

## COBORDISM OF $(k)$ -FRAMED MANIFOLDS

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**ABSTRACT.** The cobordism theories that arise by considering manifolds whose stable normal bundle has category  $k$  are introduced. Using these theories, we define a new filtration of the homotopy groups of spheres. We study the filtration and obtain an upper bound for the filtration of elements in the stable  $n$ -stem.

**0. Introduction.** In [3] J. Ewing and L. Smith constructed the cobordism group  $\Omega_*^{\text{hyfr}}$  of hyperframed manifolds and they established the long exact sequence

$$(0.1) \quad \cdots \rightarrow \Omega_n^{\text{fr}} \xrightarrow{F} \Omega_n^{\text{hyfr}} \xrightarrow{\Delta} \tilde{\Omega}_{n-1}^{\text{fr}}(SO) \xrightarrow{J} \Omega_{n-1}^{\text{fr}} \rightarrow \cdots$$

This sequence gives a geometric description of the kernel and the cokernel of the bistable  $J$  homomorphism defined by G. W. Whitehead.

A hyperframed manifold is a manifold together with a trivialization of the restriction of its stable normal bundle to the complement of a hypersurface. Such a manifold is precisely one whose stable normal bundle has category 2 in the sense of Svarc [13]; that is, a manifold which can be covered by two open sets over each of which the stable normal bundle is trivial.

In this paper, we study the cobordism theories that arise by considering manifolds whose stable normal bundle has category  $k$ . The study of these theories was suggested by the above-mentioned authors in [3].

We begin in §1 with the precise definition of these manifolds, which we call  $(k)$ -framed manifolds, and we describe a suitable cobordism relation between them that allows the construction of a graded abelian group  $\Omega_*^{(k)}$ , which we call the  $(k)$ -framed cobordism group. We then establish in (1.4) the existence, for each positive integer  $k$ , of a long exact sequence

$$(0.2) \quad \cdots \rightarrow \Omega_n^{(k-1)} \xrightarrow{F} \Omega_n^{(k)} \xrightarrow{\Delta} \tilde{\Omega}_{n-k}^{\text{fr}}(SO^{\wedge k}) \xrightarrow{K} \Omega_{n-1}^{(k-1)} \rightarrow \cdots$$

in which  $SO^{\wedge k}$  denotes the  $k$ -fold smash product  $SO \wedge \cdots \wedge SO$ . The sequence (0.2) contains (0.1) as a particular case. Indeed, the groups  $\Omega_*^{(0)}$  and  $\Omega_*^{(1)}$  are none other than the groups  $\Omega_*^{\text{fr}}$  and  $\Omega_*^{\text{hyfr}}$  of framed and hyperframed manifolds, respectively, and (0.1) is just (0.2) with  $k = 1$ ; in particular,  $K$  is the bistable  $J$  homomorphism. The rest of the section is devoted to giving a suitable geometric interpretation of the bordism group  $\tilde{\Omega}_*^{\text{fr}}(SO^{\wedge k})$  which is then used to construct the homomorphisms appearing in (0.2). One could then prove the exactness of (0.2) directly from

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the geometrical definitions of the homomorphisms (see [9]). However, we have chosen to give a shorter, homotopy theoretic proof in §2.

In §2 we define spectra  $M^{(k)}SO$  whose stable homotopy groups are isomorphic via the Pontrjagin-Thom construction to the  $(k)$ -framed cobordism groups  $\Omega_*^{(k)}$ . These spectra are shown in (2.3) to be related to each other by cofibration sequences

$$M^{(k-1)}SO \rightarrow M^{(k)}SO \rightarrow \Sigma^k SO^{\wedge k}.$$

The construction of  $M^{(k)}SO$  is based on the  $k$ th stage of Milnor's construction of the classifying space  $BSO$ . We remark that by considering other models for  $BSO$ , such as Segal's construction which is defined as the geometric realization of a semisimplicial set, one can then consider the corresponding manifolds and obtain potentially different theories.

In the last section we introduce a spectral sequence whose  $E^1$ -term is

$$E_{k,n}^1 = \tilde{\Omega}_n^{\text{fr}}(SO^{\wedge k})$$

and in which the bistable  $J$  homomorphism appears embedded as the first differential  $d^1: E_{1,n}^1 \rightarrow E_{0,n}^1$ . This fact suggests the use of the differentials

$$d^{k+1}: E_{k+1,n-k}^{k+1} \rightarrow E_{0,n}^{k+1}$$

to define a *higher order bistable  $J$  homomorphism*  $J^{(k)}$  which is defined on a subgroup of  $\tilde{\Omega}_{n-k}^{\text{fr}}(SO^{\wedge(k+1)})$  and takes values in a quotient of the stable  $n$ -stem  $\Omega_n^{\text{fr}}$ . We show that the successive images of  $J, J^{(1)}, J^{(2)}, \dots$  define a nontrivial filtration (cf. (3.3))

$$\text{im } J = G_n^{(1)} \subset G_n^{(2)} \subset \dots \subset \Omega_n^{\text{fr}}$$

of the stable  $n$ -stem. It is well known [11] that the elements in the image of the bistable  $J$  homomorphism are those which can be represented by framed manifolds which admit bounding framings. A similar characterization of the elements in  $\Omega_n^{\text{fr}}$  which lie in the subgroup  $G_n^{(k)}$  is given in (3.7). Thus, an element in  $\Omega_n^{\text{fr}}$  is of filtration  $\leq k$  if and only if it can be represented by a framed manifold which admits a  $(k-1)$ -framed bounding structure (see (1.2)).

G. W. Whitehead conjectured that  $J$  is surjective, i.e. that every element in  $\Omega_n^{\text{fr}}$  is of filtration one. As a consequence of the theorem of Kahn and Priddy [1] one knows that every element in the 2-primary component does have filtration one. However, more recently K. Knapp [6, 7] has shown that Whitehead's conjecture is false by exhibiting, for each prime  $p \geq 5$ , a  $p$ -primary element in  $\Omega_*^{\text{fr}}$  which is not in the image of  $J$ . The question of finding nice framed manifolds representing these elements arises, and we believe that finding their filtration would be useful in this direction. Theorem (3.5) gives a rough upper bound for the filtration of elements in  $\Omega_n^{\text{fr}}$ .

This work forms part of my doctoral dissertation and I would like to take this opportunity to thank Professor I. M. James, my supervisor in Oxford. A large part of this paper derives from conversations with Dr. L. Astey and Dr. G. B. Segal to whom a special debt of gratitude is owed.

**1.  $(k)$ -framed manifolds.** Let  $W$  be a smooth, compact and oriented  $n$ -dimensional manifold and let  $k \geq 0$  be an integer.

(1.1) DEFINITION. The manifold  $W$  is said to admit a  $(k)$ -framed structure if it may be covered by  $k + 1$  open sets over each of which the stable normal bundle  $\nu$  of  $W$  is trivial.

A  $(k)$ -framed manifold of dimension  $n$  is a quadruple  $(W, \mathcal{V}, \Psi; k)$ , where  $\mathcal{V} = \{V_0, \dots, V_k\}$  is a covering of  $W$  by open sets and  $\Psi = \{\psi_0, \dots, \psi_k\}$  is a family of framings of the restrictions  $\nu|_{V_j}$  of the stable normal bundle of  $W$ , i.e.  $\psi_j$  is a specific trivialization of the bundle  $\nu|_{V_j}$ .

REMARK. Note that the  $V_j$  are not required to be connected or nonempty.

According to Definition (1.1) a manifold  $W$  admits a  $(0)$ -framed structure if and only if its stable normal bundle is trivial and  $(0)$ -framed manifolds are just framed manifolds. A manifold which admits a  $(1)$ -framed structure is one for which there exists a covering by two open sets over each of which the stable normal bundle is trivial. An easy verification shows that such manifolds are precisely the manifolds which admit a hyperframed structure in the sense of [3].

*Cobordism of  $(k)$ -framed manifolds.* We begin by defining a cobordism relation for  $(k)$ -framed manifolds.

(1.2) DEFINITION. A closed  $n$ -dimensional  $(k)$ -framed manifold  $(M, \mathcal{U}, \Phi; k)$  is said to be a  $(k)$ -framed boundary if there exists a compact  $(n + 1)$ -dimensional  $(k)$ -framed manifold  $(W, \mathcal{V}, \Psi; k)$  with  $\partial W = M$  and such that  $U_j = V_j \cap M$  and the framing  $\psi_j$  agrees with  $\phi_j$  on  $U_j$ .

Note that the disjoint union of two closed  $n$ -dimensional  $(k)$ -framed manifolds admits an obvious  $(k)$ -framed structure; the inverse of a  $(k)$ -framed manifold  $(M, \mathcal{U}, \Phi; k)$  is defined naturally as the  $(k)$ -framed manifold  $(M, \mathcal{U}, -\Phi; k)$ , where  $-\Phi = \{-\phi_0, \dots, -\phi_k\}$  and  $-\phi_j$  is the negative framing.

(1.3) DEFINITION. Two closed  $n$ -dimensional  $(k)$ -framed manifolds  $(M, \mathcal{U}, \Phi; k)$  and  $(M', \mathcal{U}', \Phi'; k)$  are said to be  $(k)$ -framed cobordant if the disjoint union  $(M \sqcup M', \mathcal{U} \sqcup \mathcal{U}', \Phi \sqcup -\Phi'; k)$  is a  $(k)$ -framed boundary.

One sees in the usual way that the relation of  $(k)$ -framed cobordism is an equivalence relation. We shall denote the set of equivalence classes of closed  $n$ -dimensional  $(k)$ -framed manifolds by  $\Omega_n^{(k)}$ . Taking the disjoint union of  $(k)$ -framed manifolds is clearly compatible with the  $(k)$ -framed cobordism relation (1.3), thus giving to  $\Omega_n^{(k)}$  the structure of an abelian group. By regarding the empty manifold as an  $n$ -dimensional  $(k)$ -framed manifold one sees that its class is the identity of  $\Omega_n^{(k)}$ . We shall call  $\Omega_n^{(k)}$  the  $n$ -dimensional  $(k)$ -framed cobordism group. Hence one has a graded abelian group  $\Omega_*^{(k)}$  for every nonnegative integer.

We can now state the theorem which fits our groups into a long exact sequence.

(1.4) THEOREM. For every positive integer  $k$  there is a long exact sequence

$$\dots \rightarrow \Omega_n^{(k-1)} \xrightarrow{F} \Omega_n^{(k)} \xrightarrow{\Delta} \tilde{\Omega}_{n-k}^{\text{fr}}(SO^{\wedge k}) \xrightarrow{K} \Omega_{n-1}^{(k-1)} \rightarrow \dots$$

in which  $SO^{\wedge k} = SO \wedge \dots \wedge SO$  is the  $k$ -fold smash product of  $SO$  with itself and the homomorphisms involved will be defined below.

In the particular case where  $k = 1$  the groups  $\Omega_*^{(0)}$  and  $\Omega_*^{(1)}$  are just the groups  $\Omega_*^{\text{fr}}$  and  $\Omega_*^{\text{hyfr}}$  of framed and hyperframed manifolds, respectively [3]. Moreover,  $K$  is just the bistable  $J$  homomorphism and the exact sequence of (1.4) reduces to the sequence of [3].

The homomorphisms of (1.4) will be described explicitly after giving a new geometrical interpretation of the groups  $\tilde{\Omega}^{\text{fr}}(SO^{\wedge k})$ . This will occupy the rest of this section. It is possible then to give a proof of (1.4) based only on our interpretation of the groups  $\tilde{\Omega}^{\text{fr}}(SO^{\wedge k})$  and the description of the homomorphisms (see [9]). However, for the sake of brevity we give a homotopy theoretic proof in the next section.

In the statement of Theorem (1.4) the reduced framed bordism groups  $\tilde{\Omega}_*^{\text{fr}}(SO^{\wedge k})$  play a crucial role. In what follows we reinterpret these groups in a way which is adequate for defining  $\Delta$  and  $K$  and which is of independent interest.

*The multiframed bordism groups.* For fixed nonnegative integers  $k$  and  $n$  let us consider  $(k + 2)$ -tuples  $(M; \alpha_0, \dots, \alpha_k)$  in which  $M$  is a closed  $n$ -dimensional manifold and where  $\alpha_0, \dots, \alpha_k$  are (global) framings of  $M$ . We say that such a  $(k + 2)$ -tuple bounds if there exists a  $(k + 2)$ -tuple  $(W; \beta_0, \dots, \beta_k)$  with  $\partial W = M$  and where  $\beta_0, \dots, \beta_k$  are (global) framings of  $W$  which agree with  $\alpha_0, \dots, \alpha_k$  on  $M$ . With this definition at hand we can construct in the usual way a cobordism group. We refer to this group as the  $n$ -dimensional *multiframed bordism group* consisting of manifolds with  $k + 1$  global framings. Our terminology is in accordance with [11], where the group consisting of triples  $(M; \alpha_0, \alpha_1)$  is considered and is called the *biframed bordism group*. Let us now consider the bordism group  $\Omega_n^{\text{fr}}(SO^{\times k})$  of the  $k$ -fold cartesian product of  $SO$  with itself. By definition this group consists of bordism classes of triples  $(M, \alpha, f)$ , where  $(M, \alpha)$  is a framed manifold and  $f: M \rightarrow SO^{\times k}$  is a map. For such a triple  $(M, \alpha, f)$  denote by  $f_j: M \rightarrow SO$  the composition of  $f$  with the projection on the  $j$ th factor and use the maps  $f_1, \dots, f_k$  to twist the framing  $\alpha$  of  $M$  to obtain framings  $\alpha_1, \dots, \alpha_k$  of  $M$ .

(1.5) PROPOSITION. *The correspondence  $(M, \alpha, f) \mapsto (M; \alpha, \alpha_1, \dots, \alpha_k)$  defines an isomorphism of  $\Omega_n^{\text{fr}}(SO^{\times k})$  with the  $n$ -dimensional multiframed bordism group of manifolds with  $k + 1$  global framings.*

PROOF. It is easily verified that the correspondence defines a homomorphism of groups. Its inverse is constructed as follows. For a given  $(k + 2)$ -tuple  $(M, \alpha_0, \dots, \alpha_k)$  there are  $k$  maps  $\alpha_0/\alpha_j: M \rightarrow SO$  which measure the difference between the framings  $\alpha_j$  and  $\alpha_0$  for  $1 \leq j \leq k$ . These maps give rise to a map  $f: M \rightarrow SO^{\times k}$  and the correspondence  $(M; \alpha_0, \dots, \alpha_k) \mapsto (M, \alpha_0, f)$  defines the required inverse.

*The groups  $\tilde{\Omega}_n^{\text{fr}}(SO^{\wedge k})$ .* Let us denote by  $W^k(SO)$  the “fat wedge” of  $k$  copies of  $SO$ , i.e. the subspace of  $SO^{\times k}$  consisting of all  $(x_1, \dots, x_k)$  such that  $x_j$  is the basepoint of  $SO$  for some  $j$ . We then have the cofibration sequence

$$W^k(SO) \rightarrow SO^{\times k} \rightarrow SO^{\wedge k}$$

which induces the long exact sequence

$$\dots \rightarrow \Omega_n^{\text{fr}}(W^k(SO)) \rightarrow \Omega_n^{\text{fr}}(SO^{\times k}) \rightarrow \tilde{\Omega}_n^{\text{fr}}(SO^{\wedge k}) \rightarrow \dots$$

Consider for each  $1 \leq f \leq k$  the projection  $SO^{\times k} \rightarrow SO^{\times(k-1)}$  which replaces the  $j$ th coordinate by the basepoint. As is well known the homomorphisms induced by these projections define a splitting

$$\Pi: \Omega_*^{\text{fr}}(SO^{\times k}) \rightarrow \Omega_*^{\text{fr}}(W^k(SO)).$$

Hence we have the split short exact sequence

$$O \rightarrow \Omega_*^{\text{fr}}(W^k(SO)) \rightarrow \Omega_*^{\text{fr}}(SO^{\times k}) \rightarrow \tilde{\Omega}_*^{\text{fr}}(SO^{\wedge k}) \rightarrow 0$$

in which  $\tilde{\Omega}_*^{\text{fr}}(SO^{\wedge k})$  is naturally identified with the subgroup  $\ker \Pi$  of  $\Omega^{\text{fr}}(SO^{\times k})$ .

(1.6) PROPOSITION. *Under the description of  $\Omega_*^{\text{fr}}(SO^{\times k})$  given in (1.5) the group  $\tilde{\Omega}_*^{\text{fr}}(SO^{\wedge k})$  is identified with the subgroup of the multiframed bordism group consisting of all those classes represented by  $(k+2)$ -tuples  $(M; \alpha_0, \dots, \alpha_k)$  for which*

(i)  $(M, \alpha_0)$  is a framed boundary, and

(ii) for each  $1 \leq j \leq k$  the  $(k+1)$ -tuple  $(M; \alpha_0, \dots, \hat{\alpha}_j, \dots, \alpha_k)$  represents zero in  $\Omega_*^{\text{fr}}(SO^{\times(k-1)})$ , where  $\hat{\alpha}_j$  means that  $\alpha_j$  is deleted.

The proof is an immediate consequence of (1.5) and of the definition of the splitting  $\Pi$ , and is left to the reader.

*The forgetful homomorphism.* To define the homomorphism

$$(1.7) \quad F: \Omega_n^{(k-1)} \rightarrow \Omega_n^{(k)}$$

of Theorem (1.4) one observes that a  $(k-1)$ -framed manifold  $(M, \mathcal{U}, \Phi; k-1)$  can always be regarded as a  $(k)$ -framed manifold by adding the empty set to the covering  $\mathcal{U}$  and the empty framing to  $\Phi$ .

*The boundary homomorphism.* The definition of the homomorphism

$$(1.8) \quad \Delta: \Omega_n^{(k)} \rightarrow \tilde{\Omega}_{n-k}^{\text{fr}}(SO^{\wedge k})$$

depends essentially on the following construction.

If  $M$  is a closed  $n$ -dimensional manifold and  $\mathcal{U} = \{U_0, \dots, U_k\}$  is a covering of  $M$  by open sets, we construct a covering  $\{W_0, \dots, W_k\}$  of  $M$  satisfying the following two conditions.

(i) Each  $W_j$  is a smooth manifold with smooth boundary and  $W_j \subset U_j$ .

(ii) Whenever  $i \neq j$ , the boundaries of  $W_i$  and  $W_j$  intersect transversally.

Here is a proof that such a covering exists. Choose a smooth partition of unity  $\{\rho_0, \dots, \rho_k\}$  on  $M$  subordinated to the covering  $\mathcal{U}$  and choose positive real numbers  $\varepsilon_0, \dots, \varepsilon_k$  such that the open sets  $\rho_j^{-1}(\varepsilon_j, \infty)$  cover  $M$ . Set  $W_j = \rho_j^{-1}[\varepsilon_j, \infty)$ . Then  $W_j$  is a smooth manifold with smooth boundary  $N_j = \partial W_j$  and condition (i) is satisfied.

The proof that our submanifolds can be chosen to satisfy condition (ii) is by induction, the key ingredient being an application of the version of the transversality theorem that involves a given submanifold being transversal not just to one submanifold but to each of several submanifolds. We refer to [4] for this version of the transversality theorem. Let us then suppose that we have been able to construct  $W_0, \dots, W_t$  with the property that  $N_i$  intersects  $N_j$  transversally whenever  $i \neq j$  and both  $i$  and  $j$  are less than  $t$ . Choose a collar  $K \cong N_t \times [0, 1)$  of  $N_t$  in  $W_t$  so that the collection obtained from  $\{W_j - N_j\}$  by replacing  $W_t - N_t$  by  $W_t - K$  still forms a

covering of  $M$ . Then there exists an isotopy of  $M$  which ends with a diffeomorphism  $f$  whose restriction to  $\text{cl}(W_t - K)$  is the identity and  $\partial f(W_t)$  intersects  $N_j$  transversally if  $j < t$ . It follows that  $W_0, \dots, f(W_t), \dots, W_k$  still form a covering of  $M$ .

In order to define the homomorphism  $\Delta$  of (1.8) consider a  $(k)$ -framed manifold  $(M, \mathcal{U}, \Phi; k)$  representing an element of  $\Omega_n^{(k)}$ . Let  $W_0, \dots, W_k$  be submanifolds of  $M$  constructed as above to satisfy conditions (i) and (ii). Define  $Q_0 = W_0$  and  $Q_j = \text{cl}(W_j - (Q_0 \cup \dots \cup Q_{j-1}))$ . Then  $Q_j$  is an  $n$ -dimensional differentiable submanifold (with corners) of  $M$ . The intersection  $N = Q_0 \cap \dots \cap Q_k$  is a closed  $(n - k)$ -dimensional submanifold of  $M$  and the association

$$(M, \mathcal{U}, \Phi; k) \mapsto (N, \phi_0|N, \dots, \phi_k|N)$$

defines  $\Delta$ . This requires some clarification. First, note that the normal bundle  $\nu$  of  $N$  in  $M$  is a trivial  $k$ -plane bundle; indeed, observe that  $N$  is the common boundary of the  $(n - k + 1)$ -dimensional manifolds

$$X_j = Q_0 \cap \dots \cap \hat{Q}_j \cap \dots \cap Q_k,$$

where  $\hat{Q}_j$  means that  $Q_j$  is deleted, and hence linearly independent sections  $s_0, \dots, s_{k-1}$  of  $\nu$  are defined by the inward vector fields of  $N$  in each of  $X_0, \dots, X_{k-1}$ . Thus, using this trivialization of  $\nu$  we have the induced global framings  $\phi_0|N, \dots, \phi_k|N$  of  $N$  obtained by adding the trivialization of  $\nu$  to the framings  $\phi_j$ . Secondly one must convince oneself that the  $(k + 2)$ -tuple  $(N, \phi_0|N, \dots, \phi_k|N)$  satisfies conditions (i) and (ii) of (1.6); this is accomplished by noting that for  $1 \leq j \leq k$  the  $(k + 1)$ -tuples

$$(X_j; \phi_0|X_j, \dots, \widehat{\phi_j|X_j}, \dots, \phi_k|X_j)$$

provide nullcobordisms for

$$(N, \phi_0|N, \dots, \widehat{\phi_j|N}, \dots, \phi_k|N).$$

Lastly, one has to show that the class  $[N; \phi_0|N, \dots, \phi_k|N]$  in  $\tilde{\Omega}_{n-k}^{\text{fr}}(SO^{\wedge k})$  does not depend on any of the choices made; this is a tedious but elementary verification which we leave to the reader.

*The homomorphism  $K$ .* We conclude this section by defining  $K: \tilde{\Omega}_{n-k}^{\text{fr}}(SO^{\wedge k}) \rightarrow \Omega_{n-1}^{(k-1)}$  and we shall again make use of (1.6).

We start by defining an open covering  $V_0, \dots, V_{k-1}$  of the standard sphere  $S^{k-1}$ ; roughly speaking the covering will consist of the faces of the standard cube

$$I^k = \{ (t_0, \dots, t_{k-1}) \in R^k \mid -1 \leq t_j \leq 1 \}.$$

Define  $\partial_j I^k = \{ t \in I^k \mid |t_j| = 1 \}$  and note that  $\partial_0 I^k \cup \dots \cup \partial_{k-1} I^k = \partial I^k$ . Let  $r: I^k \rightarrow D^k$  be the radial projection given by  $r(t) = t|t|/||t||$ , where  $|t| = \max |t_j|$  and  $||t|| = (\sum t_j^2)^{1/2}$ . We identify  $\partial I^k$  with  $S^{k-1}$  under radial projection. We now set  $Q_j = r(\partial_j I^k)$ ; note that each  $Q_j$  is a smooth submanifold (with corners) of  $S^{k-1}$  and that  $Q_0 \cup \dots \cup Q_{k-1} = S^{k-1}$ . We now define  $V_j$  as the result of thickening  $Q_j$  slightly; more precisely  $V_j$  is  $Q_j$  with a small collar of  $\partial Q_j$  in  $S^{k-1} - \text{int } Q_j$ . Note that each  $V_j$  has two connected components  $V_j = V_j^+ \sqcup V_j^-$  which are defined by radial projection of the sets  $\{t \in I^k \mid t_j = +1\}$  and  $\{t \in I^k \mid t_j = -1\}$ , respectively.

We now define the value of  $K$  on the class in  $\tilde{\Omega}_{n-k}^{\text{fr}}(SO^{\wedge k})$  represented by  $(M; \alpha_0, \dots, \alpha_k)$  to be the class in  $\Omega_{n-1}^{(k-1)}$  represented by the  $(k-1)$ -framed manifold  $(M \times S^{k-1}, \mathcal{U}, \Phi; k-1)$ , where  $U_j = M \times V_j$  and

$$\phi_j = (\alpha_0 \times \sigma|_{M \times V_j^+}) \amalg (\alpha_{j+1} \times \sigma|_{M \times V_j^-})$$

and  $\sigma$  is the canonical framing of  $S^{k-1}$ . The proof that  $K$  is well defined will be omitted.

**2.  $(k)$ -framed manifolds as  $(B, f)$ -manifolds.** In this section, we give a homotopy interpretation of the  $(k)$ -framed cobordism groups  $\Omega_{\star}^{(k)}$  defined previously and give a proof of Theorem (1.4).

Recall [10] that Milnor's classifying space  $BG$  of a compact Lie group  $G$  is filtered by the spaces  $B^{(k)}G = E^{(k)}G/G$ , where  $E^{(k)}G = G * \dots * G$  is the join of  $k+1$  copies of  $G$ . We refer to  $B^{(k)}G$  as the  $k$ th stage of Milnor's construction.

Let us consider the classifying space  $BSO_r$  and denote by  $i_k: B^{(k)}SO_r \rightarrow BSO_r$  the inclusion of its  $k$ th stage. Let  $\gamma_r^{(k)}$  be the pullback  $i_k^*(\gamma_r)$  of the universal  $r$ -plane bundle over  $BSO_r$ .

If  $M^{(k)}SO_r$  is the Thom complex of the bundle  $\gamma_r^{(k)}$ , then the canonical inclusion

$$j_r: B^{(k)}SO_r \rightarrow B^{(k)}SO_{r+1}$$

is such that  $j_r^*(\gamma_{r+1}^{(k)}) = \gamma_r^{(k)} \oplus 1$ . Therefore for fixed  $k \geq 0$  there are maps  $\Sigma M^{(k)}SO_r \rightarrow M^{(k)}SO_{r+1}$  which define a spectrum  $M^{(k)}SO$  whose  $r$ th term is the Thom complex  $M^{(k)}SO_r$ . Note that  $M^{(0)}SO$  is the sphere spectrum  $S^0$ .

These spectra provide the required homotopy interpretation of the groups  $\Omega_{\star}^{(k)}$  as is seen in the following:

(2.1) THEOREM. *There is an isomorphism  $\Omega_{\star}^{(k)} \cong \Pi_{\star}^s(M^{(k)}SO)$  given by the Pontrjagin-Thom construction.*

The theorem follows from the general theory of  $(B, f)$ -manifolds in the sense of Lashof [8] after noting that  $(k)$ -framed manifolds are  $(B, f)$ -manifolds. This in turn is an immediate consequence of the following proposition.

(2.2) PROPOSITION. *A compact  $n$ -dimensional manifold  $W$ , with or without boundary, admits a  $(k)$ -framed structure if and only if in the following diagram there is a lift  $\xi$  of the classifying map  $\nu$  of the stable normal bundle of  $W$ .*

$$\begin{array}{ccc} & & B^{(k)}SO \\ & \nearrow \xi & \downarrow i_k \\ W & \xrightarrow{\nu} & BSO \end{array}$$

This proposition follows from Theorem 9 of [13]; see also [5].

Note that the inclusions  $B^{(k-1)}SO_r \rightarrow B^{(k)}SO_r$  induce maps of Thom complexes which in turn give rise to maps of Thom spectra  $M^{(k-1)}SO \rightarrow M^{(k)}SO$  for every integer  $k > 0$ .

(2.3) THEOREM. *There are cofibrations of spectra*

$$M^{(k-1)}SO \rightarrow M^{(k)}SO \rightarrow \Sigma^k SO^{\wedge k},$$

where  $\Sigma^k SO^{\wedge k}$  denotes the suspension spectrum of the space  $\Sigma^k SO^{\wedge k}$ .

We remark that the particular case  $k = 1$  of (2.3) was obtained in [3].

Before proving (2.3) we establish the long exact sequences of (1.4).

PROOF OF (1.4). The diagram

$$\begin{array}{ccccccc} \cdots & \rightarrow & \Omega_n^{(k-1)} & \xrightarrow{F} & \Omega_n^{(k)} & \xrightarrow{\Delta} & \tilde{\Omega}_{n-k}^{\text{fr}}(SO^{\wedge k}) \xrightarrow{K} \Omega_{n-1}^{(k-1)} \rightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \rightarrow & \Pi_n^s(M^{(k-1)}SO) & \rightarrow & \Pi_n^s(M^{(k)}SO) & \rightarrow & \Pi_n^s(\Sigma^k SO^{\wedge k}) \rightarrow \Pi_{n-1}^s(M^{(k-1)}SO) \rightarrow \cdots \end{array}$$

in which the vertical arrows are the Pontrjagin-Thom isomorphisms of (2.1) and the lower row is the homotopy exact sequence of the cofibration (2.3), is commutative. This is an exercise in understanding the Pontrjagin-Thom construction and our definitions of the homomorphisms  $F$ ,  $\Delta$  and  $K$  given in the previous section.

PROOF OF (2.3). It is enough to prove that the quotient  $M^{(k)}SO_r/M^{(k-1)}SO_r$  is homotopy equivalent to  $\Sigma^{k+r}SO_r^{\wedge k}$ , for then the theorem follows by putting together the cofibrations

$$M^{(k-1)}SO_r \rightarrow M^{(k)}SO_r \rightarrow \Sigma^{k+r}SO_r^{\wedge k}$$

and passing to spectra.

According to Milnor [10] the canonical projection of  $E^{(k)}SO_r$  onto  $B^{(k)}SO_r$  induces a homeomorphism

$$\Sigma E^{(k-1)}SO_r \rightarrow B^{(k)}SO_r/B^{(k-1)}SO_r.$$

Since  $\Sigma E^{(k-1)}SO_r$  has the homotopy type of  $\Sigma^k SO_r^{\wedge k}$  the latter space is homotopy equivalent to  $B^{(k)}SO_r/B^{(k-1)}SO_r$ . The proof is completed by applying Lemma (2.4) to the complex  $B^{(k)}SO_r$  and its subcomplex  $B^{(k-1)}SO_r$ .

(2.4) LEMMA. *Let  $X$  be a finite CW complex and  $Y \subset X$  a closed subset. Let  $\xi$  be a real  $r$ -dimensional vector bundle over  $X$ . If  $M(\xi)$  and  $M(\xi|Y)$  are the Thom complexes of  $\xi$  and its restriction to  $Y$ , then*

$$M(\xi)/M(\xi|Y) \cong \Sigma^r(X/Y)$$

providing  $\xi|X - Y$  is trivial.

PROOF. Since  $X$  and  $Y$  are compact, we have that  $M(\xi) = E(\xi)^+$  and  $M(\xi|Y) = E(\xi|Y)^+$ , the one-point compactifications of the total spaces of  $\xi$  and  $\xi|Y$ , respectively. But

$$(E(\xi)|X - Y)^+ = E(\xi)^+/E(\xi|Y)^+$$

and since  $\xi|X - Y$  is trivial,

$$E(\xi|X - Y)^+ = \Sigma^r(X - Y)^+ = \Sigma^r(X/Y)$$

and the lemma is proved.



**3. A spectral sequence.** By Theorem (2.3) we have a Massey exact couple

$$(3.1) \quad \begin{array}{ccc} \bigoplus \Pi_*^s(M^{(k-1)}SO) & \xrightarrow{i} & \bigoplus \Pi_*^s(M^{(k)}SO) \\ \partial \nwarrow & & \swarrow \\ & \bigoplus \Pi_*^s(\Sigma^k(SO^{\wedge k})) & \end{array}$$

where in each summation  $k$  runs through all integers and where we define the spectrum  $M^{(k)}SO$  for negative values of  $k$  to be the trivial spectrum (i.e. every term of  $M^{(k)}SO$  is a point) and we regard  $SO^{\wedge k}$  as the sphere spectrum  $S^0$  when  $k = 0$ .

According to Theorem (1.4) we also have a Massey exact couple

$$\begin{array}{ccc} \bigoplus \Omega_*^{(k-1)} & \rightarrow & \bigoplus \Omega_*^{(k)} \\ \nwarrow & & \swarrow \\ & \bigoplus \tilde{\Omega}_*^{\text{fr}}(SO^{\wedge k}) & \end{array}$$

which has a good deal of geometric content and which by (2.1) is just (3.1).

By the usual procedure, the above exact couple gives rise to a first quadrant spectral sequence  $\{E_{k,n}^r\}$  for which

$$E_{k,n}^1 = \Pi_n^s(SO^{\wedge k}) = \tilde{\Omega}_n^{\text{fr}}(SO^{\wedge k}).$$

Let a filtration of the oriented cobordism ring  $\Pi_*^s(MSO)$  be defined by

$$F_k \Pi_*^s(MSO) = \text{im}(\Pi_*^s(M^{(k)}SO) \rightarrow \Pi_*^s(MSO)),$$

where  $M^{(k)}SO \rightarrow MSO$  is the map induced on Thom complexes by the inclusion of  $B^{(k)}SO$  in  $BSO$ . Then  $F_0 \subset F_1 \subset \cdots \subset \Pi_*^s(MSO)$  and, since any closed oriented manifold can be covered by a finite number of balls,  $\bigcup F_k = \Pi_*^s(MSO)$ .

We now have the following

(3.2) PROPOSITION. *The spectral sequence  $\{E_{k,n}^r\}$  converges to  $\Pi_*^s(MSO)$ . Precisely, the limit term  $E^\infty$  is the graded group associated with the filtration of  $\Pi_*^s(MSO)$  defined above:*

$$E_{k,n}^\infty = F_k \Pi_{k+n}^s(MSO) / F_{k-1} \Pi_{k+n}^s(MSO).$$

PROOF. This is formally identical with the proof of convergence of the Atiyah-Hirzebruch spectral sequence.

*The higher order bistable  $J$  homomorphisms.* Let

$$i^k: \Pi_*^s(S^0) \rightarrow \Pi_*^s(M^{(k)}SO)$$

be the  $k$ th iteration of the homomorphism  $i$  of (3.1). Define  $H_*^{(k)} = \partial^{-1}(\text{im}(i^k))$  and  $G_*^{(k)} = \ker(i^k)$ , where  $\partial$  is the boundary homomorphism of (3.1). These graded groups are subgroups of  $\Pi_*^s(SO^{\wedge(k+1)})$  and  $\Pi_*^s(S^0)$ , respectively. By the Massey theory of exact couples  $E_{k+1,*-k}^{k+1}$  may be expressed as a quotient of  $H_*^{(k)}$ , and  $E_{0,*}^{k+1}$  as the quotient of  $E_{0,*}^1 = \Pi_*^s(S^0)$  by  $G_*^{(k)}$ .

We now define the  $k$ th order bistable  $J$  homomorphism

$$J^{(k)}: H_*^{(k)} \rightarrow \Pi_*^s(S^0)/G_*^{(k)}$$

to be the canonical projection of  $H_*^{(k)}$  onto  $E_{k+1,*-k}^{k+1}$  followed by the differential  $d^{k+1}: E_{k+1,*-k}^{k+1} \rightarrow E_{0,*}^{k+1}$ . Hence, the value of  $J^{(k)}$  on a given  $x$  in  $H_*^{(k)}$  is the coset modulo  $G_*^{(k)}$  defined by an element  $y \in \Pi_*^s(S^0)$  for which  $i^k(y) = \partial(x)$ .

Our definition of  $J^{(k)}$  suggests itself upon observing that the bistable  $J$  homomorphism appears embedded in our spectral sequence as the first differential  $d^1: E_{1,*}^1 \rightarrow E_{0,*}^1$ . Thus, our  $J^{(0)}$  is precisely the bistable  $J$  homomorphism, and our terminology for the  $J^{(k)}$  is in accordance with the fact that they are essentially defined as the higher order differentials.

Let us now consider the filtration of the stable homotopy groups of spheres defined by the groups  $G_*^{(k)}$ .

$$(3.3) \quad 0 = G_*^{(0)} \subset G_*^{(1)} \subset \cdots \subset \Pi_*^s(S^0).$$

The proof of the following proposition consists of a straightforward verification using the definition of  $J^{(k)}$ .

(3.4) PROPOSITION. *The image of  $J^{(k)}$  is precisely the subgroup  $G_*^{(k+1)}/G_*^{(k)}$  of  $\Pi_*^s(S^0)/G_*^{(k)}$ .*

In particular, it follows from (3.4) that  $G_*^{(1)}$  is  $\text{im } J \subset \Pi_*^s(S^0)$ . Thus the groups  $G_*^{(k)}$  of (3.3) are the intermediate stages of a filtration of  $\Pi_*^s(S^0)$  which starts with  $\text{im } J$ ; we shall see shortly that the filtration is convergent.

As a consequence of the theorem of Kahn and Priddy one knows [11] that every 2-primary element of  $\Pi_*^s(S^0)$  has filtration exactly one. On the other hand our filtration acquires significance due to the result of K. Knapp [7] that for every prime  $p \geq 5$  there exist  $p$ -primary elements in  $\Pi_*^s(S^0)$  of filtration greater than one, namely the elements  $\beta_{p+1}$  considered by L. Smith in [12]. It is an interesting problem to determine the filtrations of these elements, or indeed of arbitrary elements of the stable  $n$ -stem  $\Pi_n^s(S^0)$ .

An upper bound for the filtration of elements in the stable  $n$ -stem is given by the following

(3.5) THEOREM. *Every element of the stable  $n$ -stem has filtration at most  $[n/4] + 1$ . In particular, the filtration (3.3) is convergent.*

We shall give two proofs of (3.5). The first is a homotopy theoretic proof, while the second is geometric and depends upon Theorem (3.7) which is of independent interest as it gives a geometric characterization of the filtration (3.3).

Before giving the first proof of (3.5) we remark that the construction that yields the spectra  $M^{(k)}SO$  and the cofibrations of (2.3) also applies when one replaces  $SO$  by other classical groups. In particular, one can construct spectra  $M^{(k)}\text{Spin}$  and cofibrations

$$(3.6) \quad M^{(k)}\text{Spin} \rightarrow M^{(k+1)}\text{Spin} \rightarrow \Sigma^{k+1}\text{Spin}^{\wedge(k+1)}.$$

This suggests the problem of studying the filtration of  $\Pi_*^s(S^0)$  arising from  $\text{Spin}$ .

FIRST PROOF OF (3.5). As remarked earlier it is a consequence of the Kahn-Priddy theorem that every 2-primary element of  $\Pi_*^s(S^0)$  has filtration one. Hence (3.5) holds at the prime 2. Therefore it remains to prove (3.5) at odd primes.

Since  $\Sigma^{k+1}\text{Spin}^{\wedge(k+1)}$  is  $4k + 3$  connected, it follows from the homotopy exact sequence of the cofibration (3.6) that

$$j: \Pi_n^s(M^{(k)}\text{Spin}) \rightarrow \Pi_n^s(M^{(k+1)}\text{Spin})$$

is an isomorphism for  $4k + 2 \geq n$ . It follows that in the commutative diagram

$$\begin{array}{ccc} \Pi_n^s(S^0) & \xrightarrow{j^k} & \Pi_n^s(M^{(k)}\text{Spin}) \\ \searrow j_0 & & \downarrow \cong \\ & & \Pi_n^s(M\text{Spin}) \end{array}$$

the vertical arrow is an isomorphism. We now localize this diagram at an odd prime. Then  $j_0 = 0$  by [2], and so  $j^k = 0$  for  $4k + 2 \geq n$ . The assertion of (3.5) follows from the commutativity of the diagram

$$\begin{array}{ccc} \Pi_n^s(S^0) & \xrightarrow{j^k} & \Pi_n^s(M^{(k)}\text{Spin}) \\ \searrow j^k & & \downarrow \\ & & \Pi_n^s(M^{(k)}SO) \end{array}$$

where the vertical arrow is induced by the projection of the double covering.

We observe that if we had used  $SO$  throughout the proof of (3.5) we would have obtained the weaker result that every element of the stable  $n$ -stem has filtration at most  $[n/2] + 1$ .

(3.7) THEOREM. *The elements in  $\Omega_*^{\text{fr}}$  whose filtration does not exceed  $k$  are precisely those which are represented by a framed manifold which admits the structure of a  $(k - 1)$ -framed boundary.*

PROOF. Let  $\alpha \in \Omega_*^{\text{fr}}$  have filtration not exceeding  $k$ . By definition of the filtration (3.3) and by (2.1) this means that  $\alpha$  lies in the kernel of the forgetful homomorphism  $\Omega_*^{\text{fr}} \rightarrow \Omega_*^{(k)}$ . Hence, if  $\alpha$  is represented by the framed manifold  $(M, \phi)$ , then there exists a  $(k)$ -framed manifold  $(W, \mathcal{V}, \Psi; k)$  with  $\partial W = M$  and such that  $V_0$  contains  $M$  and the intersection  $V_j \cap M$  is empty for  $j > 0$ . It is easy to see that there exist two compact manifolds  $W_0 \subset V_0$  and  $W_1 \subset V_1 \cup \dots \cup V_k$  such that  $W = W_0 \cup W_1$ ,  $\partial W_1 = W_0 \cap W_1$  and  $\partial W_0 = M \amalg \partial W_1$ . Hence the manifold  $W_0$  with the framing  $\psi_0|_{W_0}$  is a cobordism between  $(M, \phi)$  and  $(\partial W_1, \psi_0|_{\partial W_1})$ ; thus the latter framed manifold represents  $\alpha$  and  $\partial W_1$  clearly admits the structure of a  $(k - 1)$ -framed boundary, namely the one obtained by restricting  $\psi_j$  to  $V_j \cap \partial W_1$  for  $1 \leq j \leq k$ .

The converse is easily proved by attaching an outwards collar  $M \times [0, 1]$  along the boundary to a  $(k - 1)$ -framed manifold bounded by a representative  $M$  of  $\alpha$ .

In view of (3.7), Theorem (3.5) is equivalent to the following geometrical statement.

(3.8) COROLLARY. *Every element of the stable  $n$ -stem  $\Omega_n^{\text{fr}}$  is represented by a framed manifold which admits a bounding  $[n/4]$ -framed structure.*

SECOND PROOF OF (3.5). We prove the equivalent (3.8). The idea of the proof was kindly pointed out by G. Pastor.

As we remarked before, it is a consequence of the theorem of Kahn and Priddy that every element in the 2-primary component of  $\Omega_*^{\text{fr}}$  is in  $\text{im } J$ . In addition, it is well known that every element of  $\Omega_n^{\text{fr}}$  is in  $\text{im } J$  if  $n \leq 6$ . Hence, let  $\alpha \in \Omega_n^{\text{fr}}$  be an odd

torsion element with  $n > 6$ . We now consider the forgetful homomorphism  $\Omega_n^{\text{fr}} \rightarrow \Omega_n^{\text{Spin}}$ . According to [2] its image consists of elements of order two; it follows that  $\alpha$  is mapped trivially into  $\Omega_n^{\text{Spin}}$ . Thus, if the framed manifold  $(M, \phi)$  represents  $\alpha$ , then  $M$  bounds a Spin manifold on which one can perform surgery to obtain a 3-connected  $(n + 1)$ -dimensional manifold  $W$  bounded by  $M$ . There is a well-known upper bound for the category of a space (in the sense of Lusternik-Schnirelmann) given in terms of its dimension and its connectivity (see (5.1) of [5]) which for a  $(t - 1)$ -connected CW complex  $X$  reads

$$(3.9) \quad \text{cat } X \leq 1 + \dim X/t.$$

Since our  $W$  has the homotopy type of an  $n$ -dimensional complex, this formula gives  $\text{cat } W \leq 1 + n/4$  which implies that  $W$  may be covered by  $[n/4] + 1$  open sets each contractible in  $W$ . This completes the second proof of (3.5).

Observe that in this last proof the crucial points are the formula (3.9) and the fact that the framed manifold  $(M, \phi)$  bounds a 3-connected manifold. It follows that a straightforward application of (3.9) yields

(3.10) THEOREM. *If an element  $\alpha$  of the stable  $n$ -stem can be represented by a framed manifold which bounds a  $(t - 1)$  connected manifold, then the filtration of  $\alpha$  does not exceed  $[n/t] + 1$ .*

The question of whether a given element of  $\Pi_*^s(S^0)$  is represented by a framed manifold which bounds a  $(t - 1)$  connected manifold is equivalent to the problem of determining the kernel of  $\Pi_*^s(S^0) \rightarrow \Pi_*^s(MO[t])$ , where  $MO[t]$  is the Thom spectrum for bundles which are trivial over the  $t - 1$  skeleton. This seems to be a very difficult question.

## REFERENCES

1. J. F. Adams, *The Kahn-Priddy theorem*, Proc. Cambridge Philos. Soc. **73** (1973), 45–55.
2. D. W. Anderson, E. H. Brown, Jr., and F. P. Peterson, *The structure of the Spin cobordism ring*, Ann. of Math. (2) **86** (1967), 271–298.
3. J. Ewing and L. Smith, *Cobordism of hyperframed manifolds and the stable J homomorphism*, Manuscripta Math. **22** (1977), 171–197.
4. M. W. Hirsh, *Differential topology*, Springer-Verlag, Berlin and New York, 1976.
5. I. M. James, *On category, in the sense of Lusternik-Schnirelmann*, Topology **17** (1978), 331–348.
6. K. Knapp, *On the bi-stable J-homomorphism*, Algebraic Topology (Aarhus, 1978), Lecture Notes in Math., Vol. 763, Springer-Verlag, Berlin, Heidelberg, and New York, 1979.
7. ———, *Some applications of K-theory to framed bordism: E invariant and transfer*, Habilitationsschrift, Bonn, 1979.
8. R. Lashof, *Poincare duality and cobordism*, Trans. Amer. Math. Soc. **109** (1963), 257–277.
9. E. Micha, *On the Pontrjagin-Thom construction*, Ph. D. Thesis, Oxford, 1981.
10. J. Milnor, *Construction of universal bundles, II*, Ann. of Math. (2) **63** (1956), 430–436.
11. N. Ray, *A geometrical observation on the Arf invariant of a framed manifold*, Bull. London Math. Soc. **4** (1972), 163–164.
12. L. Smith, *On realizing complex bordism modules*, Applications to the Homotopy of Spheres, Amer. J. Math. **92** (1970), 793–856.
13. A. S. Svarc, *The genus of a fibre space*, Amer. Math. Soc. Transl. **55** (1966), 49–140.

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