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GLOBAL SOLVABILITY AND REGULARITY FOR $\bar\partial$ ON AN ANNULUS BETWEEN TWO WEAKLY PSEUDO-CONVEX DOMAINS

BY

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ABSTRACT. Let M_1 and M_2 be two bounded pseudo-convex domains in $\mathbf{C}^n$ with smooth boundaries such that $\overline{M}_1 \subset M_2$. We consider the Cauchy-Riemann operators $\bar\partial$ on the annulus $M = M_2 \setminus \overline{M}_1$. The main result of this paper is the following: Given a $\bar\partial$-closed (p, q) form α, $0 < q < n$, which is C^∞ on $\overline{M}$ and which is cohomologous to zero on M, there exists a $(p, q-1)$ form u which is C^∞ on $\overline{M}$ such that $\bar\partial u = \alpha$.

0. Introduction. Let M_1 and M_2 be two bounded pseudo-convex domains in $\mathbf{C}^n$ with smooth boundaries such that $\overline{M}_1 \subset M_2$. We consider the annulus M between M_1 and M_2, i.e., $M = M_2 \setminus \overline{M}_1$. The Cauchy-Riemann equation $\bar\partial$ on M is a system of overdetermined first-order differential equations. We ask the following question: Given a (p, q) form α, where $0 < q < n$, when can one solve the equation

$$\bar\partial u = \alpha \tag{0.1}$$

and if α is smooth up to the boundary of M, does there exist a solution u of (0.1) which is also smooth up to the boundary?

A necessary condition for α to be solvable is that α must satisfy the compatibility condition

$$\bar\partial \alpha = 0 \tag{0.2}$$

since $\bar\partial^2 = 0$.

In this paper, we shall prove that if $n \geqslant 3$, α has L^2 coefficients and satisfies (0.2), and α is orthogonal to a finite-dimensional space (i.e., the harmonic (p, q) forms), then there exists a $(p, q-1)$ form u such that (0.1) holds. Furthermore, if α is smooth up to the boundary of M, then we can find a u that is smooth up to the boundary also and u satisfies (0.1) (Theorems 1, 2 and 3).

Our method is to use the $\bar\partial$-Neumann problem with weights which was used by Hörmander [3] and Kohn [4] to solve the equation (0.1) on weakly pseudo-convex domains. The $\bar\partial$-Neumann problem was a method to solve the equation $\bar\partial$ with solutions smooth up to the boundary. If one can show that the subelliptic estimate holds for the $\bar\partial$-Neumann problem, then one can conclude that the harmonic forms are finite dimensional and one can solve (0.1) provided α has L^2 coefficients and satisfies (0.2) and α is orthogonal to the harmonic space. One can find a solution u

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smooth up to the boundary if α is smooth up to the boundary. Moreover, one also obtains local regularity [**2**, **8**]. In the case of an annulus, some of the important known results are

(1) If we assume M_1 and M_2 are both *strictly* pseudo-convex and $n \geqslant 3$, then M satisfies condition $z(q)$ and the $\bar{\partial}$-Neumann problem satisfies the subelliptic $\frac{1}{2}$ estimate (see Kohn [**5**], Hörmander [**3**] and Folland and Kohn [**2**]).

(2) If we assume M_1 and M_2 are weakly pseudo-convex and *real analytic* and $0 < q < n - 1$, then it is proved by Kohn [**6**] and Dirridj and Fornaess [**1**] that the subelliptic estimate holds for the $\bar{\partial}$-Neumann problem on M.

If we assume only that M_1 and M_2 are weakly pseudo-convex, the subelliptic estimate does not hold in general and we do not get local regularity as was the case for strictly pseudo-convex domains. In this paper we shall show that global regularity holds for the $\bar{\partial}$-Neumann problem with weights (Theorem 1) and we obtain global solvability and regularity for (0.1).

This paper is arranged as follows. In §I we give some definitions and notations. We give a brief introduction to the $\bar{\partial}$-Neumann problem with weights in §II and state the main theorems in this paper (Theorems 1, 2 and 3). In §III we prove the basic a priori estimate (3.1). The estimate is similar (but weaker) to the basic estimate obtained by Hörmander in [**3**] on weakly pseudo-convex domains. However, it is sufficient for us to solve the $\bar{\partial}$-Neumann problem with weights and obtain the finite dimensionality of the harmonic forms (Lemmas 3.1 and 3.2), which gives the solvability of $\bar{\partial}$ in the L^2 sense. Based on the estimate (3.1), one can prove global regularity for the weighted $\bar{\partial}$-Neumann problem and obtain regularity for $\bar{\partial}$. Since the proof from the a priori estimate to global regularity essentially follows the line in Kohn and Nirenberg [**8**] and Kohn [**4**], we omit the proofs here and only state the results in §IV, which completes the proof of Theorem 1. Theorems 2 and 3 follow from it. A large part of this paper is devoted to the proof of estimate (3.1).

It is easy to see that our results can be generalized to the annulus between two pseudo-convex manifolds M_1 and M_2 as long as there exist real-valued functions whose complex hessian is positive definite in a neighborhood of the boundaries of M_1 and M_2. The proof is the same and we shall only stay in $\mathbf{C}^n$ in this paper.

I. Preliminaries and notations. We denote the boundaries of M, M_1 and M_2 by bM, bM_1 and bM_2, respectively. Let ρ be the defining function for M, i.e., $\rho = 0$ on bM and $\rho < 0$ in M and $\rho > 0$ outside $\overline{M}$ (the sign of ρ is specified). We normalize ρ such that $|d\rho| = 1$ on bM. We recall some definitions.

DEFINITION 1.1. Let M be a domain in $\mathbf{C}^n$ and ρ be a defining function of M. If z_0 is a point on the boundary of M, we shall say M is pseudo-convex (pseudo-concave) at z_0 if, for every $(a^1,\ldots,a^n) \in \mathbf{C}^n$ such that $\sum \rho_{z_i}(z_0)a^i = 0$, then

$$\sum \rho_{z_i\bar{z}_j}(z_0)a_i\bar{a}_j \geqslant 0 \qquad (\leqslant 0). \tag{1.1}$$

It is easy to see that the domain M which we consider here is pseudo-convex at bM_2 and pseudo-concave at bM_1.

DEFINITION 1.2. Let $\lambda \in C^2(\overline{M})$. For every $P \in \overline{M}$, we define the complex hessian at P to be the hermitian form, if $(z_1,\dots,z_n)$ are the coordinates for $\mathbf{C}^n$,

$$\lambda_{z_i\bar z_j} dz_i \otimes d\bar z_j \tag{1.2}$$

and λ is called strongly plurisubharmonic (plurisuperharmonic) if the complex hessian is positive (negative) definite.

An example of a strongly plurisubharmonic (plurisuperharmonic) function is $|z|^2$ $(-|z|^2)$. In this paper we shall fix the function $\lambda \in C^\infty(\overline{M})$ and let t be any nonnegative real number.

We denote by $\mathfrak{a}^{p,q}(M)$ the space of all the (p, q) forms on M which are smooth up to the boundary. $L^2_{(p,q)}(M)$ denotes all the (p, q) forms with L^2 coefficients. We denote by $L^2(M, t\lambda)$ the space of functions on M which are square integrable with respect to the density $e^{-t\lambda}$; therefore $f \in L^2(M, t\lambda)$ if and only if

$$\int_M |f|^2 e^{-t\lambda}\, dx < \infty \tag{1.3}$$

and the norm of f is defined by (1.3) and we denote it by $\|f\|^2_{(t)} = \langle f, f\rangle_{(t)}$. Note that the norm $\|\ \|_{(t)}$ is equivalent to $\|\ \|_0 = \|\ \|$ for every t.

We denote by $L^2_{(p,q)}(M, t\lambda)$ the space of all (p, q) forms with coefficients in $L^2(M, t\lambda)$. The norm of a (p, q) form is defined, as usual, as the sum of the norms of its coefficients.

For each nonnegative integer s, we define the Sobolev s-norm $\|\ \|_s$ by

$$\|f\|_s = \int_M \sum_{|\alpha|\leqslant s} |D^\alpha f|^2\, dx, \tag{1.4}$$

where

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_{2n}}\right)^{\alpha_{2n}} \quad \text{for } \alpha = (\alpha_1,\dots,\alpha_{2n}),$$

a multiple integer, and $|\alpha| = \sum \alpha_j$, where $x_1,\dots,x_{2n}$ are real coordinates for M.

The completion of $C^\infty(M)$ in the norm $\|\ \|_s$ is the Hilbert space W^s. We use $W^s_{p,q}(M)$ to denote the space of (p, q) forms with coefficients in the Sobolev s-space. We also define the norm $\|\ \|_{-1}$ by

$$\|f\|_{-1} = \sup(|(f, g)|/\|g\|_1),$$

where the supremum is taken over all functions $g \in C_0^\infty(M)$. The norm $\|\ \|_{-1}$ is weaker than the L^2 norm $\|\ \|$ in the sense that any sequence of functions which are bounded in the $\|\ \|$ norm has a subsequence which is convergent in the norm $\|\ \|_{-1}$.

We shall also simply use notations $W^s_{p,q}$, $\mathfrak{a}^{p,q}$, $L^2_{p,q}$ when there is no danger of confusion.

II. The $\bar\partial$-Neumann problem with weights and the main theorems. In this section we shall give a brief introduction to the $\bar\partial$-Neumann problem with weight t (for details see [4]). Let $(z_1,\dots,z_n)$ be the complex coordinates for $\mathbf{C}^n$. Then any (p, q) form $\phi \in \mathfrak{a}^{p,q}(M)$ can be expressed as

$$\phi = \sum_{I,J}{}' \phi_{I,J}\, dz^I \wedge d\bar z^J, \tag{2.1}$$

where $I = (i_1,\dots,i_p)$ and $J = (j_1,\dots,j_q)$ are multi-indices and $dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$, $d\bar{z}^J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$. The notation Σ' means the summation over strictly increasing multi-indices and $\phi_{I,J}$ is defined for arbitrary I and J so that they are antisymmetric.

The operator $\bar{\partial}$: $\mathfrak{a}^{p,q} \to \mathfrak{a}^{p,q+1}$ is expressed by

$$\bar{\partial}\phi = \sum_{I,J}{}' \sum_k \frac{\partial \phi_{I,J}}{\partial \bar{z}^k} dz^k \wedge dz^I \wedge d\bar{z}^J. \tag{2.2}$$

The formal adjoint ϑ: $\mathfrak{a}^{p,q} \to \mathfrak{a}^{p,q+1}$ of $\bar{\partial}$ under the usual L^2 norm is defined to be the operator such that

$$\langle \vartheta\psi, \phi\rangle = \langle \psi, \bar{\partial}\phi\rangle \tag{2.3}$$

for every $\phi \in \mathfrak{a}^{p,q}(M)$ and ϕ is compactly supported in M. Therefore ϑ can be expressed explicitly by

$$\vartheta\phi = (-1)^P \sum_{I,K}{}' \sum_j \frac{\partial \phi_{I,jK}}{\partial z_j} dz^I \wedge d\bar{z}^K. \tag{2.4}$$

It is easy to check that $\bar{\partial}^2 = \vartheta^2 = 0$. Let ϑ_t be the formal adjoint of $\bar{\partial}$ under the $L^2(M, t\lambda)$ norm, i.e.,

$$\langle \vartheta_t\psi, \phi\rangle_{(t)} = \langle \psi, \bar{\partial}\phi\rangle_{(t)} \tag{2.5}$$

for every compactly supported $\phi \in \mathfrak{a}^{p,q}(M)$. We have the following relation between ϑ and ϑ_t:

$$\vartheta_t = e^{t\lambda}\vartheta e^{-t\lambda}. \tag{2.6}$$

We take the L^2 closure of $\bar{\partial}$, still denoted by $\bar{\partial}$, as usual. The Hilbert space adjoint of $\bar{\partial}$ in $L^2(M)$ and $L^2(M, t\lambda)$ is denoted by $\bar{\partial}^*$ and $\bar{\partial}_t^*$, respectively. We use the notation Dom( ) to denote the domain. Then the smooth forms in Dom($\bar{\partial}^*$) and Dom($\bar{\partial}_t^*$) must satisfy some "natural" boundary conditions. For if $\phi \in \mathfrak{a}^{p,q}$ and $\psi \in \mathfrak{a}^{p,q-1}$, we have

$$\langle \vartheta_t\phi, \psi\rangle_{(t)} = \langle \phi, \bar{\partial}\psi\rangle_{(t)} + \int_{bM} \langle \sigma(\vartheta, d\rho)\phi, e^{-\lambda t}\psi\rangle\, ds, \tag{2.7}$$

where $\sigma(\vartheta, d\rho)$ denotes the symbol of ϑ in the $d\rho$ direction and dS is the volume element on bM. Therefore if $\phi \in \mathfrak{a}^{p,q} \cap \text{Dom}(\bar{\partial}_t^*)$, we must have

$$\sigma(\vartheta, d\rho)\phi = 0 \quad \text{on } bM \tag{2.8}$$

which is a condition independent of t. Define the space $\mathcal{D}^{p,q} \subset \mathfrak{a}^{p,q}$ by

$$\mathcal{D}^{p,q} = \left\{ \phi \in \mathfrak{a}^{p,q} | \sigma_p(\vartheta, dp)\phi_p = 0 \text{ for all } p \in bM \right\}.$$

Then $\mathcal{D}^{p,q}$ consists of all the smooth forms in Dom($\bar{\partial}_t^*$).

On $\mathcal{D}^{p,q}$ we define the hermitian form Q^t: $\mathcal{D}^{p,q} \times \mathcal{D}^{p,q} \to \mathbf{C}$ by

$$Q^t(\phi, \psi) = \langle \bar{\partial}\phi, \bar{\partial}\psi\rangle_{(t)} + \langle \vartheta_t\phi, \vartheta_t\psi\rangle_{(t)} + \langle \phi, \psi\rangle_{(t)}. \tag{2.9}$$

Let $\tilde{\mathcal{D}}^{p,q}$ be the Hilbert space obtained by completing $\mathcal{D}^{p,q}$ under the norm $Q^t(\phi, \phi)^{1/2}$. Again $\tilde{\mathcal{D}}^{p,q}$ is independent of t since the norms $Q^t(\phi, \phi)^{1/2}$ are equivalent to $Q^0(\phi,\phi)^{1/2} = Q(\phi,\phi)^{1/2}$. Since $Q^t(\phi,\phi) \geqslant \|\phi\|^2_{(t)}$, for every $\alpha \in L^2_{p,q}(M)$ there exists a unique $\phi^t \in \tilde{\mathscr{D}}^{p,q}$ such that

$$Q^t(\phi^t,\psi) = \langle \alpha, \psi\rangle_{(t)} \quad \text{for all } \psi \in \mathscr{D}^{p,q}. \tag{2.10}$$

The Friedrichs representative of Q^t, denoted by F^t, is defined by $F^t\phi^t = \alpha$. By the identity of weak and strong extensions of first-order differential operators, we have another description of F^t, i.e.,

$$F^t = \overline{\bar\partial\bar\partial^*_t} + \overline{\bar\partial^*_t\bar\partial} + I = \Box^t_{p,q} + I, \tag{2.11}$$

where the second equality is by definition and

$$\begin{aligned}\operatorname{Dom}(F^t) &= \{\phi \in \tilde{\mathscr{D}}^{p,q} \mid \bar\partial\phi \in \operatorname{Dom}(\bar\partial^*) \text{ and } \bar\partial^*\phi \in \operatorname{Dom}(\bar\partial)\}\\ &= \operatorname{Dom}(\Box^t_{p,q}).\end{aligned} \tag{2.12}$$

Note that $\operatorname{Dom}(F^t)$ is also independent of t.

We define the space of the harmonic (p,q) forms $\mathscr{H}_{p,q} = \mathscr{H}$ and weighted harmonic (p,q) forms $\mathscr{H}^t_{p,q} = \mathscr{H}^t$ by

$$\mathscr{H} = \{\phi \in \tilde{\mathscr{D}}^{p,q} \mid \bar\partial\phi = \bar\partial^*\phi = 0\}, \tag{2.13}$$

$$\mathscr{H}^t = \{\phi \in \tilde{\mathscr{D}}^{p,q} \mid \bar\partial\phi = \bar\partial^*_t\phi = 0\}. \tag{2.14}$$

We can now formulate the $\bar\partial$-Neumann problem of weight t with respect to the function λ as follows:

Find a bounded operator $N^t_{p,q}\colon L^2_{p,q} \to L^2_{p,q}$ ($N^t_{p,q} = N^t$ is called the $\bar\partial$-Neumann operator with weight t) such that N^t satisfies the following properties:

(i) $\mathscr{R}(N^t) \subset \operatorname{Dom}(F^t)$ and $\mathscr{R}(N^t) \perp \mathscr{H}^t$ (in the $L^2_{p,q}(M, t\lambda)$ norm), where $\mathscr{R}(\ )$ denotes the range.

(ii) $\mathscr{N}(N^t) = \mathscr{H}^t$, where $\mathscr{N}(\ )$ denotes the null space.

(iii) $\bar\partial N^t = N^t\bar\partial$, $\bar\partial^*_t N^t = N^t\bar\partial^*_t$ and $N^t\Box^t_{p,q} = \Box^t_{p,q}N^t = I - H^t$, where H^t is the projection from $L^2_{p,q}(M, t\lambda)$ onto $\mathscr{H}^t$.

(iv) For every $\alpha \in L^2_{p,q}(M, t\lambda)$, we have the orthogonal decomposition

$$\alpha = \bar\partial\bar\partial^*_t N^t\alpha + \bar\partial^*_t\bar\partial N^t\alpha + H^t\alpha. \tag{2.15}$$

If (2.15) holds, then it is easy to see that if $\bar\partial\alpha = 0$ and $H^t\alpha = 0$, then

$$\alpha = \bar\partial\bar\partial^*_t N^t\alpha. \tag{2.16}$$

Therefore $u_t = \bar\partial^*_t N^t\alpha$ is a solution of (0.1). Notice that $u_t \in L^2_{p,q-1}(M, t\lambda)$ and $u_t \perp \operatorname{Ker}\bar\partial_t$. It is this property that links the L^2 solution of (0.1) to the $\bar\partial$-Neumann problem with weights. In our application, we need the following proposition which is proved in [4].

PROPOSITION 2.1. *If $\alpha \in L^2_{p,q}(M)$ and $\bar\partial\alpha = 0$ in the L^2 sense and if*

$$\langle \alpha, \psi\rangle = 0 \quad \textit{for every } \psi \in \mathscr{D}^{p,q} \textit{ with } \vartheta\psi = 0, \tag{2.17}$$

then $\alpha \perp \mathscr{H}^t$ in the $\langle\ \rangle_{(t)}$ norm.

We shall prove that such N^t exists when t is large enough and $q > 0$, $n \geqslant 3$. We also prove the global regularity for N^t. We state the main theorems of this paper.

THEOREM 1. *Let M_1 and M_2 be two bounded pseudo-convex domains in $\mathbf{C}^n$ such that $\overline{M}_1 \subset M_2$, $n \geqslant 3$ and $q > 0$. Let λ be a smooth function on $\overline{M}$ such that $\lambda = |z|^2$ in a neighborhood of bM_2 and $\lambda = -|z|^2$ in a neighborhood of bM_1. Then there exists a number T_0 such that the $\bar{\partial}$-Neumann operator with weight t, N^t, exists for each $t \geqslant T_0$. Furthermore, for each s, there exists a number T_s such that*

$$N^t\big(W^s_{p,q}(M)\big) \subset W^s_{p,q}(M), \qquad H^t\big(W^s_{p,q}(M)\big) \subset W^s_{p,q}(M)$$

whenever $t \geqslant T_s$. When $t \geqslant T_0$, $\mathscr{H}^t_{p,q}$ is finite dimensional, its dimension is independent of t and it represents the $\bar{\partial}$-cohomology of M.

An immediate consequence of Theorem 1 is the following L^2 solvability of (0.1).

THEOREM 2. *Under the same hypotheses as Theorem 1 and $0 < q < n$, given any $\alpha \in L^2(M, t\lambda) = L^2_{p,q}$ and $\bar{\partial}\alpha = H^t\alpha = 0$, there exists a $u^t \in L^2_{p,q-1}(M, t\lambda)$ such that $u^t \perp \operatorname{Ker}\bar{\partial}$ in the $\langle\ \rangle_{(t)}$ norm and u satisfies (0.1). In fact, one can choose $u^t = \bar{\partial}^*_t N^t\alpha$.*

In view of Proposition 2.1, we have the following corollary.

COROLLARY 1. *Under the same hypotheses as Theorem 1 and $0 < q < n$, given any $\alpha \in W^s_{p,q}(M)$, $\bar{\partial}\alpha = 0$ in the L^2 sense and $\langle \alpha, \psi \rangle = 0$ for every $\psi \in \mathscr{D}^{p,q}$ with $\vartheta\psi = 0$, there exists a $u \in W^s_{p,q}(M)$ such that u satisfies (0.1). In fact we can choose $u_t = \partial^*_t N^t\alpha$ for any $t \geqslant T_s$.*

From Corollary 1 and the Sobolev embedding theorem, if $\alpha \in \mathfrak{a}^{p,q}(M)$ and α satisfies the hypotheses of Corollary 1, then we can find a solution $u^{t_m} \in C^m(\overline{M})$ as long as one chooses t_m large enough for every m. We actually can obtain a smooth solution u which was proved in [7].

THEOREM 3. *Under the same hypothesis as in Theorem 1 and $0 < q < n$, if $\alpha \in \mathfrak{a}^{p,q}(M)$, $\bar{\partial}\alpha = 0$ and $\langle \alpha, \psi \rangle = 0$ for every $\psi \in \mathscr{D}^{p,q}$ and $\vartheta\psi = 0$, then there exists a $u \in \mathfrak{a}^{p,q-1}(M)$ such that $\bar{\partial}u = \alpha$.*

In order to prove Theorem 1, our starting point is to prove an a priori estimate in the next section. It would be interesting for one to know if the finite-dimensional weighted harmonic forms $\mathscr{H}^t_{p,q}$ in Theorem 1 actually vanish.

III. Basic estimates and the existence of N^t. We start by proving the a priori estimate.

PROPOSITION 3.1. *Let M and λ be the same as in Theorem 1. Then there exist constants c, T and for each $t \geqslant T$, a constant C_t such that*

$$t\|\phi\|^2_{(t)} \leqslant cQ^t(\phi, \phi) + C_t\|\phi\|^2_{-1} \quad \text{for every } \phi \in \mathscr{D}^{p,q}. \tag{3.1}$$

PROOF. By using a partition of unity $\{\zeta_i\}_{i=1}^N$, $\sum_{i=1}^N \zeta_i^2 = 1$, it suffices to prove the estimate (3.1) when ϕ is supported in a small neighborhood U. If $\overline{U} \subset M$, then by the ellipticity of Q^t in the interior of M we have

$$\|\phi\|^2_1 \leqslant cQ^t(\phi, \phi) \quad \text{for every } \phi \in \mathscr{D}^{p,q}. \tag{3.2}$$

Thus by a well-known inequality in Sobolev space, we have

$$t\|\phi\|_{(t)}^2 \leqslant c\|\phi\|_1^2 + C_t\|\phi\|_{-1}^2 \tag{3.3}$$

which implies (3.1), when $\bar U \cap bM = \varnothing$ and $\phi \in C_0^\infty(U)$.

If ϕ is supported in a neighborhood U of bM_2, since M is pseudo-convex at every point of bM_2 and λ is strongly plurisubharmonic on U (shrink U if necessary), the estimate (3.1) is proved in Hörmander [3] with $C_t = 0$. Thus we only have to prove (3.1) when ϕ is supported in a neighborhood U such that $U \cap bM_1 \neq \varnothing$. We choose U so small such that

(1) $\lambda = -|z|^2$ on U,

(2) there exists a boundary complex frame on U, which means there exists a set of orthonormal complex vector fields $L_1, \ldots, L_n$ such that

$$L_i(\rho) = 0 \quad \text{if } i = 1, \ldots, n-1 \quad \text{and} \quad L_n(\rho) = 1 \quad \text{on } bM_1 \cap U. \tag{3.4}$$

Let $\omega^1, \ldots, \omega^n$ be the bases of (1,0) forms on U which are dual to $L_1, \ldots, L_n$. Any (p,q) forms ϕ on $U \cap \bar M$ can be expressed in terms of ω as follows:

$$\phi = \sum_{I,J}{}' \phi_{IJ}\omega^I \wedge \bar\omega^J,$$

where the notations I, J, Σ' are the same as in (2.1) and $\omega^I \wedge \bar\omega^J = \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \wedge \bar\omega^{j_1} \wedge \cdots \wedge \bar\omega^{i_q}$. Let $\mathscr{D}^{p,q}(U \cap \bar M)$ denote the space of (p,q) forms in $\mathscr{D}^{p,q}$ which has support in $U \cap \bar M$. Under our coordinate system, $\mathscr{D}^{p,q}(U \cap \bar M)$ can be characterized by

$$\mathscr{D}^{p,q}(U \cap \bar M) = \{\phi \in \mathfrak{a}^{p,q} | \phi \text{ has support in } U \cap \bar M \text{ and } \phi_{I,J} = 0 \text{ on } bM \text{ if } n \in J\}. \tag{3.5}$$

In these bases, $\bar\partial\phi$ and $\vartheta_t\phi$ can be written as

$$\bar\partial\phi = \sum_{I,J}{}' \sum_j \frac{\partial\phi_{I,J}}{\partial\bar\omega^j}\bar\omega^j \wedge \omega^I \wedge \bar\omega^J + \cdots = A\phi + \cdots, \tag{3.6}$$

$$\vartheta_t\phi = (-1)^{p-1}\sum_{I,K}\sum_j \delta_j^t\phi_{I,jK}\omega^I \wedge \bar\omega^K + \cdots = B\phi + \cdots, \tag{3.7}$$

where $\delta_j^t = e^{t\lambda}(\partial/\partial\omega_j)e^{-t\lambda}$ and dots indicate terms where no derivatives occur and the second equalities in (3.6) and (3.7) are definitions of A and B.

For $\phi \in \mathscr{D}^{p,q}(U \cap \bar M)$, we have

$$\|\bar\partial\phi\|_{(t)}^2 + \|\vartheta_t\phi\|_{(t)}^2 = \sum_{I,J,L}{}' \sum_{jl} \mathscr{E}_{lL}^{jJ}\langle \bar L_j(\phi_{I,J}), \bar L_l(\phi_{I,J})\rangle_{(t)} + \sum_{I,K}{}' \sum_{j,k}\langle \delta_j^t\phi_{I,jK}, \delta_k^t\phi_{I,kK}\rangle_{(t)} + R(\phi), \tag{3.8}$$

where $R(\phi)$ involves terms that can be controlled by $O(\|A\phi\|_{(t)} + \|B\phi\|_{(t)})\|\phi\|_t$, where $\mathscr{E}_{lL}^{jJ} = 0$ unless $j \notin J$, $l \notin L$ and $\{j\} \cup J = \{l\} \cup L$ in which case $\mathscr{E}_{lL}^{jJ}$ is the sign of permutation $\binom{jJ}{lL}$. Rearranging the terms in (3.8), we have

$$\|\bar\partial\phi\|_{(t)}^2 + \|\vartheta_t\phi\|_{(t)}^2 = \sum_{I,J}{}' \sum_j \|\bar L_j(\phi_{I,J})\|_{(t)}^2 - \sum_{I,K}{}' \sum_{jk}\langle \bar L_k(\phi_{I,jK}), \bar L_j(\phi_{I,kK})\rangle_{(t)} + \sum_{I,K}{}' \sum_{j,k}\langle \delta_j^t\phi_{I,jK}, \delta_k^t\phi_{I,kK}\rangle_{(t)} + R(\phi). \tag{3.9}$$

We now apply integration by parts to the terms $\langle \delta_j^t \phi_{I,jK}, \delta_k^t \phi_{I,iK} \rangle_{(t)}$. Notice that

(3.10) (i) $\phi_{I,J} = 0$ if $n \in J$ on bM.

(3.11) (ii) $L_j(\rho) = \bar{L}_k(\rho) = L_j(\phi_{I,J}) = \bar{L}_k(\phi_{I,J}) = 0$ on bM if $j,k < n$ and $n \in J$.

Let f_j be bounded smooth functions such that $\langle \delta_j^t u, v \rangle_{(t)} = \langle u, -\bar{L}_j v \rangle_{(t)} + \langle u, f_j v \rangle_{(t)}$ for every $u, v \in C_0^\infty(U \cap M)$. Then

$$
\begin{aligned}
\langle \delta_j^t \phi_{I,jK}, \delta_k^t \phi_{I,kK} \rangle_{(t)} &= \langle -\bar{L}_k \delta_j^t \phi_{I,jK}, \phi_{I,kK} \rangle_{(t)} + \langle \delta_j^t \phi_{I,jK}, \bar{f}_k \phi_{I,kK} \rangle_{(t)} \\
&= \langle -\delta_j^t \bar{L}_k \phi_{I,jK}, \phi_{I,kK} \rangle_{(t)} + \left\langle \left[\delta_j^t, \bar{L}_k \right] \phi_{I,jK}, \phi_{I,kK} \right\rangle_{(t)} \\
&\quad + \langle \delta_j^t \phi_{I,jK}, \bar{f}_k \phi_{I,kK} \rangle_{(t)} \qquad (3.12) \\
&= \langle \bar{L}_k \phi_{I,jK}, \bar{L}_j \phi_{I,kK} \rangle_{(t)} + \left\langle \left[\delta_j^t, \bar{L}_k \right] \phi_{I,jK}, \phi_{I,kK} \right\rangle_{(t)} \\
&\quad - \langle \bar{L}_k \phi_{I,kK}, f_j \phi_{I,kK} \rangle_{(t)} + \langle \delta_j^t \phi_{I,jK}, \bar{f}_k \phi_{I,kK} \rangle_{(t)}.
\end{aligned}
$$

In the above calculations, no boundary terms arise because of (3.4) and (3.5). Introducing the notation

$$\|\bar{L}\phi\|_{(t)}^2 = \sum_{I,J}{}' \sum_j \left\| \bar{L}_j(\phi_{I,J}) \right\|_{(t)}^2 + \|\phi\|_{(t)}^2$$

and applying integration by parts to the last terms of (3.12), we have

$$\left| \langle \delta_j^t \phi_{I,jK}, \bar{f}_k \phi_{I,kK} \rangle_{(t)} \right| \leqslant C \|\bar{L}\phi\|_{(t)} \|\phi\|_{(t)}. \tag{3.13}$$

In order to calculate the commutator $[\delta_j^t, \bar{L}_k]$, we introduce the following notations: Let u_{ij} be the coefficients of $\partial\bar{\partial}u$, i.e.,

$$\partial\bar{\partial}u = \sum_{i,j} u_{ij} \omega^i \wedge \bar{\omega}^j.$$

Assume c_{jk}^i are the smooth functions such that

$$\bar{\partial}\omega^i = \sum_{j,k} c_{jk}^i \bar{\omega}^j \wedge \omega^k.$$

Then u_{ij} can be calculated as follows:

$$\partial\bar{\partial}u = \partial\left(\sum \bar{L}_k(u) \bar{\omega}^k \right) = \sum \left(L_j \bar{L}_k(u) \omega^j \wedge \bar{\omega}^k + \sum \bar{L}_i(u) \bar{c}_{jk}^i \right) \omega^j \wedge \bar{\omega}^k.$$

From the fact that $\partial\bar{\partial} + \bar{\partial}\partial = 0$, we have

$$u_{jk} = L_j \bar{L}_k(u) + \sum \bar{L}_i(u) \bar{c}_{jk}^i = \bar{L}_k L_j(u) + \sum L_i(u) c_{kj}^i. \tag{3.14}$$

When $u = \rho$, since $\omega_n = \partial\rho$, we have

$$\partial\bar{\partial}\rho = \overline{\bar{\partial}\omega^n} = \sum_{j,k} \bar{c}_{jk}^n \omega^j \wedge \bar{\omega}^k = \rho_{jk} \omega^j \wedge \bar{\omega}^k,$$

where $(\bar{c}_{jk}^n) = (\rho_{jk})$ is the Levi matrix.

From (2.14) we have

$$[\delta_j^t, \bar L_k]u = [L_j - tL_j(\lambda), \bar L_k]u = [L_j, \bar L_k]u + tL_k\bar L_j(\lambda)u$$
$$= \sum_i c_{kj}^i L_i(u) - \sum_i \bar c_{jk}^i \bar L_i(u) + t\bar L_k L_j(\lambda)u$$
$$= t\lambda_{jk}u + \sum c_{kj}^i \delta_i^t(u) - \sum \bar c_{jk}^i \bar L_i(u). \tag{3.15}$$

Substitute (3.15) in (3.12) and observe that the terms $\langle c_{kj}^i\delta_i^t\phi_{I,jk}, f_k\phi_{I,kK}\rangle_{(t)}$ can be estimated by $O(\|\bar L\phi\|_{(t)}\|\phi\|_{(t)})$ if $i < n$ by using integration by parts. If $i = n$, and $j, k < n$, we have

$$\langle c_{kj}^n\delta_n^t\phi_{I,jK}, f_k\phi_{I,kK}\rangle_{(t)} = \int_{bM} c_{kj}^n\phi_{I,jK}\bar\phi_{I,kK}e^{-t\lambda}\,dS + \cdots$$
$$= \int_{bM}\rho_{jk}\phi_{I,jK}\bar\phi_{I,kK}e^{-t\lambda}\,ds + \cdots, \tag{3.16}$$

where dots mean terms which can be estimated by $O(\|\bar L\phi\|_{(t)}\|\phi\|_{(t)})$. Therefore

$$\|\bar\partial\phi\|_{(t)}^2 + \|\vartheta_t\phi\|_{(t)}^2 = \sum_{I,J}{}' \sum_j \|\bar L_j\phi_{IJ}\|_{(t)}^2 + t\sum_{I,K}{}' \sum_{j,k}\langle\lambda_{jk}\phi_{I,jK}, \phi_{I,kK}\rangle_{(t)}$$
$$+ \sum_{I,K}\sum_{j,k<n}\int_{b\Omega}\rho_{jK}\phi_{I,jK}\bar\phi_{I,kK}e^{-t\lambda}\,dS + R(\phi) + E(\phi), \tag{3.17}$$

where $E(\phi)$ consists of all the terms that can be estimated by $O(\|\bar L\phi\|_{(t)}\|\phi\|_{(t)})$.

This estimate is the same one obtained in Hörmander [3] with the exception that L_j's are a special boundary complex. We now apply integration by parts to the term $\|\bar L_j(\phi_{I,J})\|_{(t)}^2$. In order not to get boundary terms, we assume $j < n$.

$$\|\bar L_j(\phi_{I,J})\|_{(t)}^2 = \langle -\delta_j^t\bar L_j\phi_{I,J}, \phi_{I,J}\rangle_{(t)} + \langle\bar L_j\phi_{I,J}, f_j\phi_{I,J}\rangle_{(t)}$$
$$= \langle\delta_j^t\phi_{I,J}, \delta_j\phi_{I,J}\rangle_{(t)} + \left\langle[\bar L_j, \delta_j^t]\phi_{I,J}, \phi_{I,J}\right\rangle_{(t)}$$
$$-\langle\delta_j^t\phi_{I,J}, \bar f_j\phi_{I,J}\rangle_{(t)} + \langle\bar L_j\phi_{I,J}, f_j\phi_{I,J}\rangle_{(t)}. \tag{3.18}$$

Using the arguments as before, we substitute (3.15) into (3.18) and have for $j < n$

$$\|\bar L_j(\phi_{I,J})\|_{(t)}^2 = \|\delta_j^t\phi_{I,J}\|^2 - \int_{bM}\rho_{jj}|\phi_{I,J}|^2e^{-t\lambda}\,dS - t\langle\lambda_{jj}\phi_{I,J}, \phi_{I,J}\rangle + \cdots, \tag{3.19}$$

where dots mean terms that can be estimated by $O(\|\bar L\phi\|_{(t)}\|\phi\|_{(t)})$. Substituting (3.19) into (3.17), we have

$$\|\bar\partial\phi\|_{(t)}^2 + \|\vartheta_t\phi\|_{(t)}^2 = \sum_{I,J}{}'\|\bar L_n(\phi_{I,J})\|_{(t)}^2 + \sum_{I,J}{}' \sum_{j<n}\|\delta_j^t\phi_{I,J}\|_{(t)}^2$$
$$+ t\sum_{I,K}{}' \sum_{j,k}\langle\lambda_{jk}\phi_{I,jK}, \phi_{I,kK}\rangle_{(t)} - t\sum_{I,J}{}' \sum_{l<n}\langle\lambda_{ll}\phi_{I,J}, \phi_{I,J}\rangle_{(t)}$$
$$+ \sum_{I,K}{}' \sum_{j,k}\int_{bM}\rho_{jk}\phi_{I,kK}\bar\phi_{I,kK}e^{-t\lambda}\,ds - \sum_{I,J}{}' \sum_{l<n}\int_{bM}\rho_{ll}|\phi_{I,J}|^2e^{-t\lambda}\,dS$$
$$+ E(\phi) + R(\phi). \tag{3.20}$$

We first discuss the terms $E(\phi)$ and $R(\phi)$. Let $\varepsilon > 0$ be small and C_ε be large constants which might vary in the following arguments. By integration by parts to the terms in $E(\phi)$ which involves $\bar{L}_j, j < n$, we have

$$|E(\phi)| \leqslant \varepsilon\left(\sum_{I,J}{}' \|\bar{L}_n(\phi_{I,J})\|^2_{(t)} + \sum_{I,J'} \sum_{j<n} \|\delta^t_j \phi_{I,J}\|^2_{(t)}\right) + C_\varepsilon \|\phi\|^2_{(t)} \tag{3.21}$$

and

$$\begin{aligned} |R(\phi)| &\leqslant \frac{\varepsilon}{1-\varepsilon}\left(\|A\phi\|^2_{(t)} + \|B\phi\|^2_{(t)}\right) + C_\varepsilon \|\phi\|^2_{(t)} \\ &\leqslant \varepsilon\left(\|\bar{\partial}\phi\|^2_{(t)} + \|\vartheta_t \phi\|^2_{(t)}\right) + C_\varepsilon \|\phi\|^2_{(t)}. \end{aligned} \tag{3.22}$$

Combining (3.21) and (3.22), (3.20) now reads

$$\begin{aligned} \|\bar{\partial}\phi\|^2_{(t)} + \|\vartheta_t \phi\|^2_{(t)} \geqslant (1-2\varepsilon)&\left(\sum_{I,J}{}' \|\bar{L}_n(\phi_{I,J})\|^2_{(t)} + \sum_{I,J}{}' \sum_{j<n} \|\delta^t_j \phi_{I,J}\|^2_{(t)}\right) \\ &+ (1-\varepsilon)(T_1 + T_2) - C_\varepsilon \|\phi\|^2_{(t)}, \end{aligned} \tag{3.23}$$

where

$$T_1 = t\sum_{I,K}{}' \sum_{j,k} \langle \lambda_{jk}\phi_{I,jK}, \phi_{I,kK}\rangle - t\sum_{I,J}{}' \sum_{l<n} \langle \lambda_{ll}\phi_{I,J}\phi_{I,J}\rangle_{(t)},$$

$$T_2 = \sum_{I,K}{}' \sum_{j,k<n} \int_{bM} \rho_{jk}\phi_{I,jK}\bar{\phi}_{I,kK} e^{-t\lambda}\, dS$$

$$-\sum_{I,J}{}' \sum_{l<n} \int_{bM} \rho_{ll}|\phi_{IJ}|^2 e^{-t\lambda}\, dS.$$

It is easy to see the term T_2 is nonnegative for $(\rho_{ij})_{i,j=1}^{n-1} \leqslant 0$ which implies

$$\left(\rho_{ij} - \sum_{l=1}^{n-1} \rho_{ll}\delta_{ij}\right)_{i,j=1}^{n-1} \geqslant 0. \tag{3.24}$$

Let $T_1 = tP + tQ$, where

$$P = \sum_{\substack{I,K\\ n\notin K}} \sum_{j,k<n} \langle \lambda_{jk}\phi_{I,jK}, \phi_{I,kK}\rangle_{(t)} - \sum_{\substack{I,J\\ n\notin J}} \sum_{l<n} \langle \lambda_{ll}\phi_{I,J}, \phi_{I,J}\rangle,$$

$$Q = \sum_{I,K} \sum_{\substack{i=n \text{ or}\\ j=n}} \langle \lambda_{ik}\phi_{I,jK}, \phi_{I,kK}\rangle_{(t)} + \sum_{\substack{I,J\\ n\in K}} \sum_{j<n} \langle \lambda_{jk}\phi_{I,jK}, \phi_{I,kK}\rangle_{(t)}$$

$$- \sum_{\substack{I,J\\ n\in J}} \sum_{l<n} \langle \lambda_{ll}\phi_{I,J}, \phi_{I,J}\rangle_{(t)}.$$

Since $(\lambda_{jk})_{j,k=1}^{n}$ is negative definite, its $(n-1)\times(n-1)$ submatrix $(\lambda_{jk})_{j,k=1}^{n-1}$ is also negative definite, and $(\lambda_{jk} - \delta_{jk}\Sigma_{l<n}\lambda_{ll})_{i,j=1}^{n-1}$ is positive definite when $n \geqslant 3$.

We denote by $d_{ij} = \lambda_{ij} - \delta_{ij}\Sigma_{l<n}\lambda_{ll}$ and $d(x)$ the smallest eigenvalue of $(d_{ij})_{i,j=1}^{n-1}$ at the point $x \in U \cap \bar{M}$. Then $d(x) \geqslant d_0 > 0$ for some positive number d_0 and all $x \in U \cap \bar{M}$. Therefore

$$P \geqslant d_0 \sum_{\substack{I,J\\ n\notin J}} \|\phi_{I,J}\|^2_{(t)}. \tag{3.25}$$

Every term in Q has the form $\langle \lambda_{jk}\phi_{I,J}, \phi_{I,L}\rangle_{(t)}$, where $n \in J$ or $n \in L$. Assuming $n \in J$, using the inequality $\|u\|^2_{(t)} \leqslant \varepsilon\|u\|_1^2 + C_\varepsilon\|u\|^2_{-1}$, we have

$$(3.26) \qquad |\langle \lambda_{jk}\phi_{I,J}, \phi_{I,L}\rangle_{(t)}| \leqslant \|\lambda_{jk}\phi_{I,J}\|_{(t)}\|\phi_{I,L}\|_{(t)} \leqslant \varepsilon\|\phi_{I,J}\|_1^2 + C_{\varepsilon,t}\|\phi_{I,J}\|^2_{-1} + \|\phi_{I,L}\|_{(t)}.$$

Since $\phi_{I,J}$ vanishes on the boundary when $n \in J$, the calculations (3.12) and (3.18) can be applied to $\bar L_1, \dots, \bar L_n$. Using (3.26), we have

$$(3.27) \qquad \|\bar L_i\phi_{I,J}\|^2_{(t)} \leqslant \|\delta_i^t\phi\|^2_{(t)} + O\left(\|\bar L\phi_{I,J}\|_{(t)}\|\phi\|_{(t)}\right) + \varepsilon\|\phi\|_1^2 + C_{\varepsilon,t}\|\phi\|^2_{-1},$$

$$(3.28) \qquad \|\delta_i^t\phi\|^2_{(t)} \leqslant \|\bar L_i(\phi_{I,J})\|^2_{(t)} + O\left(\|\bar L\phi_{I,J}\|_{(t)}\|\phi\|_{(t)}\right) + \varepsilon\|\phi\|_1^2 + C_{\varepsilon,t}\|\phi\|^2_{-1},$$

where $i = 1, \dots, n$.

Combining (3.27) and (3.28), we obtained when $n \in J$

$$(3.29) \qquad \|\phi_{I,J}\|_1^2 = \sum_i \|\bar L_i(\phi_{I,J})\|^2_{(t)} + \sum_i \|\delta_i^t(\phi_{I,J})\|^2_{(t)} + \|\phi\|^2_{(t)} \leqslant 4\left(\|\bar L_n(\phi_{I,J})\|^2_{(t)} + \sum_{j<n} \|\delta_j^t\phi_{I,J}\|^2_{(t)}\right) + C_t\|\phi\|^2_{-1}.$$

From (3.23), (3.25), (3.26) and (3.29), there exist constants T, c, independent of t, and for each $t \geqslant T$, a constant C_t such that

$$(3.30) \qquad \sum_{I,J}{}' \|\bar L_n(\phi_{I,J})\|^2_{(t)} + \sum_{I,J}{}' \sum_{j<n} \|\delta_j^t\phi_{I,J}\|^2_{(t)} + \sum_{\substack{I,J\\ n\in J}} \|\phi_{I,J}\|_1^2 + t\sum_{\substack{I,J\\ n\notin J}} \|\phi_{I,J}\|^2_{(t)} \leqslant cQ^t(\phi, \phi) + C_t\|\phi\|^2_{-1}.$$

By an interpolation theorem in Sobolev space, we have for $n \in J$

$$(3.31) \qquad \|\phi_{I,J}\|^2_{(t)} \leqslant \left(t^{-1}\|\phi_{I,J}\|_1^2 + C_t\|\phi_{I,J}\|^2_{-1}\right).$$

Substituting (3.31) into (3.30), we have the desired inequality (with a larger C_t)

$$t\|\phi\|^2_{(t)} \leqslant cQ^t(\phi, \phi) + C_t\|\phi\|^2_{-1} \quad \text{for every } \phi \in \mathscr{D}^{p,q}(U \cap \bar M)$$

and the proposition is proved.

An immediate consequence of the basic estimate (3.1) is the following lemma whose proof can also be found in [**4, 8**].

Lemma 3.1. *If $q \geqslant 1$ and t is sufficiently large, then $\mathscr{H}^t$ is finite dimensional and there exists $c > 0$ such that for all $\phi \in \tilde{\mathscr{D}}^{p,q}$ with $\phi \perp \mathscr{H}^t$, we have*

$$(3.32) \qquad \|\phi\|^2_{(t)} \leqslant c\left(\|\bar\partial\phi\|^2_{(t)} + \|\vartheta_t\phi\|^2_{(t)}\right).$$

Proof. If $h \in \mathscr{H}_t^{p,q}$, then from (3.1) we have $(t - c)\|h\|^2_{(t)} \leqslant C_t\|h\|^2_{-1}$. Since $L^2(M, t\lambda)$ is compact in $W^{-1}(M, t\lambda)$, we have $\mathscr{H}_t^{p,q}$ is finite dimensional when $t > c$.

To prove (3.32), we assume that (3.32) does not hold and deduce a contradiction. If for every $\nu \in N$ there exists a $\phi_\nu \perp \mathscr{H}_t^{p,q}$, then $\|\phi_\nu\|_{(t)} = 1$ such that

$$(3.33) \qquad \|\phi_\nu\|^2_{(t)} \geqslant \nu\left(\|\bar\partial\phi_\nu\|^2_{(t)} + \|\vartheta_t\phi_\nu\|^2_{(t)}\right).$$

Combining this and (3.1), we have $\|\phi_\nu\|^2_{(t)} \leqslant C_t\|\phi_\nu\|^2_{-1}$ which implies ϕ_ν converges in L^2 to ϕ where $\phi \perp \mathscr{H}_t^{p,q}$. By (3.33) we have that $\phi \in \mathscr{H}_t^{p,q}$, a contradiction. Thus (3.32) must hold for all $\phi \perp \mathscr{H}_t^{p,q}$.

LEMMA 3.2. *The range of* $\square^t$ *is closed if* t *is sufficiently large and* $q \geqslant 1$. *In this case* N^t *exists and satisfies the properties* (i)–(iv) *defined in* §II.

PROOF. From Lemma 3.1, for every $\phi \perp \mathscr{H}^t$ and $\phi \in \operatorname{Dom}(F^t)$, we have

$$\begin{aligned}\|\phi\|^2_{(t)} &\leqslant c\left(\|\bar\partial\phi\|^2_{(t)} + \|\vartheta_t\phi\|^2_{(t)}\right)\\ &= c\langle\square^t\phi, \phi\rangle_{(t)} \leqslant c\left(\|\square^t\phi\|_{(t)}\|\phi\|_{(t)}\right)\end{aligned}$$

which implies

$$\|\phi\|^2_{(t)} \leqslant c\|\square^t\phi\|^2_{(t)} \tag{3.34}$$

and that the range of $\square^t$ is closed follows from (3.34). By the open mapping theorem, the range of $\square^t$ is isomorphic to $(H^t)^\perp$. We now define N^t as follows: If $\alpha \perp \mathscr{H}^t$, let ϕ be the unique element in Dom F^t such that $\square^t\phi = \alpha$. Define $N^t\alpha = \phi$. We extend N^t linearly to $L^2_{p,q}$ by requiring $N^t(\mathscr{H}^t) = 0$. It is easy to check that N^t satisfies properties (i)–(iv).

IV. Global regularity up to the boundary. From the estimate (3.1) we can derive a priori estimates for F^t and $\square^t$ in the Sobolev s-space when t is large (how large t should be depends on s). Using the method of elliptic regularization, one can pass from a priori estimates to obtain global regularity for F^t and $\square^t$ as was done in Kohn and Nirenberg [**8**] and Kohn [**4**]. Notice in Kohn's paper, the basic a priori estimate does not have the term $C_t\|\phi\|^2_{-1}$. The norm $\|\phi\|_{-1}$ is weaker than the L^2 norm, therefore it is harmless as was proved in the paper of Kohn and Nirenberg [**8**, Theorem 2′]. We conclude the results in the following theorems.

THEOREM 4.1. *Let* M *and* λ *be the same as in Theorem* 1. *For every nonnegative integer* s, *there exist constants* T_s *and* C_s *such that for every* $t \geqslant T_s$, *if* $\alpha \in W^s_{p,q}(M)$ *and* $\phi^t \in \tilde{\mathscr{D}}^{p,q}$ *such that* $Q^t(\phi^t, \psi) = (\alpha, \psi)_{(t)}$ *for all* $\psi \in \mathscr{D}^{p,q}$, *then* $\phi^t \in W^s_{p,q}(M)$ *and* $\|\phi^t\|_s \leqslant C_s\|\alpha\|_s$.

THEOREM 4.2. *For each nonnegative integer* s, *there exists a number* T_s *such that if* $q > 0$ *and* $t \geqslant T_s$, *then* $\mathscr{H}^t_{p,q} \subset W^s_{p,q}$ *and if* $\alpha \perp \mathscr{H}^t_{p,q}$, *then there exists a unique* $\phi \in W^s_{p,q} \cap \tilde{\mathscr{D}}^{p,q}$ *and* $\phi \perp \mathscr{H}^t$ *such that* $\square^t\phi = \alpha$.

Combining Theorems 4.1 and 4.2, we have proved Theorems 1, 2 and 3.

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