TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 291, Number 1, September 1985

ON LIPSCHITZ HOMOGENEITY OF THE HILBERT CUBE

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ABSTRACT. The main contribution of this paper is to prove the conjecture of [Vä] that the Hilbert cube Q is Lipschitz homogeneous for any metric d_s, where s is a decreasing sequence of positive real numbers s_k converging to zero, $d_s(x,y) = \sup\{s_k|x_k - y_k|: k \in N\}$, and $R(s) = \sup\{s_k/s_{k+1}: k \in N\} < \infty$. In addition to other results, we shall show that for every Lipschitz homogeneous compact metric space X there is a constant $\lambda < \infty$ such that X is homogeneous with respect to Lipschitz homeomorphisms whose Lipschitz constants do not exceed λ. Finally, we prove that the hyperspace 2^I of all nonempty closed subsets of the unit interval is not Lipschitz homogeneous with respect to the Hausdorff metric.

1. Introduction. Let (X_1, d_1) and (X_2, d_2) be metric spaces. A surjective map $f: X_1 \to X_2$ is called a Lipschitz homeomorphism if there exists a real number $K \geq 1$ such that $K^{-1}d_1(x,y) \leq d_2(fx, fy) \leq Kd_1(x,y)$ for all points x, y of X_1. Let bilip denote the least such constant K. In case $\operatorname{bilip} f \leq L$, we say that f is an L-Lipschitz homeomorphism. A metric space X is called Lipschitz homogeneous if for all points x and y of X there is a Lipschitz homeomorphism $f: X \to X$ with $f(x) = y$. It is well known that the Hilbert cube $Q = [-1,1]^N$ is topologically homogeneous as was shown by [**Ke**] in 1931. It is natural to ask whether Q is Lipschitz homogeneous with respect to some suitable metric. Let Q_s be the Hilbert cube equipped with the metric d_s, where s is a decreasing sequence of positive real numbers converging to zero and $d_s(x,y) = \max\{s_k|x_k - y_k|: k \in N\}$. Then d_s is a natural metric for the setting of Lipschitz maps. J. Väisälä proved in [**Vä**] that if Q_s is Lipschitz homogeneous, then

$$R(s) = \sup\{s_k/s_{k+1}: k \in N\} < \infty.$$

We have proved the converse that was left open.

On the other hand, it was also shown in [**Vä**] that if one takes $s = \{1/k: k \in N\}$, then there is a constant $L < \infty$ such that for all $x, y \in Q$ there exists an L-Lipschitz homeomorphism $f: Q_s \to Q_s$ with $f(x) = y$. However, we shall prove that *every* compact Lipschitz homogeneous metric space has this property.

2. Notation. Let S denote the set of all decreasing sequences s of positive real numbers s_k converging to zero and for which $R(s) < \infty$. We will think of the cube Q_s as an infinite product of the line segments $I_k^s = [-s_k, s_k]$ with the natural metric d_s given by $d_s(x,y) = (\|x-y\|_{c_0}) = \max\{|x_k - y_k|: k \in N\}$. Then Q_s can be considered as a metric subspace of the Banach space $c_0(N)$. For each $k \in N$,

Received by the editors February 17, 1984.
1980 *Mathematics Subject Classification.* Primary 54E40.
Key words and phrases. Hilbert cube, homogeneous, Lipschitz homeomorphism, Q-manifold, hyperspace.

let $Q_s^k = \prod\{I_j^s : j \geq k\}$. Hence, $Q_s = I_1^s \times \cdots \times I_k^s \times Q_s^{k+1}$ for all $k \geq 1$. For a compact metric space X and $\varepsilon > 0$, the maximum cardinality of an ε-discrete subset of X will be denoted by $N(X, \varepsilon)$. (A subset D of a metric space (X, d) is called ε-discrete if $d(x, y) \geq \varepsilon$ for all distinct elements $x, y \in D$.) The symbol $\overline{B}(A, \varepsilon)$ denotes the *closed* ε-neighborhood of a set A. The set of all Lipschitz homeomorphisms of X onto itself will be denoted by $L(X)$, and for a real number $r \geq 1$, we write $L_r(X) = \{f \in L(X) : \operatorname{bilip} f \leq r\}$.

3. Lipschitz homogeneity of the cubes Q_s. In this section we establish the following result, which is the main contribution of the paper.

3.1. THEOREM. *If $s \in S$, then Q_s is Lipschitz homogeneous.*

PROOF. We shall say that a point $x \in Q_s$ is in good position if $x_k \geq s_k - s_{k+1}/4$ for each $k \in N$. The meaning of this convention will become clear later. First we shall show that every point of Q_s can be mapped to a good position by a Lipschitz homeomorphism of Q_s. Indeed, let $x \in Q_s$. For each $k \in N$, let f_k be a PL-homeomorphism $I_k^s \to I_k^s$ such that $|f_k x_k| \geq s_k - s_{k+1}/4$. Note that we can assume $\operatorname{bilip} f_k \leq s_k/(s_{k+1}/4) \leq 4R(s)$. If $f_k x_k \geq 0$, let g_k be the identity map of I_k^s. Otherwise let g_k be the reflection of I_k^s relative to the point 0. Then the map $h : Q_s \to Q_s$, given by

$$h(q_1, q_2, \ldots) = (g_1 f_1 q_1, g_2 f_2 q_2, \ldots),$$

is a $4R(s)$-Lipschitz homeomorphism of Q_s such that $(hx)_k \geq s_k - s_{k+1}/4$ for every $k \in N$.

To show that Q_s is Lipschitz homogeneous it will be sufficient to prove that an arbitrary point of Q_s can be mapped to $\overline{0} = (0, 0, \ldots)$ by a Lipschitz homeomorphism of Q_s. To that end, let $q \in Q_s$. By the above, we can assume that q is in good position. For each $k \in N$, we define a map φ_k of $I_k^s \times I_{k+1}^s$ onto itself as follows. (The square $I_k^s \times I_{k+1}^s$ has the maximum metric.) Let ϕ_k be a map $\partial(I_k^s \times I_{k+1}^s) \to \partial(I_k^s \times I_{k+1}^s)$ such that ϕ_k is the identity on

$$\partial(I_k^s \times I_{k+1}^s) - [(\{s_k\} \times [0, s_{k+1}]) \cup ([s_k - s_{k+1}, s_k] \times \{s_{k+1}\})]$$

and ϕ_k maps linearly the segments $\{s_k\} \times [0, s_{k+1}/2]$, $\{s_k\} \times [s_{k+1}/2, s_{k+1}]$ and $[s_k - s_{k+1}, s_k] \times \{s_{k+1}\}$ onto the segments

$$\{s_k\} \times [0, s_{k+1}], \quad [s_k - s_{k+1}/2, s_k] \times \{s_{k+1}\} \quad \text{and} \quad [s_k - s_{k+1}, s_k - s_{k+1}/2] \times \{s_{k+1}\},$$

respectively. Then ϕ_k is defined on $\partial(I_k^s \times I_{k+1}^s)$. Let the restriction of φ_k to $A_k = [s_k - s_{k+1}, s_k] \times [0, s_{k+1}]$ be the radial extension of ϕ_k relative to the point $(s_k - s_{k+1}, 0)$ and let φ_k be the identity on the complement of A_k. The reader will find Figure 1 helpful.

Then φ_k is a PL-homeomorphism and the action of φ_k on B_2 is given by

$$\varphi_k(x, y) = (\tfrac{3}{2}x - y - (s_k - s_{k+1})/2,\ x - (s_k - s_{k+1})).$$

It follows that φ_k is a Lipschitz homeomorphism; let $\operatorname{bilip} \varphi_k = L$. (We note that L is a universal constant not depending on the sequence s.)

For each $k \in N$, we define a map $g_k : Q_s \to Q_s$ by setting

$$g_k = \mathrm{id}_{I_1^s \times \cdots \times I_{k-1}^s} \times \varphi_k \times \mathrm{id}_{Q_s^{k+2}}.$$

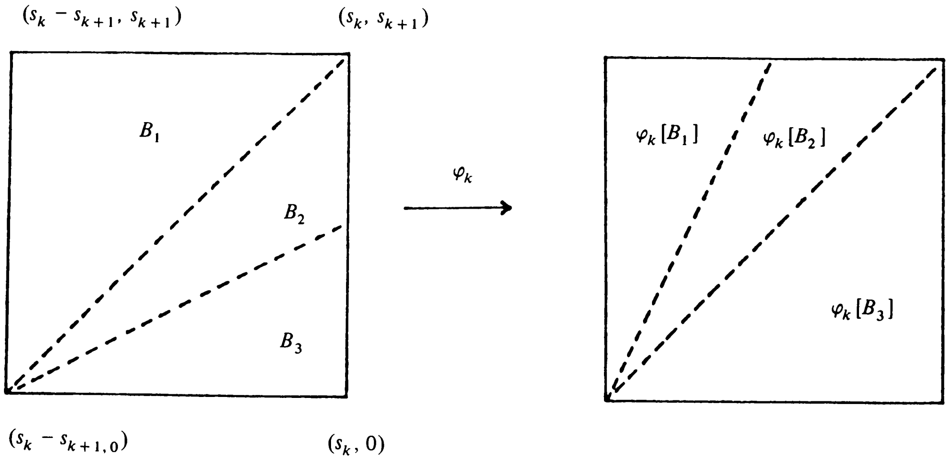

FIGURE 1

The important property of g_k is that it acts "backwards", i.e. the direction of the map is from I^s_{k+1}-coordinates to I^s_k-coordinates. Further, g_k leaves I^s_j-coordinates fixed for $j < k$. Now having defined the maps g_k, let $f_n = g_n g_{n-1} \cdots g_1$ for $n \in N$. Recall that our point q was in good position in Q_s. Hence, (q_1, q_2) is in good position in $I^s_1 \times I^s_2$ (in the obvious sense) and thus f_1 maps q to a point $(r_1, u, q_3, q_4, \ldots)$, where $s_1 - s_2 \leq r_1 \leq s_1 - s_2/4$ and $s_2 - s_3/4 \leq u \leq s_2$ because φ_1 is level-raising for I^s_2-coordinates. Consequently $((f_1 q)_2, (f_1 q)_3)$ is in good position in $I^s_2 \times I^s_3$ (see Figure 2). As above we infer that $((f_2 q)_3, (f_2 q)_4)$ is in good position in $I^s_3 \times I^s_4$ and so on. In general, an inductive argument shows that f_n maps q to a point $(r_1, r_2, \ldots, r_n, u, q_{n+2}, q_{n+3}, \ldots)$, where $s_i - s_{i+1} \leq r_i \leq s_i - s_{i+1}/4$ for $1 \leq i \leq n$ and $s_{n+1} - s_{n+2}/4 \leq u \leq s_{n+1}$.

The sequence $\{f_n \colon n \in N\}$ forms a Cauchy sequence in the space of continuous maps of Q_s into itself, equipped with the supremum metric. (Just note that the distance of f_n and f_{n+k} is less than $2s_n$ for all $k \in N$.) It follows that $f = \lim f_n$ is

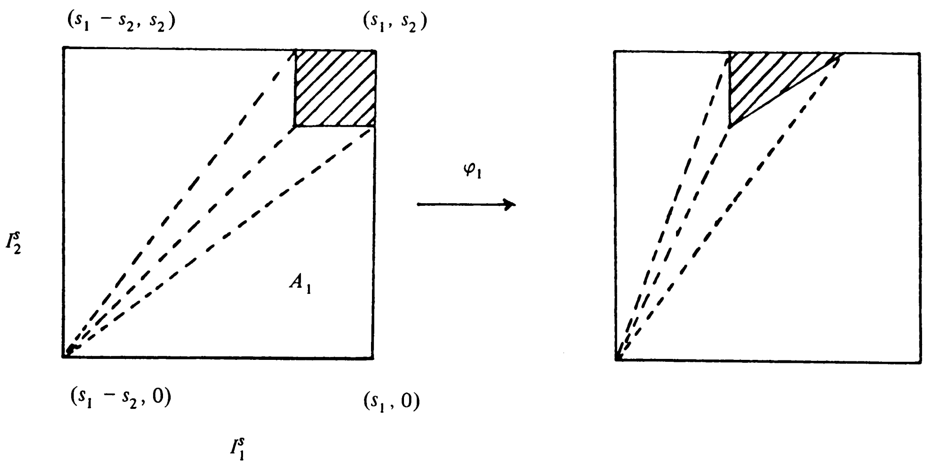

FIGURE 2

a continuous map of Q_s into itself. As Q_s is compact, it can easily be shown that f is surjective. We shall prove that f is a Lipschitz homeomorphism.

The proof that f is Lipschitz depends on the following inequality, whose validity is left as an illuminating exercise to the reader. Let x and y be any points of $I_k^s \times I_{k+1}^s$, and let $x' = \varphi_k(x)$, $y' = \varphi_k(y)$. Then

$$(*) \qquad |x'_{k+1} - y'_{k+1}| \leq \max\{|x_k - y_k|, 2|x_{k+1} - y_{k+1}|\}.$$

The idea behind this inequality can be seen from Figure 3.

Now let x and y be arbitrary points of Q_s. Put $\|x - y\|_{c_0} = \alpha$. We shall prove by induction that $|(f_k x)_{k+1} - (f_k y)_{k+1}| \leq 2\alpha$ for all $k \geq 0$, where we put $f_0 = \mathrm{id}_{Q_s}$. Clearly this holds for $k = 0$. Suppose that the claim holds for $0 \leq k \leq n$. Since by the induction hypothesis $|(f_n x)_{n+1} - (f_n y)_{n+1}| \leq 2\alpha$, f_n does not affect I_{n+2}^s-coordinates and $|x_{n+2} - y_{n+2}| \leq \alpha$, $(*)$ yields

$$\begin{aligned} |(f_{n+1}x)_{n+2} - (f_{n+1}y)_{n+2}| &= |(g_{n+1}f_n x)_{n+2} - (g_{n+1}f_n y)_{n+2}| \\ &\leq \max\{|(f_n x)_{n+1} - (f_n y)_{n+1}|, 2|x_{n+2} - y_{n+2}|\} \leq 2\alpha. \end{aligned}$$

Hence, the claim is valid for all $k \in N$. Since the maps φ_k are L-Lipschitz, we obtain $|(f_k x)_i - (f_k y)_i| \leq 2L\alpha$ for all $k \in N$ and $i \leq k$. Therefore, $\|f_k x - f_k y\|_{c_0} \leq 2L\alpha$ for every $k \in N$ and thus

$$\|fx - fy\|_{c_0} \leq 2L\|x - y\|_{c_0}.$$

To prove that f is bilipschitzian, it will be sufficient to show that f satisfies a left-hand side inequality $K^{-1}\|x - y\|_{c_0} \leq \|fx - fy\|_{c_0}$ for some $K \geq 1$. We shall prove this for $K = 2L$. To accomplish this, we derive from $(*)$ a similar inequality for φ_k^{-1}. This is possible since φ_k and φ_k^{-1} have similar expressions on A_k. (Switch I_k^s and I_{k+1}^s, if necessary.) Let x and y be any points of $I_k^s \times I_{k+1}^s$ and let $x' = \varphi_k(x)$, $y' = \varphi_k(y)$. Then

$$(**) \qquad |x_k - y_k| \leq \max\{|x'_{k+1} - y'_{k+1}|, 2|x'_k - y'_k|\}.$$

Now let $x, y \in Q_s$, $x \neq y$, and let k_0 be the least index k for which $\|x - y\|_{c_0} = |x_k - y_k| = \alpha$. The argument given below shows that we can assume $k_0 > 1$. Since $\varphi_{k_0-1}^{-1}$ is L-Lipschitz, either

$$|(f_{k_0-1}x)_{k_0-1} - (f_{k_0-1}y)_{k_0-1}| \geq \alpha/L \quad \text{or} \quad |(f_{k_0-1}x)_{k_0} - (f_{k_0-1}y)_{k_0}| \geq \alpha/L.$$

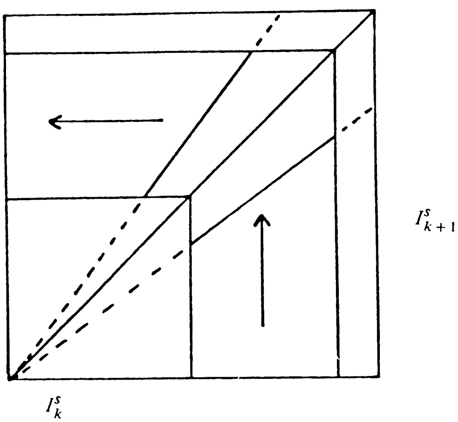

FIGURE 3

In the first case, $\|fx - fy\|_{c_0} \geq \alpha/L$ since $I^s_{k_0-1}$-coordinates are not changed by g_j for $j \geq k_0$. Thus, suppose that the latter case occurs. From (**) we obtain that either

(i) $|(f_{k_0}x)_{k_0} - (f_{k_0}y)_{k_0}| \geq \alpha/2L$, or

(ii) $|(f_{k_0}x)_{k_0+1} - (f_{k_0}y)_{k_0+1}| \geq \alpha/L$.

If (i) is true, then we are done for the same reason as in the first case. Case (ii) is a starting point for an inductive process where (ii) appears at each step. However, case (ii) can occur only finitely many times, since

$$|(f_kx)_{k+1} - (f_ky)_{k+1}| \leq 2s_{k+1} \to 0.$$

It follows that in any case $\|fx - fy\|_{c_0} \geq \alpha/2L$. Thus, $\operatorname{bilip} f \leq 2L$.

To finish the proof, note that f maps q to the point $(r_1, r_2, \ldots)$, where $s_i - s_{i+1} \leq r_i \leq s_i - s_{i+1}/4$. For each $k \in N$, let $h_k\colon I^s_k \to I^s_k$ be a $4R(s)$-Lipschitz homeomorphism for which $h_k(r_k) = 0$. Let $h\colon Q_s \to Q_s$ be the map given by

$$h(x_1, x_2, \ldots) = (h_1x_1, h_2x_2, \ldots).$$

Then hf is an $8R(s)L$-Lipschitz homeomorphism of Q_s with $hf(q) = \overline{0}$, as required.

REMARK. In particular, Q is Lipschitz homogeneous with respect to the metric $\rho(x,y) = \sup\{2^{-k}|x_k - y_k|\colon k \in N\}$.

4. The number of essentially different Lipschitz homogeneous metrics d_s for Q. It is natural to ask what is the number of all Lipschitz equivalence classes of metrics d_s, where $s \in S$. This question will be answered by using a completely computational application of the following "comparison principle". Recall that for a compact metric space X and an $\varepsilon > 0$, $N(X, \varepsilon)$ denotes the maximum cardinality of an ε-discrete subset of X.

COMPARISON PRINCIPLE. *Let X and Y be compact metric spaces. If there is an L-Lipschitz homeomorphism $\varphi\colon X \to Y$, then for each $\varepsilon > 0$, we have $N(Y, L\varepsilon) \leq N(X, \varepsilon)$. Thus, if for each $L \geq 1$ there is an $n \in N$ with $N(Y, L/n) > N(X, 1/n)$, then Y is not Lipschitz homeomorphic to X.*

For the proof note that if D is an $L\varepsilon$-discrete subset of Y, then $\varphi^{-1}[D]$ is an ε-discrete subset of X.

4.1. LEMMA. *For each $s \in S$ and each $\varepsilon > 0$, we have*

$$N(Q_s, \varepsilon) = \prod_{k \in N} ([2s_k/\varepsilon] + 1),$$

where $[r]$ denotes the greatest integer p with $p \leq r$.

PROOF. For each $k \in N$ let $D_k = \{-s_k + i\varepsilon\colon 0 \leq i < [2s_k/\varepsilon] + 1\}$. Then D_k is an ε-discrete subset of $[-s_k, s_k]$ with $|D_k| = [2s_k/\varepsilon] + 1$ and hence $D = D_1 \times D_2 \times D_3 \times \cdots$ is an ε-discrete subset of Q_s with

$$|D| = \prod_{k \in N} ([2s_k/\varepsilon] + 1).$$

(Note that the above product makes sense since there is an n such that $2s_k/\varepsilon < 1$ for $k \geq n$.) Suppose that D' is any ε-discrete subset of Q_s. For each $k \in N$, we define a map $f_k\colon D' \to D_k$ by setting

$$f_k(x) = \max\{y \in D_k\colon y \leq x_k\}.$$

Let $f\colon D' \to D$ be the map given by $f(x) = (f_1(x), f_2(x), f_3(x), \ldots)$. Then f is injective. Indeed, suppose that $x, y \in D'$ and that $f(x) = f(y)$. For each $k \in N$, we have

$$x_k, y_k \in [f_k(x), f_k(x) + \varepsilon[$$

and consequently $\|x - y\|_{c_0} < \varepsilon$ which implies $x = y$ since D' is ε-discrete. Thus, $|D'| \leq |D|$.

4.2. LEMMA. *Let $r, s \in S$ and suppose that $s_k/r_k \geq 2^{\alpha k}$ for all $k \in N$, where $\alpha > 0$. Then Q_r is not Lipschitz homeomorphic to Q_s.*

PROOF. To derive a contradiction, let $L \geq 1$. Choose an $m' \in N$ large enough to satisfy the following conditions: (i) $2^{\alpha m'/4} \geq L$; (ii) $m' \geq 4/\alpha$. Then choose an n such that $1 \leq k \leq m'$ implies $r_k/2^{-n-1} \geq 1$ and let $m = \max\{k \in N\colon r_k/2^{-n-1} \geq 1\}$. Clearly m satisfies the conditions (i)–(ii) and $2^{n+1}r_m \geq 1$. By Lemma 4.1,

$$N(Q_r, 2^{-n}) = \prod_{k \in N} ([2^{n+1}r_k] + 1)$$

and

$$N(Q_s, 2^{-n}L) = \prod_{k \in N} ([2^{n+1}(s_k/L)] + 1).$$

We shall apply the Comparison Principle. For each $k > m$ we have $[2^{n+1}r_k] + 1 = 1$ and thus

$$\begin{aligned}\frac{N(Q_s, 2^{-n}L)}{N(Q_r, 2^{-n})} &\geq \prod_{k=1}^{m} \frac{[2^{n+1}(s_k/L)] + 1}{[2^{n+1}r_k] + 1} \geq \prod_{k=1}^{m} \frac{2^{n+1}(s_k/L)}{2^{n+1}r_k + 1} \geq \prod_{k=1}^{m} \frac{s_k/r_k}{2L} \\ &\geq 2^{-m}L^{-m} \prod_{k=1}^{m} 2^{\alpha k} = 2^{-m}L^{-m}2^{m(m+1)\alpha/2} \\ &\geq (2^{-\alpha m/2}2^{(m+1)\alpha-2})^{m/2} = 2^{(m\alpha/2+\alpha-2)m/2} > 1,\end{aligned}$$

a contradiction with the assumption that there is an L-Lipschitz homeomorphism of Q_r onto Q_s.

4.3. COROLLARY. *There are 2^ω essentially different Lipschitz homogeneous metrics d_s, where $s \in S$.*

PROOF. For each $\alpha > 0$, put $s_k(\alpha) = 2^{-\alpha k}$. Then $0 < \alpha < \beta$ implies $s_k(\alpha)/s_k(\beta) = 2^{(\beta-\alpha)k}$ and Lemma 4.2 shows that $Q_{s(\beta)}$ is not Lipschitz homeomorphic to $Q_{s(\alpha)}$.

5. Lipschitz Q-manifolds. As we have shown that the cubes Q are Lipschitz homogeneous for $s \in S$, it is reasonable to introduce the following notion.

5.1. DEFINITION. A separable metric space X is called a Lipschitz Q-manifold if there is an $s \in S$ such that each point of X has a neighbourhood which is Lipschitz homeomorphic to Q_s.

It is well known that a connected Hilbert cube manifold is homogeneous. This can be proved by using the facts that Q is locally homeomorphic to $Q \times [0, 1[$ and that every homeomorphism between two Z-sets of Q can be extended to a homeomorphism of Q. There is no theory of Lipschitz Z-sets but we can obtain a

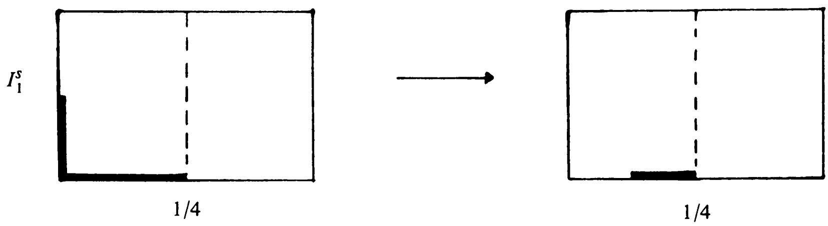

FIGURE 4

proof of the Lipschitz homogeneity of connected Lipschitz Q-manifolds by using an elementary isotoping device augmenting the proof of 3.1.

To show that X is Lipschitz homogeneous, it will be sufficient to prove that any point $p \in X$ has a neighbourhood U such that any $q \in U$ can be mapped to p by a Lipschitz homeomorphism of X. Since Q_s is locally Lipschitz homeomorphic to $[0,1[\times Q_s$ (this follows from 3.1 by an argument similar to that of 12.1 in [**Ch**]) it is easily seen that proving the Lipschitz homogeneity of X amounts to establishing the following variant of 3.1.

5.2. THEOREM. *For any two points x and y of $[0,1/4[\times Q_s$ there is a Lipschitz homeomorphism $F\colon [0,1[\times Q_s \to [0,1[\times Q_s$ such that $F(x) = y$ and F is the identity on $[3/4,1[\times Q_s$.*

PROOF. We shall show that any point x of $[0,1/4[\times Q_s$ can be mapped to the point $(1/8,\bar{0})$ by a map F that satisfies the conditions of 5.2. Since $x_1 < 1/4$ there is a Lipschitz homeomorphism of $[0,1[\times I_1^s$ which takes x to a point $(1/8, x_2')$ and which is the identity on $[1/4,1[\times I_1^s$. Hence, we can assume that $(x_1,x_2) \in \{1/8\} \times I_1^s$. (Figure 4 shows a typical situation when we map a boundary point of $[0,1/4[\times I_1^s$, i.e. $x_1 = 0$. The map can be realized by taking a suitable PL-homeomorphism of the boundary of $[0,1/4[\times I_1^s$ and then extending it radially.) From now on we shall work with the projection $x' = (x_2, x_3, \ldots)$ of x on Q_s.

As in the proof of 3.1, we use the stretching maps $f_k\colon I_k^s \to I_k^s$ such that $|f_k x_k'| \geq s_k - s_{k+1}/4$ for all $k \in N$. Since the constants bilip f_k are uniformly bounded there is a Lipschitz homeomorphism $F_1\colon [0,1[\times Q_s \to [0,1[\times Q_s$ which (Lipschitz) isotopes $f_1 \times f_2 \times f_3 \times \cdots$ to the identity so that $F_1(t,q_1,q_2,\ldots) = (t, f_1q_1, f_2q_2,\ldots)$ for every $(t,q) \in [0,1/4] \times Q_s$ and F_1 is the identity on $[3/4,1[\times Q_s$. Write $F_1(x) = (1/8, x_1'', x_2'', \ldots)$.

Following the notation from the proof of 3.1, we must avoid using reflections g_k since they cannot be isotoped to the identity. Write $Q_s = (I_1^s \times I_2^s) \times (I_3^s \times I_4^s) \times \cdots$. For each $k \in N$, (x_{2k-1}'', x_{2k}'') lies in one of the small four corners of $I_{2k-1}^s \times I_{2k}^s$. Let g_k be a Lipschitz homeomorphism of $I_{2k-1}^s \times I_{2k}^s$ which maps the corner of (x_{2k-1}'', x_{2k}'') into the one which is in good position. These maps are to be obtained by rotating the boundary $\partial(I_{2k-1}^s \times I_{2k}^s)$ (with a suitable squeezing continuing the earlier effect of the maps f_k) and then radially extending to the whole of $I_{2k-1}^s \times I_{2k}^s$. Now one can easily find a Lipschitz homeomorphism $F_2\colon [0,1[\times Q_s \to [0,1[\times Q_s$ which (Lipschitz) isotopes $g_1 \times g_2 \times \cdots$ to the identity so that

$$F_2(t,q_1,q_2,q_3,q_4,\ldots) = (t, (g_1(q_1,q_2))_1, (g_1(q_1,q_2))_2, (g_2(q_3,q_4))_1, \ldots)$$

for all $(t,q) \in [0,1/4] \times Q_s$ and F_2 is the identity on $[3/4,1[\times Q_s$.

Next we shall handle the maps φ_k. For each $t \in [1/4, 3/4]$, define $\varphi_k(t)\colon I_k^s \times I_{k+1}^s \to I_k^s \times I_{k+1}^s$ as in the proof of 3.1 but replace A_k by the set

$$A_k(t) = [s_k - 2(3/4 - t)s_{k+1}, s_k] \times [s_{k+1} - 2(3/4 - t)s_{k+1}, s_{k+1}].$$

Thus, the sets $A_k(t)$ are linearly squeezed to their corners. On each slice $\{t\} \times Q_s$, where $t \in [1/4, 3/4]$, the maps $\varphi_k(t)$ induce a $2L$-Lipschitz homeomorphism $f_t\colon \{t\} \times Q_s \to \{t\} \times Q_s$. Thus, we easily see that we obtain a Lipschitz homeomorphism F_3 of $[0,1[\times Q_s$ such that the map f from the proof of 3.1 is isotoped by F_3 to the identity in $[1/4, 3/4] \times Q_s$. Let $F_4\colon [0,1[\times Q_s \to [0,1[\times Q_s$ do the same for the map h. Then $F = F_4 \cdots F_1$ is the required map.

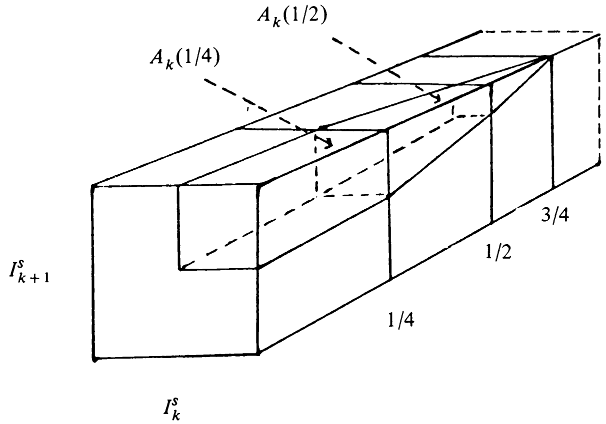

FIGURE 5

5.3. COROLLARY. *A connected Lipschitz Q-manifold is Lipschitz homogeneous.*

PROOF. See the proof of the following proposition.

Indeed, Theorem 5.2 yields a slightly stronger "basic cube" result for Lipschitz Q-manifolds.

5.4. PROPOSITION. *Let X be a connected metric space. Suppose that for each $x \in X$ there exists an $s_x \in S$ and a neighbourhood U_x of x such that U_x is Lipschitz homeomorphic to $Q_{s(x)}$. Then X is a Lipschitz Q-manifold.*

PROOF. As a locally separable connected metrizable space, X is separable. Let $p, q \in X$. Use the connectedness of X to find a simple chain $\{U_1, \dots, U_n\}$ of open subsets of X and points $x_1, \dots, x_{n+1} \in X$ such that $p = x_1 \in U_1$, $q = x_{n+1} \in U_n$ and $x_{i+1} \in U_i \cap U_{i+1}$ for $1 \le i \le n$, with Lipschitz homeomorphisms

$$\varphi_i\colon U_i \to [0,1[\times Q_{s_i}, \qquad s_i \in S,$$

where $\varphi_i(x_i)$, $\varphi_i(x_{i+1}) \in [0, 1/4[\times Q_{s(i)}$. By 5.2 we can find a Lipschitz homeomorphism f_i of $[0,1[\times Q_{s(i)}$ such that $f_i\varphi_i(x_i) = \varphi_i(x_{i+1})$ and f_i is the identity on $[3/4, 1[\times Q_{s(i)}$. Define a map $g_i\colon X \to X$ by setting $g_i(x) = \varphi_i^{-1} f_i \varphi_i(x)$ for $x \in U_i$ and $g_i(x) = x$ for $x \in X - U_i$. Let $K_i = \varphi_i^{-1}[[0, 3/4] \times Q_{s(i)}]$, and put $\delta_i = d(X - U_i, K_i) > 0$ and $\mu_i = \operatorname{diam} K_i$. The map g_i being a Lipschitz homeomorphism on U_i and the identity on $X - K_i$, suppose that $x \in K_i$ and $y \in X - U_i$.

Then

$$1+\frac{\mu_i}{\delta_i} \geq \frac{d(x,y)+\mu_i}{d(x,y)} \geq \frac{d(g_ix,g_iy)}{d(x,y)} \geq \frac{d(y,K_i)}{d(y,K_i)+\mu_i} \geq \frac{1}{1+\mu_i/\delta_i},$$

which shows that g_i is a Lipschitz homeomorphism of X. Now $g_ng_{n-1}\cdots g_1$ maps p to q. Thus, X is Lipschitz homogeneous and the claim of 5.4 follows.

The use of simple chains in the proof of 5.4 can be motivated by the fact that this proof can easily be modified to show that a connected Lipschitz Q-manifold is *strongly Lipschitz n-homogeneous* for every $n \in N$. It follows that for each $s \in S$, Q_s is strongly Lipschitz n-homogeneous. It is well known that if A_1, A_2 are compact subsets of the pseudointerior $\{x \in Q_s\colon |x_k| < s_k \,\forall\, k \in N\}$ and $f\colon A_1 \to A_2$ is a homeomorphism, then f can be extended to a homeomorphism of Q_s. The same does *not* hold for Lipschitz homeomorphisms as can be seen from the following example.

5.5. EXAMPLE. Define a sequence $\{\alpha_n\}$ of positive integers by setting $\alpha_1 = 2$, $\alpha_{n+1} = \alpha_n + n^2$. Let $a_n = s_n(1-2^{-\alpha_n - n})$ and let $b_n = s_n(1-2^{-\alpha_n})$. We define two compact subsets A_1, A_2 of the pseudointerior of Q_s:

$$A_1 = \{x \in Q_s\colon |x_k| \leq a_k \,\forall\, k \in N\} \quad \text{and} \quad A_2 = \{x \in Q_s\colon |x_k| \leq b_k \,\forall\, k \in N\}.$$

Clearly the map $f\colon A_1 \to A_2$ given by $(fx)_k = (b_k/a_k)x_k$ is a Lipschitz homeomorphism. Suppose that it has an extension to a Lipschitz homeomorphism $\overline{f}$ of Q_s. Let $L = \operatorname{bilip} \overline{f}$ and let $n \geq 3$ be such that $2^n > L$. Notice that the set $E = \overline{B}(A_1, 2^{-\alpha_n - n}s_n)$ has the form $F \times Q_s^n$, where F is the set of all points x of $I_1^s \times \cdots \times I_{n-1}^s$ such that

$$|x_k| \leq a_k + 2^{-\alpha_n - n}s_n < s_k \quad \text{for } 1 \leq k < n.$$

Now $I_1^s \times \cdots \times I_{n-1}^s - F$ is homotopy equivalent to S^{n-2} and so is $Q_s - E = (I_1^s \times \cdots \times I_{n-1}^s - F) \times Q_s^n$. Since $\overline{f}$ is an L-Lipschitz homeomorphism, we have

$$\overline{B}(A_2, 2^{-\alpha_n - n}s_n/L) \subset \overline{f}[E] \subset \overline{B}(A_2, 2^{-\alpha_n - n}s_nL).$$

The inequalities $2^{-\alpha_n - n}L < 2^{-\alpha_n}$ and $2^{-\alpha_{n+1}} < 2^{-\alpha_n - n}/L$ imply that

$$\overline{B}(A_2, 2^{-\alpha_n - n}s_n/L) = F' \times Q_s^{n+1} \quad \text{and} \quad \overline{B}(A_2, 2^{-\alpha_n - n}s_nL) = F'' \times Q_s^{n+1},$$

where both F' and F'' are products $[-r_1, r_1] \times \cdots \times [-r_n, r_n]$ with $r_k < s_k$ for $1 \leq k \leq n$. Thus, there is a natural retraction

$$\varphi\colon Q_s - \overline{B}(A_2, 2^{-\alpha_n - n}s_n/L) \to Q_s - B(A_2, 2^{-\alpha_n - n}s_nL) \qquad (L > 1!)$$

induced by the retraction of $(I_1^s \times \cdots \times I_n^s) - F'$ onto $\overline{(I_1^s \times \cdots \times I_n^s) - F''}$. Thus, we obtain a retraction

$$\varphi'\colon Q_s - \overline{f}[E] \to Q_s - B(A_2, 2^{-\alpha_n - n}s_nL).$$

The induced map $\varphi'_*\colon H_{n-1}(Q_s - \overline{f}[E], Z) \to H_{n-1}(Q_s - B(A_2, 2^{-\alpha_n - n}s_nL), Z) = Z$ of the singular groups is an epimorphism and hence $H_{n-1}(Q_s - \overline{f}[E], Z) \neq \{0\}$, which contradicts the fact that $H_{n-1}(Q_s - E, Z) = \{0\}$.

6. The existence of uniform constants of Lipschitz homogeneity. As can be seen from our proofs for 3.1 and 5.4, for each compact connected Lipschitz Q-manifold there is a uniform constant $\lambda < \infty$ such that the Lipschitz homogeneity of the manifold is witnessed by Lipschitz homeomorphisms f with $\operatorname{bilip} f \leq \lambda$. The following result shows that the same holds for any Lipschitz homogeneous compact metric space.

6.1. THEOREM. *Let X be a Lipschitz homogeneous compact metric space. Then there is a constant $\lambda < \infty$ such that for each pair x, y of points of X there exists an $f \in L_\lambda(X)$ with $f(x) = y$.*

PROOF. For each $x \in X$ and each $n \in N$, let $L_n(X;x)$ denote the set of all $y \in X$ such that there is an $f \in L_n(X)$ with $f(x) = y$. First we shall show that the sets $L_n(X;x)$ are closed. Suppose that $y \in X$ and that $\{x_n\}$ is a sequence of elements of $L_n(X;x)$ such that $x_n \to y$. By the definition of $L_n(X;x)$, for each $k \in N$ there is an $f_k \in L_n(X;x)$ with $f_k(x) = x_k$. Since X is compact, $\{f_k(z)\colon k \in N\}$ is relatively compact for all $z \in X$. On the other hand, $\{f_k\}$ is an equicontinuous subset of $C_c(X)$ (= the set of all continuous self-maps of X with the topology of compact convergence) since it is even equilipschitzian. By Ascoli's theorem $\{f_k\}$ is relatively compact in $C_c(X)$. Hence, there is a subsequence $\{f_{k_j}\}$ of $\{f_k\}$ such that the maps f_{k_j} converge uniformly to some $f \in C(X)$. It follows that f is an n-Lipschitz homeomorphism of X. (Note that f is surjective since X is compact.) But

$$f(x) = \lim f_{k_j}(x) = \lim x_{k_j} = y$$

and hence $y \in L_n(X;x)$. Consequently the sets $L_n(X;x)$ are closed.

Since X is Lipschitz homogeneous, we have $X = \bigcup_{n\in N} L_n(X;x)$ for every $x \in X$. By the Baire Category Theorem there is an $n_x \in N$ such that $U_x = \operatorname{int} L_{n(x)}(X;x) \neq \varnothing$. Accordingly, choose a point $y_x \in U_x$. Since $y_x \in L_{n(x)}(X;x)$, there is an $f_x \in L_{n(x)}(X)$ with $f_x(x) = y_x$. As f_x is continuous, we can find an open neighbourhood V_x of x such that $f_x[V_x] \subset U_x$. Put $m_x = n_x^4$. It follows that for every pair z, w of points of V_x there is an $f \in L_{m(x)}(X)$ with $f(z) = w$. Indeed, let $z, w \in V_x$. Then $f_z(z)$, $f_x(w) \in U_x$ and hence (since $U_x \subset L_{n(x)}(X;x)$) there exist $g, h \in L_{n(x)}(X)$ such that $g(x) = f_x(z)$ and $h(x) = f_x(w)$. Consequently the map $f_x^{-1}hg^{-1}f_x$ takes z to w and

$$\operatorname{bilip}(f_x^{-1}hg^{-1}f_x) \leq n_x^4.$$

Now $\{V_x\colon x \in X\}$ is an open cover of X and has a finite subcover $\{V_{x_1}, \ldots, V_{x_n}\}$. For each pair (i,j), where $1 \leq i,\ j \leq n$, choose a Lipschitz homeomorphism $f_{ij}\colon X \to X$ such that $f_{ij}(x_i) = x_j$, and let

$$m = \max\{\operatorname{bilip} f_{ij}\colon 1 \leq i,\ j \leq n\}.$$

Finally, put $\lambda = m \cdot m_{x_1} \cdots m_{x_n}$. Then λ satisfies the condition of 6.1. In fact, let $x, y \in X$. Choose i, j so that $x \in V_i$ and $y \in V_j$. By the previous paragraph there exist $g \in L_{m(x_i)}(X)$, $h \in L_{m(x_j)}(X)$ with $g(x) = x_i$ and $h(y) = x_j$. Then $h^{-1}f_{ij}g$ maps x to y and

$$\operatorname{bilip}(h^{-1}f_{ij}g) \leq m_{x_j} \cdot m \cdot m_{x_i} \leq \lambda,$$

as desired.

6.2. REMARK. By an argument that is easier but similar to that given for 6.1, one can show that if $G \subset L(X)$ is a compact group of Lipschitz homeomorphisms of X, metrized by the metric $\tilde{\sigma}$ for which

$$\tilde{\sigma}(f,g) = \max\{\sigma(f,g), \sigma(f^{-1}, g^{-1})\} \quad \text{and} \quad \sigma(f,g) = \sup\{d(fx, gx)\colon x \in X\},$$

then there is a uniform constant $\lambda < \infty$ such that $G \subset L_\lambda(X)$. It was shown in [**DW**] that there are compact connected 2-manifolds without boundary (any having genus > 1 is good) X such that no compact subgroup of $H(X)$ acts transitively on X, or, equivalently, X has no compatible metric for which the isometries are transitive. Anyhow, it has been noted in [**HJ**] that for any compact homogeneous metric space X and any $\varepsilon > 0$, there is a compatible metric of X for which X is Lipschitz homogeneous with a uniform constant $\leq 1+\varepsilon$.

7. The hyperspace 2^I of closed nonempty subsets of the unit interval. It was shown by Schori and West in 1975 that 2^I is homeomorphic to Q. Their result implied that 2^I is homogeneous. The following question arises: is 2^I even Lipschitz homogeneous with respect to its standard Hausdorff metric? It is not too difficult to show that the answer to this question is negative. Recall that the Hausdorff metric (denote it by d) of 2^I is defined by setting $d(A_1, A_2) < \varepsilon$ iff $A_1 \subset B(A_2, \varepsilon)$ and $A_2 \subset B(A_1, \varepsilon)$, where $A_1, A_2 \in 2^I$ and $\varepsilon > 0$.

7.1. LEMMA. *For each n, we have $N(2^I, 1/n) = 2^{n+1} - 1$.*

PROOF. Let $A = \{i/n\colon 0 \leq i \leq n\}$. Then $\mathcal{F} =$ the power set $P(A)$ minus the empty set is a $(1/n)$-net of 2^I. Suppose that $\mathcal{D}$ is a $(1/n)$-discrete subset of 2^I. Define a map $F\colon \mathcal{D} \to \mathcal{F}$ by setting $x \in F(D)$ iff either $x = 0$ and $x \in D$ or $x = i/n$, $i > 0$, and

$$](i-1)/n, i/n] \cap D \neq \emptyset.$$

(Then $d(D, F(D)) < 1/n$.) Suppose that $D, D' \in \mathcal{D}$ and $F(D) = F(D')$. It easily follows that $d(D, D') < 1/n$ and hence $D = D'$. Thus, F is injective which implies $|\mathcal{D}| \leq |\mathcal{F}| = 2^{n+1} - 1$.

7.2. PROPOSITION. *2^I is not Lipschitz homogeneous.*

PROOF. We shall prove that there is no Lipschitz homeomorphism of 2^I mapping the point $\{0\}$ of 2^I to the point I. To derive a contradiction, suppose that $\varphi\colon 2^I \to 2^I$ is a Lipschitz homeomorphism with $\varphi(\{0\}) = I$ and let $\operatorname{bilip}\varphi = L$. Choose an $n \in N$ with $n \geq 5L^2$. Denoting by $\overline{B}_d(\{0\}, 1/n)$ the closed $(1/n)$-neighbourhood of $\{0\}$ in 2^I, notice that $\overline{B}_d(\{0\}, 1/n) = 2^{[0,1/n]}$ and by the above lemma

$$N\left(\overline{B}_d(\{0\}, 1/n), 1/n^2\right) = 2^{n+1} - 1.$$

By the Comparison Principle $N(\varphi[\overline{B}_d(\{0\}, 1/n)], L/n^2) \leq 2^{n+1}$. Since φ is L-Lipschitz, we have $\overline{B}_d(I, 1/nL) \subset \varphi[\overline{B}_d(\{0\}, 1/n)]$. Thus, we obtain

$$(*) \qquad N\left(\overline{B}_d(I, 1/nL), L/n^2\right) \leq 2^{n+1}.$$

To see that $(*)$ is not true, let $D = \{i/Ln\colon 0 \leq i \leq r = [Ln]\}$, and for $0 \leq i < r$, let D_i be an L/n^2-discrete subset of $]i/Ln, (i+1)/Ln[$ of cardinality $[n/L^2]$. Choose subsets $D_i' \subset D_i$ for which $|D_i'| \geq [n/L^2] - 2$ and for which $D \cup D_0' \cup \cdots \cup D_{r-1}'$ is an (L/n^2)-discrete subset of I. (Just throw out the least and greatest elements of D_i.) Note that any $C \in 2^I$ containing D belongs to $\overline{B}_d(I, 1/nL)$. Especially each set $D \cup F_0 \cup \cdots \cup F_{r-1}$, where $F_i \in P(D_i')$, is an element of $\overline{B}_d(I, 1/nL)$. But the collection of all such sets is (L/n^2)-discrete in 2^I and their number is

$$\prod_{k=0}^{r-1} |P(D_k')| \geq \prod_{k=0}^{r-1} 2^{n/L^2-3} = (2^{n/L^2-3})^r > 2^{n+1},$$

a contradiction.

Another question which arises in the context of 2^I is whether 2^I is Lipschitz homeomorphic to (Q,ρ), where ρ is the metric given by $\rho(x,y) = \sum 2^{-k}|x_k - y_k|$. The answer is negative, even if I is replaced by any Peano continuum. (It was shown in [**CS**] that if X is a nondegenerate Peano continuum, then 2^X is homeomorphic to Q.)

7.3. LEMMA. *Let X be a connected metric space with* $\operatorname{diam} X \geq 1$. *Then* $N(X, 1/2n) \geq n$.

PROOF. Let $x_0 \in X$ be arbitrary. Since X is connected and $\operatorname{diam} X \geq 1$, for each $j \leq n-1$ there is an x_j such that $d(x_0, x_j) = j/2n$. The set $\{x_0, \dots, x_{n-1}\}$ is $(1/2n)$-discrete, because $1 \leq i < j \leq n-1$ implies $d(x_i, x_j) \geq d(x_j, x_0) - d(x_i, x_0) \geq 1/2n$.

Now let X be any Peano continuum. We can assume that $\operatorname{diam} X \geq 1$. Let s be the sequence $\langle 2^{-k} \rangle$. Note that if $x, y \in Q$ and $\rho(x,y) \geq 2^{-n}$, then there is an $i \leq n+1$ with $|x_i - y_i| \geq 2^{-n-1}$. Therefore, $s_i|x_i - y_i| \geq 2^{-2n-2}$ and hence $d_s(x,y) \geq 2^{-2n-2}$. It follows that any 2^{-n}-discrete subset of (Q,ρ) is 2^{-2n-2}-discrete in Q_s and so

$$N((Q,\rho), 2^{-n}) \leq N(Q_s, 2^{-2n-2}) \leq 2^{(n+3)(2n+3)},$$

where we have used Lemma 4.1. However, $N(2^X, 2^{-n})$ grows much faster. Indeed, by Lemma 7.3, $N(X, 2^{-n}) \geq 2^{n-1}$ and thus $N(2^X, 2^{-n}) \geq 2^{2^{n-1}} - 1$. Consequently,

$$N(2^X, 2^{-n}L)/N((Q,\rho), 2^{-n}) \to \infty \quad \text{as } n \to \infty$$

for any $L \geq 1$ and the Comparison Principle shows that 2^X is not Lipschitz homeomorphic to (Q,ρ).

ACKNOWLEDGEMENT. The author is grateful to J. Liukkonen, J. Väisälä and J. Luukkainen for several useful comments.

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