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THE CAUCHY PROBLEM FOR $u_t = \Delta u^m$ WHEN $0 < m < 1$

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ABSTRACT. This paper deals with the Cauchy problem for the nonlinear diffusion equation $\partial u/\partial t - \Delta(u|u|^{m-1}) = 0$ on $(0,\infty) \times \mathbf{R}^N, u(0,\cdot) = u_0$ when $0 < m < 1$ (fast diffusion case). We prove that there exists a global time solution for any locally integrable function u_0: hence, no growth condition at infinity for u_0 is required. Moreover the solution is shown to be unique in that class. Behavior at infinity of the solution and L^∞_{loc}-regularizing effects are also examined when $m \in (\max\{(N-2)/N, 0\}, 1)$.

1. Introduction. This paper deals with the Cauchy problem

$$\frac{\partial u}{\partial t} = \Delta(u|u|^{m-1}) \quad \text{on } (0,\infty) \times \mathbf{R}^N, \tag{1.1}$$

$$u(0,\cdot) = u_0, \tag{1.2}$$

where

$$0 < m < 1 \tag{1.3}$$

and

$$u_0 \in L^1_{\mathrm{loc}}(\mathbf{R}^N). \tag{1.4}$$

Equation (1.1) has been suggested as a mathematical model for a lot of physical problems. We will not recall them here and we refer to the survey [15] where the very extensive literature on (1.1) is summarized.

Our goal is to emphasize some features of (1.1) when $m < 1$. Thus our main result claims that the Cauchy problem (1.1), (1.2) has a *global* solution in time for *any* $u_0 \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$. This is in sharp contrast with the case $m \geq 1$ where some growth condition at infinity is required on u_0 to provide even a local solution in time, namely

If $m = 1$, there exists $c > 0$ such that

$$\int_{\mathbf{R}^N} e^{-c|x|^2} |u_0(x)|\, dx < \infty.$$

If $m > 1$

$$\sup_{R \geq 1} R^{-(N+2/(m-1))} \int_{\{x;|x|\leq R\}} u_0(x) < \infty.$$

Note that these conditions are necessary to obtain nonnegative solutions. The necessity is proved in [**2**] and the sufficiency in [**8**] for $m > 1$ (see [**12**] for $m = 1$).

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Here we prove that, if $0 < m < 1$, there is a *local* estimate of the form

$$\int_{\{x;|x|\leq R\}} |u(t,x)|\,dx \leq F\left(t, R, \int_{\{x;|x|\leq 2R\}} |u_0(x)|\,dx\right), \tag{1.5}$$

where F is bounded on bounded subsets of $\mathbf{R}^3$ (see (2.4) for its precise form).

Combined with classical monotonicity properties, this suffices to establish existence for any $u_0 \in L^1_{\text{loc}}(\mathbf{R}^N)$ independently of the behavior of $u_0(x)$ for $|x|$ large. The estimate (1.5) relies essentially on an idea introduced in [**3**]. It has been extensively used for semilinear problems in [**3**, 4] and also recently in [**7**]. The same kind of local estimates enable us to prove *uniqueness* of strong solutions for all $0 < m < 1$ (see [**8**, **10**] when $m > 1$).

In the case when

$$(N-2)^+/N = \max((N-2)/N, 0) < m < 1, \tag{1.6}$$

we obtain here more precise results on the solutions of (1.1).

(i) We exhibit a regularizing effect from $L^1_{\text{loc}}(\mathbf{R}^N)$ into $L^\infty_{\text{loc}}(\mathbf{R}^N)$, that is, an estimate of the form

$$\sup_{|x|\leq R} |u(t,x)| \leq F\left(t, R, \int_{\{x;|x|\leq 2R\}} |u_0(x)|\,dx\right) \quad \forall t > 0. \tag{1.7}$$

Conditions under which solutions become bounded are also discussed.

(ii) We look at the behavior for $|x|$ large of the nonnegative solutions.

The precise results and more comments are to be found in the next section. Their proofs are the content of §3.

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2. The results. By a solution of (1.1), we mean a function u satisfying

$$u \in C([0,\infty[; L^1_{\text{loc}}(\mathbf{R}^N)), \tag{2.1}$$

$$\frac{\partial u}{\partial t} - \Delta(u|u|^{m-1}) = 0 \quad \text{in } \mathcal{D}'(]0,\infty[\times\mathbf{R}^N). \tag{2.2}$$

Since $0 < m < 1$, (2.1) implies that $|u|^m$ is locally integrable on $[0,\infty[\times\mathbf{R}^N$. Therefore (2.2) makes sense in the space of distributions on $]0,\infty[\times\mathbf{R}^N$. Throughout this paper, we shall write $u(t,x)$ or $u(t)$ to designate such a function.

Let $B_R = \{x \in \mathbf{R}^N; |x| \leq R\}$. We will write

$$\int_{B_R} f = \int_{B_R} f(x)\,dx.$$

THEOREM 2.1. *Let $u_0 \in L^1_{\text{loc}}(\mathbf{R}^N)$. Then there exists u satisfying* (2.1), (2.2) *and*

$$u(0,\cdot) = u_0, \tag{2.3}$$

$$\forall t > 0, \forall R > 0 \quad \int_{B_R} |u(t)| \le C \left[\int_{B_{2R}} |u_0| + t^\alpha R^{-\gamma} \right], \tag{2.4}$$

where

$$\alpha = 1/(1-m), \quad \gamma = 2/(1-m) - N \tag{2.5}$$

and $C = C(N, m)$.

In the case when

$$(N-2)^+/N < m < 1, \tag{2.6}$$

the solutions obtained above are locally bounded. More precisely:

THEOREM 2.2. *Assume* (2.6) *holds. Let* $u_0 \in L^1_{\text{loc}}(\mathbf{R}^N)$. *Then there exists a solution* u *of* (2.1), (2.2) *with* $u(0,\cdot) = u_0$ *and such that* $\forall t, R > 0$

$$\sup_{x \in B_R} |u(t,x)| \le C \left[t^{-\theta} \left[\int_{B_{4R}} |u_0| \right]^{2\theta/N} + (t/R^2)^\alpha \right], \tag{2.7}$$

where $\theta^{-1} = m - 1 + 2/N$ *and* $C = C(m, N)$.

By slightly adapting the proof of Theorems 2.1 and 2.2 in §3, it will follow that if (2.6) holds, the solution obtained, $u(t)$, is bounded in $\mathbf{R}^N$ for each $t > 0$ if for some $p > 0$

$$||u_0||_p \equiv \sup_{\xi \in \mathbf{R}^N} \int_{|x-\xi| \le p} |u_0(x)|\, dx < +\infty. \tag{2.8}$$

We prove that this condition is necessary for nonnegative solutions. The same result was established for $m > 1$ in [**8**, **2**] (for $m = 1$, it is obvious from the usual representation of the solution in terms of u_0).

We next state a uniqueness result for (1.1), (1.2). For technical reasons, we are compelled to deal with strong solutions by which we mean a function u satisfying (2.1), (2.2) and

$$\frac{\partial u}{\partial t} \in L^1_{\text{loc}}(]0, \infty[\times \mathbf{R}^N). \tag{2.9}$$

THEOREM 2.3. *Let* $u, \hat{u}$ *satisfy* (2.1), (2.2) *and* (2.9). *Then* $u(0,\cdot) = \hat{u}(0,\cdot) \Rightarrow u \equiv \hat{u}$.

REMARKS. We will see that, if $u_0 \ge 0$ and (2.6) holds, by construction $u(t) > 0$ on $\mathbf{R}^N$ and

$$-\frac{\theta u}{t} \le u_t \le \frac{\alpha u}{t} \tag{2.10}$$

for all $t > 0$ (this comes from the results in [**1**]). As a consequence, by standard regularity results $u \in C^\infty(]0, \infty[\times \mathbf{R}^N)$ and (2.9) is more than satisfied. If u does not have a sign, using again the arguments in [**1**] one can still conclude that (2.9) holds if u is continuous and $\partial u/\partial t$ is a measure. Methods to prove continuity of u can be found in [**11**, **16**]. To prove that $\partial u/\partial t$ is a measure would require a localization of the results in [**6**] concerning the estimate of $\int_{\mathbf{R}^N} |\partial u/\partial t|$ in terms of $\int_{\mathbf{R}^N} |u_0|$. We refer to [**9**] for uniqueness results when $u_0 \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$.

Concerning the behavior of $u(t,x)$ for large $|x|$ we have

THEOREM 2.4. *Assume that* (2.6) *holds, and let u be a nonnegative function satisfying* (2.1), (2.2) *and* (2.9). *Then if $u \not\equiv 0$ one has for each $t > 0$:*

$$\liminf_{|x|\to\infty} |x|^{2\alpha}\, u(t,x) \geq (2m\gamma t)^{\alpha}, \tag{2.11}$$

where α, γ are defined in (2.5).

REMARKS. Property (2.11) is in striking contrast with the behaviour of nonnegative solutions of (1.1) when $m > 1$: In fact in this case a compactly supported datum u_0 gives rise to a (spatially) compactly supported solution $u(t,x)$ (see [**14**]). However in the linear case $m = 1$ it is well known that the minimal rate of spatial decay for nonnegative, nonzero solutions is precisely that of the fundamental solution [**19**]. Notice that (2.11) does not involve the initial datum u_0. Its sharpness can be checked on the explicit solution given in [**5**]

$$U_a(t,x) = t^{-\theta}\{a + (2m\gamma)^{-1} \cdot |x|^2 \cdot t^{-2\theta/N}\}^{-\alpha}, \tag{2.12}$$

where θ is given in (2.7) and a is an arbitrary positive constant. The total mass $M(a) = \int_{\mathbf{R}^N} U_a(t,x)\,dx$ is a time invariant which depends only on a for given m and N. One easily checks in (2.12) that the behaviour for large $|x|$ does not depend on M and realizes the equality in (2.11). In the case when $N = 1$ and $u_0 \in L^1(\mathbf{R}^N)$ stronger results about this asymptotic behaviour have been recently obtained in [**17**]. In particular, equality holds in (2.11) under these assumptions when u_0 is compactly supported.

3. The proofs.

LEMMA 3.1. *Let $u, \hat{u}$ satisfy* (2.1), (2.2) *with $u \geq \hat{u}$. Then, for all $R > 0$ and $t, s \geq 0$*

$$\int_{B_R} [u(t) - \hat{u}(t)] \leq C\left[\int_{B_{2R}} [u(s) - \hat{u}(s)] + |t-s|^{\alpha} R^{-\gamma}\right], \tag{3.1}$$

where α, γ are given in (2.5) *and $C = C(m,N)$.*

REMARK. Actually, this lemma proves that *any* nonnegative solution of (2.1), (2.2) satisfies the local estimate (2.4) and not only the one we will construct below. Moreover, the proof we are going to give could be applied as well to solutions of the same equation in open cylinders containing $[t,s] \times \overline{B_{2R}}$. Existence on the whole space is not required. Note that in view of the explicit solutions (2.12), exponents α, γ in (3.1) are sharp.

PROOF. From (2.1), (2.2) applied to u and $\hat{u}$, we have for any $\psi \in C_0^\infty(\mathbf{R}^N)$ and $\alpha \in C_0^\infty(0,\infty)$

$$-\int_0^\infty \int_{\mathbf{R}^N} \alpha'\psi(u - \hat{u}) = \int_0^\infty \int_{\mathbf{R}^N} \alpha\Delta\psi(u|u|^{m-1} - u|\hat{u}|^{m-1})$$

which implies

$$\frac{d}{dt}\int_{\mathbf{R}^N} \psi(u(t) - \hat{u}(t)) = \int_{\mathbf{R}^N} \Delta\psi(u|u|^{m-1} - \hat{u}|\hat{u}|^{m-1}) \tag{3.2}$$

in $\mathcal{D}'(0,\infty)$ and therefore in $L^1_{\rm loc}(0,\infty)$ as well. Since

$$(r|r|^{m-1} - s|s|^{m-1}) \le 2^{1-m}(r-s)^m \quad \forall r \ge s,$$

(3.2) implies

$$\left|\frac{d}{dt}\int \psi(u(t)-\hat u(t))\right| < 2^{1-m}\int |\Delta\psi|(u-\hat u)^m.$$

We set $v = u - \hat u$. By Hölder's inequality, we obtain

$$\left|\frac{d}{dt}\int \psi v(t)\right| \le C(\psi)\left[\int \psi v(t)\right]^m, \tag{3.3}$$

where

$$C(\psi) = \left[2\int_{\mathbf{R}^N} |\Delta\psi|^\alpha \psi^{-\alpha m}\right]^{1-m}.$$

By integrating the differential inequality (3.3), we are led to

$$\forall s,t \ge 0 \quad \left[\int \psi v(t)\right]^{1-m} \le \left[\int \psi v(s)\right]^{1-m} + (1-m)C(\psi)|t-s|. \tag{3.4}$$

If one can choose $\psi \in C_0^\infty(\mathbf{R}^N)$ such that

$$0 \le \psi \le 1, \qquad \psi = 1 \text{ on } B_R, \quad \psi = 0 \text{ outside } B_{2R} \tag{3.5}$$

and

$$C(\psi) \le C(m,N)R^{-\gamma(1-m)} \tag{3.6}$$

one obtains the desired estimate (3.1). By setting $\psi(x) = \psi_0(x/R)$, since by change of variable

$$C(\psi) = R^{-2+N(1-m)}C(\psi_0) = R^{-\gamma(1-m)}C(\psi_0),$$

we are reduced to the case $R = 1$. We then choose for instance $\psi_0 = \varphi^k$, where k is an integer $\ge 2\alpha$ and $\varphi \in C_0^\infty(\mathbf{R}^N)$ satisfies (3.6) with $R = 1$. Then we verify

$$C(\psi_0) \le C'(m,N)\int (|\Delta\varphi|^\alpha + |\nabla\varphi|^{2\alpha}) \le C(m,N).$$

PROOF OF THEOREM 2.1. The starting point is the following classical result (see [**15**]): for any $u_0 \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, there exists

$$u \in C([0,\infty[; L^1(\mathbf{R}^N)) \cap L^\infty([0,\infty[\times \mathbf{R}^N)$$

solution of (2.2) with $u(0,\cdot) = u_0$. Moreover the mapping $u_0 \mapsto u$ is nondecreasing.

Now, let $u_0 \in L^1_{\rm loc}(\mathbf{R}^N)$ and $u_0^+ = \sup(u_0, 0)$, $u_0^- = \sup(-u_0, 0)$. We denote by (v^n) and (v_p) two sequences of nonnegative functions of $L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ increasing to u_0^+ and u_0^- respectively.

Let p be fixed and let us denote by u^n (resp. w^n) the solution of (2.1), (2.2) with $u^n(0,\cdot) = v^n - v_p$ (resp. $w^n(0,\cdot) = v^n$). By monotonicity $u^n \le w^n$, which implies $(u^n)^+ \le w^n$ since $w^n \ge 0$. By Lemma 3.1 applied with $u = w^n$ and $\hat u \equiv 0$, we have for all $R, t > 0$

$$\int_{B_R} (u^n)^+(t) \le \int_{B_R} w^n(t) \le C\left[\int_{B_{2R}} u_0^+ + t^\alpha R^{-\gamma}\right]. \tag{3.7}$$

Since $n \mapsto u^n$ is nondecreasing, it follows that, when $n \uparrow \infty$, $u^n(t)$ increases for all t to some function $u_p(t) \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$. Obviously u_p is also a solution of (2.2) and satisfies

$$(3.8) \qquad \int_{B_R} (u_p)^+(t) \le C\left[\int_{B_{2R}} u_0^+ + t^\alpha R^{-\gamma}\right].$$

It remains to show that u_p satisfies (2.1). We remark that, for all $s, t \ge 0$

$$(3.9) \qquad \begin{aligned} \int_{B_R} |u_p(t) - u_p(s)| \le \int_{B_R} [u_p(t) - u^n(t)] + \int_{B_R} |u^n(t) - u^n(s)| \\ + \int_{B_R} [u_p(s) - u^n(s)]. \end{aligned}$$

By Lemma 3.1 applied with u replaced by u^k and $\hat{u}$ by u^n $(k \ge n)$ and after letting k tend to ∞, we obtain

$$(3.10) \qquad \int_{B_R} [u_p(t) - u^n(t)] \le C\left[\int_{B_{2R}} [u_p(s) - u^n(s)] + |t-s|^\alpha R^{-\gamma}\right].$$

By (3.9), (3.10) and the fact that $u^n \in C([0,\infty[; L^1(\mathbf{R}^N))$

$$\limsup_{t \to s} \int_{B_R} |u_p(t) - u_p(s)| \le (1 + C)\int_{B_{2R}} [u_p(s) - u^n(s)].$$

We obtain the continuity of $u_p(t)$ in $L^1_{\mathrm{loc}}(\mathbf{R}^N)$ by letting n go to ∞ in this inequality.

Now we let p tend to ∞ and by a monotone (decreasing) process exactly similar to the previous one, we obtain a solution u to (2.1), (2.2) with $u(0, \cdot) = u_0$. As in (3.8), we have

$$\int_{B_R} u^-(t) \le C\left[\int_{B_{2R}} u_0^- + t^\alpha R^{-\gamma}\right].$$

This together with (3.8) and $u \le u_p$ yields the estimate (2.4).

The proof of Theorem 2.2 is based on the result of Theorem 2.1 and the deep pointwise estimate established by Aronson and Bénilan [**1**], namely:

LEMMA 3.2. *Assume* (2.6) *holds. Let* $u_0 \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, $u_0 \ge 0$, *and let* u *be the nonnegative solution of* $u(0) = u_0$ *and*

$$(3.11) \qquad u \in C([0,\infty[; L^1(\mathbf{R}^N)), \qquad \frac{\partial u}{\partial t} - \Delta u^m = 0 \quad \text{in } \mathcal{D}'((0,\infty) \times \mathbf{R}^N).$$

Then

$$(3.12) \qquad \Delta u^m \ge -\theta u / t \quad \text{in } (0,\infty) \times \mathbf{R}^N$$

with $\theta = N/(2 + N(m-1))$.

REMARK. Estimate (3.12) can be understood as a strong pointwise inequality since $u > 0$ and $u \in C^\infty((0,\infty) \times \mathbf{R}^N)$ as proved in [**1**].

Now, the L^∞_{loc}-regularising effect will be a consequence of (2.4), (3.12) and the next lemma which is of independent interest.

LEMMA 3.3. *Let v be nonnegative, smooth and satisfying*

$$-\Delta v^m \le \Lambda v \quad \text{in } \mathcal{D}'(\mathbf{R}^N), \tag{3.13}$$

where $(N-2)^+/N < m < 1$ and $\Lambda > 0$. Then, for all $R > 0$

$$\|v\|_{L^\infty(B_R)} \le C\left[\Lambda^\theta\left[\int_{B_{2R}} v\right]^{2\theta/N} + R^{-N}\int_{B_{2R}} v\right], \tag{3.14}$$

where $\theta^{-1} = m - 1 + 2/N$ and $C = C(m, N)$.

REMARK. The method for obtaining L^∞-estimates by using the pointwise estimate (3.12) has already been used in [**8**] where a result similar to Lemma 3.3 is stated (see Proposition 1.3) to treat the case when $m > 1$. However the local character of (3.14) (which has not been explicitly established in [**8**]) does not carry over to the solution of (3.11) if $m > 1$. Obviously, this is due to the lack of local estimates like (2.4) in that case.

PROOF. It is sufficient to prove the estimate (3.14) for $R = 1$. Indeed, if v satisfies (3.13), then so does w defined by $w(x) = R^{2/(1-m)}\, v(Rx)$. Now (3.14) is nothing but the same estimate applied to w with $R = 1$.

Let $\psi \in C_0^\infty(\mathbf{R}^N)$, $0 \le \psi \le 1$. Multiplying (3.13) by $v^{p-1}\psi^2$, where $p > 1$, gives after integration by parts

$$\int \nabla v^m \nabla(v^{p-1}\psi^2) \le \Lambda \int v^p\psi^2.$$

We deduce

$$\frac{4(p-1)m}{(m+p-1)^2}\int |\nabla\, v^{(m+p-1)/2}|^2\psi^2 \le \Lambda\int v^p\psi^2 + \frac{m}{m+p-1}\int v^{m+p-1}\Delta\psi^2. \tag{3.15}$$

Now we assume $N \ge 3$ and we denote by C *any* constant depending only on N and m. Using that

$$|\nabla(v^{(m+p-1)/2}\psi)|^2 \le C|\nabla v^{(m+p-1)/2}|^2\psi^2 + Cv^{m+p-1}|\nabla\psi|^2, \tag{3.16}$$

by Sobolev's imbedding, we have with $s = N/(N-2)$

$$\left[\int v^{s(m+p-1)}\psi^{2s}\right]^{1/s} \le C\left[\frac{(m+p-1)^2}{(p-1)m}\Lambda\int v^p\psi^2 + \frac{m+p-1}{p-1}\int v^{m+p-1}|\Delta\psi^2|\right]. \tag{3.17}$$

By Hölder's inequality, for all $p \ge sm > 1$ (by 2.6), we have

$$\left[\int v^{s(m+p-1)}\psi^{2s}\right]^{1/s} \le C\left[p\Lambda\int v^p\psi^2 + \left[\int v^p\psi^2\right]^{(m+p-1)/p} C(\psi)\right], \tag{3.18}$$

where

$$C(\psi) = \left[\int \frac{|\Delta\psi^2|^{p/(1-m)}}{\psi^{2(m+p-1)/(1-m)}}\right]^{(1-m)/p}.$$

This inequality shows that v can be estimated in $L^{s(m+p-1)}_{\mathrm{loc}}$ in terms of bounds of v in L^p_{loc}. Repeating this estimate will provide an estimate of v in L^∞_{loc} in terms of an L^{sm}-norm of v. We will indicate below how one can start from an L^1-norm of v.

Now, for any $k \geq 0$, we set

$$\text{(3.19)} \qquad \begin{cases} B_k = \{x \in \mathbf{R}^N; |x| \leq 1 + 2^{-k}\}, \\ p_{k+1} = sp_k - (1-m)s, \quad p_0 = 1, \\ a_k = \int_{B_k} v^{p_k}. \end{cases}$$

For $k \geq 1$, we choose $p = p_k$ and $\psi = \psi_k^{p/(1-m)}$ in (3.18), where $\psi_k \in C_0^\infty$, $0 \leq \psi_k \leq 1$, $\psi_k = 1$ on B_{k+1} and $\psi_k = 0$ outside B_k so that $||\nabla\psi_k||_\infty \leq C2^k$ and $||\Delta\psi_k||_\infty \leq C4^k$. This implies

$$\text{(3.20)} \qquad C(\psi) \leq C\left[\int |\Delta\psi_k|^{\theta_k} + |\nabla\psi_k|^{2\theta_k}\right]^{1/\theta_k} \leq C4^k.$$

Hence by (3.18), (3.20) and the fact that $p_k \leq s^k$ for all $k \geq 1$

$$\text{(3.21)} \qquad a_{k+1} \leq C^{k+1}(\Lambda^s a_k^s + a_k^{p_{k+1}/p_k}).$$

Since $p_{k+1}/p_k < s$, the sequence $b_k = \max(a_k, 1)$ satisfies

$$\text{(3.22)} \qquad b_{k+1} \leq C^{k_1}(\Lambda+1)^s b_k^s \quad \forall k \geq 1 \text{ (we impose } C \geq 1).$$

We will prove below that this inequality also holds for $k = 0$. Let us assume it and continue. By induction

$$\text{(3.23)} \qquad b_{k+1} \leq C^{k+1+ks+\cdots+s^k}(\Lambda+1)^{s+s^2+\cdots+s^{k+1}} b_0^{s^{k+1}}.$$

But, from (3.19) we also obtain by induction

$$p_{k+1} = s^{k+1} - (1-m)(s + s^2 + \cdots + s^{k+1}).$$

Hence passing to the limit in (3.23) yields

$$\text{(3.24)} \qquad \limsup_{k\to\infty}(b_{k+1})^{1/p_{k+1}} \leq C(\Lambda+1)^\theta b_0^{2\theta/N},$$

where $\theta^{-1} = m - 1 + 2/N$; that is

$$\text{(3.25)} \qquad \max(||v||_{L^\infty(B_1)}, 1) \leq C(\Lambda+1)^\theta \max\left(\int_{B_2} v, 1\right)^{2\theta/N}.$$

Now, let $\lambda = \int_{B_2} v$ and $v = \lambda\hat{v}$ so that $1 = \int_{B_1} \hat{v}$. We verify that $\hat{v}$ satisfies the estimate $-\Delta\hat{v}^m \leq \Lambda\lambda^{1-m}\hat{v}$.

Applying (3.25) with v replaced by $\hat{v}$ and Λ by $\Lambda\lambda^{1-m}$ yields

$$||\hat{v}||_{L^\infty(B_1)} \leq C(\Lambda\lambda^{1-m} + 1)^\theta \leq C(\Lambda^\theta\lambda^{(1-m)\theta} + 1)$$

which implies

$$||v||_{L^\infty(B_1)} \leq C\left[\Lambda^\theta\left[\int_{B_2} v\right]^{(1-m)\theta+1} + \int_{B_2} v\right].$$

This is the desired estimate (3.14) since $(1-m)\theta + 1 = 2\theta/N$.

To complete the proof when $N \geq 3$, we need to establish (3.22) for $k = 0$. For this, we use Hölder's inequality and (3.18) with $p = p_1 = sm$ (recall that $p_2 = s(sm + m - 1)$):

$$\int v^{p_1}\psi^2 = \int v^{s/(s+1)}\psi^{2/(s+1)}v^{p_2/(s+1)}\psi^{2s/(s+1)}$$
$$\leq \left[\int v\psi^{2/s}\right]^{s/(s+1)} \left[\int v^{p_2}\psi^{2s}\right]^{1/(s+1)}$$
$$\leq C\left[\int v\psi^{2/s}\right]^{s/(s+1)} \left[\Lambda\int v^{p_1}\psi^2 + \left[\int v^{p_1}\psi^2\right]^{p_2/sp_1} C(\psi)\right]^{s/(s+1)}.$$

We deduce that

$$\left[\max\left\{\int v^{p_1}\psi^2, 1\right\}\right]^{1/(s+1)} \leq C(\Lambda + C(\psi))^{s/(s+1)}\left[\int v\psi^{2/s}\right]^{s/(s+1)}. \tag{3.26}$$

We now choose $\psi = \psi_0^{p_1/(1-m)}$ with $\psi_0 \in C_0^\infty$, $0 \leq \psi_0 \leq 1$, $\psi_0 = 1$ on $B_{3/2}$ and $\psi_0 = 0$ outside B_2 to obtain

$$\max\left\{\int_{B_{3/2}} v^{p_1}, 1\right\} \leq C(\Lambda + 1)^s\left[\int_{B_2} v\right]^s$$

which is (3.22) with $k = 0$.

If $N = 2$ for all $w \in C_0^\infty(\mathbf{R}^N)$ we have by Sobolev's inequality

$$\int w^2 \leq C\left[\int |\nabla w|\right]^2.$$

Hence, for $w > 0$ and all $\delta > 0$

$$\int w^{2\delta} \leq C\left[\int |\nabla w^\delta|\right]^2 \leq C\delta^2\left[\int w^{\delta-1}|\nabla w|\right]^2 \leq C\delta^2\left[\int w^{2(\delta-1)}\right]\left[\int |\nabla w|^2\right].$$

For w with support in B_2, we then have by Hölder's inequality

$$\int w^{2\delta} \leq C\delta^2\left[\int w^{2\delta}\right]^{(\delta-1)/\delta}\int |\nabla w|^2,$$

which implies

$$\left[\int w^{2\delta}\right]^{1/\delta} \leq C\delta^2\int |\nabla w|^2. \tag{3.27}$$

This inequality allows us to argue as when $N \geq 3$. Here we set $p_0 = 1$, $p_{k+1} = 2p_k + m - 1$ and $b_k = \max\{\int_{B_k} v^{p_k}, 1\}$.

By (3.15) and (3.27) applied with $w = v^{(m+p_k-1)/2}\psi$, $k \geq 1$, and $\delta = \delta_k = (2p_k + m - 1)/(p_k + m - 1)$, we obtain, as above, $b_{k+1} \leq C^{k+1}(\Lambda + 1)^2 b_k^2$.

We check independently that this also holds for $k = 0$. After iterating this estimate, we deduce

$$\limsup_{k\to\infty} b_{k+1}^{1/p_{k+1}} \leq C(\Lambda + 1)^\theta b_0^{\theta'},$$

where

$$\theta = \lim_{k\to\infty} \frac{2+2^2+\cdots+2^{k+1}}{p_{k+1}} = \frac{2}{m}, \qquad \theta' = \lim_{k\to\infty} \frac{2^{k+1}}{p_{k+1}} = \frac{1}{m}.$$

We finish by homogeneity as for $N \geq 3$.

The case $N = 1$ is easier and left to the reader.

PROOF OF THEOREM 2.2. Let $u_0 \in L^1_{\text{loc}}(\mathbf{R}^N)$. By Theorem 2.1, we know there exists a solution u of (2.1), (2.2) with $u(0,\cdot) = u_0$. This solution has been constructed as the monotone limit of solutions u_p^n in

$$C([0,\infty[; L^1(\mathbf{R}^N)) \cap L^\infty((0,\infty)\times\mathbf{R}^N)$$

with $u_p^n(0,\cdot) = v^n - v_p \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, v^n increasing to u_0^+ and v_p to u_0^-. Moreover $|u_p^n(t)|$ is bounded above by the nonnegative solution with initial datum $v^n + v_p$, which increases to $|u_0|$.

According to these remarks, it is sufficient to prove (2.7) when $u_0 \geq 0$ and $u_0 \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$.

If u is a solution of (2.2), then so is $w(t,x) = R^{2\alpha}T^{-\alpha}u(Tt, Rx)$ for any $R, T > 0$ and $\alpha = 1/(1-m)$. By homogeneity, we check that (2.7) is nothing but the same inequality applied to w at $t = 1$.

Now w satisfies the estimate (3.12). In particular $-\Delta w^m(1) \leq Aw(1)$ so that by Lemma 3.3

$$\sup_{|x|\leq 1} w(1,x) \leq C\left[A^\theta\left[\int_{|x|\leq 2} w(1)\right]^{2\theta/N} + \int_{|x|\leq 2} w(1)\right], \tag{3.28}$$

where we again denote by C any constant depending only on m and N. Now by (2.4):

$$\int_{|x|\leq 2} w(1) \leq C\left[\int_{|x|\leq 4} w(0) + 1\right]. \tag{3.29}$$

Finally (3.28), (3.29) yield

$$\sup_{|x|\leq 1} w(1,x) \leq C\left[A^\theta\left[\int_{|x|\leq 4} w(0)\right]^{2\theta/N} + \int_{|x|\leq 4} w(0) + 1\right].$$

Going back to the original variables this reads

$$\sup_{x\in B_R} |u(t,x)| \leq C\left[t^{-\theta}\left(\int_{B_{4R}} |u_0|\right)^{2\theta/N} + R^{-N}\int_{B_{4R}} |u_0| + (t/R^2)^\alpha\right].$$

This together with the next consequence of Young's inequality,

$$R^{-N}\int_{B_{4R}} |u_0| \leq C(m,N)\left[t^{-\theta}\left(\int_{B_{4R}} |u_0|\right)^{2\theta/N} + (t/R^2)^\alpha\right],$$

provides the desired estimate (2.7).

PROOF OF THEOREM 2.3. Let $u, \hat{u}$ satisfy (2.1), (2.2), (2.9). Thanks to the regularity assumption (2.9), one can apply Kato's inequality [**13**]

$$-\Delta|u|u|^{m-1} - \hat{u}|u|^{m-1}| \leq -\text{sign}(u-\hat{u})\Delta(u|u|^{m-1} - \hat{u}|\hat{u}|^{m-1}). \tag{3.30}$$

Hence, using (2.2) again, we have

$$\frac{\partial}{\partial t}|u-\hat{u}| - \Delta|u|u|^{m-1} - \hat{u}|\hat{u}|^{m-1}| \leq 0 \quad \text{in } \mathcal{D}'((0,\infty)\times \mathbf{R}^N). \tag{3.31}$$

For any $\psi \in C_0^\infty(\mathbf{R}^N)$, $0 \leq \psi \leq 1$, we then obtain

$$\begin{aligned}\frac{d}{dt}\int \psi|u-\hat{u}|(t) &\leq \int |\Delta\psi|\,|u|u|^{m-1} - \hat{u}|\hat{u}|^{m-1}|(t) \leq C\int |\Delta\psi|\,|u-\hat{u}|^m(t)\\ &\leq C(\psi)\left[\int \psi|u-\hat{u}|(t)\right]^m,\end{aligned}$$

where

$$C(\psi) = \left[2\int_{\mathbf{R}^N} |\Delta\psi|^\alpha \psi^{-\alpha m}\right]^{1-m}.$$

Therefore, we are led to a differential inequality of type (3.3) except that it is unilateral here. As in the proof of Lemma 3.1, we deduce

$$\int_{B_R} |u-\hat{u}|(t) \leq C\left[\int_{B_{2R}} |u-\hat{u}|(0) + R^N(t/R^2)^{1/(1-m)}\right].$$

Applying this estimate to $u(t,x+\xi)$, $\hat{u}(t,x+\xi)$, where $\xi \in \mathbf{R}^N$ is fixed but otherwise arbitrary, and writing $B_R(\xi) = \{x \in \mathbf{R}^N : |x-\xi| < R\}$ we get

$$\int_{B_R(\xi)} |u-\hat{u}|(t) \leq C\left[\int_{B_{2R}(\xi)} |u-\hat{u}|(0) + R^N(t/R^2)^{1/(1-m)}\right]. \tag{3.32}$$

Assume now that $u(0) = \hat{u}(0)$. Then if $N < 2/(1-m)$ (i.e., $m > (N-2)^+/N$) uniqueness follows by just letting $R \to \infty$ for each fixed $t > 0$ in (3.32). Otherwise we can argue as follows. Set

$$w(t,x) = \int_0^t |u|u|^{m-1} - \hat{u}|\hat{u}|^{m-1}|(s,x)\,ds.$$

Integrating in time in (3.31) leads to $|u(t)-\hat{u}(t)| \leq \Delta w(t,x)$ in $\mathcal{D}'(\mathbf{R}^N)$ for each $t > 0$.

Thus w is subharmonic and therefore for all $\xi \in \mathbf{R}^N$

$$w(t,\xi) \leq \frac{C}{R^N}\int_{B_R(\xi)} w(t,x)\,dx \quad \forall R > 0 \tag{3.33}$$

for some $C = C(N)$. The uniqueness result now follows by noticing that, as a consequence of (3.32), the averages on the right-hand side of (3.33) tend to zero as $R \to \infty$. In fact, denoting by C some constant depending only on m and N, one has

$$\begin{aligned}\int_{B_R(\xi)} w(t,x)\,dx &\leq C\int_0^t \int_{B_R(\xi)} |u-\hat{u}|^m\\ &\leq C\int_0^t R^{N(1-m)}\left(\int_{B_R(\xi)} |u-\hat{u}|(s)\right)^m ds\\ &\leq CR^{N(1-m)}\int_0^t (R^{N-2\alpha}s^\alpha)^m\,ds \leq Ct^{1/(1-m)}R^{N-2m/(1-m)}.\end{aligned}$$

We now briefly discuss boundedness of solutions.

PROPOSITION 3.1. *Assume* (2.6) *holds. Then the solution obtained in Theorem* 2.2 *is bounded for all* $t > 0$ *if*

$$||u_0||_p = \sup_{\xi \in \mathbf{R}^N} \int_{B_p(\xi)} |u_0(x)| \, dx < +\infty.$$

Moreover this condition is necessary if $u_0 \geq 0$ *and* $0 < m < 1$.

PROOF. We can obviously reduce ourselves to the case $p = 1$.

We begin with the sufficiency part. Since the solution with initial value $|u_0|$ bounds the absolute value of the solution with initial value u_0, it suffices to deal with nonnegative solutions and without loss of generality we can assume $p = 1$. Let $\xi \in \mathbf{R}^N$ be arbitrary. We apply Theorem 2.2 to $u(t, x + \xi)$ for fixed $t > 0$ to obtain

$$\sup_{x \in B_R(\xi)} u(t,x) \leq C \left[t^{-\theta} \left(\int_{B_{4R}(\xi)} u_0 \right)^{2\theta/N} + (t/R^2)^{1/(1-m)} \right].$$

Now the result follows by choosing $R = 1/4$ and letting ξ vary over $\mathbf{R}^N$.

As to the necessity, we use the two-sided estimate (3.1) with $\hat{u} = 0$ and $u(t, x)$ replaced again by $u(t, x + \xi)$ to get

$$\int_{B_1(\xi)} u_0 \leq C \left[\int_{B_2(\xi)} u(t) + t^{1/(1-m)} \right]$$

which yields, if for instance $u(1) \in L^\infty(\mathbf{R}^N)$,

$$\sup_{\xi \in \mathbf{R}^N} \int_{B_1(\xi)} u_0 \leq C[||u(1)||_\infty + 1].$$

REMARK. When $u_0 \geq 0$ and $m \geq 1$, condition (2.8) is known to be necessary and sufficient for solutions of (1.1), (1.2) to be globally bounded at each positive time. For $m > 1$, sufficiency has been proved in [**8**] whereas necessity follows from the Harnack type inequality obtained in [**2**]. When $m = 1$, the result follows easily from the usual representation

$$u(t,x) = \frac{C}{t^{N/2}} \int_{\mathbf{R}^N} e^{-|x-y|^2/4t} u_0(y) \, dy \geq \frac{C}{et^{N/2}} \int_{|x-y|^2 \leq 4t} u_0(y) \, dy.$$

Note that Proposition 3.1 contains as a particular case the well-known regularizing effect from L^1 into L^∞ (see for instance [**18**]).

PROOF OF THEOREM 2.4. The idea is to show that any nonnegative solution of (2.1), (2.2), (2.9) is, for $|x|$ large, bigger than some of the similarity solutions

$$U_\mu(t,x) = \mu^{2\alpha} U(t, \mu x) \qquad (\alpha = 1/(1-m)),$$

where $U(t,x) \equiv U_1(t,x)$, i.e. (see (2.12))

$$U(t,x) = t^{-\theta}(1 + b|x|^2 \, t^{-2\theta/N})^{-\alpha}$$

with $\theta^{-1} = m - 1 + 2/N$, $b = (2m\gamma)^{-1}$, γ and α as in (2.5).

By the uniqueness result of Theorem 2.3, u is necessarily the solution obtained in Theorem 2.2. One knows that, if $u_0 \geq 0$, $u_0 \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ and $m > (N-2)^+/N$, then the corresponding solution is positive for all $t > 0$ (see [**1**]). Hence, by construction, $u > 0$ for all $t > 0$. Since it is also bounded and satisfies $u_t = \operatorname{div}(u^{-(1-m)} \nabla u)$, by classical arguments u is C^∞ on $(0, \infty) \times \mathbf{R}^N$ (see [**12**]).

Now we use the following comparison lemma.

LEMMA 3.4. *Let* $0 \le \tau < T$ *and* $S =]\tau, T[\times \{x \in \mathbf{R}^N; |x| > 1\}$. *Assume* v, w *are nonnegative and* C^∞ *on a neighborhood of* $\overline{S}$ *and satisfy*

$$\frac{\partial v}{\partial t} = \Delta v^m, \quad \frac{\partial w}{\partial t} = \Delta w^m \quad \text{on } S, \tag{3.34}$$

$$v(t,x) \le w(t,x), \qquad \tau < t < T, |x| = 1, \tag{3.35}$$

$$v(\tau,x) \le w(\tau,x), \qquad |x| \ge 1. \tag{3.36}$$

Then

$$v \le w \quad \text{in } S. \tag{3.37}$$

Let us assume this lemma and continue.

Fix $0 < \tau < T$. Since $u(t,x)$ is continuous and positive in S, there exists $\delta = \delta(\tau, T) > 0$ such that

$$\delta = \min u(t,x), \qquad \tau \le t \le T, |x| \le 1. \tag{3.38}$$

We now select $\mu > 0$ such that

$$U_\mu(t-\tau, x) \le \delta \quad \text{when } \tau \le t \le T, |x| \ge 1/2. \tag{3.39}$$

To this aim, according to the definition of U_μ we need

$$\mu^{2\alpha}(t-\tau)^{-\theta}\{1 + b\mu^2|x|^2(t-\tau)^{-2\theta/N}\}^{-\alpha} \le \delta \tag{3.40}$$

or

$$\delta^{m-1} \le \mu^{-2}(t-\tau)^{\theta(1-m)} + b|x|^2(t-\tau)^{-1}$$

for $\tau \le t \le T$ and $|x| \ge 1/2$ which is implied by

$$\delta^{m-1} \le \mu^{-2}(t-\tau)^{\theta(1-m)} + b(t-\tau)^{-1}/4. \tag{3.41}$$

But this function of t is bounded below by $Cb^{N(1-m)/2}\mu^{-N/\theta}$, where $C = C(m,N) > 0$. Thus (3.41) is satisfied if we choose μ such that

$$\mu \le Cb^{\theta(1-m)/2}\delta^{(1-m)\theta/N}.$$

Since $U_\mu(t-\tau,x) = 0$ for $t = \tau$, $|x| \ge 1$, by (3.39), (3.38) and Lemma 3.4

$$U_\mu(t-\tau,x) \le u(t,x) \quad \forall \tau < t < T, |x| \ge 1,$$

whence

$$\liminf_{|x|\to\infty} |x|^{2\alpha} u(t,x) \ge \liminf_{|x|\to\infty} |x|^{2\alpha} U_\mu(t-\tau,x) = \left[\frac{2mN(t-\tau)}{(1-m)\theta}\right]^\alpha. \tag{3.42}$$

Since the right-hand side of (3.42) does not depend on μ, we can let τ tend to 0 and $T \to \infty$, so that (2.11) holds (note that $\gamma = N/(1-m)\theta$).

PROOF LEMMA 3.4. Let $\psi \in C_0^\infty(\mathbf{R}^N)$, $\psi \ge 0$. By Kato's inequality,

$$\frac{\partial}{\partial t}(v-w)^+ \le \Delta(v^m - w^m)^+ \quad \text{on } S$$

so that, thanks to (3.35)

$$\frac{\partial}{\partial t}\int_{|x|\ge 1}(v-w)^+\psi \le \int_{|x|\ge 1}(v^m - w^m)^+\Delta\psi \le C(\psi)\left[\int_{|x|\ge 1}|v-w|\right]^m,$$

where again $C(\psi) = 2[\int_{\mathbf{R}^N} |\Delta\psi|^\alpha \psi^{-\alpha m}]^{1-m}$.

Repeating the arguments in Lemma 3.1 and using (3.36), we deduce

$$\int_{1<|x|<R} (v-w)^+(t) \leq C(t-\tau)^\alpha R^{-\gamma} \quad \forall t \in (\tau, T), \forall R > 1.$$

We let R tend to ∞ to conclude.

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