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THE TRACTION PROBLEM FOR INCOMPRESSIBLE MATERIALS

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ABSTRACT. The traction problem for incompressible materials is treated as a bifurcation problem, where the applied loads are served as parameters. We take both the variational approach and the classical power series approach. The variational approach provides a natural, unified way of looking at this problem. We obtain a count of the number of equilibria together with the determination of their stability. In addition, it also lays down the foundation for the Signorini-Stoppelli type computations. We find second order sufficient conditions for the existence of power series solutions. As a consequence, the linearization stability follows, and it clarifies in some sense the role played by the linear elasticity in the context of the nonlinear elasticity theory. A systematic way of calculating the power series solution is also presented.

1. Introduction. In this paper, we analyze another variation of the basic problem studied in Chillingworth, Marsden and Wan [**2, 3**], in which only volume-preserving deformations are admissible.

Let $B \subset R^3$, $0 \in B$, be an open bounded set with smooth boundary ∂B. Set $\mathcal{U} = \{\phi | \phi: \overline{B} \to R^3$ of Sobolev class H^s, $\phi(0) = 0\}$ with $s > \frac{3}{2} + 1$. Denote by $\mathcal{C}_{\text{vol}} = \{\phi \in \mathcal{U} | J(\phi) = 1\}$ the space of volume preserving deformations, where $J(\phi) = \det F$, $F = D\phi$. The first Piola-Kirchhoff stress tensor P is given by $-JpF^{-T} + \partial W/\partial F$ $(F^{-T} = (F^{-1})^T)$, where $W = W(X, C)$, $C = F^T F$, stands for a smooth stored energy function, and p is an undetermined hydrostatic pressure. For simplicity in notation, we often drop the variable X. Denote by M_3 the space of all 3×3 matrices with inner product $\langle A, B \rangle = \operatorname{Trace} A^T B$, $\text{sym} = \{E \in M_3 | E^T = E\}$, $\text{skew} = \{K \in M_3 | K^T = -K\}$.

As in [**2, 3**], we assume the following conditions throughout this paper:

(H1) The undeformed state is stress-free with pressure zero; $p_0 1 = (\partial W/\partial F)(I_B) = 0$.

(H2) $c = 4(\partial^2 W/\partial C^2)(I_B)$ is positive definite on traceless symmetric matrices, i.e. there exists a constant $c_1 > 0$, such that $\langle c(X)E, E \rangle \geq c_1 |E|^2$ for all $E \in \text{sym}$ with $\operatorname{tr} E = 0$.

The conditions (H1) and (H2) are equivalent to: $W(C)$ has a nondegenerate local minimum at I on the manifold of positive symmetric matrices C, $\det C = 1$, with the normalization $p_0 = 0$. Let $W^* = W - p_0(\det F - 1)$, so that $\partial W^*/\partial F = \partial W/\partial F - Jp_0 F^{-T}$. Thus, replacing W by W^* if necessary, we can always assume that $p_0 = 0$.

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Let $\mathcal{L} = \{(b,\tau) | b: B \to R^3$ of class H^{s-2}, $\tau: \partial B \to R^3$ of class $H^{s-3/2}$, $\int_B b\,dV + \int_{\partial B} \tau\,dA = 0\}$. Given any load $l = (b,\tau) \in \mathcal{L}$, a configuration $\phi \in \mathcal{C}_{\mathrm{vol}}$ is called an *equilibrium solution* for the load l iff there exists $p \in H^{s-1}$ (and hence unique[2] such that

$$\text{(E)} \qquad \begin{aligned} -\operatorname{Div} P &= b \quad \text{in } B, \\ PN &= \tau \quad \text{on } \partial B. \end{aligned}$$

Here, $\operatorname{Div} P = \sum_{j=1}^3 P_{ij,j} = \sum_{j=1}^3 \partial P_{ij}/\partial X_j$. An equilibrium solution $\phi \in \mathcal{C}_{\mathrm{vol}}$ is said to be *stable* iff ϕ is a local minimum of the function V on $\mathcal{C}_{\mathrm{vol}}$, where $V(\phi) = \int_B W(\phi)\,dV - \langle l, \phi\rangle$, $\langle l,\phi\rangle = \int_B \langle b,\phi\rangle\,dV + \int_{\partial B}\langle \tau,\phi\rangle\,dA$.

Set $\Phi(p,\phi) = (-\operatorname{Div} P, PN)$. Let $SO(3)$, the proper orthogonal group on R^3, act on ϕ, l by composition and on p trivially. Then, Φ is $SO(3)$-equivariant and $SO(3)I_B = SO(3)$ is a set of trivial solutions.

Again, as in [**2, 3**], the basic problem is

(T) *Describe the equilibrium solutions for loads λl, with $\lambda > 0$, small, and l near some fixed load l_0.*

Specifically, one needs

(a) to count the number of solutions,

(b) to determine their stabilities.

For general reference on elasticity theory, please see [**5, 8–10, 14, 15**].

The variational approach in [**2, 3**] proves to be very powerful and successful in dealing with such a traction problem for compressible materials. The symmetry group $SO(3)$ plays an important role, and a reduced bifurcation equation is obtained through applying the Liapunov-Schmidt procedure twice. Indeed, old results become unified and many new results are obtained. In §2, we aim to carry out a similar but more complicated analysis for the incompressible materials. It turns out the loads should be classified into the same types as in [**2**]. One obtains the same bifurcation diagrams for type 0 and type 1 loads. One also gets the same upper bound for types 2, 3, and 4 loads in "nondegenerate" cases, circles of solutions for parallel loads, and homogeneous solutions for "homogeneous" loads (cf. [**1**]).

The Signorini Scheme [**11, 12, 15, 16**] enables us to find power series solutions near identity when the applied load possesses no axis of equilibria. This scheme has been generalized in Marsden and Wan [**11**]. A perturbation scheme for incompressible materials can be found in Green and Spratt [**7**]. In the last section, we aim to extend the results in [**11**] to traction problems for incompressible materials. Again, necessary conditions (i.e. Signorini's compatibility conditions, incompressibility conditions) are obtained by formal expansions. A geometric reformulation of the first order conditions enables us to generalize a result of Tolotti [**14**]. Motivated by the methods in [**3**], one can get a second order sufficient condition, via results from §2. As a consequence, the linearization stability result (cf. [**11, 6**]) follows, and it clarifies in some sense the role played by the linear elasticity in the context of the nonlinear elasticity theory. Based on this second order sufficient condition, we work out a generalized Signorini Scheme for incompressible materials.

As a preparation for §§2 and 3, we conclude this section with a study of the "linearized" problem of our problem (E) in terms of a map $\Phi(p,\phi)$.

[2] It suffices to show that $\operatorname{Div}(JqF^{-T}) = 0$ and $(JqF^{-T})N = 0$ imply $q = 0$. For $\operatorname{Div}(JF^{-T}) = 0$, we have $Dq = 0$ in B. Thus $Dq = 0$ on B and $q = 0$ on ∂B imply $q = 0$.

Let $\mathcal{Q} = \{q: \overline{B} \to R | q \in H^{s-1}\}$ and $\mathcal{V} = \{u \in \mathcal{U} | \operatorname{div} u = 0\}$.

1.1. DEFINITION. Let $L = \mathcal{Q} \times \mathcal{V} \to \mathcal{L}$ be the *linear map* defined by

$$L(q,u) = (-\mathrm{Div}[-q1 + c(e)], [-q1 + c(e)]N),$$

with $e = \frac{1}{2}(\nabla u + \nabla u^T)$.

1.2. PROPOSITION (CF. [9]). *Under assumptions* (H1) *and* (H2) *on* W, *we have*

(a) $\ker L = \{(0, KX) | K \in \text{skew}\}$,

(b) $\operatorname{Im} L = \mathcal{L}_e =$ *the space of all equilibrated loads.*

For $l = (b, \tau) \in \mathcal{L}$, $(k(l))_{ij} = \int_B b_i X_j \, dV + \int_{\partial B} \tau_i X_j \, dA$. The load l is said to be *equilibrated* or $l \in \mathcal{L}_e$ iff $k(l) \in \text{sym}$. It is easy to see that $l \in \mathcal{L}_e$ iff $\langle l, KX \rangle = 0$ for all $K \in \text{skew}$.

1.3. LEMMA. $\langle l, w \rangle = 0$ *for all* w, $\operatorname{tr} \nabla w = 0$ *iff* $l = \binom{-\mathrm{Div}\, q}{qN}$ *for some scalar function* q $(\in H^{s-1})$.

PROOF. Suppose $l = \binom{-\mathrm{Div}\, q1}{qN}$ for some q. Then

$$\langle l, w \rangle = \left\langle \begin{pmatrix} -\mathrm{Div}\, q1 \\ qN \end{pmatrix}, w \right\rangle = \langle q1, \nabla w \rangle = \langle q, \operatorname{tr} \nabla w \rangle = 0.$$

On the other hand, assume that $l = \binom{b}{\tau}$. $\langle l, w \rangle = 0$ for all w, with $\operatorname{tr}(\nabla w) = 0$. First claim τ must be normal to ∂B, i.e. $\tau = g(X)N$. Indeed choose vector fields w_n so that $\operatorname{tr}(\nabla w_n) = 0$, $\|w_n\|_0 \to 0$ as $n \to \infty$, and $w_n|_{\partial B} =$ tangent component τ_t of τ on ∂B.

From $\int \langle b, w_n \rangle \, dV + \int \langle \tau, w_n \rangle \, dA = 0$, $\int \langle b, w_n \rangle \, dV \to 0$ as $n \to \infty$, one has $\int \langle \tau, w_n \rangle \, dV = \int \langle \tau_t, \tau_t \rangle \, dA = 0$. Thus, $\tau_t = 0$. Next, $\binom{b + \mathrm{Div}\, g1}{0}$ kills all w, such that $\operatorname{tr} \nabla w = 0$. From Hodge theory, $b + \mathrm{Div}\, g1 = \nabla \varphi = \mathrm{Div}\, \varphi 1$, with $\varphi(\partial B) = 0$. Consequently, $b = -\mathrm{Div}(g - \varphi)1$. To finish the proof it suffices to take $q = g - \varphi$. Q.E.D.

PROOF. (a) Clearly, $\ker L \supset \{(0, KX) | K \in \text{skew}\}$. To see the other inclusion, let $(q, u) \in \ker L$. Thus, $0 = \langle L(q,u), u \rangle = \langle -q1 + c(e), \nabla u \rangle = \langle c(e), \nabla u \rangle = \langle c(e), e \rangle$ by divergence theorem and symmetry of c. From positive definiteness of c on traceless matrices, $e = (\nabla u + \nabla u^T)/2 = 0$. Therefore, $u = KX$ for some $K \in \text{skew}$ (cf. Fichera [**5**]). Now $\mathrm{Div}\, q1 = 0$ and $qN = 0$ imply $q = 0$.

(b) $\langle L(q,u), KX \rangle = \langle -q1 + c(\nabla u), K \rangle = 0$ for all $K \in \text{skew}$. Thus, $L(q,u) \in \mathcal{L}_e$. On the other hand, let $l \in \mathcal{L}_e$. By the Korn second inequality, and the equivalence of the norm $\|\nabla u\|_0$ and $\|u\|_1$ on u such that $\int u \, dV = 0$, one gets $\langle c(\nabla u), \nabla u \rangle = \langle c(e), e \rangle \geq c_1 |e|^2 \geq c_2 \|u\|_1^2$ on u such that $\int u_{i,j} \, dV = \int u_{j,i} \, dV \int u \, dV = 0$, and $\operatorname{div} u = 0$. Therefore, by the Lax-Milgram theorem, there exists $u \in \mathcal{V} \cap \mathcal{U}_{\mathrm{sym}}$ such that $\langle c(\nabla u), \nabla w \rangle = \langle l, w \rangle$ for all $w \in \mathcal{V} \cap \mathcal{U}_{\mathrm{sym}}$. Since $\langle l, KX \rangle = 0$ for all $K \in \text{skew}$, $\langle c(\nabla u), \nabla w \rangle = \langle l, w \rangle$ for all $w \in \mathcal{V}$. Hence

$$\left\langle \begin{pmatrix} -\mathrm{Div}\, c(\nabla u) \\ c(\nabla u)N \end{pmatrix} - l, w \right\rangle = 0 \quad \text{for all } w \in \mathcal{V}.$$

Therefore, by Lemma 1.3 there exists q, such that

$$\begin{pmatrix} -\mathrm{Div}\, c(\nabla u) \\ c(\nabla u)N \end{pmatrix} - l = \begin{pmatrix} -\mathrm{Div}\, q1 \\ qN \end{pmatrix}$$

or

$$l = \begin{pmatrix} -\operatorname{Div}(-q1 + c(\nabla u)) \\ (-q1 + c(\nabla u))N \end{pmatrix}. \quad \text{Q.E.D.}$$

2. The variational approach. (A) *Existence and stability via a potential on* $SO(3)$. We begin our variational approach by giving some preliminary results.

2.1. PROPOSITION (CF. EBIN AND MARSDEN [**4**]). $\mathcal{C}_{\text{vol}}$ *is a submanifold of* $\mathcal{U}$ *with tangent space at* $\phi \in \mathcal{C}_{\text{vol}}$ *given by* $T_\phi \mathcal{C}_{\text{vol}} = \{u \in \mathcal{U} | \operatorname{tr}(\nabla u F^{-1}) = 0\}$.

PROOF. Consider the smooth map $J: \mathcal{U} \to H^{s-1}$, defined by $J(\phi) = \det F$. Clearly, $J^{-1}(1) = \mathcal{C}_{\text{vol}}$. Let $\phi \in \mathcal{C}_{\text{vol}}$. Then $DJ(u) = \operatorname{tr}(F^{-1}\nabla u)$. Denote $\mathcal{U} = \mathcal{V} \oplus G = \{u \in \mathcal{U} | \operatorname{div} u = 0\} \oplus \{\nabla\phi - \phi(0) | \phi = 0 \text{ on } \partial B\}$, obtained from the Hodge decomposition of vector fields. At $\phi = I_B$, $\ker DJ = \mathcal{V}$ and $DJ|G$ is an isomorphism onto (i.e. a Dirichlet problem), J is split surjecive. To obtain split surjectivity at $\phi \in \mathcal{C}_{\text{vol}}$, one uses the smooth right translations $R_\phi(u) = u \circ \phi$. Hence $\mathcal{U} = T_\phi \mathcal{C}_{\text{vol}} \oplus \{F^{-T}\nabla\phi - F^{-T}\nabla\phi(0) | \phi | \partial B = 0\}$. Q.E.D.

It is convenient to let Skew $= \{(0, KN) | K \in \text{skew}\}$ and $\mathcal{L} = \mathcal{L}_e \oplus \text{skew}$. Thus, $\mathcal{U}_{\text{sym}} = \text{skew}^\perp = \{u \in \mathcal{U} | \int u_{i,j}\, dV = \int u_{j,i}\, dV\}$. Recall from the following lemma from [**2**]

2.2. LEMMA. (a) $\mathcal{U} = \{KX | K \in \text{skew}\} \oplus \mathcal{U}_{sym}$.

(b) *For some neighborhood* U *of* 0 *in* $\mathcal{U}_{\text{sym}}$, *the map* $\rho: SO(3) \times \{I + U\} \to \mathcal{C}$. $\rho(Q, I + u) = Q^{-1}(I + u)$ *defines a tubular nbd of* $SO(3)$ *in* $\mathcal{C}$.

(c) Skew $= \mathcal{U}_{\text{sym}}^\perp$.

For $\mathcal{U}_{\text{sym}} \supset G = \{\nabla\phi - \phi(0) | \phi = 0 \text{ on } \partial B\}$, it follows easily that

2.3. LEMMA. $\{I + \mathcal{U}_{\text{sym}}\} \cap \mathcal{C}_{\text{vol}}$ *is a submanifold of* $I + \mathcal{U}_{\text{sym}}$ *near* I, *with tangent space* $\mathcal{U}_{\text{sym}} \cap \mathcal{V} = \{u \in \mathcal{U}_{\text{sym}} | \operatorname{div} u = 0\}$.

The variational approach (cf. Remark 2.12) starts from the following. For scalar or tensor fields A_i on B, write $\langle A_1, A_2 \rangle_V = \int_B \langle A_1, A_2 \rangle\, dV$.

2.4. PROPOSITION. *The deformation* $\phi \in \mathcal{C}_{\text{vol}}$ *is an equilibrium solution with a load* λl *iff there is a* p *such that* (p, ϕ) *is a critical point of the function* $\tilde{V}$ *on* $\mathcal{Q} \times \mathcal{C}$, *where* $\tilde{V}(p, \phi) = \int W(\phi)\, dV - \langle \lambda l, \phi \rangle - \langle p, \det F - 1 \rangle_V$.

PROOF. For $D_\phi \langle p, \det F - 1 \rangle_V(u) = \langle p, J \operatorname{tr}(F^{-1}\nabla u) \rangle_V = \langle JpF^{-T}, \nabla u \rangle_V$,

$$\begin{aligned} D_\phi \tilde{V}(u) &= \left\langle \frac{\partial W}{\partial F}, \nabla u \right\rangle_V - \langle \lambda l, u \rangle - \langle JpF^{-T}, \nabla u \rangle_V \\ &= \langle \Phi(p, \phi) - \lambda l, u \rangle \quad \text{(by divergence theorem).} \end{aligned}$$

Thus, $D_p \tilde{V} = 0$, $D_\phi \tilde{V} = 0$ iff $\Phi(p, \phi) = \lambda l$, $\phi \in \mathcal{C}_{\text{vol}}$. Q.E.D.

To consider equilibrium solutions near $SO(3)$, by Lemma 2.2(b) one needs to examine the critical points $(Q, p, \phi) \in SO(3) \times \mathcal{Q} \times (I + U)$ of the function

$$\tilde{V}_\rho(Q, p, \phi) = \tilde{V}(p, \rho(Q, \phi)) = \int W(\phi)\, dV - \langle \lambda Q l, \phi \rangle - \langle p, \det F - 1 \rangle_V.$$

2.5. PROPOSITION. *$(Q, p, \phi) \in SO(3) \times \mathcal{Q} \times (I+U)$ is a critical point of $\tilde{V}_\rho$ on $SO(3) \times \mathcal{Q} \times (I+U)$ iff*

(a) $\Phi(p, \phi) = \lambda Ql$ (mod Skew),

(b) $J(F) = 1$, *and*

(c) $\langle \lambda WQl, \phi \rangle = 0$ *for all* $W \in$ skew.

PROOF. From the equation $D_\phi \tilde{V}_\rho(u) = \langle \Phi(p, \phi) - \lambda Ql, u \rangle$, we see that $D_\phi \tilde{V}_\rho = 0$ on $\mathcal{U}_{\text{sym}}$ iff $\Phi(p, \phi) \equiv \lambda Ql$ mod Skew. Clearly $D_p \tilde{V}_\rho = 0$ iff $J(F) = 1$. For $D_Q \tilde{V}_\rho(WQ) = -\langle \lambda WQl, \phi \rangle$, $D_Q \tilde{V}_\rho = 0$ iff $\langle \lambda WQl, \phi \rangle = 0$ for all $W \in$ skew. Q.E.D.

Since, $\{I + \mathcal{U}_{\text{sym}}\} \cap \mathcal{C}_{\text{vol}}$ is a submanifold near I by Lemma 2.3, we can linearize the map $\mathcal{Q} \times \{I + \mathcal{U}_{\text{sym}}\} \cap \mathcal{C} \to \mathcal{L}$ mod Skew, $(p, \phi) \to \Phi(p, \phi)$ mod Skew, to obtain a linear map $\mathcal{Q} \times \mathcal{U}_{\text{sym}} \cap \mathcal{U} \to \mathcal{L}$ mod skew, $(q, u) \to L(q, u)$ mod skew (cf. Definition 1.1). This linear map is an isomorphism by Proposition 1.2. Hence, by the inverse mapping theorem, the equation $\Phi(p, \phi) \equiv \lambda Ql$ mod Skew can be solved uniquely near $(0, I) \in \mathcal{Q} \times (I+U) \cap \mathcal{C}_{\text{vol}}$, provided $\lambda \geq 0$ is small, and l is near l_0. Let us denote this solution by $(P_Q(\lambda l), \phi_Q(\lambda l))$ or simply by (P_Q, ϕ_Q) when no confusion may arise.

Now, we are ready to carry out the Liapunov-Schmidt procedure.

2.6. DEFINITION. Define $f\colon SO(3) \to R$ by $f(Q) = \tilde{V}_p(Q, p_Q, \phi_Q)$ for $\lambda \geq 0$ small, and l near l_0.

The following is a standard corollary of Proposition 2.5.

2.7. COROLLARY. *(Q, P_Q, ϕ_Q) is a critical point of $\tilde{V}_p$ iff Q is a critical point of f on $SO(3)$.*

2.8. PROPOSITION. *Assume the elastic tensor $c(X)\colon \text{sym} \to \text{sym}$ satisfies the condition* (H2) *in* §1. *Then, there exists a constant $c_2 > 0$ such that $\langle c(e), e \rangle \geq c_2 \|Du\|^2$ for all $u \in \mathcal{U}_{\text{sym}} \cap \mathcal{V}$, where $e = \frac{1}{2}(\nabla u + \nabla u^T)$.*

This proposition follows from the second Korn inequality. By Lemma 2.2(b), one may study V by looking at $V \circ \rho$.

2.9. PROPOSITION. *V_ρ has a local minimum on $SO(3) \times \{I + \mathcal{U}_{\text{sym}}\} \cap \mathcal{C}_{\text{vol}}$ at (Q, ϕ_Q) iff f has a local minimum at Q.*

PROOF. It suffices to show that, to each (Q, ϕ_Q), there exists a neighborhood $\mathcal{U}_1 \times \mathcal{U}_2$ of (Q, ϕ_Q) such that $V_p|_{\tilde{Q} \times \mathcal{U}_2}$ has a nondegenerate local minimum at $\phi_{\tilde{Q}}$ on $(I + \mathcal{U}_{\text{sym}}) \cap \mathcal{C}_{\text{vol}}$. Let $\phi_{\tilde{Q}} + u \in (I + \mathcal{U}_{\text{sym}}) \cap \mathcal{C}_{\text{vol}}$. $1 = J(\phi_{\tilde{Q}} + u) = 1 + \text{tr}(\tilde{F}^{-1}\nabla u) + O(\|\nabla u\|^2)$, implies $\text{tr}(F^{-1}\nabla u) = O(\|\nabla u\|^2)$. $P_Q = 0$, when $\lambda l = 0$, so $P_Q = o(1)$. Thus

$$|\langle P_Q, \text{tr}\, \tilde{F}^{-1}\nabla u \rangle| \leq \|P_Q\|\, \|\text{tr}\, \tilde{F}^{-1}\nabla u\| = o(\|Du\|^2).$$

For $u \in \mathcal{U}_{\text{sym}}$, $\langle \partial W/\partial F, \nabla u \rangle_V - \langle \lambda \tilde{Q}l, u \rangle = \langle P_Q, \text{tr}\, \tilde{F}^{-1}\nabla u \rangle$. Thus $\langle \partial W/\partial F, \nabla u \rangle_V - \langle \lambda \tilde{Q}l, u \rangle = o(\|Du\|^2)$.

$$\begin{aligned} V_\rho|_{\tilde{Q} \times \mathcal{U}_2}(\phi_{\tilde{Q}} + u) - V_\rho|_{\tilde{Q} \times \mathcal{U}_2}(\phi_{\tilde{Q}}) &= \left[\left\langle \frac{\partial W}{\partial F}, \nabla u \right\rangle_V - \langle \lambda \tilde{Q}l, u \rangle \right] + \int \left[\frac{1}{2} \frac{\partial^2 W}{\partial F^2}(\nabla u)^2 + O(|\nabla u|^3) \right] dV \\ &= o(\|Du\|^2) + \int \left[\frac{1}{2} \frac{\partial^2 W}{\partial F^2}(\nabla u)^2 + O(\|\nabla u\|^3) \right] dV \\ &\geq c\|Du\|^2 - K\|Du\|^3 > 0 \end{aligned}$$

for small $\mathcal{U}_1, \mathcal{U}_2$, λ and $\|l - l_0\|$, by Proposition 2.8 and continuity arguments. Q.E.D.

2.10. COROLLARY (OF THE PROOF). *Index of* V *at* $Q^{-1}\phi_Q$ = *Index of* f *at* Q.

Summarizing:

2.11. THEOREM. *The equilibrium solutions of the incompressible traction problem are in* 1-1 *correspondence* $(Q^{-1}\phi_Q \leftrightarrow Q)$ *with the critical points of a function* f *on* $SO(3)$. *The stable solutions correspond to local minima of* f.

2.12. REMARKS. (a) It follows easily from the equation $\langle \partial W/\partial F, \nabla u\rangle_V - \langle \lambda l, u\rangle - \langle JpF^{-T}, \nabla u\rangle_V = \langle \Phi(p,\phi) - \lambda l, u\rangle$ that equilibrium solutions are critical points of V on $\mathcal{C}_{\text{vol}}$. Indeed, *they are the same.* To see the converse, we need a variant (or extension) of Lemma 1.3.

(b) Let $\phi \in \mathcal{C}_{\text{vol}}$. Then, $\langle l, u\rangle = 0$ for all $\operatorname{Tr}(F^{-1}\nabla u) = 0$ iff

$$l = \begin{pmatrix} -\operatorname{Div} JqF^{-T} \\ JqF^{-T}N \end{pmatrix}$$

for some function q. Proof:

$$\begin{aligned}\langle l, u\rangle &= \int b(X)\cdot u(X)\,dv + \int \tau(X)\cdot u(X)\,dA \\ &= \int b(\phi^{-1}x)\cdot u(\phi^{-1}x)\,dv + \int \tau(\phi^{-1}x)\left(\frac{dA}{da}\right)\cdot u(\phi^{-1}x)\,da.\end{aligned}$$

For $\operatorname{div} u(\phi^{-1}x) = 0$ iff $\operatorname{Tr}(F^{-1}\nabla u) = 0$, by Lemma 1.3,

$$\begin{aligned}&b(\phi^{-1}x) = -\operatorname{div} q(\phi^{-1}x) = -F^{-T}\nabla q, \\ &\tau(\phi^{-1}x)\,(dA/da) = q(\phi^{-1}x)n.\end{aligned}$$

Thus, $b(X) = -F^{-T}\nabla q = -\operatorname{Div}(JqF^{-T})$ by $\operatorname{Div}(JF^{-T}) = 0$, and $\tau(X) = q(X)nda/dA = JqF^{-T}N$ by $nda = JF^{-T}NdA$.

(c) Now, let $\phi \in \mathcal{C}_{\text{vol}}$ be a critical point of V. Thus

$$D_\phi V(u) = \langle \partial W/\partial F, \nabla u\rangle - \langle \lambda l, u\rangle = \left\langle \begin{pmatrix} -\operatorname{Div} \partial W/\partial F \\ (\partial W/\partial F)N \end{pmatrix} - \lambda l, u \right\rangle = 0$$

on $\operatorname{tr}(F^{-1}\nabla u) = 0$. By the above lemma,

$$\begin{pmatrix} -\operatorname{Div} \partial W/\partial F \\ (\partial W/\partial F)N \end{pmatrix} - \lambda l = \begin{pmatrix} -\operatorname{Div} JqF^{-T} \\ JqF^{-T}N \end{pmatrix}$$

for some q, or $\Phi(q,\phi) = \lambda l$.

(d) Combining this result with Proposition 2.4, we see immediately that the pressure p is really a Lagrange multiplier for the optimization of V on the manifold $J = 1$.

(B) *Load classification and a second order potential on* S_{A_0}. As before, we make the expansions

$$\phi_Q(\lambda l) = 1 + \lambda u_Q(l) + O(\lambda^2)$$

and

$$P_Q(\lambda l) = \lambda q_Q(l) + O(\lambda^2).$$

2.13. LEMMA. *$L(q_Q(l), u_Q(l)) = (Ql)_e$, where $u_Q(l) \in \mathcal{U}_{\text{sym}} \cap \mathcal{V}$, L is as in Definition* 3.1, *and $(Ql)_e$ is the equilibrated part of Ql according to the decomposition* $\mathcal{L} = \mathcal{L}_e \otimes$ skew.

PROOF. By Proposition 2.5(a)

$$\Phi(\lambda q_Q(l) + O(\lambda^2), I + \lambda u_Q(l) + O(\lambda^2)) \equiv \lambda Ql \mod \text{skew}.$$

Thus, $L(q_Q(l), u_Q(l)) = (\lambda Ql)_e$. Q.E.D.

The load classifications enter into this approach through the following

2.14. PROPOSITION. *All the critical points of f are near $S_{A_0} = \{Q | k(Ql_0) \in \text{sym}\}$ for $l = l_0$ and $\lambda > 0$ small.*

PROOF. Otherwise, there exists $\lambda_n \searrow 0$, $Q_n \to Q \notin S_{A_0}$, by Proposition 3.7(c). $(\lambda_n W Q_n l_0, I + \lambda_n u_{Q_n}(l_0) + O(\lambda_n^2)) = 0$.

Hence, one must have $\langle WQl_0, I\rangle = \langle W, k(Ql_0)\rangle = 0$ for all $W \in$ skew, which is absurd. Q.E.D.

Since the map Φ is $SO(3)$-equivariant, applying De Silva's Lemma if necessary, we may assume in what follows that $l_0 \in \mathcal{L}_e$.

Thus, loads are classified into types $0, 1, \ldots, 4$ according to the different topological types of the associated S_{A_0}. The corresponding critical manifolds S_{A_0} are 4 points, $S^1 \cup 2$ points. $Rp^2 \cup 1$ points, $S^1 \cup S^1$ and $SO(3)$ respectively. This classification of loads is the same as that in [**2**] for the compressible materials.

Now, we shall derive a second order potential on S_{A_0} as in [**2, 3**].

2.15. LEMMA. $\langle Ql, u_Q(l)\rangle = \langle (Ql)_e, u_Q(l)\rangle = \langle c(\nabla u_Q(l)), \nabla u_Q(l)\rangle_V$.

PROOF. This follows from the divergence theorem and $\langle q_Q(l)1, \nabla u_Q(l)_V\rangle = 0$.

2.16. PROPOSITION.

$$f(Q) = c + \lambda\left[-\langle l, Q^T 1\rangle - \frac{\lambda}{2}\langle c(\nabla u_Q^0), \nabla u_Q^0\rangle + O(\lambda^2) + O(\lambda\|l - l_0\|)\right],$$

where $L(q_Q, \nabla u_Q^0) = (Ql_0)_e$, $u_Q^0 \in \mathcal{V}$ and $c = \int W(I)\,dV$.

PROOF. Recall that $\phi_Q = I + \lambda u_Q(l) + O(\lambda^2)$. Thus,

$$\begin{aligned} f(Q) &= \int W(\phi_Q)\,dV - \langle \lambda Ql, \phi_Q\rangle \\ &= \left[c + \lambda \int \frac{\partial W}{\partial F}(I)(u_Q(l))\,dV + \frac{\lambda^2}{2}\int \frac{\partial^2 W}{\partial F^2}(u_Q(l))^2\,dV + O(\lambda^3)\right] \\ &\quad - \langle \lambda Ql, I + \lambda u_Q(l) + O(\lambda^2)\rangle. \end{aligned}$$

Thus, by Lemma 2.15,

$$f(Q) = c + \lambda\left[\langle -l, Q^T 1\rangle - \frac{\lambda}{2}\langle c(\nabla u_Q(l)), \nabla u_Q(l)\rangle + O(\lambda^2)\right].$$

For, $\nabla u_Q(l) = \nabla u_Q(l_0) + O(|l - l_0|)$, $u_Q(l_0) = u_Q^0 + KX$ for some $K \in$ skew,

$$\langle c(\nabla u_Q(l)), \nabla u_Q(l)\rangle = \langle c(\nabla u_Q^0), \nabla u_Q^0\rangle + O(|l - l_0|).$$

Hence,

$$f(Q) = c + \lambda \left[-\langle l, Q^T 1 \rangle - \frac{\lambda}{2} \langle c(\nabla u_Q^0), \nabla u_Q^0 \rangle + O(\lambda^2) + O(\lambda \| l - l_0 \|) \right]. \quad \text{Q.E.D.}$$

2.17. DEFINITION. Let $f^*(Q) = (f(Q) - c)/\lambda$ for $\lambda > 0$. Thus

$$f^*(Q) = -\langle l, Q^T 1 \rangle - \frac{\lambda}{2} \langle c(\nabla u_Q^0), \nabla u_Q^0 \rangle + O(\lambda^2) + O(\lambda \| l - l_0 \|).$$

From this formula of f^*, one is led to examine the function $-\langle l_0, Q^T 1 \rangle$ on $SO(3)$ (i.e. $\lambda = 0$).

For convenience, we collect some relevant facts from [**2**].

2.18. PROPOSITION. (a) *Q is a critical point of $-\langle l_0, Q^T 1 \rangle$ iff $Q \in S_{A_0}$.*

(b) *S_{A_0} is a submanifold in $SO(3)$ and $-\langle l_0, Q^T 1 \rangle$ is nondegenerate on $(T_Q S_{A_0})^\perp \subset T_Q SO(3)$ with index that of $QA_0 - \operatorname{Tr}(QA_0)1$ ($A_0 = k(l_0)$).*

This proposition suggests that one should carry out another Liapunov-Schmidt reduction.

Thus, take any normal bundle of S_{A_0} in $SO(3)$ with fibres orthogonal to $T_Q S_{A_0}$. Denote by $Q + n(Q)$, $Q \in S_{A_0}$, the critical point of f^* restricting the fibre through Q. Therefore $n(Q) = \lambda y + O(\lambda^2)$, where $y \in T_Q S_{A_0}^\perp$.

2.19. DEFINITION. Set $\tilde{f}(Q) = f^*(Q + n(Q))$ for $Q \in S_{A_0}$.

2.20. PROPOSITION.

$$\tilde{f}(Q) = -\langle l, Q^T 1 \rangle - \frac{\lambda}{2} \langle c(\nabla u_Q^0), \nabla u_Q^0 \rangle + O(\lambda^2) + O(\lambda \| l - l_0 \|).$$

PROOF. It suffices to observe $\langle l, (Q + n(Q))^T 1 \rangle = \langle l, Q^T 1 \rangle + O(\lambda |l - l_0|) + O(\lambda^2)$, which follows from the fact $\langle l_0, y \rangle = 0$. Q.E.D.

Let us now state a refinement of Theorem 2.11:

2.21. THEOREM. *For $\lambda > 0$ small and l near l_0, the solutions to the incompressible traction problem are in 1-1 correspondence to the critical points of $\tilde{f}(Q)$, $Q \in S_{A_0}$. Indeed, if Q is a critical point of $\tilde{f}$ on S_{A_0}, then $(Q + n(Q))^{-1} \phi_{Q+n(Q)}$ is a solution to the traction problem. Furthermore, the index of V at $(Q + n(Q))^{-1} \phi_{Q+n(Q)}$ equals the index of $(QA_0 - \operatorname{Tr}(QA_0)1)$ plus the index of $\tilde{f}$ at Q.*

(C) *The nondegenerate cases, parallel loads, and homogeneous loads.* Since the reduced bifurcation function given by Proposition 2.16 is basically the same as that in [**3**], the same arguments provide the following results.

2.22 THEOREM. *For $\lambda > 0$ small, l near l_0 and l_0 of type 0, the incompressible traction problem with load λl has exactly 4 solutions, exactly one of them is stable.*

2.23. THEOREM. *Let $l = l(c)$ depend on $c \in R^2$ with $l(0) = l_0$ a type 1 load. Under generic conditions, the incompressible traction problem for load $\lambda l(c)$ has the following bifurcation diagram (near S^1). Furthermore, it is also universal.*

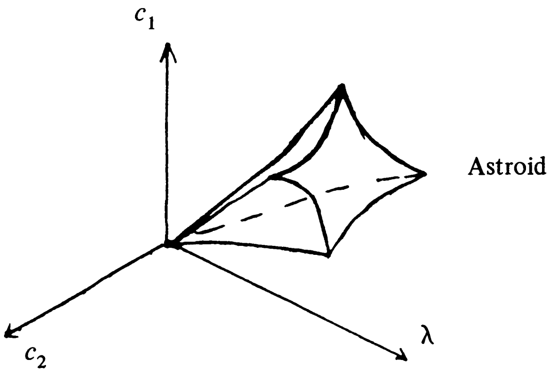

FIGURE 1

2.24. THEOREM. *Under generic conditions, the incompressible traction problem for load* λl, $\|l - l_0\|/\lambda$, $\lambda > 0$ *small, has*

(a) *at most* 13 *solutions near* Rp^2 *for type* 2 *loads,*

(b) *at most* 8 *solutions near* $S^1 \cup S^1$ *for type* 3 *loads,*

(c) *at most* 40 *solutions near* $SO(3)$ *for type* 4 *loads.*

Recall that a *nontrivial parallel* load l is a load in the form $l(X) = f(X)a$, where $f: B \to R$, $0 \neq a \in R^3$ and $\int f(X)X\,dV + \int f(X)X\,dA \neq 0$. For such a load, the potential function V is S^1-invariant. Thus, the traction problem becomes degenerate and cannot be covered by previous theorems.

2.25. THEOREM. *Let* l_0 *be an equilibrated nontrivial parallel load. Then, for* $\lambda > 0$ *small, there exist exactly two circles of equilibrium solutions to our incompressible traction problem. One of them is stable.*

A load l_0 is said to be "*homogeneous*" if it is in the form $l_0 = \binom{0}{AN}$, with $A \in \text{sym}$. Let us consider the traction problem of an *isotropic homogeneous, incompressible* material for a *homogeneous* load. Calculations show the reduced bifurcation equation may always be trivial, and thus one has a degenerate traction problem. We take a direct approach in this case.

2.26. THEOREM. *Let* l_0 *be a "homogeneous" load. For* $\lambda > 0$ *small, the solution set of a traction problem for an isotropic, homogeneous, incompressible material, consists of homogeneous configurations only and is diffeomorphic to* $S_{A_0} = \{Q \in SO(3) | QA_0 \in \text{sym}\}$.

In other words, no bifurcations occur (cf. [**1**]). From the representation theorem for the Piola-Kirchhoff stress tensor P, one has

2.27. LEMMA. $P(p, F) = J(-p + \partial W/\partial F)F^{-T} \in \text{sym}$ *for* $F \in M$, *where* $M = \text{sym} \cap \{F | \det F = 1\}$ *is a manifold near* I.

PROOF OF THEOREM 2.26. Let $Q \in S_{A_0} = \{Q | QA_0 \in \text{sym}\}$, where $A_0 = k(l_0) = A\,\text{vol}(B)$. Applying the inverse function theorem to the map $(p, F) \to P(p, F)$ near $(0, I) \in R^3 \times M$ one finds $\lambda p_{\lambda Q}$, $1 + \lambda E_{\lambda Q} \in M$ such that

$$P(\lambda p_{\lambda Q}, 1 + \lambda E_{\lambda Q}) = \lambda QA$$

for λ small. Define $\psi_Q(X) = Q^{-1}(X + \lambda E_{\lambda Q} X)$. Clearly $\psi_Q(X)$ is homogeneous. $P(\lambda P_{\lambda Q}, Q^{-1}(1 + \lambda E_{\lambda Q})) = Q^{-1} P(\lambda P_{\lambda Q}, 1 + \lambda E_{\lambda Q}) = Q^{-1}(\lambda Q A) = \lambda A$. Hence, $-\operatorname{Div} P = 0$, $PN = \lambda AN$ and ψ_Q solves the traction problem. Indeed $Q \to \psi_Q$ is a diffeomorphism. By Theorem 2.21,

$$\{\psi | Q \in S_{A_0}\} \subseteq \{\text{all solutions near } SO(3)\}$$
$$\subseteq \{(Q + n(Q))^{-1} \phi_{Q+n(Q)} | Q \in S_{A_0}\} \approx S_{A_0}.$$

The proof is now completed by topological considerations (i.e. invariance of domain, connectedness). Q.E.D.

3. A power series approach. In this section, let us consider the problem: *Given $l(\lambda) \in \mathcal{L}$, $l(\lambda) \to 0$ as $\lambda \to 0$, under what conditions does there exist $\phi(\lambda) \in \mathcal{C}_{\text{vol}}$, $p(\lambda) \to 0$ and $\phi(\lambda) \to I$ as $\lambda \to 0$, such that $\Phi(p(\lambda), \phi(\lambda)) = l(\lambda)$?* If they exist, how do we find them? Here and what follows, $l(\lambda)$, $p(\lambda)$ and $\phi(\lambda)$ are smooth functions in λ. Let us write $l(\lambda) = \lambda l_1 + \lambda^2 l_2 + \cdots$ (more precisely, $l(\lambda) = \lambda l_1 + \lambda^2 l_2 + \cdots + \lambda^n l_n + O(\lambda^{n+1})$ for any n). $p(\lambda) = \lambda p_1 + \lambda^2 p_2 + \cdots$ and $\phi(\lambda) = I + \lambda u_1 + \lambda^2 u_2 + \cdots$. Throughout this section, $l(\lambda)$ is assumed to be given.

(A) *Necessary conditions.* Let $(p(\lambda), \phi(\lambda))$ be a solution to the above problem.

3.1. *Signorini's compatibility conditions.* From Proposition 2.4 $\langle l(\lambda), W^T \phi(\lambda)\rangle = \langle W l(\lambda), \phi(\lambda)\rangle = 0$ for all $W \in$ skew. Thus, $k(l(\lambda), \phi(\lambda)) \in$ sym or

$$2 \operatorname{skew}(k(l(\lambda), \phi(\lambda)) \approx \int l(\lambda) \times \phi(\lambda) = 0,$$

where "$\times$" denotes the usual cross product in R^3, and $\int l \times \phi = \int_B b \times \phi \, dV + \int_{\partial B} \tau \times \phi \, dA$. From the equation $\int l(\lambda) \times \phi(\lambda) = 0$, we obtain:

(C_0) $\quad\lambda^1$ order, $\quad \int l_1 \times I = 0 \quad (\text{so, } l_1 \in \mathcal{L}_e)$,

(C_1) $\quad\lambda^2$ order, $\quad \int l_1 \times u_1 + \int l_2 \times I = 0$,

(C_{n-1}) $\quad\lambda^n$ order, $\quad \int l_1 \times u_{n-1} + \int l_2 \times u_{n-2} + \cdots \int l_n \times I = 0$.

3.2. *Incompressibility conditions.* Since $1 = J(I + \lambda \nabla u_1 + \lambda^2 \nabla u_2 + \cdots)$ by looking at λ^n order terms, we have:

(I_1) $\quad\lambda^1$ order, $\quad \operatorname{tr}(\nabla u_1) = 0$,

(I_2) $\quad\lambda^2$ order, $\quad \operatorname{tr} \nabla u_2 - \frac{1}{2} \operatorname{tr}(\nabla u_1 \nabla u_1) = 0$,

$\vdots$

(I_n) $\quad\lambda^n$ order, $\quad \operatorname{tr} \nabla u_n - \operatorname{tr}(\nabla u_1 \nabla u_{n-1}) + \mathcal{H}(\nabla u_1, \ldots, \nabla u_{n-2}) = 0$,

where $\mathcal{H}$ is a polynomial in $\nabla u_1, \ldots, \nabla u_{n-2}$ of degree n ($\deg \nabla u_i = i$).

3.3. *Linear equations.* Expand $\Phi = \Phi_1 + \Phi_2 + \Phi_3 + \cdots$ near $(0, I)$ as a (formal) Taylor series with $\Phi_1 = L$, $\Phi_2 = \Phi_2((q, u), (q, u))$, etc.

Hence,

$$\begin{aligned}
&\Phi(\lambda p_1 + \lambda^2 p_2 + \cdots, I + \lambda u_1 + \lambda^2 u_2 + \cdots) \\
&\quad = L(\lambda p_1 + \lambda^2 p_2 + \cdots, \lambda u_1 + \lambda^2 u_2 + \cdots) \\
&\qquad + \Phi_2(\lambda p_1 + \lambda^2 p_2 + \cdots, \lambda u_1 + \lambda^2 u_2 + \cdots) + \cdots \\
&\quad = \lambda l_1 + \lambda^2 l_2 + \cdots.
\end{aligned}$$

Comparing orders in λ, we get

$$\begin{aligned}
&(\mathrm{L}_1) \quad \lambda^1 \text{ order}, \quad L(p_1, u_1) = l_1,\\
&(\mathrm{L}_2) \quad \lambda^2 \text{ order}, \quad L(p_2, u_2) + \Phi_2(p_1, u_1)^2 = l_2,\\
&\qquad \vdots\\
&(\mathrm{L}_n) \quad \lambda^n \text{ order}, \quad L(p_n, u_n) + 2\Phi_2((p_1, u_1), (p_{n-1}, u_{n-1}))\\
&\qquad\qquad + \mathcal{K}((p_1, u_1), \dots, (p_{n-2}, u_{n-2})) = l_n,
\end{aligned}$$

where $\mathcal{K}$ is a polynomial in $(p_1, u_1), \dots, (p_{n-2}, u_{n-2})$.

Now, we will reformulate the 1st order necessary conditions in geometric terms.

Recall, $S_A = \{Q \in SO(3) | k(Ql) \in \mathrm{sym}\}$ for $k(l) = A$. $T_Q S_A = \{KQ | Kk(Ql) = k(KQl) \in \mathrm{sym}\}$, the tangent space of S_A at Q. For $k(Ql) \in \mathrm{sym}$, $L(p_Q(l), u_Q(l)) = Ql$. $u_Q(l) \in \mathcal{U}_{\mathrm{sym}} \cap \mathcal{V}$.

3.4. LEMMA. *$2\langle l_2, Q^T X\rangle + \langle c(\nabla u_Q(l_1)), \nabla u_Q(l_1)\rangle$ restricted to S_{A_1} has a critical point $Q \in S_{A_1}$ iff $0 = \langle l_2, (KQ)^T X\rangle + \langle c(\nabla u_Q(l_1)), \nabla u_{KQ}(l_1)\rangle$ for all $KQ \in T_Q S_{A_1}$.*

3.5. LEMMA. *Let $A \in \mathrm{sym}$. Then, $\langle KA, W\rangle = \langle K, WA\rangle$ is a symmetric bilinear form with kernel $\{K \in \mathrm{skew} | KA \in \mathrm{sym}\}$.*

The proofs of these two lemmas are elementary and we leave them to the readers.

3.6. THEOREM. *Let $l_1 \in \mathcal{L}_e$ (i.e. condition (C_0) holds). There exist p_1, u_1, $L(p_1, u_1) = l_1$, with $\operatorname{tr} \nabla u_1 = 0$. $\int l_1 \times u_1 + \int l_2 \times I = 0$ iff $2\langle l_2, Q^T X\rangle + \langle c(\nabla u_Q(l_1)), \nabla u_Q(l_1)\rangle$ restricted to S_{A_1} has a critical point $Q \in S_{A_1}$.*

PROOF. For $\operatorname{tr} \nabla u_I = 0$ by the divergence theorem,

$$\begin{aligned}
\langle c(\nabla u_I(l_1)), \nabla u_W(l_1)\rangle &= \langle c(\nabla u_W(l_1)), \nabla u_I(l_1)\rangle\\
&= \langle -q_W(l_1) + c(\nabla u_W(l_1)), \nabla u_I(l_1)\rangle = \langle W l_1, u_I(l_1)\rangle.
\end{aligned}$$

The "only iff" part: $\int l_1 \times u_1 + \int l_2 \times I = 0$ implies $\langle W l_1, u_1\rangle + \langle W l_2, I\rangle = 0$ for all $W \in \mathrm{skew}$, and $u_I = u_1 + KX$ for some $K \in \mathrm{skew}$.

$$\langle W l_1, u_I\rangle = \langle W l_1, u_1\rangle + \langle W l_1, KX\rangle = \langle W l_1, u_1\rangle + \langle K, k(W l_1)\rangle.$$

Thus, $\langle l_2, W^T X\rangle + \langle c(\nabla u_I(l_1)), \nabla u_W(l_1)\rangle = (\langle w l_2, I\rangle + \langle W l_1, u_1\rangle) + \langle K, k(W l_1)\rangle = 0$ for $W l_1 \in \mathrm{sym}$, $W \in \mathrm{skew}$.

By Lemma 3.4, the proof of the "only if" part is now completed.

The "if" part: one needs to find $K \in \mathrm{skew}$ so that, for $u_1 = u_I - KX$, $\langle W l_1, u_1\rangle + \langle W l_2, I\rangle = 0$ for all $W \in \mathrm{skew}$.

$\langle W l_1, u_I\rangle + \langle l_2, W^T X\rangle = 0$ for $W A_1 \in \mathrm{sym}$ (for $\langle c(\nabla u_I(l_1)), \nabla u_W(l_1)\rangle = \langle W l_1, u_I\rangle$) by hypothesis.

By Lemma 3.5 there exists $K \in \mathrm{skew}$ such that $\langle W l_1, u_I\rangle + \langle l_2, W^T X\rangle = \langle W A_1, K\rangle$ for all $W \in \mathrm{skew}$.

Therefore,

$$\begin{aligned}
\langle W l_1, u_1\rangle + \langle W l_2, I\rangle &= \langle W l_1, u_I\rangle - \langle W l_1, KX\rangle + \langle W l_2, I\rangle\\
&= \langle W l_1, u_I\rangle + \langle l_2, W^T X\rangle - \langle W A_1, K\rangle = 0
\end{aligned}$$

for all $W \in \mathrm{skew}$. Q.E.D.

3.7. COROLLARY (EXTENSION OF TOLOTTI [**14**]). *There exist at least 4 rotations Q in $SO(3)$ such that conditions* (C_0), (C_1), (L_1) *and* (I_1) *hold for* $Ql_1 = l_1^*$ *and* $Ql_2 = l_2^*$, *i.e.* $l_1^* \in \mathcal{L}_e$ *and* $\int l_1^* \times u_1 + \int l_2^* \times I = 0$ *for some* (p_1, u_1) *with* $L(p_1, u_1) = l_1^*$, $\operatorname{tr} \nabla u_1 = 0$.

PROOF. Let Q be a critical point of $2\langle l_2, Q^T X\rangle + \langle c(\nabla u_Q(l_1)), \nabla u_Q(l)\rangle$ on S_{A_1}.
By Lemma 3.4, $\langle l_2, (KQ)^T X\rangle + \langle c(\nabla u_Q(l_1)), \nabla u_{KQ}(l_1)\rangle = 0$ for all $Kk(Ql_1) \in$ sym.
Since, $u_{KQ}(l_1) = u_K(Ql_1) = u_K(l_1^*)$,

$$u_Q(l_1) = u_I(Ql_1) = u_I(l_1^*),$$

$$\langle l_2^*, K^T X\rangle + \langle c(\nabla u_I(l_1^*)), \nabla u_K(l_1^*)\rangle = 0 \quad \text{for all } Kk(l_1^*) \in \text{sym}.$$

By Theorem 3.6 and Lemma 3.4 (with $Q = I$, $l_1 = l_1^*$, $l_2 = l_2^*$), the corollary follows by observing at least 4 such critical points can be found. Q.E.D.

(B) *A first order sufficient condition, and a Signorini Scheme.* Let $l_1 \in \mathcal{L}_e$ (i.e. an equilibrated load). The load l_1 has an axis $a \in R^3$, $\|a\| = 1$, of equilibria if $R_\theta l_1 \in \mathcal{L}_e$ for any rotation R_θ with axis a. It can be shown that l_1 has no axis of equilibria iff the function $\langle l_1, Q^T I\rangle$ on $SO(3)$ has a nondegenerate critical point at $Q = I$ (cf. [**2**]).

3.8. THEOREM. *Suppose* $l'(0) = l_1 \in \mathcal{L}_e$ *has no axis of equilibria. Then there exist unique* $p(\lambda), \phi(\lambda) \in \mathcal{C}_{\text{vol}}$ *such that* $\Phi(p(\lambda), \phi(\lambda)) = l(\lambda)$, $p(\lambda) \to 0$, $\phi(\lambda) \to I$, *as* $\lambda \to 0$.

PROOF. It suffices to observe $f = \text{constant} - \lambda\langle l_1, Q^T I\rangle + O(\lambda^2)$, and $\langle l_1, Q^T I\rangle$ has a nondegenerate critical point on $SO(3)$ at $Q = I$. Q.E.D.

It is convenient to state and prove the following lemmas in which $l_1 \in \mathcal{L}_e$ may possess axis of equilibria. These lemmas will also be used in (C).

3.9. LEMMA. *Let* $K \in$ skew. *Then,* $\int l_1 \times KX = 0$ *iff* $Kk(l_1) \in$ sym, *i.e.* $K \in T_I S_{A_1}$.

3.10. LEMMA. *Suppose* $u_1, \ldots, u_{n-1}$ *satisfy* (I_1), $\ldots$, $(I)_{n-1}$. *Then, there exist* $\phi = 1 + \lambda u_1 + \lambda^2 u_2 + \cdots + \lambda^{n-1} u_{n-1} + \cdots$ *such that* $J(\phi) = 1$.

PROOF. Let π be a projection of $\mathcal{U}$ onto the tangent space $\mathcal{V} = \{u \in \mathcal{U} | \operatorname{tr} \nabla u = 0\}$ of $\mathcal{C}_{\text{vol}}$ at I. By the inverse mapping theorem there exists unique $\phi = 1 + \lambda u_1^* + \lambda^2 u_2^* + \cdots$ in $\mathcal{C}_{\text{vol}}$ such that

$$\pi(\lambda u_1^* + \lambda^2 u_2^* + \cdots) = \pi(\lambda u_1 + \lambda^2 u_2 + \cdots + \lambda^{n-1} u_{n-1}).$$

Thus, $\pi(u_1^*) = \pi(u_1), \ldots, \pi(u_{n-1}^*) = \pi(u_{n-1})$. Using ($I_1$), $\ldots$, (I_{n-1}), we obtain $u_1^* = u_1, \ldots, u_{n-1}^* = u_{n-1}$. Q.E.D.

3.11. LEMMA. *Let* $l_1 \in \mathcal{L}_e$. *Suppose* $(p_1, u_1), \ldots, (p_{n-1}, u_{n-1})$ *satisfies* (C_1), (L_1), (I_1), $\ldots$, (C_{n-2}), (L_{n-2}), (I_{n-2}), (L_{n-1}) *and* (I_{n-1}). *Then,* (C_{n-1}) *is a solvability condition of* (L_n) *and* (I_n) *for* p_n, u_n.

PROOF. Choose $\phi = 1 + \lambda u_1 + \lambda^2 u_2 + \cdots + \lambda^{n-1} u_{n-1} + \lambda^n u_n^* + \cdots$ by Lemma 3.10. Set

$$\Phi(\lambda p_1 + \lambda^2 p_2 + \cdots + \lambda^{n-1} p_{n-1}, I + \lambda u_1 + \lambda^2 u_2 + \cdots + \lambda^{n-1} u_{n-1} + \lambda^n u_n^* + \cdots)$$
$$= \lambda l_1 + \lambda^2 l_2 + \cdots + \lambda^{n-1} l_{n-1} + \lambda^n l_n^* + \cdots.$$

So u_n^* satisfies (I_n^*), (L_n^*): $L(0,u_n^*)+\mathcal{K}((p_1,u_1),\dots,(p_{n-1},u_{n-1}))=l_n^*$, and ($\mathrm{C}_{n-1}^*$): $\int l_1\times u_{n-1}+\cdots+\int l_n^*\times I=0$. One needs to find $p_n, u_n=u_n^*+v$ such that $\operatorname{tr}\nabla v=0$ and $L(p_n,v)=l_n-l_n^*$. Since $\int l_1\times u_{n-1}+\cdots+\int l_n\times I=0$, $\int l_1\times u_{n-1}+\cdots+\int l_n^*\times I=0$ $\int(l_n-l_n^*)\times I=0$ or $l_n-l_n^*\in\mathcal{L}_e$. Consequently, such p_n,u_n can be found by Proposition 1.2. Q.E.D.

Now, we are ready to present the Signorini Scheme for incompressible materials.

3.12. THEOREM. *Let $l_1\in\mathcal{L}_e$ have no axis of equilibria. Suppose $(p_1,u_1),\dots,(p_{n-1},u_{n-1})$ $(n\geq 1)$ satisfies* (C_1), (L_1), (I_1), $\dots$, (C_{n-1}), (L_{n-1}) *and* (I_{n-1}). *Then* (C_n), (L_n) *and* (I_n) *define p_n,u_n.*

PROOF. By Lemma 3.11, there exists $(p_n,\tilde{u}_n)$ which satisfies (L_n), (I_n). One needs to find $K\in$ skew, $u_n=\tilde{u}_n+KX$, so that (C_n) holds, i.e.

$$\int l_1\times u_n+\cdots+\int l_{n+1}\times I=0,$$

or

$$\int l_1\times KX=-\left(\int l_1\times\tilde{u}_n+\cdots+\int l_{n+1}\times I\right).$$

For l_1 has no axis of equilibria, $T_IS_{A_1}=\{0\}$, by Lemma 3.9, the map $K\to\int l_1\times KX$ is an isomorphism. Therefore, such a $K\in$ skew can be found and indeed, it must be unique. Q.E.D.

(C) *A second order sufficient condition, and linearization stability.* Now we will extend Theorem 3.12 so that the load l_1 may possess an axis of equilibria (i.e. $T_IS_{A_1}\neq\{0\}$).

3.13. THEOREM. *Suppose $2\langle l_2,Q^TI\rangle+\langle c(\nabla u_Q),\nabla u_Q\rangle$ has a nondegenerate critical point at I restricted to S_{A_1}. Then, there exist $p(\lambda),\phi(\lambda)\in C_{\mathrm{vol}}$ such that $\Phi(p(\lambda),\phi(\lambda))=l(\lambda)$, $p(\lambda)\to 0$, $\phi(\lambda)\to I$, as $\lambda\to 0$.*

PROOF. It suffices to observe, by Proposition 2.16,

$$\tilde{f}=\text{constant}-\lambda\langle l_2,Q^T1\rangle-\frac{\lambda}{2}\langle c(\nabla u_Q),\nabla u_Q\rangle+O(\lambda^2).\quad\text{Q.E.D.}$$

The above theorem enables us to relate the linear elasticity theory (cf. [**9**]) to that of nonlinear elasticity theory in a satisfactory and expected way (cf. [**15, 16**]). The appropriate notion one may introduce is the following

3.14. DEFINITION (CF. [**6**]). Given $l_1\in\mathcal{L}_e$. The load l_1 is said to be *linearization stable in $\mathcal{L}_e$* if there exist $p(\lambda),\phi(\lambda)\in\mathcal{C}_{\mathrm{vol}}$, $\Phi(p(\lambda),\phi(\lambda))=l(\lambda)$, $p(\lambda)\to 0$, $\phi(\lambda)\to I$ as $\lambda\to 0$ for some $l(\lambda)\in\mathcal{L}_e$, $l'(0)=l_1$. If $l_1\in\mathcal{L}_e$ is linearization stable in $\mathcal{L}_e$, then there exist p_1,u_1, such that $L(p_1,u_1)=l_1$, $\operatorname{tr}\nabla u_1=0$ with $\int l_1\times u_1=0$. Conversely, we have

3.15. THEOREM. *A load $l_1\in\mathcal{L}_e$ is linearization stable in $\mathcal{L}_e$ if there exist p_1,u_1 such that $L(p_1,u_1)=l_1$, $\operatorname{tr}\nabla u_1=0$, $\int l_1\times u_1=0$.*

In other words, (C_1), (I_1), (L_1) are the only obstructions for the solvability of the equation $\Phi(p(\lambda),\phi(\lambda))=l(\lambda)$, with $l'(0)$ given.

PROOF. By Theorem 3.6, $\langle c(\nabla u_Q),\nabla u_Q\rangle$ has a critical point at $Q=I$. Take $l_2\in\mathcal{L}_e$ so that $2\langle l_2,Q^TI\rangle+\langle c(\nabla u_Q),\nabla u_Q\rangle$ is nondegenerate at I along S_{A_1}. By

Theorem 3.13 there exist $p(\lambda), \phi(\lambda) \in \mathcal{C}_{\mathrm{vol}}$, $\Phi(p(\lambda), \phi(\lambda)) = \lambda l_1 + \lambda^2 l_2$, $p(\lambda) \to 0$, $\phi(\lambda) \to I$ as $\lambda \to 0$. Q.E.D.

(D) *A generalized Signorini Scheme.* Here, we present a scheme for the solution found in Theorem 3.13. This scheme extends the Signorini Scheme given by Theorem 3.12.

3.16. THEOREM. *Let the hypothesis in Theorem* 3.13 *be fulfilled. Suppose* $(p_1, u_1), \ldots, (p_{n-2}, u_{n-2})$ $(n \geq 1)$ *and* $p_{n-1}, u_{n-1} \bmod KX$, $K \in T_I S_{A_1}$, *are determined through* $(\mathrm{C}_1), (\mathrm{I}_1), (\mathrm{L}_1), \ldots, (\mathrm{C}_{n-1}), (\mathrm{I}_{n-1}), (\mathrm{L}_{n-1})$. *Then, equations* $(\mathrm{C}_n), (\mathrm{I}_n)$ *and* (L_n) *define* u_{n-1} *and* $p_n, u_n \bmod KX$, $K \in T_I S_{A_1}$.

Our proof consists of brute force computations. Basically, Lemmas 3.17–3.19 are collections of relevant facts similar to that in [**11**].

3.17. LEMMA. (a) $2\langle l_2, Q^T X\rangle + \langle c(\nabla u_Q), \nabla u_Q\rangle$ *has a critical point on* S_{A_1} *at* $Q = I$ *iff* $\langle l_2, K^T X\rangle + \langle c(\nabla u_Q), \nabla u_Q\rangle = 0$ *for all* $K \in T_I S_{A_1}$.

(b) *The Hessian is* $\langle l_2, K^2 X\rangle + \langle c(\nabla u_K), \nabla u_K\rangle + \langle c(\nabla u_I), \nabla u_{K^2}\rangle$ *for all* $K \in T_I S_{A_1}$.

Expand $P = P_1 + P_2 + \cdots$ and $\Phi = \Phi_1 + \Phi_2 + \cdots$ as formal series at $(0, I)$ and $(0, I_B)$ respectively. Thus, $\Phi_1 = L$.

$$\begin{aligned} P &= -JqF^{-T} + P^* \\ &= -(1 + \operatorname{tr} H + \cdots)q(1 - H^T + \cdots) + \frac{\partial P^*}{\partial F}(H) + \frac{1}{2}\frac{\partial^2 P^*}{\partial F^2}(H)^2 + \cdots \\ &= \left[-q + \frac{\partial P^*}{\partial F}(H)\right] + \left[q(H^T - \operatorname{tr} H) + \frac{1}{2}\frac{\partial^2 P^*}{\partial F^2}(H)^2\right] + \cdots, \qquad P^* = \frac{\partial W}{\partial F}. \end{aligned}$$

Hence, $P_1 = -q + (\partial P^*/\partial F)(H)$ and $P_2 = q(H^T - \operatorname{tr} H) + \frac{1}{2}(\partial^2 P^*/\partial F^2)(H)^2$.

Applying the divergence theorem, we get

3.18. LEMMA. (a) $\langle L(q, u), w\rangle = \langle P_1(q, \nabla u), \nabla w\rangle$,

(b) $\langle \Phi_2(q, u)^2, w\rangle = \langle P_2(q, \nabla u)^2, \nabla w\rangle$.

Now, set $W = W(D)$, $D = (F^T F - 1)/2$. Thus $P^* = \partial W/\partial F$, $S^* = \partial W/\partial D$ and $P^*(F) = FS^*(D)$. AT $F = I$, computations show

3.19. LEMMA. (a) $(\partial P^*/\partial F)(H) = (\partial S^*/\partial D)(H)$,

(b)

$$\frac{\partial^2 P^*}{\partial F^2}(H, K) = H\frac{\partial S^*}{\partial D}(K) + K\frac{\partial S^*}{\partial D}(H) + \frac{\partial S^*}{\partial D}(K^T H) + \frac{\partial^2 S^*}{\partial D^2}(H, K).$$

3.20. LEMMA. (a) $\Phi_2(0, KX)^2 = L(0, \frac{1}{2}K^T KX)$.

(b) $2\Phi_2((p_1, u_1), (0, KX)) = Kl_1 + L(0, K^T u_1)$ *for* $K \in T_I S_{A_1}$.

PROOF. (a)

$$\begin{aligned} \langle 2\Phi_2(0, KX)^2, w\rangle &= \left\langle \frac{\partial S^*}{\partial D}(K^T K), \nabla w\right\rangle \quad \text{(by Lemmas 3.18(b) and 3.19(b))} \\ &= \langle L(0, K^T KX), w\rangle \quad \text{(by Lemmas 3.18(a) and 3.19(a))} \end{aligned}$$

for all $w \in \mathcal{U}$.

(b)

$$\begin{aligned}\langle 2\Phi_2((p_1,u_1),(0,KX)),w\rangle &= \langle 2\Phi_2((p,0),(0,KX)) + 2\Phi_2((0,u_1)(0,KX)),w\rangle \\ &= \left\langle p_1K^T + K\frac{\partial S^*}{\partial D}(H) + \frac{\partial S^*}{\partial D}(K^TH), \nabla w\right\rangle \\ &\qquad \text{(by Lemmas 3.18(b) and 3.19(b))} \\ &= \left\langle K\left(-p_1 + \frac{\partial S^*}{\partial D}(H)\right) + \frac{\partial S^*}{\partial D}(K^TH), \nabla w\right\rangle \\ &= \langle Kl_1 + L(0,K^Tu_1),w\rangle, \quad \text{for all } w\in\mathcal{U}. \quad \text{Q.E.D.}\end{aligned}$$

3.21. LEMMA. *$\langle Wl_1, K^Tu_1\rangle - \langle Wl_2, KX\rangle$ is symmetric in W and K, where $W, K \in T_IS_{A_1}$.*

PROOF. $\int l_1 \times u_1 + \int l_2 \times I = 0$ means $k(l_1,u_1) + k(l_2,I) \in$ sym. Thus, $\langle \tilde{K}^T, k(l_1,u_1)\rangle + \langle \tilde{K}^T, k(l_2,I)\rangle = 0$ for all $\tilde{K} \in$ skew, or $\langle \tilde{K}l_1,u_1\rangle + \langle \tilde{K}l_2, X\rangle = 0$.
Let $\tilde{K} = WK - KW$. Then one obtains

$$\langle WKl_1,u_1\rangle + \langle WKl_2,X\rangle = \langle KWl_1,u_1\rangle + \langle KWl_2,X\rangle,$$

or

$$\langle Kl_1,W^Tu_1\rangle - \langle Kl_2,WX\rangle = \langle Wl_1,K^Tu_1\rangle - \langle Wl_2,KX\rangle. \quad \text{Q.E.D.}$$

Now, we are ready to prove Theorem 3.16 in steps (α), (β), (γ).

(α) Let $n > 2$. One needs to show that there exists a unique $K \in T_IS_{A_1}$ such that $u_{n-1} = u^*_{n-1} + KX$ (u^*_{n-1} given by hypothesis), and a corresponding u_n, p_n obtained by Lemma 3.11, solve (L_n), (I_n) and (C_n).

For each $K \in T_IS_{A_1}$, from (I_n), (L_n) for p_n, u_n and (I^*_n), (L^*_n) for p^*_n, u^*_n (given by Lemma 3.11)

$$L(p_n - p^*_n, u_n - u^*_n) + 2\Phi_2((p_1,u_1),(0,KX)) = 0,$$

$$L(p_n - p^*_n + p_K, u_n - u^*_n + u_K + K^Tu_1) = 0 \quad \text{(by Lemma 3.20(b))}.$$

For, $\operatorname{tr}(\nabla u_n - \nabla u^*_n + \nabla u_K + K^TH) = \operatorname{tr}(\nabla u_n - \nabla u^*_n + K^TH) = 0$ by (I_n), (I^*_n). Thus, $u_n - u^*_n + u_K + K^Tu_1 + \tilde{K}X = 0$ and $p_n - p^*_n + p_K = 0$ for some $\tilde{K} \in$ skew. Putting in (C_n), one has

$$\int l_1 \times (u^*_n - u_K - K^Tu_1 - \tilde{K}X) + \int l_2 \times (u^*_{n-1} + KX) \\ + \int l_3 \times u_{n-2} + \cdots + \int l_{n+1} \times I = 0.$$

or

(1) $$-k(l_1, u_K + K^Tu_1 + \tilde{K}X) + k(l_2,KX) + M \in \text{sym},$$

with $M = k(l_1,u^*_n) + k(l_2,u^*_{n-1}) + k(l_3,u_{n-2}) + \cdots + k(l_{n+1},I)$. Since, $\langle W, k(l_1,\tilde{K}X)\rangle = \langle -Wl_1, \tilde{K}X\rangle = \langle -WA_1, \tilde{K}\rangle = 0$ for $W \in T_IS_{A_1}$

(2) $\langle W, -k(l_1, u_k + K^Tu_1)\rangle + \langle W, k(l_2,KX)\rangle + \langle W, M\rangle = 0$ for all $W \in T_IS_{A_1}$.

Now, $\langle W, -k(l_1,u_K + K^Tu_1)\rangle + \langle W, k(l_2,KX)\rangle = \langle Wl_1,u_K\rangle + \langle Wl_1,K^Tu_1\rangle - \langle Wl_2,KX\rangle$ is a nondegenerate form by Lemmas 3.21 and 3.20(b).

Thus, there exists a unique $K \in T_I S_{A_1}$ such that equation (2) holds. Thus, one gets the uniqueness part of K in the theorem. To obtain the existence part, one seeks $\tilde{K} \in$ skew so that

$$u_{n-1} = u_{n-1}^* + KX, \quad u_n = u_n^* - u_K - K^T u_1 - \tilde{K}X, \quad p_n = p_n^* - p_K$$

satisfies (C_n). (Conditions (I_n) and (L_n) are fulfilled automatically.) Consider the equation for $\tilde{K}$:

$$\begin{aligned} k(l_1, \tilde{K}X) &= -k(l_1, u_k + K^T u_1) + k\langle l_2, KX\rangle + M \quad \mathrm{mod\,sym} \\ &= N \quad \mathrm{mod\,sym}. \end{aligned}$$

From $\langle \tilde{K}, k(W^T l_1)\rangle = \langle k(l_1, \tilde{K}(X)), W\rangle$, the solvability condition for $\tilde{K}$ is $\langle W, N\rangle = 0$ for all $W \in T_I S_{A_1}$, which is precisely the equation (2).

(β) The proof for $n = 2$ is basically the same as in (α), where one needs Lemma 3.20(a).

Indeed, one has

$$L(p_2 - p_2^*, u_2 - u_2^*) + 2\Phi_2((p_1, u_1), (0, KX)) - \Phi_2(0, KX)^2 = 0$$

and

$$L(p_2 - p_2^* + p_K, u_2 - u_2^* + u_K + K^T u_1 - \tfrac{1}{2}K^T KX) = 0$$

by Lemma 3.20. For, $\mathrm{tr}(\nabla u_2 - \nabla u_2^* + \nabla u_K + K^T H - \frac{1}{2}K^T K) = 0$ by (I_2), (I_2^*). Thus $u_2 - u_2^* + u_K + K^T u_1 - \frac{1}{2}K^T KX + \tilde{K}X = 0$ and $p_2 - p_2^* + p_K = 0$ for some $\tilde{K} \in$ skew. Putting in (C_2), for $\int l_1 \times K^T KX = 0$ or $k(l_1, K^T KX) \in$ sym (notice $K \in T_I S_{A_1}$). One has the same equations for $K, \tilde{K}$ as equation (1).

The rest of the proof is the same as that in (α).

(γ) When $n = 1$ this is Theorem 3.6. Q.E.D.

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