_I, \Lambda^2 \Lambda^{-1} \sum_I A_{IJ}^{(k)} \phi_I \right\rangle \right| \\ &= \left| \sum_J \left\langle \sum_I A_{IJ}^{(k)} \phi_I, \sum_1^{2n-1} \left(\frac{\partial^2}{\partial t^2} + 1 \right) \Lambda^{-1} \sum_I A_{IJ}^{(k)} \phi_I \right\rangle \right|, \end{aligned}$$

where t_1, \dots, t_{2n-1}, r is a boundary coordinate.

To bound the above expression by $C(Q(\phi, \phi) + \|\phi\|^2)$, it suffices to bound $|\sum_J \langle \sum_I A_{IJ}^{(k)} \phi_I, D^2 \Lambda^{-1} \sum_I A_{IJ}^{(k)} \phi_I \rangle|$, where $D \in \{L_1, \dots, L_{n-1}, \bar{L}_1, \dots, \bar{L}_{n-1}, L_n - \bar{L}_n\}$.

First we consider $D = \bar{L}_i$, $i = 1, \dots, n-1$.

We need only consider $\langle \phi_I, \bar{L}^2 \Lambda^{-1} \phi_J \rangle$ where $I \cap J$ has at least $q - 2$ elements, otherwise one of the coefficients $A_{IJ}^{(k)}$ is zero.

Now for $i = 1, \dots, k$

$$\begin{aligned} |\langle \phi_I, \bar{L}_i^2 \Lambda^{-1} \phi_J \rangle| &\leq C(|\langle S^0 \phi_I, \bar{L}_i \phi_J \rangle| + \|\phi\|^2) \\ &\quad (\text{where } S^0 \text{ is a pseudodifferential operator of order zero}) \\ &\leq C'(Q(\phi, \phi) + \|\phi\|^2), \end{aligned}$$

where we used Schwarz's inequality and (2.3) in the third line.

For $i = k+1, \dots, n-1$, the inequality follows from assumption (2) of the theorem.

Now consider $|\langle \phi_I, L_i^2 \Lambda^{-1} \phi_J \rangle|$, $i = 1, \dots, n-1$.

$$\begin{aligned} |\langle \phi_I, L_i^2 \Lambda^{-1} \phi_J \rangle| &= |\langle \bar{L}_i \phi_I, L_i \Lambda^{-1} \phi_J \rangle| + O(\|\phi\|^2) \quad (\text{by Lemma 2.4}) \\ &\leq C(Q(\phi, \phi) + \|\phi\|^2) \end{aligned}$$

by the same reason as above.

Finally,

$$\begin{aligned} &\left| \sum_J \left\langle \sum_I A_{IJ}^{(k)} \phi_I, (L_n - \bar{L}_n)(L_n - \bar{L}_n) \Lambda^{-1} \sum_K A_{KJ}^{(k)} \phi_K \right\rangle \right| \\ &= \left| \sum_J \left\langle \sum_I A_{IJ}^{(k)} \phi_I, (L_n - \bar{L}_n) S^0 \sum_K A_{KJ}^{(k)} \phi_K \right\rangle \right| \\ &\quad (S^0 \text{ a pseudodifferential operator of order zero}) \\ &\leq \left| \sum_J \left\langle \sum_I A_{IJ}^{(k)} \phi_I, L_n \sum_K A_{KJ}^{(k)} S^0 \phi_K \right\rangle \right| \\ &\quad + \left| \sum_J \left\langle S^0 \sum_I A_{IJ}^{(k)} \phi_I, \bar{L}_n \sum_K A_{KJ}^{(k)} \phi_K \right\rangle \right| + O(\|\phi\|^2). \end{aligned}$$

The second term in the above expression is bounded by $C(Q(\phi, \phi) + \|\phi\|^2)$ by (2.3), and

$$\begin{aligned} &\left| \sum_J \left\langle \sum_I A_{IJ}^{(k)} \phi_I, L_n \sum_K A_{KJ}^{(k)} S^0 \phi_K \right\rangle \right| = \left| \sum_J \left\langle \bar{L}_n \sum_I A_{IJ}^{(k)} \phi_I, \sum_K A_{KJ}^{(k)} S^0 \phi_K \right\rangle \right| \\ &\quad + \int_{\text{b}\Omega} \sum_J \left(\sum_I A_{IJ}^{(k)} \phi_I \right) \left(\overline{\sum_K A_{KJ}^{(k)} S^0 \phi_K} \right) dS + O(\|\phi\|^2). \end{aligned}$$

Using (2.3) once more, we see that the first term is bounded by $C(Q(\phi, \phi) + \|\phi\|^2)$. It remains to estimate the second term.

By writing $u_J = \sum_K A_{KJ}^{(k)} S^0 \zeta \phi_K$, where $\zeta \in C_0^\infty(U')$, $U \subset U'$, and $\zeta = 1$ on U , we obtain

$$\int_{\text{b}\Omega} \sum_J \left(\sum_I A_{IJ}^{(k)} \phi_J \right) \left(\overline{\sum_K A_{KJ}^{(k)} S^0 \phi_K} \right) dS = \int_{\text{b}\Omega} \sum_{IJ} A_{IJ}^{(k)} \phi_I \bar{u}_J dS.$$

Since $A^{(k)}$ is positive semidefinite,

$$\begin{aligned} \sum_{IJ} A_{IJ}^{(k)} \phi_I \bar{u}_J &\leq \left(\sum_{IJ} A_{IJ}^{(k)} \phi_I \bar{\phi}_J \right)^{1/2} \left(\sum_{IJ} A_{IJ}^{(k)} u_I \bar{u}_J \right)^{1/2} \\ &\leq C \left(\sum_{IJ} A_{IJ}^{(k)} \phi_I \bar{\phi}_J \right)^{1/2} \left(\sum_I |u_I|^2 \right)^{1/2}, \end{aligned}$$

and

$$\begin{aligned} \sum_I |u_I|^2 &= \sum_{IK} u_I A_{IK}^{(k)} \overline{\zeta S^0 \phi_K} \\ &\leq C \left(\sum_I |u_I|^2 \right)^{1/2} \left(\sum_{IK} A_{IK}^{(k)} (\zeta S^0 \phi_I) (\overline{\zeta S^0 \phi_K}) \right)^{1/2}, \end{aligned}$$

which imply

$$\sum_I |u_I|^2 \leq C' \sum_{IK} A_{IK}^{(k)} (\zeta S^0 \phi_I) (\overline{\zeta S^0 \phi_K}),$$

and, hence,

$$\sum_{IJ} A_{IJ}^{(k)} \phi_I \bar{u}_J \leq C'' \left(\sum_{IJ} A_{IJ}^{(k)} \phi_I' \bar{\phi}_J \right)^{1/2} \left(\sum_{IK} A_{IK}^{(k)} (\zeta S^0 \phi_I) (\overline{\zeta S^0 \phi_K}) \right)^{1/2}.$$

Therefore,

$$\int \sum_{IJ} A_{IJ}^{(k)} \phi_I \bar{u}_J \leq C''' \left(\int \sum_{IJ} A_{IJ}^{(k)} \phi_I \bar{\phi}_J + \int \sum_{IJ} A_{IJ}^{(k)} (\zeta S^0 \phi_I) (\zeta S^0 \phi_J) \right).$$

By (2.3) the first term on the right side is bounded by $C(Q(\phi, \phi) + \|\phi\|^2)$, and the second term by $C(Q(\zeta S^0 \phi, \zeta S^0 \phi) + \|\zeta S^0 \phi\|^2)$, which is in turn bounded by $C'(Q(\phi, \phi) + \|\phi\|^2)$.

Combining all these estimates, we get (2.5).

PROOF OF (2.6). If f satisfies $\|f\phi\|_\varepsilon^2 \leq C(Q(\phi, \phi) + \|\phi\|^2)$, then

$$\begin{aligned} \left\| \sum_{j \leq k} \varepsilon_{jH}^I (L_j f) \phi_I \right\|_{\varepsilon/2}^2 &= \left\langle \sum_{j \leq k} \varepsilon_{jH}^I (L_j f) \phi_I, \Lambda^\varepsilon \sum_{j \leq k} \varepsilon_{jH}^I (L_j f) \phi_I \right\rangle \\ &= \left\langle \sum_{j \leq k} \varepsilon_{jH}^I L_j (f \phi_I), \Lambda^\varepsilon \sum_{j \leq k} \varepsilon_{jH}^I (L_j f) \phi_I \right\rangle \\ &\quad - \left\langle f \sum_{j \leq k} \varepsilon_{jH}^I L_j \phi_I, \Lambda^\varepsilon \sum_{j \leq k} \varepsilon_{jH}^I (L_j f) \phi_I \right\rangle. \end{aligned}$$

Now,

$$\begin{aligned}
& \left| \left\langle \sum_{j < k} \varepsilon_{jH}^I L_j(f\phi_I), \Lambda^\varepsilon \sum_{j \leq k} \varepsilon_{jH}^I (L_j f)\phi_I \right\rangle \right| \\
&= \left| \left\langle \sum_{j \leq k} \varepsilon_{jH}^I f\phi_I, \Lambda^\varepsilon \bar{L}_j \sum_{j \leq k} \varepsilon_{jH}^I (L_j f)\phi_I \right\rangle \right| + O(\|f\phi\|_\varepsilon \|\phi\|) \\
&= \left| \left\langle \sum_{j \leq k} \varepsilon_{jH}^I \Lambda^\varepsilon(f\phi_I), \bar{L}_j \sum_{j \leq k} \varepsilon_{jH}^I (L_j f)\phi_I \right\rangle \right| + O(\|f\phi\|_\varepsilon \|\phi\|) \\
&\leq O\left(\|f\phi\|_\varepsilon \left(\sum_{j \leq k} \|\bar{L}_j \phi_I\|\right) + \|f\phi\|_\varepsilon \|\phi\|\right) \\
&\leq C(Q(\phi, \phi) + \|\phi\|^2),
\end{aligned}$$

where, in the second line, we used Lemma 2.4 and the fact that if P, P' are pseudodifferential operators of order s, t , respectively, then $[P, P']$ is a pseudodifferential operator of order $s + t - 1$. In the fifth line we used (2.3).

For the second term,

$$\begin{aligned}
& \left| \left\langle f \sum_{j \leq k} \varepsilon_{jH}^I L_j \phi_I, \Lambda^\varepsilon \sum_{j \leq k} \varepsilon_{jH}^I (L_j f)\phi_I \right\rangle \right| \\
&\leq \left| \left\langle \sum_{j \leq k} \varepsilon_{jH}^I L_j \phi_I, \Lambda^\varepsilon \sum_{j \leq k} \varepsilon_{jH}^I (L_j f)f\phi_I \right\rangle \right| + O\left(\left\|\sum_{j \leq k} \varepsilon_{jH}^I L_j \phi_I\right\| \|\phi\|\right) \\
&\leq \left| \left\langle \sum_{j \leq k} \varepsilon_{jH}^I L_j \phi_I, \sum_{j \leq k} \varepsilon_{jH}^I (L_j f)\Lambda^\varepsilon f\phi_I \right\rangle \right| + O\left(\left\|\sum_{j \leq k} \varepsilon_{jH}^I L_j \phi_I\right\| \|\phi\|\right) \\
&\leq C\left(\left\|\sum_{j \leq k} \varepsilon_{jH}^I L_j \phi_I\right\|^2 + \|f\phi\|_\varepsilon^2 + \|\phi\|^2\right).
\end{aligned}$$

Finally, we may assume that $n \notin I$ and, hence,

$$\begin{aligned}
\left\|\sum_{j \leq k} \varepsilon_{jH}^I L_j \phi_I\right\|^2 &= \left\|\sum_{j \in I} \varepsilon_{jH}^I L_j \phi_I - \sum_{\substack{j \in I \\ k < j < n}} \varepsilon_{jH}^I L_j \phi_I\right\|^2 \\
&\leq \left\|\sum_{j \in I} \varepsilon_{jH}^I L_j \phi_I\right\|^2 + \sum_{\substack{j \in I \\ k < j < n}} \|L_j \phi_I\|^2 \leq C(Q(\phi, \phi) + \|\phi\|^2),
\end{aligned}$$

where we get the third line because the first expression in the second line is one square term in the expansion of $\bar{\delta}^*$, and the second term is bounded because of (2.3). This completes the proof.

PROOF OF (2.7). Suppose (d_{IJ}) is a matrix such that for each J ,

$$\left\| \left\| \sum_I d_{IJ} \phi_I \right\| \right\|_\varepsilon^2 \leq C(Q(\phi, \phi) + \|\phi\|^2) \quad \text{for some } \varepsilon > 0.$$

Write $u_J = \sum_I d_{IJ} \phi_I$. Solving for ϕ_I in terms of the u_J 's by Cramer's rule, we get

$$\det(d_{IJ}) \phi_I = \sum_J f_J u_J,$$

where the f_J are linear combinations of $(n-1)$ products of the d_{IJ} . Hence,

$$\| \det(d_{IJ}) \phi_I \|_\varepsilon^2 \leq C \sum_J \left\| \sum_I d_{IJ} \phi_I \right\|_\varepsilon^2 \leq C'(Q(\phi, \phi) + \|\phi\|^2)$$

by assumption.

REMARK 1. This theorem provides a way to compute some ideals and conclude the subellipticity if, after a finite number of computations, we get 1 in one of these increasing ideals if the Levi-form satisfied the required conditions.

REMARK 2. In Theorem 2.1 we require $k \geq n - q$. It seems from the proof that there is no restriction on k . Actually, if $k < n - q$ it is impossible for $1 \in \bigcup_m I_m$. This is because the term $A_{II}^{(k)}$, where $I = (n - q, \dots, n - 1)$, is identically equal to zero. Hence, the determinant of the positive semidefinite matrix is zero throughout a neighbourhood. Thus, the ideals could not generate any function which is not zero on the boundary. When $k = n - q$, $A^{(k)} \geq 0$ implies that the submatrix $(c_{ij})_{i,j=1,\dots,n-q}$ of the Levi-form is positive semidefinite. If only some $(n-q) \times (n-q)$ minor of Levi-form is positive semidefinite, it is not sufficient to conclude that the corresponding $A^{(k)} \geq 0$.

REMARK 3. The assumption that some choice of L_1, \dots, L_n and k gives a positive semidefinite matrix $A^{(k)}$ depends very much on the choice of L_1, \dots, L_n , as we can see from the following easy example.

Let $r = \operatorname{Re} z_3 + |z_1|^2 - |z_2|^2$, where we are working on $(0, 2)$ forms. Let

$$L_i = r_{z_i} \frac{\partial}{\partial z_3} - r_{z_3} \frac{\partial}{\partial z_i}, \quad i = 1, 2, \quad L_3 = \frac{\partial}{\partial z_3}.$$

We get $A^{(k)} \geq 0$ by choosing $k = 1$. Actually, $1 \in \bigcup I_m$. If we transform coordinates by setting

$$\begin{aligned} w_1 &= z_1 + z_2, & w_2 &= z_1 - z_2, & w_3 &= z_3, \\ \tilde{L}_i &= r_{w_i} \frac{\partial}{\partial w_3} - r_{w_3} \frac{\partial}{\partial w_i}, & i &= 1, 2, & \tilde{L}_3 &= \frac{\partial}{\partial w_3}, \end{aligned}$$

then $A^{(k)} = 0$ by choosing $k = 1$.

EXAMPLE. If the domain Ω is defined by

$$r = \operatorname{Re} z_n + g(z_1, \dots, z_p) = \sum_{p+1}^{n-1} |f_i(z_i)|^2,$$

where $f_i, g \in C^\infty$ and $(\partial^2 g / \partial z_i \partial \bar{z}_j)_{i,j=1,\dots,p}$ is positive semidefinite. Then the Levi matrix is in the form

$$\left(\begin{array}{c|c} C & 0 \\ \hline 0 & D \end{array} \right)$$

where C is positive semidefinite and $D \leq 0$ is diagonal. Then the associated matrix is positive semidefinite. Also, condition (2) in Theorem 2.1 is satisfied (see first proof of Theorem 2.5).

More generally, if the Levi-form of L_i is either nonnegative or nonpositive in a whole neighbourhood of x_0 for each $k+1 \leq i \leq n-1$, then condition (2) of Theorem 2.1 is satisfied.

COROLLARY 2.3. *Let Ω be a domain in \mathbb{C}^n with C^∞ boundary, $x_0 \in \text{b}\Omega$. Suppose L is a C^∞ vector field tangential on $\text{b}\Omega$ such that:*

- (1) *The Levi-form of L is nonnegative in a neighbourhood $U \cap \bar{\Omega}$ of x_0 .*
- (2) *L is of finite type at x_0 (i.e., there exists a polynomial p such that*

$$(p(L, \bar{L})(\partial\bar{\partial}r, L\bar{L}))(x_0) \neq 0).$$

Then a subelliptic estimate holds for $(p, n-1)$ forms at x_0 .

PROOF. We use assumption (1) to verify conditions (1) and (2) of Theorem 2.1. Let L_1, \dots, L_n be a C^∞ basis for $T^{1,0}$ on Ω , L_1, \dots, L_{n-1} tangential and $L_n = L$.

To show condition (1) we set $k = 1$. Then $A^{(k)}$ is a singleton matrix (c_{11}) , and $c_{11} \geq 0$ by our assumption.

To show condition (2) we once again expand $Q(\phi, \phi)$.

We may write

$$\phi = \sum \phi_i \bar{\omega}_1 \wedge \dots \wedge \hat{\omega}_i \wedge \dots \wedge \bar{\omega}_n = \sum \phi_i \omega_i,$$

where \hat{i} means i is omitted.

$$\begin{aligned} Q(\phi, \phi) &= \left\| \sum_1^n (-1)^i \bar{L}_i \phi_i \right\|^2 + \sum_{\substack{ij \\ i \neq j}} \|L_i \phi_j + (-1)^{i+j+1} \bar{L}_j \phi_i\|^2 + O(\|\phi\|^2) \\ &= \sum_1^n \|\bar{L}_i \phi_i\|^2 + \sum_{i \neq j} \|L_i \phi_j\|^2 + O(\|\phi\|^2) + R(\phi). \end{aligned}$$

There is no boundary term because for $i \neq j$, $n \in \hat{i}$ or \hat{j} . Here,

$$R(\phi) = \sum_{ij} \sum_1^n \langle \bar{L}_k \phi_i, f_{ijk} \phi_j \rangle + \sum_{ij} \sum_{k=1}^{n-1} \langle L_k \phi_i, g_{ijk} \phi_j \rangle,$$

where $f_{ijk}, g_{ijk} \in C^\infty$ in a neighbourhood of x_0 .

Set

$$F(\phi) = \sum_1^n \|\bar{L}_i \phi_i\|^2 + \sum_{i \neq j} \|L_i \phi_j\|^2.$$

We want to show $|R(\phi)| \leq \varepsilon F(\phi) + C\|\phi\|^2$, which in turn establishes condition (2) of Theorem 2.1.

We separate the estimation of $\langle \bar{L}_k \phi_i, f_{ijk} \phi_j \rangle$ into three cases. We may assume $k \leq n-1$. If $k = n$ the expression can be dealt with similarly to one of the following expressions:

- (i) $k = i$,
- (ii) $k \neq i$, $k \neq j$,
- (iii) $k \neq i$, $k = j$.

For the first two cases $|\langle \bar{L}_k \phi_{\hat{i}}, f_{ijk} \phi_{\hat{j}} \rangle|$ is bounded by $\varepsilon F(\phi) + C\|\phi\|^2$ because one of the $\|\bar{L}_k \phi_{\hat{i}}\|^2$ or $\|L_k \phi_{\hat{j}}\|^2$ is in F . For case (iii) we note that if $i \neq j$, then either $n \in \hat{i}$ or $n \in \hat{j}$. Suppose $n \in \hat{i}$. We have

$$\|L_k \phi_{\hat{i}}\|^2 = \|\bar{L}_k \phi_{\hat{i}}\|^2 + O\left(\sum_{k=1}^n \|\bar{L}_k \phi_{\hat{i}}\| \|\phi\|\right).$$

So case (iii) is also settled.

The terms

$$\langle L_k \phi_{\hat{i}}, g_{ijk} \phi_{\hat{j}} \rangle = \langle \phi_{\hat{i}}, \bar{L}_k(g_{ijk} \phi_{\hat{j}}) \rangle + O(\|\phi\|^2),$$

hence, they are estimated as above.

We finish the proof by showing that finite type at x_0 implies that $1 \in I_m$ for some m .

The matrix $A^{(k)} = (c_{11})$, therefore $c_{11} = \det(c_{11}) \in I_1$. $L_1(c_{11}) \in I_2$, and by conjugation $\bar{L}_1(c_{11}) = \overline{L_1(c_{11})} \in I_2$. By induction $D_1 D_2 \cdots D_m c_{11} \in I_{m+1}$, where D_i is either L_1 or \bar{L}_1 . If L_1 is of finite type, then for some polynomial p of order m we have $(p(L_1, \bar{L}_1)c_{11})(x_0) \neq 0$, which implies $1 \in I_{m+1}$ and thus proves the corollary.

REMARK. We can also prove Corollary 2.3 directly. We need to show that

$$|||c_{11}\phi|||_{1/2}^2 \leq C(Q(\phi, \phi) + \|\phi\|^2)$$

and, if $|||f\phi|||_{\varepsilon/2}^2 \leq C(Q(\phi, \phi) + \|\phi\|^2)$,

$$|||L_1(f)\phi|||_{\varepsilon/2}^2 \leq C'(Q(\phi, \phi) + \|\phi\|^2)$$

and

$$|||\bar{L}_1(f)\phi|||_{\varepsilon/2}^2 \leq C''(Q(\phi, \phi) + \|\phi\|^2).$$

These are simplifications of (2.5) and (2.6), and this reduced form is easy to prove.

PROOF OF THEOREM 2.2. We only sketch the proof since the technique is exactly the same as the proof of Theorem 2.1.

First of all, for $\phi \in D_U^{0,q}$,

$$\begin{aligned} Q(\phi, \phi) &= \|\bar{\partial}\phi\|^2 + \|\bar{\partial}^* \phi\|^2 \\ &\geq C \left(\sum_K' \left\| \sum_{jJ} \varepsilon_{jJ}^K \bar{L}_j \phi_J \right\|^2 + \sum_H' \left\| \sum_{jJ} \varepsilon_{jH}^J L_j \phi_J \right\|^2 + O(\|\phi\|^2) \right) \\ &= C \left(\sum_{\substack{j \notin J \\ j > k}} \|\bar{L}_j \phi_J\|^2 + \sum_{\substack{j \in J \\ \text{or } j \leq k}} \|L_j \phi_J\|^2 \right. \\ &\quad \left. + \int_{\text{b}\Omega} \sum B_{IJ}^{(k)} \phi_I \bar{\phi}_J dS + R(\phi) + O(\|\phi\|^2) \right), \end{aligned}$$

where $|R(\phi)| \leq \varepsilon Q(\phi, \phi) + C\|\phi\|^2$, using assumption (2). Here we integrated by parts on those $\|\bar{L}_j \phi_J\|^2$ where $j \leq k$. Finally, we have

$$(2.8) \quad \sum_{\substack{j \notin J \\ j > k}} \|\bar{L}_j \phi_J\|^2 + \sum_{\substack{j \in J \\ \text{or } j \leq k}} \|L_j \phi_J\|^2 + \int_{\text{b}\Omega} \sum B_{IJ}^{(k)} \phi_I \bar{\phi}_J dS \leq C(Q(\phi, \phi) + \|\phi\|^2).$$

Analogous to (2.5) and (2.6) we have

$$(2.9) \quad \left\| \left\| \sum_I B_{IJ}^{(k)} \phi_I \right\| \right\|_{1/2}^2 \leq C(Q(\phi, \phi) + \|\phi\|^2) \quad \text{for all } \phi \in D_U^{0,q}.$$

Let

$$(2.10) \quad \begin{aligned} m_i &= \varepsilon_{jI}^J(\bar{L}_j f) && \text{if } \langle jI \rangle = J \text{ and } j \leq k, \\ &= 0 && \text{otherwise,} \end{aligned}$$

where f satisfies $\|f\phi\|_\varepsilon^2 \leq C(Q(\phi, \phi) + \|\phi\|^2)$ and J is a fixed $(q+1)$ -tuple. Then

$$\left\| \left\| \sum m_I \phi_I \right\| \right\|_{\varepsilon/2}^2 \leq C'(Q(\phi, \phi) + \|\phi\|^2).$$

(2.4) and (2.7) are still valid here. Combining (2.4), (2.7), (2.9), and (2.10), we prove Theorem 2.2 in the same way as Theorem 2.1.

COROLLARY 2.4. *Let Ω be a domain in \mathbf{C}^n with C^∞ boundary, and let $x_0 \in \text{b}\Omega$. Suppose L is a C^∞ vector field tangential on $\text{b}\Omega$ such that:*

- (1) *The Levi-form of L is nonpositive in a neighbourhood $U \cap \bar{\Omega}$ of x_0 .*
- (2) *L is of finite type at x_0 .*

Then a subelliptic estimate holds for 0 forms at x_0 .

PROOF.

$$\begin{aligned} Q(\phi, \phi) &= \|\bar{\partial}\phi\|^2 = \sum_1^n \|\bar{L}_i \phi\|^2 \\ &= \sum_2^n \|\bar{L}_i \phi\|^2 + \|L_1 \phi\|^2 - \int_{\text{b}\Omega} c_{11} |\phi|^2 dS \\ &\quad + O\left(\sum_1^{n-1} \|L_i \phi\| \|\phi\| + \sum_1^n \|\bar{L}_i \phi\| \|\phi\|\right). \end{aligned}$$

Hence, we have

$$\|L_1 \phi\|^2 + \sum_2^n \|\bar{L}_i \phi\|^2 - \int_{\text{b}\Omega} c_{11} |\phi|^2 dS \leq C(Q(\phi, \phi) + \|\phi\|^2).$$

Statements (1) and (2) of Theorem 2.2 are satisfied because of the above inequality, and finite type implies $1 \in J_m$. This proves the corollary.

We now prove a theorem in the case when the Levi-form is smoothly diagonalisable in a neighbourhood. This is a very restrictive case, but we can see how the method works here.

THEOREM 2.5. *Let Ω be a domain in \mathbf{C}^n with C^∞ boundary, and let L_1, \dots, L_n be C^∞ vector fields as before. Let U be a neighbourhood of x_0 such that the Levi-form with respect to L_1, \dots, L_n is diagonal. We call the diagonal elements $\lambda_1(z), \dots, \lambda_{n-1}(z)$. Suppose $\lambda_1(z), \dots, \lambda_{n-q}(z) \geq 0$ in U and, for each $j > n - q$, $\lambda_j(z) \geq 0$ or ≤ 0 in U . Furthermore, if for $j = 1, \dots, n - q$, there exist polynomials p_j such that $(p_j(L_j, \bar{L}_j)\lambda_j)(x_0) \neq 0$, then a subelliptic estimate holds for (p, q) forms at x_0 .*

PROOF. We only sketch the proof here. First we calculate $Q(\phi, \phi)$. Almost the same as before,

$$(2.11) \quad \sum_{\substack{j \notin J \\ \text{or } j \in \{1, \dots, n-q, n\}}} \|\bar{L}_j \phi_J\|^2 + \sum_{\substack{j \in J \\ n-q < j < n}} \|L_j \phi_J\|^2 + \sum_{\substack{j \in J \\ j \leq k}} \int_{\text{b}\Omega} \lambda_j |\phi_J|^2 dS \leq C(Q(\phi, \phi) + \|\phi\|^2).$$

The terms $|\bar{L}_j \phi_J| |\phi_K|$ are all bounded by $\varepsilon Q(\phi, \phi) + C\|\phi\|^2$, using the assumption that the Levi-form is diagonal and the λ_i 's are of the same sign in U .

If $n \in J$ then we have $\|\phi_J\|_1^2 \leq C(Q(\phi, \phi) + \|\phi\|^2)$. For each ϕ_J where $n \notin J$ we claim there exists a $j \leq n - q$ such that

$$|||\lambda_j \phi_J|||_{1/2}^2 \leq C(Q(\phi, \phi) + \|\phi\|^2).$$

To find such a j we notice that if J is a q -tuple not containing n , then $\{1, \dots, n - q\} \cap J$ is nonempty. We pick a $j \in \{1, \dots, n - q\} \cap J$. For this j , clearly,

$$|\langle L_i(\lambda_j \phi_J), \lambda_j \phi_J \rangle| \leq C(Q(\phi, \phi) + \|\phi\|^2) \quad \text{for } i = 1, \dots, n - 1$$

and

$$\begin{aligned} |\langle L_n(\lambda_j \phi_J), \lambda_j \phi_J \rangle| &\leq |\langle \lambda_j \phi_J, \bar{L}_n(\lambda_j \phi_J) \rangle| + \int_{\text{b}\Omega} |\lambda_j \phi_J|^2 dS + C\|\phi\|^2 \\ &\leq C(Q(\phi, \phi) + \|\phi\|^2), \end{aligned}$$

where we used (2.11) and the fact that $|\lambda_j| \leq \text{constant}$ in a neighbourhood of x_0 . Hence,

$$|||\lambda_j \phi_J|||_{1/2}^2 \leq C(Q(\phi, \phi) + \|\phi\|^2).$$

The next step is to show that if

$$|||f \phi_J|||_\varepsilon^2 \leq C(Q(\phi, \phi) + \|\phi\|^2),$$

then

$$|||\bar{L}_j(f) \phi_J|||_{\varepsilon/2}^2 \leq C(Q(\phi, \phi) + \|\phi\|^2)$$

and

$$|||L_j(f) \phi_J|||_{\varepsilon/2}^2 \leq C(Q(\phi, \phi) + \|\phi\|^2).$$

We will not prove this because it is similar to the proof of (2.6), but we used the fact, by (2.11),

$$\|\bar{L}_j \phi_J\|^2 \leq C(Q(\phi, \phi) + \|\phi\|^2)$$

and

$$\|L_j \phi_J\|^2 = \|\bar{L}_j \phi_J\|^2 + \int \lambda_j |\phi_J|^2 + R(\phi) \leq C(Q(\phi, \phi) + \|\phi\|^2).$$

Finally, using the above two facts, it is easily seen that for all polynomials p ,

$$|||(p(L_j, \bar{L}_j)\lambda_j)\phi_J|||_\varepsilon^2 \leq C(Q(\phi, \phi) + \|\phi\|^2),$$

where ε depends on the degree of p .

By assumption there exist p_j 's such that $p_j(L_j, \bar{L}_j)\lambda_j(x_0) \neq 0$. Hence, for each J ,

$$|||\phi_J|||_\varepsilon^2 \leq C(Q(\phi, \phi) + \|\phi\|^2) \quad \text{for some } \varepsilon.$$

Taking the smallest ε for all J 's, we get

$$|||\phi|||_\varepsilon^2 \leq C(Q(\phi, \phi) + \|\phi\|^2).$$

This completes the proof.

There are still many problems in proving the existence of subelliptic estimates in nonpseudoconvex domains. It seems difficult to find a nice general way of defining ideals so that it can apply to all domains. Rather, we know a way to tackle the problem. The difficulty mainly lies in finding a way of expressing $Q(\phi, \phi)$ so that we have a good estimate to work with.

In the case $q = n - 1$ it is still not known whether a subelliptic estimate holds at $x_0 = 0$ in domains such as

$$r = \operatorname{Re} z_3 - |z_1^2 - z_2^3|^2 + |z_1|^{10} + |z_2|^{10},$$

where the nonnegative C^∞ vector field

$$L = 3z_2^2 \frac{\partial}{\partial z_1} + 2z_1 \frac{\partial}{\partial z_2} - \frac{1}{2}(3z_2^2 r_{z_1} + 2z_1 r_{z_2}) \frac{\partial}{\partial z_3}$$

vanishes at the origin.

Moreover, there should be some geometrical meaning to the ideals we have defined similar to those in pseudoconvex domains. We guess that they measure the "order of contact" (in a suitable sense) of $(n - q)$ -dimensional varieties with the boundary in the nonnegative direction.

3. Necessary conditions for subellipticity. In this section we first use an explicit sequence of forms to give another proof of a theorem by Derridj [3]. Then we use a similar sequence to prove a proposition to show that the method may be very useful in studying the necessary conditions for subellipticity in many nonpseudoconvex domains. Finally, we remark on a theorem of Catlin [1], showing that if there is an $(n - 1)$ -dimensional manifold lying on the boundary, then the solution to the $\bar{\partial}$ -Neumann problem is not hypoelliptic.

THEOREM 3.1. *Let Ω be a domain in \mathbb{C}^n with C^∞ boundary. If the Levi-form is nondegenerate at $x_0 \in \partial\Omega$, then a subelliptic estimate holds for (p, q) forms at x_0 if and only if the Levi-form has at least $n - q$ positive eigenvalues or at least $q + 1$ negative eigenvalues.*

PROOF. This is a theorem of Derridj [3], but we use another method to prove it.

We need to show that if the Levi-form has $n - q - 1$ positive eigenvalues and q negative eigenvalues at x_0 , then there is no subelliptic estimate at x_0 for $(0, q)$ forms.

We prove this by contradiction. Assuming the subelliptic estimate of order ε holds at $x_0 = 0$, we construct a sequence u_m of $(0, q)$ forms. By comparing both sides of the inequality $\|u_m\|_\varepsilon^2 \leq C(Q(u_m, u_m) + \|u_m\|^2)$, we conclude that $\varepsilon \leq 0$.

First of all, by a change of coordinates we may assume that

$$r = 2 \operatorname{Re} z_n - \sum_{i=1}^q |z_i|^2 + \sum_{j=q+1}^{n-1} |z_j|^2 + \lambda(z),$$

where $\lambda(z) \leq C(|z_n||z'| + |z|^3)$ and $z' = (z_1, \dots, z_{n-1})$. (The reader may refer to [3 or 4] for this.) We define a basis of holomorphic vector fields L_1, \dots, L_n as follows:

$$L_i = r_{z_i} \frac{\partial}{\partial z_n} - r_{z_n} \frac{\partial}{\partial z_i}, \quad i = 1, \dots, n-1, \quad L_n = \frac{\partial}{\partial z_n}.$$

Let $\omega_1, \dots, \omega_n$ be a basis of $(1, 0)$ forms dual to L_1, \dots, L_n . Define

$$u_m = \left(\frac{1}{z_n - \sum_{i=1}^q |z_i|^2 - 1/m} \right)^p \phi(x_1) \phi(y_1) \cdots \phi(x_n) \phi(y_n) \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_q,$$

where p is a fixed large integer and $\phi \in C_0^\infty(\mathbf{R})$ will be chosen later. Clearly $u_m \in D_U^{0,q}$.

$$\begin{aligned} Q(u_m, u_m) + \|u_m\|^2 &= \|\bar{\partial}^* u_m\|^2 + \|\bar{\partial} u_m\|^2 + \|u_m\|^2 \\ &= \|L_1 u_m\|^2 + \cdots + \|L_q u_m\|^2 \\ &\quad + \|\bar{L}_{q+1} u_m\|^2 + \cdots + \|\bar{L}_n u_m\|^2 + \|u_m\|^2. \end{aligned}$$

Here we use u_m to represent both the function and the q form, the usage of which should be clear from the context.

For $i = 1, \dots, q$,

(3.1)

$$\begin{aligned} L_i u_m &= r_{z_i} \frac{\partial u_m}{\partial z_n} - r_{z_n} \frac{\partial u_m}{\partial z_i} \\ &= \frac{-p}{(z_n - \sum |z_i|^2 - 1/m)^{p+1}} \left(r_{z_i} + \bar{z}_i \left(1 + \frac{\partial \lambda}{\partial z_n} \right) \right) \phi(x_1) \cdots \phi(y_n) \\ &\quad + \frac{1}{2} r_{z_i} \frac{1}{(z_n - \sum |z_i|^2 - 1/m)^p} \phi(x_1) \\ &\quad \quad \quad \cdots \phi(y_{n-1}) (\phi'(x_n) \phi(y_n) - i \phi(x_n) \phi'(y_n)) \\ &\quad - \frac{1}{2} \left(1 + \frac{\partial \lambda}{\partial z_n} \right) \frac{1}{(z_n - \sum |z_i|^2 - 1/m)^p} \phi(x_1) \\ &\quad \quad \quad \cdots \hat{\phi}(x_i) \hat{\phi}(y_i) \cdots \phi(y_n) (\phi'(x_i) \phi(y_i) - i \phi(x_i) \phi'(y_i)). \end{aligned}$$

Integrating both sides of (3.1) with respect to x_i, y_i 's, we get

$$\|L_i u_m\|^2 \leq \text{sum of three terms.}$$

We calculate the first term, calling it I .

$$I = \int_{\Omega} \left| \frac{-p}{(z_n - \sum |z_i|^2 - 1/m)^{p+1}} \right|^2 \left| r_{z_i} + \bar{z}_i \left(1 + \frac{\partial \lambda}{\partial z_n} \right) \right|^2 |\phi(x_1) \cdots \phi(y_n)|^2 dx dy.$$

We notice that $r_{z_i} = -\bar{z}_i + \mu_i(z)$, where $\mu_i(z) = O(|z_n| + |z|^2)$, so

$$r_{z_i} + \bar{z}_i (1 + \partial \lambda / \partial z_n) = O(|z_n| + |z|^2).$$

Using the implicit function theorem, we see that

$$x_n = \frac{r}{2} + \sum_{i=1}^q \frac{|z_i|^2}{2} - \sum_{j=q+1}^{n-1} \frac{|z_j|^2}{2} + \psi(z', y_n, r),$$

where $\psi(z', y_n, r) = O(|y_n||z'| + |r||z'| + |(r, z', y_n)|^3)$.

With a change of coordinates

$$\tilde{x}_i = m^{1/2}x_i, \quad \tilde{y}_i = m^{1/2}y_i, \quad i = 1, \dots, n-1, \quad \text{and} \quad \tilde{x}_n = mx_n, \quad \tilde{y}_n = my_n,$$

we have

$$I \leq Cm^{2p-n-1} \int \left| \frac{1}{\tilde{z}_n - \sum |z_i|^2 - 1} \right|^{2p+2} |f(\tilde{z})|^2 \left| \phi\left(\frac{\tilde{x}_1}{m^{1/2}}\right) \cdots \phi\left(\frac{\tilde{y}_n}{m}\right) \right|^2 d\tilde{x} d\tilde{y},$$

where $f \in C^\infty(\mathbf{C}^n)$, $|f(z)| \leq C'|\tilde{z}|$ for a constant C' .

$$I \leq Cm^{2p-n-1} \int_{x_n=-\infty}^{\sum |\tilde{z}_i|^2/2 - \sum |\tilde{z}_j|^2/2 + \xi(\tilde{z}', \tilde{y}_n)} \frac{1}{((\tilde{x}_n - \sum |\tilde{z}_i|^2 - 1)^2 + \tilde{y}_n^2)^{p+1}} |\tilde{z}|^2 \cdot \left| \phi\left(\frac{\tilde{x}_1}{m^{1/2}}\right) \cdots \phi\left(\frac{\tilde{y}_n}{m}\right) \right|^2 d\tilde{x} d\tilde{y}.$$

For suitable choice of ϕ ,

$$|\xi(z', y_n)| \leq 1/2 \quad \text{when } z \in \text{supp}(\phi(\tilde{x}_1/m^{1/2}) \cdots \phi(\tilde{y}_n/m)).$$

Integrating with respect to x_n , we get

$$\begin{aligned} I &\leq Cm^{2p-n-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{((\sum_{i=1}^n |\tilde{z}_i|^2 + 1)^2 + \tilde{y}_n^2)^p} \\ &\quad \cdot \left| \phi\left(\frac{\tilde{x}_1}{m^{1/2}}\right) \cdots \phi\left(\frac{\tilde{y}_n}{m}\right) \right|^2 d\tilde{x}' d\tilde{y} \\ &\leq Cm^{2p-n-1}, \end{aligned}$$

where we get the last line because $1/(x_1^2 + \cdots + x_{n-1}^2 + y_1^2 + \cdots + y_n^2 + 1)^p$ is an integrable function in \mathbf{R}^{2n-1} for sufficiently large p .

Using the same argument, we can compute the integral of the last two terms on the right side of (3.1). The integral of the second term is bounded by $\text{const } m^{2p-n-2}$; the last, by $\text{const } m^{2p-n-1}$. Hence,

$$\|L_i u_m\|^2 \leq Cm^{2p-n-1}, \quad i = 1, \dots, q.$$

For $j = q+1, \dots, n-1$,

$$\begin{aligned} \bar{L}_j u_m &= \bar{r}_{z_j} \frac{\partial u_m}{\partial \bar{z}_n} - \bar{r}_{z_n} \frac{\partial u_m}{\partial \bar{z}_j} \\ &= \frac{1}{2} \bar{r}_{z_j} \left(\frac{1}{z_n - \sum |z_i|^2 - 1/m} \right)^p \phi(x_1) \\ &\quad \cdots \phi(y_{n-1}) (\phi'(x_n) \phi(y_n) + i \phi(x_n) \phi'(y_n)) \\ &\quad - \frac{1}{2} \left(1 + \frac{\partial \lambda}{\partial z_n} \right) \left(\frac{1}{z_n - \sum |z_j|^2 - 1/m} \right)^p \phi(x_1) \\ &\quad \cdots \hat{\phi}(x_j) \hat{\phi}(y_j) \cdots \phi(y_n) (\phi'(x_j) \phi(y_j) + i \phi(x_j) \phi'(y_j)). \end{aligned}$$

Just as in the computation of $\|L_i u_m\|^2$, we get

$$\|\bar{L}_j u_m\|^2 \leq \text{const } m^{2p-n-1}, \quad j = q+1, \dots, n-1.$$

$\|\bar{L}_n u_m\|^2$ and $\|u_m\|^2$ are also bounded by $\text{const } m^{2p-n-1}$ with the same argument. Hence,

$$(3.2) \quad Q(u_m, u_m) + \|u_m\|^2 \leq C m^{2p-n-1}.$$

We proceed to compute $\|u_m\|_\varepsilon^2$. We use new coordinates $(x_1, \dots, x_{n-1}, y_1, \dots, y_n, r)$, where r is the defining function for Ω . Let $(\xi_1, \dots, \xi_{2n-1})$ be the Fourier transform variable dual to (x_1, \dots, y_n) .

$$\begin{aligned} \|u_m\|_\varepsilon^2 &= \int (1 + |\xi|^2)^\varepsilon \left| \int \frac{1}{(r/2 - \sum_{i=1}^{n-1} |z_i|^2/2 + \psi - 1/m + i y_n)^p} \right. \\ &\quad \left. \cdot e^{-i\xi \cdot (x', y)} \phi(x_1) \cdots \phi(y_n) dx' dy \right|^2 d\xi dr \\ &\geq \int |\xi_{2n-1}|^{2\varepsilon} \left| \int \frac{1}{(r/2 - \sum |z_i|^2/2 + \psi - 1/m + i y_n)^p} \right. \\ &\quad \left. \cdot e^{-i\xi_{2n-1} y_n} \phi(x_n) \phi(y_n) dy_n \right|^2 \\ &\quad \cdot |\phi(x_1) \cdots \phi(y_{n-1})|^2 d\xi dr dx' dy', \end{aligned}$$

where we used Plancherel's theorem on ξ_1, \dots, ξ_{2n-2} in the second line. We use similar transformations as before, where:

$$\begin{aligned} \tilde{x}_i &= m^{1/2} x_i, \quad \tilde{y}_i = m^{1/2} y_i, \quad i = 1, \dots, n-1, \\ \tilde{x}_n &= m x_n, \quad \tilde{y}_n = m y_n, \quad \tilde{\xi}_{2n-1} = \xi_{2n-1}/m, \quad r = mr. \end{aligned}$$

Then

$$\|u_m\|_\varepsilon^2 \geq \text{const } m^{2\varepsilon+2p-n-1} I_m,$$

where

$$\begin{aligned} I_m &= \int |\tilde{\xi}_{2n-1}|^{2\varepsilon} \left| \int \frac{1}{(\tilde{r}/2 - \sum |\tilde{z}_i|^2/2 - \tilde{\psi}(\tilde{z}', \tilde{y}_n, \tilde{r}) - 1 + i \tilde{y}_n)^p} \right. \\ &\quad \left. \cdot e^{-i\tilde{\xi}_{2n-1} \tilde{y}_n} \phi(x_n) \phi\left(\frac{\tilde{y}_n}{m}\right) d\tilde{y}_n \right|^2 \\ &\quad \cdot \left| \phi(\tilde{x}_1/m^{1/2}) \cdots \phi\left(\frac{\tilde{y}_{n-1}}{m^{1/2}}\right) \right|^2 d\tilde{\xi}_{2n-1} d\tilde{r} d\tilde{x}' d\tilde{y}' \end{aligned}$$

where

$$\tilde{\psi}(\tilde{z}', \tilde{y}_n, \tilde{r}) = m\psi\left(\frac{\tilde{z}'}{m^{1/2}}, \frac{\tilde{y}_n}{m}, \frac{\tilde{r}}{m}\right).$$

Clearly, for fixed $(\tilde{z}', \tilde{y}_n, \tilde{r})$, $\psi \rightarrow 0$ as $m \rightarrow \infty$. Hence, for each $(\tilde{\xi}_{2n-1}, \tilde{x}', \tilde{y}', \tilde{r})$, we have by the Lebesgue dominated convergence theorem,

$$\begin{aligned} &\int \frac{1}{(\tilde{r}/2 \sum |\tilde{z}_i|^2/2 - 1 - \tilde{\psi} + i \tilde{y}_n)^p} e^{-i\tilde{\xi}_{2n-1} \tilde{y}_n} \phi(x_n) \phi\left(\frac{\tilde{y}_n}{m}\right) d\tilde{y}_n \\ &\rightarrow \int \frac{1}{(\tilde{r}/2 - \sum |\tilde{z}_i|^2/2 - 1 + i \tilde{y}_n)^p} e^{-i\tilde{\xi}_{2n-1} \tilde{y}_n} d\tilde{y}_n. \end{aligned}$$

Using Fatou's lemma we get

$$\begin{aligned} \liminf I_m &\geq \int |\tilde{\xi}_{2n-1}|^{2\varepsilon} \left| \int \frac{1}{(\tilde{r}/2 - \sum |\tilde{z}_i|^2/2 - 1 + i\tilde{y}_n)^p} \right. \\ &\quad \left. \cdot e^{-i\tilde{\xi}_{2n-1}\tilde{y}_n} \right|^2 d\tilde{\xi}_{2n-1} d\tilde{r} d\tilde{x}' d\tilde{y}' \\ &\geq C > 0, \end{aligned}$$

which means that

$$|||u_m|||_\varepsilon^2 \geq Cm^{2\varepsilon+2p-n-1}$$

for infinitely many m , where $C > 0$.

Comparing this with (3.2), using $|||u_m|||_\varepsilon^2 \leq C(Q(u_m, u_m) + \|u_m\|^2)$, we get $2\varepsilon + 2p - n - 1 \leq 2p - n - 1$, which implies $\varepsilon \leq 0$. This finishes the proof.

REMARK. In fact we can use this method to prove that in C^∞ domains

$$r = 2 \operatorname{Re} z_n - \sum_{i=1}^q |z_i|^{p_i} + \sum_{j=q+1}^{n-1} |z_j|^{p_j},$$

where p_i, p_j 's are positive even integers, there is no subelliptic estimate for $(0, q)$ forms. We can also add a C^∞ function ψ satisfying

$$|\psi| \leq C \left| \sum_{s_1, \dots, s_n} z_1^{s_1} \cdots z_n^{s_n} \right|,$$

where s_i 's satisfy some 'weight' condition to the defining function r to give the same conclusion.

We can use a similar sequence of u_m 's to study the reason why in some domains subelliptic estimates higher than a certain order cannot hold when there is a manifold with some order of contact with the boundary. As an example we prove the following:

PROPOSITION 3.2. *Let Ω be a domain in \mathbf{C}^n given by*

$$r = 2 \operatorname{Re} z_n - \sum_{i=1}^k |f_i(z')|^2 + \psi(z''),$$

where $z' = (z_1, \dots, z_q)$, $z'' = (z_{q+1}, \dots, z_{n-1})$, f_i 's are holomorphic functions, $\psi: \mathbf{C}^{n-q-1} \rightarrow \mathbf{R}$ is C^∞ , and $|\psi(z'')| \leq C|z''|^\eta$.

If a subelliptic estimate of order ε holds at $z = 0$ for $u \in D_U^{0, n-1}$, then $\varepsilon \leq 1/\eta$.

To prove Proposition 3.2 we need the following two lemmas.

LEMMA 3.3. *Let f_i 's be holomorphic functions in \mathbf{C}^n , where $1 \leq i \leq k$ and $f_i(0) = 0$. Then there exists a closed neighbourhood M of 0, $\delta_0 > 0$, such that for all $0 < \delta < \delta_0$ we have either of the following:*

- (i) $m\{z \in M: \sum_{i=1}^k |f_i(z)|^2 < \delta\} \sim \delta^a$ for some $a > 0$;
- (ii) $m\{z \in M: \sum_{i=1}^k |f_i(z)|^2 < \delta\} \sim \delta^a |\log \delta|$ for some $a > 0$;

where (i) or (ii) holds independent of δ and $x \sim y$ means that $C_1 y \leq x \leq C_2 y$ where $C_1, C_2 > 0$.

PROOF. Let $V = \{z \in \mathbf{C}^n: \text{there exists } i \leq k \text{ such that } f_i(z) = 0\}$. Then $0 \in V$. Now by Hironaka's desingularization theorem, there exists an open neighbourhood

M of 0, an n -dimensional complex manifold \tilde{M} , and a proper holomorphic mapping $\varphi: \tilde{M} \rightarrow M$ with the following properties:

(1) $\varphi^{-1}(V)$ is the union of submanifolds of \tilde{M} , intersecting at normal crossings, i.e., if x_0 is a point in the intersection of two submanifolds, then there exists a neighbourhood W of x_0 and a local coordinate (z_1, \dots, z_n) such that $\varphi^{-1}(V) \cap W = \{z \in W: z_1 = 0 \text{ or } z_2 = 0\}$.

(2) $\varphi|_{\varphi^{-1}(M-V)}$ is biholomorphic.

Let $g = \prod f_i^2$; then $V = V(g)$, where $V(g)$ is the variety defined by the single holomorphic function g .

The mapping $g \circ \varphi: \tilde{M} \rightarrow \mathbf{C}$ is a holomorphic function vanishing only on $\varphi^{-1}(V \cap M)$. Therefore at any point $x_0 \in \varphi^{-1}(V \cap M)$, locally using (1) we can write

$$g \circ \varphi(z) = h(z) z_1^p z_2^q,$$

where h is nonvanishing and p, q are integers such that $p > 0$ and $q \geq 0$. Hence

$$\prod (f_i^2 \circ \varphi) = \left(\prod f_i^2 \right) \circ \varphi = h z_1^p z_2^q,$$

so

$$f_i^2 \circ \varphi = h_i z_1^{p_i} z_2^{q_i}, \quad 1 \leq i \leq k,$$

where h_i 's are nonvanishing, and p_i and q_i 's are nonnegative integers.

We proceed to estimate the measure of the required set.

$$\begin{aligned} (3.3) \quad m \left\{ z \in M: \sum_{i=1}^k |f_i|^2 < \delta \right\} &= \int_{\sum |f_i|^2 < \delta} 1 \, dz = \int_{\sum |f_i^2 \circ \varphi| < \delta} |\det \varphi| \, dz' \\ &= \int_{\sum |h_i z_1^{p_i} z_2^{q_i}| < \delta} |\det \varphi| \, dz'. \end{aligned}$$

Since h_i 's are nonvanishing, for some $K_1, K_2 > 0$ we have the following:

$$\begin{aligned} \left\{ z: \sum |z_1^{p_i} z_2^{q_i}| < K_1 \delta \right\} &\subset \left\{ z: \sum |h_i z_1^{p_i} z_2^{q_i}| < \delta \right\} \\ &\subset \left\{ z: \sum |z_1^{p_i} z_2^{q_i}| < K_2 \delta \right\}. \end{aligned}$$

Also we have

$$\begin{aligned} \bigcap_{i=1}^k \left\{ z: |z_1^{p_i} z_2^{q_i}| < \frac{\delta}{k} \right\} &\subset \left\{ z: \sum_i |z_1^{p_i} z_2^{q_i}| < \delta \right\} \\ &\subset \bigcap_{i=1}^k \left\{ z: |z_1^{p_i} z_2^{q_i}| < \delta \right\}. \end{aligned}$$

In view of what we need, it suffices to find the integral of (3.2) on the set $\bigcap_{i=1}^k \{z: |z_1^{p_i} z_2^{q_i}| < \delta\}$.

By property (2) of the mapping φ , we know that, locally $\det \varphi = F z_1^a z_2^b$, where F is nonvanishing, and a, b are nonnegative integers. Hence, we have reduced the problem to estimating

$$I = \int_{\bigcap |z_1^{p_i} z_2^{q_i}| < \delta} |z_1^a z_2^b| \, dx \, dy.$$

Here $|z_i| < \varepsilon$ for all $i = 1, \dots, n$.

$$I = \text{const} \int_{\bigcap |z_1^{p_i} z_2^{q_i}| < \delta} |z_1^a z_2^b| dx_1 dy_1 dx_2 dy_2.$$

Introducing the polar coordinate $z_i = r_i e^{i\theta_i}$, $i = 1, 2$, we have

$$\begin{aligned} I &= \text{const} \int_{\bigcap r_1^{p_i} r_2^{q_i} < \delta} r_1^{a+1} r_2^{b+1} dr_1 dr_2 \\ &= \text{const} \int_{r_1=0}^{\varepsilon} \int^{\min_{1 \leq i \leq k} [\delta^{1/q_i} r_1^{-p_i/q_i}, \varepsilon]} r_1^{a+1} r_2^{b+1} dr_1 dr_2 \\ &= \text{const} \int_{r_1=0}^{\varepsilon} r_1^{a+1} \left(\min_{1 \leq i \leq k} [\delta^{1/q_i} r_1^{-p_i/q_i}, \varepsilon] \right)^{b+2} dr_1. \end{aligned}$$

We can separate the above integral into several parts, taking in each part the minimum in the bracket to integrate. Since the integrand is of the form $\delta^a r_1^b$, it is easy to see that the dominant term is δ^a or $\delta^a |\log \delta|$, where $a > 0$, which means we have either

$$C_1 \delta^a \leq I \leq C_2 \delta^a \quad \text{or} \quad C_3 \delta^a |\log \delta| \leq I \leq C_4 \delta^a |\log \delta|,$$

where C_i 's > 0 are independent of δ .

Therefore for each point $x_0 \in \varphi^{-1}(V \cap M)$, there exists a neighbourhood W_{x_0} such that $m\{z \in U: \varphi^{-1}(z) \in W_{x_0}, \sum |f_i|^2 < \delta\}$ is of the order δ^a or $|\delta^a \log \delta|$.

Since φ is proper, we can choose a finite number of neighbourhoods W_{x_i} such that $\bigcup W_{x_i} \supset \varphi^{-1}(V \cap M)$, and there exists a dominant term out of the finite number of neighbourhoods. Away from the variety V , $\sum |f_i|^2$ is small. So we get what is required by choosing δ_0 small enough.

REMARK 1. It is easy to see that the power a of δ is independent of the neighbourhood M , as long as M is sufficiently small.

REMARK 2. We use Hironaka's theorem to prove this lemma, but there should be a more elementary way to prove this.

LEMMA 3.4. Let f_i , where $i = 1, \dots, k$, be as in Lemma 3.3, and let M be the neighbourhood in the conclusion of Lemma 3.3, where we have the estimate of the measure of the set $\{z \in M: \sum_{i=1}^k |f_i|^2 < \delta\}$ when $\delta < \delta_0$.

Then there exists a constant $C > 0$, independent of $m \geq A$, such that for q large,

$$\int_M \frac{1}{(m \sum |f_i|^2 + 1)^q} dx \leq C \int_M \frac{1}{(m \sum |f_i|^2 + 1)^{q+1}} dx.$$

PROOF. This is a straightforward calculation of the integral using Lemma 3.3.

We know that

$$\int_M |g|^p dx = p \int_0^\infty \alpha^{p-1} \lambda(\alpha) d\alpha,$$

where $\lambda(\alpha) = m\{x \in M: |g(x)| > \alpha\}$. Hence,

$$I = \int_M \frac{1}{(m \sum |f_i|^2 + 1)^{q+1}} dx = q \int_0^1 \alpha^q \lambda_m(\alpha) d\alpha,$$

where

$$\lambda_m(\alpha) = \left\{ x \in M: \frac{1}{m \sum |f_i|^2 + 1} > \alpha \right\} = \left\{ x \in M: \sum |f_i|^2 < \frac{1-\alpha}{\alpha m} \right\}.$$

Then $\lambda_m(\alpha) = \gamma((1-\alpha)/\alpha m)$, where $\gamma(a) = \{x \in M: \sum |f_i(x)|^2 < a\}$. Now

$$I = q \int_0^1 \alpha^q \gamma\left(\frac{1-\alpha}{\alpha m}\right) d\alpha.$$

Put $y = (1-\alpha)/\alpha m$.

$$\begin{aligned} I &= qm \int_0^\infty \frac{1}{(my+1)^{q+2}} \gamma(y) dy \\ &= qm \left(\int_0^{\delta_0} \frac{1}{(my+1)^{q+2}} \gamma(y) dy + \int_{\delta_0}^\infty \frac{1}{(my+1)^{q+2}} \gamma(y) dy \right). \end{aligned}$$

Now $\gamma(y) \sim y^a$ or $y^a |\log y|$ for $y < \delta_0$.

Case 1. $\gamma(y) = y^a$ for $y < \delta_0$.

$$\int_0^{\delta_0} \frac{1}{(my+1)^{q+2}} \gamma(y) dy = \int_0^{\delta_0} \frac{y^a}{(my+1)^{q+2}} dy = m^{-a-1} \int_0^{m\delta_0} \frac{y^a}{(y+1)^{q+2}} dy.$$

When m is large,

$$C_1 \leq \int_0^{m\delta_0} \frac{y^a}{(y+1)^{q+2}} dy \leq C_2,$$

where $C_1, C_2 > 0$ and the above inequality holds independent of m when $q > a-1$. Therefore,

$$\int_0^{\delta_0} \frac{1}{(my+1)^{q+2}} \gamma(y) dy \sim m^{-a-1}.$$

For the second integral,

$$\begin{aligned} \int_{\delta_0}^\infty \frac{1}{(my+1)^{q+2}} \gamma(y) dy &= \frac{1}{m} \int_{m\delta_0}^\infty \frac{1}{(y+1)^{q+2}} \gamma\left(\frac{y}{m}\right) dy \\ &\leq \frac{\text{const}}{m} \int_{m\delta_0}^\infty \frac{1}{(y+1)^{q+2}} dy \leq \text{const } m^{-2-q}. \end{aligned}$$

Hence,

$$\int_0^\infty \frac{1}{(my+1)^{q+2}} \gamma(y) dy \sim m^{-a-1} \quad \text{when } q > a-1.$$

Therefore, $I \sim m^{-a}$.

The above argument also shows that

$$\int_M \frac{1}{(m \sum |f_i|^2 + 1)^q} dx \sim m^{-a-1} \quad \text{if } q > a,$$

and the assertion is established in this case.

Case 2. $\gamma(y) = y^a |\log y|$ for $y < \delta_0$.

$$\begin{aligned} \int_0^{\delta_0} \frac{1}{(my+1)^{q+2}} \gamma(y) dy &= - \int_0^{\delta_0} \frac{y^a \log y}{(my+1)^{q+2}} dy \\ &= m^{-a-1} \int_0^{m\delta_0} \frac{\log m - \log y}{(y+1)^{q+2}} dy \sim m^{-1-a} \log m \end{aligned}$$

for q large enough.

For $\int_{\delta_0}^{\infty} (1/(my+1)^{q+2})\gamma(y) dy$ we have the same estimate as in Case 1, so the conclusion follows in the same way.

PROOF OF PROPOSITION 3.2. Define

$$L_i = r_{z_i} \frac{\partial}{\partial z_n} - \frac{\partial}{\partial z_i}, \quad i = 1, \dots, n-1, \quad L_n = \frac{\partial}{\partial z_n},$$

and let $\omega_1, \dots, \omega_n$ be a basis of $(1, 0)$ forms dual to L_1, \dots, L_n .

$$u_m = \frac{1}{(z_n - \sum |f_i(z')|^2 - 1/m)^p} \phi(x_1) \cdots \phi(y_q) \phi(m^{1/\eta} x_{q+1}) \\ \cdots \phi(m^{1/\eta} y_{n-1}) \lambda(x_n) \phi(y_n) \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_{n-1},$$

where $\phi, \lambda \in C_0^\infty(\mathbf{R})$ are chosen later.

$$\|L_{q+1} u_m\|^2 \leq \left\| r_{z_{q+1}} \frac{\partial u_m}{\partial z_n} \right\|^2 + \left\| \frac{\partial u_m}{\partial z_{q+1}} \right\|^2, \\ r_{z_{q+1}} \frac{\partial u_m}{\partial z_n} = p \frac{\partial \psi}{\partial z_{q+1}} \frac{1}{(z_n - \sum |f_i(z')|^2 - 1/m)^{p+1}} \phi(x_1) \\ \cdots \phi(y_q) \phi(m^{1/\eta} x_{q+1}) \cdots \phi(y_n) + \text{lower order terms in } m.$$

Hence,

$$\left\| r_{z_{q+1}} \frac{\partial u_m}{\partial z_n} \right\|^2 \leq \text{const} \int \left| \frac{\partial \psi}{\partial z_{q+1}} \right|^2 \frac{1}{|x_n - \sum |f_i|^2 - 1/m + iy_n|^{2p+2}} \\ \cdot (\phi(x_1))^2 \cdots (\phi(y_n))^2 dx_1 \cdots dy_n.$$

Note that $|\psi(z'')| \leq C|z''|^\eta$ implies that, on compact sets, we have $|\partial \psi / \partial z_{q+1}| \leq \text{const}|z''|^{\eta-1}$.

With change of variables only in x_{q+1}, \dots, y_n as follows,

$$\tilde{x}_i = m^{1/\eta} x_i, \quad \tilde{y}_i = m^{1/\eta} y_i, \quad i = q+1, \dots, n-1, \quad \tilde{x}_n = m x_n, \quad \tilde{y}_n = m y_n,$$

we get

$$\left\| r_{z_{q+1}} \frac{\partial u_m}{\partial z_n} \right\|^2 \leq \text{const } m^{2p-2+(-2n+2q+4)/\eta} \\ \cdot \int |\tilde{z}''|^{2\eta-2} \frac{1}{((\tilde{x}_n - m \sum |f_i|^2 - 1)^2 + \tilde{y}_n^2)^{p+1}} \\ \cdot (\phi(x_1) \cdots \phi(y_q) \phi(\tilde{x}_{q+1}) \cdots \phi(\tilde{y}_n/m))^2 dx_1 \cdots dy_q d\tilde{x}_{q+1} \cdots d\tilde{y}_n.$$

Integrate with respect to \tilde{x}_n , which runs from $-\infty$ to ∞ to

$$\left(m \sum |f_i|^2 - m\psi(z''/m^{1/\eta}) \right) / 2,$$

and also integrate \tilde{y}_n from $-\infty$ to ∞ to get

$$\left\| r_{z_{q+1}} \frac{\partial u_m}{\partial z_n} \right\|^2 \leq \text{const } m^{2p-2+(-2n+2q+4)/\eta} \\ \cdot \int |\tilde{z}''|^{2\eta-2} \frac{1}{\frac{1}{2}(-m \sum |f_i|^2 + m\psi(z''/m^{1/\eta})) - 1)^{2p-2}} \\ \cdot (\phi(x_1) \cdots \phi(y_q) \phi(\tilde{x}_{q+1}) \cdots \phi(\tilde{y}_{n-1}))^2 dx_1 \cdots dy_q d\tilde{x}_{q+1} \cdots d\tilde{y}_{n-1}.$$

For suitable choice of ϕ ,

$$|m\psi(z''/m^{1/\eta})| < 1 \quad \text{for } z'' \in \text{supp}(\phi(\tilde{x}_{q+1}) \cdots \phi(\tilde{y}_{n-1})).$$

So we have

$$\left\| r_{z_{q+1}} \frac{\partial u_m}{\partial z_n} \right\|^2 \leq \text{const } m^{2p-2+(-2n+2q+4)/\eta} \cdot \int \frac{1}{(m \sum |f_i|^2 + 1)^{2p-2}} (\phi(x_1) \cdots \phi(y_q))^2 dx_1 \cdots dx_q dy_1 \cdots dy_q.$$

Using a similar technique, we can compute $\|\partial u_m / \partial z_{q+1}^m\|^2$ to get

$$(3.4) \quad \|L_{q+1}u_m\|^2 \leq C m^{2p-2+(-2n+2q+4)/\eta} \cdot \int \frac{1}{(m \sum |f_i|^2 + 1)^{2p-2}} (\phi(x_1) \cdots \phi(y_q))^2 dx_1 \cdots dx_q dy_1 \cdots dy_q.$$

The above inequality is also true when L_{q+1} is replaced by L_j , $j = q+2, \dots, n-1$. With similar calculations it is easy to see that

$$\|\bar{L}_n u_m\|^2 \leq \text{right side of (3.4)}$$

and

$$\|u_m\|^2 \leq \text{right side of (3.4)}.$$

Also,

$$\|L_1 u_m\|^2 \leq C m^{2p-4+(-2n+2q+4)/\eta} \int \frac{1}{(m \sum |f_i|^2 + 1)^{2p-2}} (\phi(x_2) \cdots \phi(y_q))^2 \cdot (\phi'(x_1)\phi(y_1) + \phi(x_1)\phi'(y_1))^2 dx_1 \cdots dy_q + \text{right side of (3.4)}.$$

The main term on the right side of the above inequality is the second term, which we can see from later calculations.

Combining all these estimates, we get

$$(3.5) \quad Q(u_m, u_m) + \|u_m\|^2 \leq C m^{2p-2+(-2n+2q+4)/\eta} \cdot \int \frac{1}{(m \sum |f_i|^2 + 1)^{2p-2}} (\phi(x_1) \cdots \phi(y_q))^2 dx_1 \cdots dx_q dy_1 \cdots dy_q.$$

To calculate $|||u_m|||_\varepsilon^2$ we use, as before, the coordinate $(x_1, \dots, x_{n-1}, y_1, \dots, y_n, r)$ and dual variable $\xi = (\xi_1, \dots, \xi_{2n-1})$.

$$\begin{aligned} |||u_m|||_\varepsilon^2 &= \int (1 + |\xi|^2)^\varepsilon \left| \int \frac{1}{((r - \sum |f_i|^2 - \psi)/2 - 1/m + iy_n)^p} \cdot \phi(x_1) \cdots \phi(y_q) \phi(m^{1/\eta} x_{q+1}) \cdots \phi(m^{1/\eta} y_{n-1}) \right. \\ &\quad \left. \cdot \phi(y_n) \lambda(x_n) e^{-i\xi \cdot (x', x'', y)} dx' dx'' dy \right|^2 d\xi dr \\ &\geq \int |\xi_{2n-1}^{2\varepsilon}| \left| \int \frac{1}{((r - \sum |f_i|^2 - \psi)/2 - 1/m + iy_n)^p} \phi(y_n) \lambda(x_n) e^{-i\xi_{2n-1} y_n} dy_n \right|^2 \\ &\quad \cdot (\phi(x_1) \cdots \phi(m^{1/\eta} y_{n-1}))^2 dx' dx'' dy' dy'' d\xi dr. \end{aligned}$$

Using the substitutions

$$\tilde{x}_i = m^{1/\eta} x_i, \quad \tilde{y}_i = m^{1/\eta} y_i, \quad i = q+1, \dots, n-1,$$

$$\tilde{y}_n = m y_n, \quad \tilde{\xi}_{2n-1} = (1/m) \xi_{2n-1}, \quad \tilde{r} = m r,$$

we obtain

$$|||u_m|||_\varepsilon^2 \geq m^{2\varepsilon+2p-2+(-2n+2q+2)/\eta} I_m,$$

where

$$I_m = \int |\tilde{\xi}_{2n-1}|^{2\varepsilon} \left| \int \frac{1}{((\tilde{r} - m \sum |f_i|^2 - \psi'(\tilde{z}''))/2 - 1 + i\tilde{y})^p} \cdot \phi\left(\frac{\tilde{y}_n}{m}\right) \lambda\left(x_n\left(x_1, \dots, \frac{\tilde{r}}{m}\right)\right) e^{-i\tilde{\xi}_{2n-1}\tilde{y}_n} d\tilde{y}_n \right|^2 \cdot (\phi(x_1) \cdots \phi(\tilde{y}_{n-1}))^2 dx' d\tilde{x}'' dy' d\tilde{y}'' d\tilde{\xi} d\tilde{r},$$

and $\psi'(\tilde{z}'') = m\psi(z''/m^{1/\eta})$.

We already see that for a suitable choice of ϕ ,

$$|\psi'(\tilde{z}'')| < 1 \quad \text{for } \tilde{z}'' \in \text{supp}(\phi(\tilde{x}_{q+1}) \cdots \phi(\tilde{y}_{n-1})).$$

If we put

$$g = - \left(\frac{\tilde{r} - m \sum |f_i|^2 - \psi'}{2} - 1 \right),$$

g is nonvanishing on

$$\text{supp}(\phi(\tilde{x}_{q+1}) \cdots \phi(\tilde{y}_{n-1})).$$

Using a further substitution

$$y'_n = g \tilde{y}_n, \quad \xi'_{2n-1} = (1/g) \tilde{\xi}_{2n-1},$$

$$I_m = \int \frac{|\xi'_{2n-1}|^{2\varepsilon}}{g^{2p+1-2\varepsilon}} \left| \int \frac{1}{(1 + i y'_n)^p} \phi\left(\frac{y'_n}{mg}\right) \lambda(x_n) e^{-i\xi'_{2n-1} y'_n} dy'_n \right|^2 \cdot (\phi(x_1) \cdots \phi(\tilde{y}_{2n-1}))^2 d\xi'_{2n-1} d\tilde{r} dx' d\tilde{x}'' dy' d\tilde{y}'' \\ = J_1 + J_2,$$

where J_1 integrates from $-\infty$ to $-mK$, J_2 integrates from $-mK$ to 0 for \tilde{r} , and K is suitably chosen.

For $\tilde{r} \in [-mK, 0]$, $z' \in \text{supp}(\phi(x_1) \cdots \phi(y_q))$, $\tilde{z}'' \in \text{supp}(\phi(\tilde{x}_{q+1}) \cdots \phi(\tilde{y}_{n-1}))$, and

$$|x_n| = \left| \frac{\tilde{r}/m + \sum |f_i|^2 - (1/m)\psi(m^{1/\eta}\tilde{z}'')}{2} \right| \leq C,$$

where C is a constant > 0 .

We may choose $\lambda \in C_0(\mathbf{R})$ such that $\lambda(x) = 1$ for $|x| \leq C$. Then

$$\int |\xi'_{2n-1}|^{2\varepsilon} \left| \int \frac{1}{(1 + i y'_n)^p} \phi\left(\frac{y'_n}{mg}\right) \lambda(x_n) e^{-i\xi'_{2n-1} y'_n} dy'_n \right|^2 d\xi'_{2n-1} \geq \text{const} > 0$$

for all $(x', \tilde{x}'', y', \tilde{y}'') \in \text{supp}(\phi(x_1) \cdots \phi(\tilde{y}_{n-1}))$ for $\tilde{r} \in [-mK, 0]$.

$$\begin{aligned} J_2 &\geq \text{const} \int \int_{\tilde{r}=-mK}^0 \frac{1}{((\tilde{r} - m \sum |f_i|^2 - \psi')/2 - 1)^{2p+1-2\varepsilon}} \\ &\quad \cdot (\phi(x_1) \cdots \phi(\tilde{y}_{n-1}))^2 dx' d\tilde{x}'' dy' d\tilde{y}'' d\tilde{r} \\ &\geq C_1 \int \frac{1}{(m \sum |f_i|^2 + 3)^{2p-2\varepsilon}} (\phi(x_1) \cdots \phi(y_q))^2 dx' dy' \\ &\quad - C_2 \int \frac{1}{(mK + m \sum |f_i|^2 + 3)^{2p-2\varepsilon}} (\phi(x_1) \cdots \phi(y_q))^2 dx' dy', \end{aligned}$$

where $C_1 > 0$.

Now

$$\int \frac{1}{(mK + m \sum |f_i|^2 + 3)^{2p-2\varepsilon}} (\phi(x_1) \cdots \phi(y_q))^2 dx' dy' \leq \text{const } m^{-2p+2\varepsilon}.$$

It is easily seen from Lemma 3.4 that the above term is much less than

$$\int \frac{1}{(m \sum |f_i|^2 + 3)^{2p-2\varepsilon}} (\phi(x_1) \cdots \phi(y_q))^2 dx' dy'$$

when m is large for p large enough, so

$$I_m \geq C \int \frac{1}{(m \sum |f_i|^2 + 3)^{2p-2\varepsilon}} (\phi(x_1) \cdots \phi(y_q))^2 dx' dy', \quad \text{where } C > 0,$$

and, finally,

$$(3.6) \quad |||u_m|||_{\varepsilon}^2 \geq C m^{2\varepsilon+2p-2+(-2n+2q+2)/\eta} \cdot \int \frac{1}{(m \sum |f_i|^2 + 3)^{2p-2\varepsilon}} (\phi(x_1) \cdots \phi(y_q))^2 dx' dy'.$$

Now if we choose $0 \leq \phi \leq 1$ such that

$$\begin{aligned} \phi(x) &= 1, & x &\leq R', \\ &= 0, & x &\geq R'', \end{aligned}$$

where $R'' > R' > 0$ and R', R'' small enough, then applying Lemmas 3.3 and 3.4, we have

$$\begin{aligned} &\int \frac{1}{(m \sum |f_i|^2 + 3)^{2p-2\varepsilon}} (\phi(x_1) \cdots \phi(y_q))^2 dx' dy' \\ &\geq \frac{1}{3} \int_{\substack{|x_i| \leq R' \\ |y_i| \leq R'}} \frac{1}{(m \sum |f_i|^2 + 1)^{2p-2\varepsilon}} dx' dy' \\ &\geq \text{const} \int_{\substack{|x_i| \leq R' \\ |y_i| \leq R'}} \frac{1}{(m \sum |f_i|^2 + 1)^{2p-2}} dx' dy' \\ &\geq \text{const} \int_{\substack{|x_i| \leq R'' \\ |y_i| \leq R''}} \frac{1}{(m \sum |f_i|^2 + 1)^{2p-2}} dx' dy' \\ &\geq \text{const} \int \frac{1}{(m \sum |f_i|^2 + 1)^{2p-2}} (\phi(x_1) \cdots \phi(y_q))^2 dx' dy', \end{aligned}$$

where the third line obtains because the power of m is the same in the second and third lines, using the same argument as in the proof of Lemma 3.4.

Combining (3.5), (3.6), and the above inequality, we have

$$\text{const } m^{2\varepsilon+2p-2+(-2n+2q+2)/\eta} \leq C m^{2p-2+(-2n+2q+4)/\eta},$$

where $C > 0$.

Taking $m \rightarrow \infty$ we get $\varepsilon < 1/\eta$.

REMARK. Using a similar argument we can also prove the following:

Let Ω be a domain in \mathbb{C}^n with C^∞ boundary. If there exists an $(n-1)$ -dimensional manifold with order of contact η , and a subelliptic estimate of order ε holds, then $\varepsilon \leq 1/\eta$.

The proof of the above is as follows. We may assume

$$r = 2\text{Re } z_n + O(|z_n||z'| + |z'|^\eta).$$

Then with the same L_i 's and ω_i 's as before, we define

$$u_m = \frac{1}{(z_n - 1/m)^p} \phi(m^{1/\eta} x_1) \phi(m^{1/\eta} x_{n-1}) \phi(m^{1/\eta} y_1) \phi(m^{1/\eta} y_{n-1}) \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_{n-1}.$$

We get $\varepsilon \leq 1/\eta$ using the same argument as in Theorem 3.1 and Proposition 3.2.

Finally, we prove a result on nonpseudoconvex domains using Catlin's method [1]. We first give a definition.

DEFINITION 3.5. If $\alpha \in L_2^{p,q}(\Omega)$ we define the singular support of α , denoted by $\text{sing supp}(\alpha)$, to be the closed subset of $\bar{\Omega}$, as follows. If $x \in \bar{\Omega}$ then $x \notin \text{sing supp}(\alpha)$ if there exists a neighbourhood U of x such that the restriction of α to $U \cap \bar{\Omega}$ is in C^∞ .

A REMARK TO CATLIN'S THEOREM. Let $\Omega \subset \mathbb{C}^n$ with C^∞ boundary, $x_0 \in \text{b}\Omega$. Suppose there exists an $(n-1)$ -dimensional complex manifold M passing through x_0 , and $M \subset \text{b}\Omega \cap U$ for a neighbourhood U of x_0 . Then there exists a neighbourhood U' of x_0 such that there is a $\bar{\partial}$ -closed form $\alpha \in L_2^{p,n-1}(\Omega)$ and, for any $(p, n-2)$ form u with $\bar{\partial}u = \alpha$, then $\text{sing supp}(u)$ strictly contains $\text{sing supp}(\alpha)$.

PROOF. It suffices to prove this for $p = 0$.

The construction of the form α here closely follows that of [1], except that in this case we have an explicit construction of the holomorphic function f used in α .

We may assume that $z_0 = (0, \dots, 0)$ and M is defined by $M = \{z: z_n = 0\}$.

Assume further that $\partial r / \partial x_n|_{z=0} = 1$. Define $f(z) = 1/z_n^{1/2}$. We want to show that f is holomorphic in a neighbourhood U' of 0. We prove that in a neighbourhood of 0 there exists δ such that

$$(z', x_n) \notin \Omega \quad \text{for all } z' = (z_1, \dots, z_{n-1}) \text{ and } 0 \leq x_n < \delta.$$

Actually, on $y_n = 0$, we have

$$r = x_n + O(|x_n||z'| + |x_n|^2).$$

There is no $|z'|^2$ term because r vanishes on $x_n = 0$. Now we choose $C, \delta > 0$ such that

$$|O(|x_n||z'| + |x_n|^2)| \leq \frac{1}{2}|x_n| \quad \text{when } |z'| < C, \quad |x_n| < \delta.$$

Therefore when $|z'| < C$, $0 \leq x_n < \delta$, we have $r \geq \frac{1}{2}x_n \geq 0$, which implies that $(z', x_n) \notin \Omega$ in this region and proves our assertion.

We now construct a family of $(n-1)$ -dimensional manifolds M_k in Ω such that M_k approaches M . $M_k = \{z: z_n = -1/k\}$.

Then for large k and some R we have $M_k \cap \{z: |z'| < R\} \subset U'$. Let $U = U' \cap \{z: |z'| < R\}$. Define $\phi \in C^\infty(\mathbf{R})$ with the property

$$\begin{aligned} \phi(x) &= 1, & |x| &\leq \varepsilon, \\ &= 0, & |x| &\geq 2\varepsilon, \end{aligned}$$

where ε is chosen such that $8\varepsilon < R$.

We now define a $(0, n-1)$ form as follows:

$$\alpha = \phi(|z'|^2) f(z) d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{n-1}.$$

First of all, since f is holomorphic, we have $\bar{\partial}\alpha = 0$. We want to show that $\alpha \in L^2(U)$.

$$\begin{aligned} \int_U |\phi(|z'|^2) f(z)|^2 dV &= \int_U \left| \frac{1}{z_n} \right| |\phi(|z'|^2)|^2 dV \\ &\leq C \int_{|z_n| < r} \left| \frac{1}{z_n} \right| dx_n dy_n = C'. \end{aligned}$$

Therefore $\alpha \in L^2(U)$.

Next, we want to show that there exists $u \in L_2^{0, n-2}(U)$ such that $\bar{\partial}u = \alpha$. Consider $\phi(|z'|^2) d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{n-1}$ as a $(0, n-1)$ form in \mathbf{C}^{n-1} . There exists a $\psi \in A^{0, n-2}$ in \mathbf{C}^{n-1} such that $\bar{\partial}\psi = \phi(|z'|^2) d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{n-1}$. Let $u = \psi f$ be a $(0, n-1)$ form in \mathbf{C}^n .

$$\begin{aligned} \bar{\partial}(\psi f) &= (\bar{\partial}\psi) f + \psi(\bar{\partial}f) = (\bar{\partial}\psi) f \\ &= f \phi(|z'|^2) d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{n-1} = \alpha. \end{aligned}$$

We now derive a contradiction. Assume there exists $v \in L_2^{0, n-2}(U)$ with $\bar{\partial}v = \alpha$ and $\text{sing supp}(v) = \text{sing supp}(\alpha)$. Let $W = \{z': |z'| < R\} \subset \mathbf{C}^{n-1}$. We put the standard hermitian metric $\sum_{i=1}^{n-1} dz_i \otimes d\bar{z}_i$ on W . Denote the inner product on forms induced by this metric by $\langle \cdot, \cdot \rangle$. Define $g_k: W \rightarrow U$ by $g_k(z') = (z', -1/k)$ and let $w = \phi(|z'|^2/4) d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{n-1}$ on W . Then

$$\begin{aligned} \int_W \langle g_k^* \alpha, w \rangle dV &= \int_W \langle g_k^* (\bar{\partial}v), w \rangle dV \\ &= \int_W \langle \bar{\partial}(g_k^* v), w \rangle dV = \int_W \langle g_k^* v, \theta w \rangle dV. \end{aligned}$$

We integrate the last line by parts and use the fact that w has compact support. Now

$$\text{supp}(\theta w) \subset \{z: 4\varepsilon \leq |z'|^2 \leq 8\varepsilon\} = S.$$

We see that $\alpha = 0$ in $\{z: 2\varepsilon \leq |z'|^2\}$. So the assumption on v says that v is smooth in $S \cap \bar{\Omega}$, and hence $|g_k^* v| \leq M$ in $\text{supp } \theta w$ for all k . Therefore,

$$(3.7) \quad \left| \int_W \langle g_k^* \alpha, w \rangle dV \right| \leq M \sup_W |\theta w| \left| \int_W dV \right| \leq C$$

independent of k .

On the other hand,

$$\begin{aligned} \left| \int_W \langle g_k^* \alpha, w \rangle dV \right| &= \left| 2^{n-1} \int_W \phi(|z'|^2) f\left(z', -\frac{1}{k}\right) \phi\left(\frac{|z'|^2}{4}\right) dV \right| \\ &= \left| 2^{n-1} \int_W \phi(|z'|^2) f\left(z', -\frac{1}{k}\right) dV \right| \\ &= \left| 2^{n-1} \left(-\frac{1}{k}\right)^{-1/2} \int_W \phi(|z'|^2) dV \right|. \end{aligned}$$

The third line obtains because $f(z', -1/k)$ is independent of z' . Now

$$\left| \int_W \phi(|z'|^2) dV \right| = C \quad \text{and} \quad \left| \left(-\frac{1}{k}\right)^{-1/2} \right| = k^{1/2}.$$

Hence, $\left| \int_W \langle g_k^* \alpha, w \rangle dV \right| = Ck^{1/2}$, where C is independent of k , so

$$\left| \int_W \langle g_k^* \alpha, w \rangle dV \right| \rightarrow \infty$$

as $k \rightarrow \infty$. This contradicts (3.7) and finishes our proof.

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