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$\mathrm{Aut}(F) \to \mathrm{Aut}(F/F'')$ IS SURJECTIVE FOR FREE GROUP F OF Rank $\geqslant 4$

BY

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ABSTRACT. In this article, it is shown that the group of automorphisms of the free metabelian group $\Phi(n)$ of rank $n \geqslant 4$ is not only finitely generated but in fact every automorphism of $\Phi(n)$ is induced by an automorphism of the free group of the same rank n. This contrasts sharply with the authors' earlier result [4] that any set of generators of the group of automorphisms of the free metabelian group $\Phi(3)$ of rank 3 contains infinitely many automorphisms which are not induced by an automorphism of the free group of rank 3.

0. Introduction. Contrary to original expectations, we prove the following result (which has been announced elsewhere [**2**]):

THEOREM A. *For $n \geqslant 4$, each automorphism of the free metabelian group $\Phi(n) = F(n)/F(n)''$ of rank n is induced by an automorphism of the free group $F(n)$ of rank n.*

As a corollary we have

COROLLARY TO THEOREM A. *For $n \geqslant 4$, the group $\mathrm{Aut}(\Phi(n))$ of all automorphisms of $\Phi(n)$ is finitely generated.*

Thus, the structure of $\mathrm{Aut}(\Phi(n))$, $n \geqslant 4$, is radically different from that of $\mathrm{Aut}(\Phi(3))$. However, the study of matrix groups over polynomial rings suggested the possibility of such contrasting behavior as a naturally expected pattern. To put matters in proper perspective and to present a better understanding of Theorem A, we state two theorems. The first theorem presages the second, and the second theorem is a widening of Theorem A to incorporate the previously known results for $\mathrm{Aut}(\Phi(2))$ and $\mathrm{Aut}(\Phi(3))$.

THEOREM 0.1. *Let $s_1,\ldots,s_r$ $(r \geqslant 2)$ be commuting indeterminates over $\mathbf{Z}$, the ring of integers.*

(a) $\mathrm{GL}_1(\mathbf{Z}[s_1, s_1^{-1},\ldots,s_r, s_r^{-1}]) = \pm A(r)$, *where $A(r)$ is the multiplicative group generated by $s_1,\ldots,s_r$, i.e.,* $\mathrm{GL}_1(\mathbf{Z}[s_1, s_1^{-1},\ldots,s_r, s_r^{-1}])$ *has only the trivial or obvious units.*

(b) [**3**] *Any generating set for* $\mathrm{GL}_2(\mathbf{Z}[s_1, s_1^{-1},\ldots,s_r, s_r^{-1}])$ *contains infinitely many elements which are not in the subgroup* $\mathrm{GE}_2(\mathbf{Z}[s_1, s_1^{-1},\ldots,s_r, s_r^{-1}])$ *generated by the set of all elementary 2×2 matrices and all invertible diagonal 2×2 matrices.*

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(c) (*Suslin* [7]) *For* $m \geqslant 3$, $\mathrm{GL}_m(\mathbf{Z}[s_1, s_1^{-1}, \ldots, s_r, s_r^{-1}])$ *is generated by the set of all elementary* $(m \times m)$ *matrices and all invertible diagonal* $(m \times m)$ *matrices, i.e.*,

$$\mathrm{GL}_m\left(\mathbf{Z}\left[s_1, s_1^{-1}, \ldots, s_r, s_r^{-1}\right]\right) = \mathrm{GE}_m\left(\mathbf{Z}\left[s_1, s_1^{-1}, \ldots, s_r, s_r^{-1}\right]\right).$$

THEOREM 0.2. (a) [1] $\mathrm{Aut}(\Phi(2))$ *is an extension of the group of inner automorphisms of* $\Phi(2)$ *by* $\mathrm{GL}_2(\mathbf{Z})$. *Consequently, each automorphism of* $\Phi(2)$ *is induced by an automorphism of* $F(2)$, *and* $\mathrm{Aut}(\Phi(2))$ *contains only the trivial or obvious elements.*

(b) [4] *Any generating set for* $\mathrm{Aut}(\Phi(3))$ *contains infinitely many automorphisms which are not induced by automorphisms of* $F(3)$.

(c) *For* $n \geqslant 4$, *each automorphism of* $\Phi(n)$ *is induced by an automorphism of* $F(n)$.

Thus for $n \geqslant 2$, $\mathrm{GL}_{n-1}(\mathbf{Z}[s_1, s_1^{-1}, \ldots, s_r, s_r^{-1}])$ and $\mathrm{Aut}(\Phi(n))$ play analogous roles as do the elements of $\mathrm{GE}_{n-1}(\mathbf{Z}[s_1, s_1^{-1}, \ldots, s_r, s_r^{-1}])$ and the induced automorphisms of $\Phi(n)$.

Theorems 0.1 and 0.2 motivate the problem of discovering other settings in which the same type of theorem holds. Intriguing candidates are the automorphism groups of the free groups in the variety of abelian-by-nilpotent groups or the variety of solvable groups, the automorphism groups of free algebras over a field, and the automorphism groups of polynomial rings over a field. This last setting is perhaps the most promising because of its close analogy to the groups in Theorem 0.2. We refer to [5] for a more detailed discussion of this case.

The organization of this paper is as follows: §1 contains a table of notation and background material about the relevant automorphism groups—mainly several theorems of Magnus. §2 contains an outline of the proof of Theorem B (Theorem B is a reformulation of Theorem A). §3 contains results to the effect that various automorphisms of $\Phi(n)$ of special types are induced by automorphisms of $F(n)$. Some important results of Suslin are also in this section. §4 is devoted entirely to one lemma—appropriately called the Main Lemma. It is a technical lemma which again states that automorphisms of $\Phi(n)$ of a special form are induced by automorphisms of $F(n)$. The completion of the proof of Theorem B is in the final §5.

Our proofs of the Main Lemma and all the numbered propositions from §2 onward (with the sole exception of Proposition 3.1) require the hypothesis that the rank $n \geqslant 4$. The results in [4] assure the falsity of almost all, if not all, of these results if $n < 4$.

1. Notation and background.

$F(n)$ = free group of rank n.

$\Phi(n) = F(n)/F(n)''$ = free metabelian group of rank n with basis $x_1, \ldots, x_n$.

$A(n) = F(n)/F(n)' = \Phi(n)/\Phi(n)'$ = free abelian group of rank n with basis $s_1, \ldots, s_n$, where $s_i = x_i\Phi(n)'$.

$\mathrm{Aut}(G)$ = group of all automorphisms of the group G.

IA-$\mathrm{Aut}(G)$ = group of all automorphisms of G which induce the identity map on G/G' = group of all IA-automorphisms of G.

$\mathbf{Z}[A(n)] = \mathbf{Z}[s_1, s_1^{-1}, \ldots, s_n, s_n^{-1}]$ = integral group ring of $A(n)$.

$\sigma_j = s_j - 1$, $1 \leqslant j \leqslant n$.

$\mathrm{GL}_n(R)$ = group of all invertible $n \times n$ matrices over the commutative ring R.

$\mathrm{SL}_n(R)$ = subgroup of $\mathrm{GL}_n(R)$ of all matrices with determinant 1.

$E_{ij}(n) = n \times n$ matrix with 1 in (i, j) position and zeros elsewhere.

$[I_n + aE_{ij}]$, $i \neq j$ and $a \in R$, is called an elementary matrix over R.

$\mathrm{E}_n(R)$ = subgroup of $\mathrm{SL}_n(R)$ generated by all elementary matrices over R.

$\vec{e}_i$ = n-tuple with 1 in the ith coordinate and zeros elsewhere (the value n will be clear from the context).

$\vec{u}^t$ = transpose of n-tuple $\vec{u}$ when viewed as a $1 \times n$ matrix.

Unimodular row over R = n-tuple with entries in the ring R which generate R as a module.

$\mathscr{A}(n)$ = the subgroup of $\mathrm{GL}_n(\mathbf{Z}[A(n)])$ of all matrices (a_{ij}) which represent IA-automorphisms of $\Phi(n)$, or, equivalently, which satisfy the conditions $\sum_{i=1}^{n}\sigma_i a_{ij} = \sigma_j$, $1 \leqslant j \leqslant n$.

$\mathscr{B}(n)$ = the subgroup of $\mathscr{A}(n)$ which represent IA-automorphisms of $\Phi(n)$ induced by IA-automorphisms of $F(n)$.

$F_{ijk}(n)$, $i < j$ and i, j, k distinct, = the $n \times n$ matrix with σ_j in the (i, k) position, $-\sigma_i$ in the (j, k) position and zeros elsewhere.

$F_{ijj}(n)$, $i < j$, = the $n \times n$ matrix with $-\sigma_j$ in the (i, j) position, σ_i in the (j, j) position and zeros elsewhere.

$F_{iji}(n)$, $i < j$, = the $n \times n$ matrix with σ_j in the (i, i) position, $-\sigma_i$ in the (j, i) position and zeros elsewhere.

Let I be an ideal of the commutative ring R. Then, the natural map $R \to R/I$ induces natural maps $\mathrm{GL}_n(R) \to \mathrm{GL}_n(R/I)$ and $\mathrm{SL}_n(R) \to \mathrm{SL}_n(R/I)$.

$\mathrm{GL}_n(R, I) = \mathrm{Kernel}\{\mathrm{GL}_n(R) \to \mathrm{GL}_n(R/I)\}$.

$\mathrm{SL}_n(R, I) = \mathrm{Kernel}\{\mathrm{SL}_n(R) \to \mathrm{SL}_n(R/I)\}$.

$\mathrm{E}_n(R, I)$ = normal closure in $\mathrm{E}_n(R)$ of $\{I_n + aE_{ij}(n) \colon i \neq j, a \in I\}$.

Induced matrix (c_{ij}) = matrix in $\mathrm{GL}_{n-1}(\mathbf{Z}[A(n)], \sigma_n\mathbf{Z}[A(n)])$ which can be completed to a matrix in $\mathscr{B}(n)$ of the form

$$\begin{pmatrix} (c_{ij}) & 0 \\ * & 1 \end{pmatrix}.$$

Let $t_1, \dots, t_n$ be indeterminates over $\mathbf{Z}[A(n)]$, and let $\mathscr{M}(n)$ denote the group of matrices generated by

$$\begin{pmatrix} s_i & t_i \\ 0 & 1 \end{pmatrix}, \qquad 1 \leqslant i \leqslant n.$$

$\mathscr{M}(n)$ consists of all matrices

$$\begin{pmatrix} g & a_1 t_1 + \cdots + a_n t_n \\ 0 & 1 \end{pmatrix}$$

such that $g \in A(n)$, $a_i \in \mathbf{Z}[A(n)]$ and $a_1\sigma_1 + \cdots + a_n\sigma_n = (g - 1)$. There is a well-known faithful representation [7] of $\Phi(n)$ *onto* $\mathscr{M}(n)$, called the Magnus representation, which is given by

$$x_i \to \begin{pmatrix} s_i & t_i \\ 0 & 1 \end{pmatrix}, \qquad 1 \leqslant i \leqslant n.$$

Henceforth, we shall identify $\Phi(n)$ *with* $\mathscr{M}(n)$.

Let $\mathscr{A}(n)$ denote the subgroup of $\mathrm{GL}_n(\mathbf{Z}[A(n)])$ of all matrices (a_{ij}) such that

$$\sum_{i=1}^{n} \sigma_i a_{ij} = \sigma_j, \qquad 1 \leqslant j \leqslant n. \tag{1.1}$$

Each α in $IA\text{-Aut}(\Phi(n))$ is uniquely defined by a map

$$\begin{pmatrix} s_j & t_j \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} s_j & a_{1j}t_1 + \cdots + a_{nj}t_n \\ 0 & 1 \end{pmatrix}, \qquad 1 \leqslant j \leqslant n.$$

Then, the map $\alpha \to (a_{ij})$ defines a faithful representation of $IA\text{-Aut}(\Phi(n))$ *onto* $\mathscr{A}(n)$. *Henceforth, we shall also identify* $IA\text{-Aut}(\Phi(n))$ *with* $\mathscr{A}(n)$.

The natural map $F(n) \to \Phi(n) = \mathscr{M}(n)$ induces natural maps $\mathrm{Aut}(F(n)) \to \mathrm{Aut}(\Phi(n))$ and $IA\text{-Aut}(F(n)) \to IA\text{-Aut}(\Phi(n))$. We shall prove

THEOREM B. *If $n \geqslant 4$, then the natural map $IA\text{-Aut}(F(n)) \to IA\text{-Aut}(\Phi(n))$ is surjective, i.e., each IA-automorphism of $\Phi(n)$ is induced by an IA-automorphism of $F(n)$.*

As an immediate corollary to Theorem B we have the following restatement of Theorem A.

THEOREM A′. *If $n \geqslant 4$, then the natural map $\mathrm{Aut}(F(n)) \to \mathrm{Aut}(\Phi(n))$ is surjective, i.e., each automorphism of $\Phi(n)$ is induced by an autormorphism of $F(n)$.*

PROOF OF THEOREM A′. We assume Theorem B. Then, we have the commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
0 & \to & IA\text{-Aut}(F(n)) & \to & \mathrm{Aut}\, F(n) & \to & \mathrm{Aut}(A(n)) \to 1 \\
 & & \downarrow & & \downarrow & & \| \\
0 & \to & IA\text{-Aut}(\Phi(n)) & \to & \mathrm{Aut}\,(\Phi(n)) & \to & \mathrm{Aut}(A(n)) \to 1 \\
 & & \downarrow & & & & \downarrow \\
 & & 0 & & & & 0
\end{array}$$

Theorem B easily follows. □

Let $\mathscr{B}(n)$ denote the image of the map $IA\text{-Aut}(F(n)) \to IA\text{-Aut}(\Phi(n)) = \mathscr{A}(n)$. Magnus [6] gave a set of generators for $IA\text{-Aut}(F(n))$. We shall give the corresponding set of generators of $\mathscr{B}(n)$.

PROPOSITION 1.1. *$\mathscr{B}(n)$ is generated by the set of all matrices of the following types: $I_n + F_{ijk}(n)$, $i < j$ and i, j, k distinct; $I_n + F_{ijj}(n)$, $i < j$; and $I_n + F_{iji}(n)$, $i < j$.* □

2. Outline of proof of Theorem B. *For the remainder of this paper*, $n \geqslant 4$.

Let $\alpha = I_n + (a_{ij}) \in \mathscr{A}(n)$. There is ν, $1 \leqslant \nu \leqslant n$, such that

$$\alpha \in \mathrm{GL}_n\left(\mathbf{Z}[A(n)],\ \sum_{j=\nu}^{n} \sigma_j \mathbf{Z}[A(n)]\right).$$

If $\nu < n$, we shall show that there is

$$\beta_\nu \in \mathscr{B}(n) \cap \mathrm{GL}_n\left(\mathbf{Z}[A(n)],\ \sum_{j=\nu}^{n} \sigma_j \mathbf{Z}[A(n)]\right)$$

such that $\beta_\nu^{-1}\alpha \in \mathrm{GL}_n(\mathbf{Z}[A(n)], \Sigma_{j=\nu+1}^n \sigma_j \mathbf{Z}[A(n)])$ (Proposition 5.2), and if $\nu = n$, then $\alpha \in \mathscr{B}(n)$. Thus, if $\nu < n$, there exist $\beta_\nu, \beta_{\nu+1}, \ldots, \beta_n \in \mathscr{B}(n)$ such that $\beta_n^{-1} \cdots \beta_{\nu+1}^{-1}\beta_\nu^{-1}\alpha \in \mathscr{B}(n)$, whence $\alpha \in \mathscr{B}(n)$ and Theorem B is proved.

The outline of the procedure for producing the β_ν is as follows. If $\nu < n$, we set $\sigma_{\nu+1} = \cdots = \sigma_n = 0$ in α to obtain

$$\bar{\alpha} = \left[I_n + (\bar{a}_{ij})\right] \in \mathrm{GL}_n(\mathbf{Z}[A(\nu-1)], \sigma_\nu \mathbf{Z}[A(\nu-1)]).$$

It is easy to show that there are $\Upsilon_\nu, \delta_\nu \in \mathscr{B}(n) \cap \mathrm{GL}_n(\mathbf{Z}[A(n)], \Sigma_{j=\nu}^n \sigma_j \mathbf{Z}[A(n)])$ such that

$$\bar{\Upsilon}_\nu^{-1}\bar{\alpha} = \left[I_n + (\bar{a}'_{ij})\right] \in \mathrm{GL}_n(\mathbf{Z}[A(\nu-1)], \sigma_\nu \mathbf{Z}[A(\nu-1)])$$

with $\bar{a}'_{i\nu} \in \sigma_\nu^2 \mathbf{Z}[A(\nu-1)]$, and

$$\bar{\delta}_\nu^{-1}\bar{\Upsilon}_\nu^{-1}\bar{\alpha} = I_n + \begin{pmatrix} \sigma_\nu d_{11} & \cdots & \sigma_\nu d_{1(\nu-1)} & 0 & \sigma_\nu d_{1(\nu+1)} & \cdots & \sigma_\nu d_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \sigma_\nu d_{n1} & \cdots & \sigma_\nu d_{n(\nu-1)} & 0 & \sigma_\nu d_{n(\nu+1)} & \cdots & \sigma_\nu d_{nn} \end{pmatrix}$$

where $\bar{\Upsilon}_\nu$ and $\bar{\delta}_\nu$ are the results of setting $\sigma_{\nu+1} = \cdots = \sigma_n = 0$ in Υ_ν and δ_ν, respectively.

In order to show that there is $\varepsilon_\nu \in \mathscr{B}(n) \cap \mathrm{GL}_n(\mathbf{Z}[A(n)], \Sigma_{j=\nu}^n \sigma_j \mathbf{Z}[A(n)])$ such that when we set $\sigma_{\nu+1} = \cdots = \sigma_n = 0$ in ε_ν we obtain $\bar{\varepsilon}_\nu = \bar{\delta}_\nu^{-1}\bar{\Upsilon}_\nu^{-1}\bar{\alpha}$, we need the

MAIN LEMMA. *Any element of $\mathscr{A}(n)$ of the form*

$$I_n + \begin{pmatrix} \sigma_n b_{11} & \cdots & \sigma_n b_{1(n-1)} & 0 \\ \vdots & & \vdots & \vdots \\ \sigma_n b_{(n-1)1} & \cdots & \sigma_n b_{(n-1)(n-1)} & 0 \\ -\sum_{i=1}^{n-1} \sigma_i b_{i1} & \cdots & -\sum_{i=1}^{n-1} \sigma_i b_{i(n-1)} & 0 \end{pmatrix}$$

is contained in $\mathscr{B}(n)$.

Let $\beta_\nu = \Upsilon_\nu \delta_\nu \varepsilon_\nu \in \mathscr{B}(n)$. If we set $\sigma_{\nu+1} = \cdots = \sigma_n = 0$ in $\beta_\nu^{-1}\alpha$, we obtain $\bar{\beta}_\nu^{-1}\bar{\alpha} = I_n$, whence we can conclude that $\beta_\nu^{-1}\alpha \in \mathscr{A}(n) \cap \mathrm{GL}_n(\mathbf{Z}[A(n)], \Sigma_{j=\nu+1}^n \sigma_j \mathbf{Z}[A(n)])$.

3. Preliminary results. The reader is reminded of the assumption that $n \geqslant 4$. Although Proposition 3.1 is valid for $n \geqslant 2$, the remaining propositions need the hypothesis that $n \geqslant 4$.

The first result will give information about the entries of a matrix in $\mathscr{A}(n)$. $\vec{e}_i$ denotes the n-tuple with 1 in the ith coordinate and zeros elsewhere.

PROPOSITION 3.1. *If $a_1, \ldots, a_n \in \mathbf{Z}[A(n)]$ and $\sigma_1 a_1 + \cdots + \sigma_n a_n = 0$, then the n-tuple $(a_1, \ldots, a_n)$ is a linear combination over $\mathbf{Z}[A(n)]$ of the n-tuples $\sigma_j \vec{e}_i - \sigma_i \vec{e}_j$, $i < j$.*

PROOF. When we set all $\sigma_k = 0$, $k \neq i$, in the equation $\sigma_1 a_1 + \cdots + \sigma_n a_n = 0$, we obtain $\sigma_i \hat{a}_i = 0$, where $\hat{a}_i \in \mathbf{Z}[s_i, s_i^{-1}]$ is the result of setting all $\sigma_k = 0$, $k \neq i$, in a_i.

Since $\mathbf{Z}[s_i, s_i^{-1}]$ is an integral domain, we conclude that $\hat{a}_i = 0$, i.e., $a_i \in \Sigma_{k \neq i} \sigma_k \mathbf{Z}[A(n)]$.

We use induction on n, the case $n = 2$ being straightforward. We write

$$a_n = \sigma_1 a_{1n} + \cdots + \sigma_{n-1} a_{(n-1)n} \tag{3.1}$$

where $a_{in} \in \mathbf{Z}[A(n)]$, and

$$(a_1, \ldots, a_n) + \sum_{i=1}^{n-1} a_{in}(\sigma_n \vec{e}_i - \sigma_i \vec{e}_n) = (a_1 + \sigma_n a_{1n}, \ldots, a_{n-1} + \sigma_n a_{(n-1)n}, 0).$$

From (3.1), $\sigma_1(a_1 + \sigma_n a_{1n}) + \cdots + \sigma_{n-1}(a_{n-1} + \sigma_n a_{(n-1)n}) = 0$. By induction, $(a_1 + \sigma_n a_{1n}, \ldots, a_{n-1} + \sigma_n a_{(n-1)n})$ is a linear combination over $\mathbf{Z}[A(n)]$ of the n-tuples $\sigma_j \vec{e}_i - \sigma_i \vec{e}_j$, $1 \leqslant i < j \leqslant n - 1$, and we are done. □

The next three propositions give us three classes of matrices which are contained in $\mathscr{B}(n)$. If $\vec{u}$ is a n-tuple over a ring R, then $\vec{u}^t$ will denote the transpose of $\vec{u}$, where $\vec{u}$ is regarded as a $1 \times n$ matrix over R.

PROPOSITION 3.2. *If $a \in \mathbf{Z}[A(n)]$ and i, j, k are distinct, then $[I_n + aF_{ijk}(n)]$ is contained in $\mathscr{B}(n)$.*

PROOF. Since $(F_{ijk}(n))^2 = 0$, $[I_n + aF_{ijk}(n)]$ is additive in a, and therefore it is sufficent to prove the proposition when $a \in A(n)$.

Let $(a_{\mu\nu})$ be an invertible $n \times n$ matrix with inverse $(b_{\mu\nu})$. Then

$$(a_{\mu\nu})\left[I_n + aF_{ijk}(n)\right](b_{\mu\nu}) = \left[I_n + a\begin{pmatrix} a_{1i}\sigma_j - a_{1j}\sigma_i \\ \vdots \\ a_{ni}\sigma_j - a_{nj}\sigma_i \end{pmatrix}^{(b_{k1} \cdots b_{kn})}\right].$$

We assume $[I_n + aF_{ijk}(n)] \in \mathscr{B}(n)$ and shall show that $[I_n + as_l^{\pm 1}F_{ijk}(n)] \in \mathscr{B}(n)$ for any l.

If we let $(b_{\mu\nu}) = [I_n + F_{lkk}(n)]^{\pm 1}$ when $l < k$ or $(b_{\mu\nu}) = [I_n + F_{klk}(n)]^{\pm 1}$ when $l > k$, then

$$(a_{\mu\nu})\left[I_n + aF_{ijk}(n)\right](b_{\mu\nu}) = \left[I_n + as_l^{\pm 1}F_{ijk}(n)\right] \in \mathscr{B}(n).$$

We will be done if we show that

$$\left[I_n + as_k^{\pm 1}F_{ijk}(n)\right] \in \mathscr{B}(n).$$

Since $n \geqslant 4$, there is $m \neq i, j, k$. Let $(b_{\mu\nu}) = [I_n + F_{mkk}(n)]$ if $m < k$ or $(b_{\mu\nu}) = [I_n + F_{kmk}(n)]$ if $m > k$. We may assume $[I_n + aF_{ijm}]$ and $[I_n + as_k^{-1}F_{ijm}]$ are in $\mathscr{B}(n)$. Then,

$$(a_{\mu\nu})\left[I_n + aF_{ijm}(n)\right](b_{\mu\nu})\left[I_n - aF_{ijm}(n)\right]\left[I_n + aF_{ijk}\right] = \left[I_n + as_kF_{ijk}\right] \in \mathscr{B}(n),$$

and

$$\begin{aligned}(a_{\mu\nu})\left[I_n - as_k^{-1}F_{ijm}(n)\right](b_{\mu\nu})\left[I_n + as_k^{-1}F_{ijm}\right]\left[I_n + aF_{ijk}\right] \\ = \left[I_n + as_k^{-1}F_{ijk}\right] \in \mathscr{B}(n). \quad \square\end{aligned}$$

PROPOSITION 3.3. *Any $\alpha \in \mathscr{A}(n)$ with exactly one nontrivial column is contained in $\mathscr{B}(n)$.*

PROOF. By symmetry we may assume the last column is nontrivial. Let

$$\alpha = \begin{pmatrix} & & & \sigma_2 f_{12} + \sigma_3 f_{13} + \cdots + \sigma_n f_{1n} \\ & & & -\sigma_1 f_{12} + \sigma_3 f_{23} + \cdots + \sigma_n f_{2n} \\ & I_{n-1} & & \vdots \\ & & & -\sigma_1 f_{1(n-1)} - \sigma_2 f_{2(n-1)} - \cdots - \sigma_n f_{(n-1)n} \\ 0 & \cdots & 0 & 1 - \sigma_1 f_{1n} - \sigma_2 f_{2n} - \cdots - \sigma_{n-1} f_{(n-1)n} \end{pmatrix},$$

where the form of the last column of α follows from Proposition 3.1.

$\det \alpha \in \pm A(n)$. If we set $\sigma_1 = \cdots = \sigma_{n-1} = 0$, then we obtain a matrix of determinant 1. Thus, it is clear that $\det \alpha \in A(n-1)$. Each $[I_n + F_{inn}(n)]$ is contained in $\mathscr{B}(n)$ and has determinant s_i. By multiplying α by a suitable product of the $[I_n + F_{inn}(n)]$ if necessary, we may assume $\det \alpha = 1$. Thus,

$$\alpha = \begin{pmatrix} & & & \sigma_2 f_{12} + \sigma_3 f_{13} + \cdots + \sigma_{n-1} f_{1(n-1)} \\ & & & -\sigma_1 f_{12} + \sigma_3 f_{23} + \cdots + \sigma_{n-1} f_{2(n-1)} \\ & I_{n-1} & & \vdots \\ & & & -\sigma_1 f_{1(n-1)} - \sigma_2 f_{2(n-1)} - \cdots - \sigma_{n-2} f_{(n-2)(n-1)} \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$
$$= \prod_{1 \leq i < j \leq n-1} \left[I_n + f_{ij} F_{ijn}(n) \right].$$

Since all the factors are in $\mathscr{B}(n)$ by Proposition 3.2, α is in $\mathscr{B}(n)$. □

PROPOSITION 3.4. *The following elements of $\mathscr{A}(n)$*

$$\begin{pmatrix} 1 + \sigma_1\sigma_2 f & \sigma_2^2 f & \\ & & 0 \\ -\sigma_1^2 f & 1 - \sigma_1\sigma_2 f & \\ & 0 & I_{n-2} \end{pmatrix}, \begin{pmatrix} 1 + \sigma_1\sigma_3 f & 0 & \sigma_3^2 f & \\ 0 & 1 & 0 & 0 \\ -\sigma_1^2 f & 0 & 1 - \sigma_1\sigma_3 f & \\ & 0 & & I_{n-3} \end{pmatrix},$$
$$\ldots, \begin{pmatrix} 1 + \sigma_1\sigma_n f & 0 & \sigma_n^2 f \\ 0 & I_{n-2} & 0 \\ -\sigma_1^2 f & 0 & 1 - \sigma_1\sigma_n f \end{pmatrix},$$

where $f \in \mathbf{Z}[A(n)]$, are all in $\mathscr{B}(n)$.

PROOF. By symmetry it is sufficient to prove the proposition for the first matrix listed above.

A straightforward calculation shows that

$$\begin{pmatrix} 1+\sigma_1\sigma_2 f & \sigma_2^2 f & \\ & & 0 \\ -\sigma_1^2 f & 1-\sigma_1\sigma_2 f & \\ & 0 & & I_{n-2} \end{pmatrix}$$

$$= \begin{pmatrix} & & -\sigma_n\sigma_2 f \\ & & \sigma_n\sigma_1 f \\ I_{n-1} & & \\ & & 0 \\ & & \vdots \\ & & 0 \\ 0 & \cdots\ 0 & 1 \end{pmatrix} \begin{pmatrix} s_n & 0 & \\ & & 0 \\ 0 & s_n & \\ & 0 & I_{n-2} \\ -\sigma_1 & -\sigma_2 & \end{pmatrix} \begin{pmatrix} & & \sigma_2 f \\ & & -\sigma_1 f \\ I_{n-1} & & \\ & & 0 \\ & & \vdots \\ & & 0 \\ 0 & \cdots\ 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} s_n^{-1} & 0 & \\ & & 0 \\ 0 & s_n^{-1} & \\ & 0 & \\ & & I_{n-2} \\ s_n^{-1}\sigma_1 & s_n^{-1}\sigma_2 & \end{pmatrix} \begin{pmatrix} & & -\sigma_2 f \\ & & \sigma_1 f \\ I_{n-1} & & \\ & & 0 \\ & & \vdots \\ & & 0 \\ 0 & \cdots\ 0 & 1 \end{pmatrix}.$$

Each factor of the product is in $\mathscr{B}(n)$, whence the product is in $\mathscr{B}(n)$. □

The next proposition is a minor extension of an important result of Suslin [**8**] which is partially based on the work of Vaserstein [**9**]. A *unimodular row* over a ring R is a k-tuple over R whose coordinates generate R as an R-module.

PROPOSITION 3.5. *Let* $\mathbf{Z}[A(m)] = \mathbf{Z}[s_1, s_1^{-1}, \ldots, s_m, s_m^{-1}]$, *let* $k \geqslant 3$, *and let* I *denote the ideal* $\sigma_m \mathbf{Z}[A(m)]$ *of* $\mathbf{Z}[A(m)]$.

(a) $\mathrm{SL}_k(\mathbf{Z}[A(m)], I) = \mathrm{E}_k(\mathbf{Z}[A(m)], I)$.

(b) $\mathrm{E}_k(\mathbf{Z}[A(m)], I)$ *is generated by the set of all matrices of the form*

$$\left[I_k + h\vec{u}^t \cdot \left(u_j\vec{e}_i - u_i\vec{e}_j\right)\right]$$

where $h \in I$, $\vec{u} = (u_1, \ldots, u_k)$ *is a unimodular row over* $\mathbf{Z}[A(m)]$, *and* $1 \leqslant i < j \leqslant m$.

REMARK.

$$\left[I_k + h\vec{u}^t \cdot \left(u_j\vec{e}_i - u_i\vec{e}_j\right)\right] = \left[I_k + h\left(u_i\vec{e}_i + u_j\vec{e}_j\right)^t \cdot \left(u_j\vec{e}_i - u_i\vec{e}_j\right)\right]$$
$$\cdot \prod_{l \neq i, j}\left[I_k + h\left(u_l\vec{e}_l\right)^t u_j e_i\right] \cdot \prod_{l \neq i, j}\left[I_k - h\left(u_l\vec{e}_l\right)^t u_i\vec{e}_j\right].$$

Therefore, $\mathrm{E}_k(\mathbf{Z}[A(m)], I)$ is generated by the set of all matrices of the form $[I_k + h(f\vec{e}_i + g\vec{e}_j)^t \cdot (g\vec{e}_i - f\vec{e}_j)]$ where $i < j$, $h \in I$, and $f, g \in \mathbf{Z}[A(m)]$.

PROOF. Part (b) of the proposition is [**8**, Corollary 1.4]. For the proof of part (a) we use the fact (Corollary 7.10 of [**8**]) that, for $k \geqslant 3$, $\mathrm{SL}_k(\mathbf{Z}[A(m)]) = \mathrm{E}_k(\mathbf{Z}[A(m)])$.

For $f \in \mathbf{Z}[A(m)]$,

$$(*) \qquad f = g\sigma_m + h$$

where $g \in \mathbf{Z}[A(m)]$ and $h \in \mathbf{Z}[A(m-1)]$. Let $\alpha \in \mathrm{SL}_k(\mathbf{Z}[A(m)], I)$, and assume $\alpha = \varepsilon_1\varepsilon_2 \cdots \varepsilon_t$ for suitable $\varepsilon_i \in \mathrm{E}_k(\mathbf{Z}[A(m)])$. Because of the decomposition $(*)$, we can write $\varepsilon_i = \gamma_i\delta_i$ where $\gamma_i \in \mathrm{E}_k(\mathbf{Z}[A(m)], I)$ and $\delta_i \in \mathrm{E}_k(\mathbf{Z}[A(m-1)])$. Hence $\gamma_1^{-1}\alpha = \delta_1\varepsilon_2 \cdots \varepsilon_t \in \mathrm{SL}_k(\mathbf{Z}[A(m)], I)$. Since $\mathrm{SL}_k(\mathbf{Z}[A(m)], I)$ is a normal subgroup of $\mathrm{SL}_k(\mathbf{Z}[A(m)])$, we may conjugate $\gamma_1^{-1}\alpha$ by δ_1 to conclude that $\varepsilon_2 \cdots \varepsilon_t\delta_1 \in \mathrm{SL}_k(\mathbf{Z}[A(m)], I)$. Continuing in this manner, we obtain the result that $\delta_1 \cdots \delta_t \in \mathrm{SL}_k(\mathbf{Z}[A(m)], I)$. But, $\delta_1 \cdots \delta_t \in \mathrm{SL}_k(\mathbf{Z}[A(m-1)])$, whence $\delta_1 \cdots \delta_t = I_k$.

This last fact enables us to write

$$\alpha = \gamma_1(\delta_1\gamma_2\delta_1^{-1})(\delta_1\delta_2\gamma_3\delta_2^{-1}\delta_1^{-1}) \cdots (\delta_1 \cdots \delta_{t-1}\gamma_t\delta_{t-1}^{-1} \cdots \delta_1^{-1})(\delta_1 \cdots \delta_{t-1}\delta_t)$$

where each

$$(\delta_1 \cdots \delta_{i-1}\gamma_i\delta_{i-1}^{-1} \cdots \delta_1^{-1}) \in \mathrm{E}_k(\mathbf{Z}[A(m)], I).$$

Therefore, $\alpha \in \mathrm{E}_k(\mathbf{Z}[A(m)], I)$. □

Although our interest is mainly in the case $m = k$, we remark that the proof shows that Proposition 3.5 is valid for any $m \geqslant 1$ and $k \geqslant 3$ if one interprets $\mathbf{Z}[A(m)]$ to be $\mathbf{Z}$ when $m = 0$.

4. Proof of the Main Lemma. As the title of this section indicates, the purpose is to prove the

MAIN LEMMA. *Let $n \geqslant 4$. Any element of $\mathscr{A}(n)$ of the form*

$$(4.1) \qquad I_n + \begin{pmatrix} \sigma_n b_{11} & \cdots & \sigma_n b_1(n-1) & 0 \\ \vdots & & \vdots & \vdots \\ \sigma_n b_{(n-1)1} & \cdots & \sigma_n b_{(n-1)(n-1)} & 0 \\ -\sum_{i=1}^{n-1} \sigma_i b_{i1} & \cdots & -\sum_{i=1}^{n-1} \sigma_i b_{i(n-1)} & 0 \end{pmatrix}$$

is contained in $\mathscr{B}(n)$.

The set of elements of the form (4.1) is a subgroup of $\mathscr{A}(n)$, and we have a homomorphism of this subgroup onto $\mathrm{GL}_{n-1}(\mathbf{Z}[A(n)], \sigma_n\mathbf{Z}[A(n)])$ which maps the matrix (4.1) onto the matrix

$$(4.2) \qquad [I_{n-1} + (\sigma_n b_{ij})].$$

Rather than work with the larger matrices (4.1), we will deal with the matrices (4.2) and make the following definition.

An invertible $(n-1) \times (n-1)$ matrix (c_{ij}) will be called *induced* if it can be completed to a matrix in $\mathscr{B}(n)$ of the form

$$\begin{pmatrix} (c_{ij}) & 0 \\ * & 1 \end{pmatrix}.$$

(Note that the last column is trivial.)

We can now restate the Main Lemma as

MAIN LEMMA′. *Any matrix in* $\mathrm{GL}_{n-1}(\mathbf{Z}[A(n)], \sigma_n\mathbf{Z}[A(n)])$ *is induced.*

PROOF. By the Remark after Proposition 3.5, $\mathrm{GL}_{n-1}(\mathbf{Z}[A(n)], \sigma_n\mathbf{Z}[A(n)])$ is generated by the set of all matrices of the form

$$\left[I_{n-1} + \sigma_n h(f\vec{e}_i + g\vec{e}_j)^t \cdot (g\vec{e}_i - f\vec{e}_j)\right] \tag{4.3}$$

where $i < j$ and $f, g, h \in \mathbf{Z}[A(n)]$. The proof that these generators (4.3) are induced is quite tedious and lengthy and will be done in several stages.

For the sake of simplicity as well as clarity we will restrict ourselves to the case $n = 4$. As the proof proceeds, it will become clear that the case $n > 4$ is the same in principle as $n = 4$. The point is that the general matrix in $\mathrm{GL}_{n-1}(\mathbf{Z}[A(n)])$ of the form (4.3) involves only two distinct indices $1 \leqslant i, j \leqslant (n-1)$. Hence, the case $n = 4$ already illustrates all the essential possibilities which occur. By using $n = 4$, we avoid matrices with blocks of zeros.

We first change from the consideration of matrices of form (4.3) to the consideration of matrices of a simpler form. (See (4.6) below.) To see how we arrive at this simpler form, we consider a matrix of form (4.3) with $i = 1, j = 2$.

$$\left[I_3 + \sigma_4 h(f\vec{e}_1 + g\vec{e}_2)^t \cdot (g\vec{e}_1 - f\vec{e}_2)\right] = \begin{pmatrix} 1 + \sigma_4 hfg & -\sigma_4 hf^2 & 0 \\ \sigma_4 hg^2 & 1 - \sigma_4 hfg & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g & -f & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\sigma_4 hf \\ 0 & 1 & -\sigma_4 hg \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -g & f & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \sigma_4 hf \\ 0 & 1 & \sigma_4 hg \\ 0 & 0 & 1 \end{pmatrix}.$$

The last factor is certainly induced, being the image of the element

$$\begin{pmatrix} 1 & 0 & \sigma_4 hf & 0 \\ 0 & 1 & \sigma_4 hg & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\sigma_1 hf - \sigma_2 hg & 1 \end{pmatrix},$$

which is in $\mathscr{B}(n)$ by Proposition 3.3. (Clearly, any matrix in $\mathrm{GL}_3(\mathbf{Z}[A(4)], \sigma_4\mathbf{Z}[A(4)])$ with only one nontrivial row or column must be induced.) Thus, we must prove that both

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g & -f & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\sigma_4 hf \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -g & f & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\sigma_4 hf^2 & 0 \\ 0 & 1 & 0 \\ 0 & -\sigma_4 hf^2 g & 1 \end{pmatrix} \begin{pmatrix} 1 + \sigma_4 hfg & 0 & -\sigma_4 hf \\ 0 & 1 & 0 \\ \sigma_4 hfg^2 & 0 & 1 - \sigma_4 hfg \end{pmatrix} \end{aligned} \tag{4.4}$$

and

(4.5)
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g & -f & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\sigma_4 hg \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -g & f & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ \sigma_4 hg^2 & 1 & 0 \\ -\sigma_4 hg^2 f & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \sigma_4 hgf & -\sigma_4 hg \\ 0 & \sigma_4 hgf^2 & 1 + \sigma_4 hgf \end{pmatrix}$$

are induced. But the first factors on the right side of both equations (4.4) and (4.5) are induced. Thus, we must prove that both of the second factors on the right side of (4.4) and (4.5) are induced.

In general, we must prove that the following matrices are induced:

(4.6)
$$\left[I_3 + \sigma_4\left(f\vec{e}_i + fg\vec{e}_j\right)^t\left(-g\vec{e}_i + \vec{e}_j\right)\right],$$

where $i \neq j$ and $f, g \in \mathbf{Z}[A(4)]$. (Note that h has been combined with either f or g.)

We begin with a few specific examples of matrices which are easily seen to be induced.

(4.7)
$$\begin{pmatrix} 1 - \sigma_4 f\sigma_1 & \sigma_4 f & 0 \\ -\sigma_4 f\sigma_1^2 & 1 + \sigma_4 f\sigma_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} s_2^{-1} & 0 & 0 \\ s_2^{-1}\sigma_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \sigma_4 f s_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} s_2 & 0 & 0 \\ -\sigma_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is induced, as is

$$\begin{pmatrix} 1 - \sigma_4 f\sigma_1 & 0 & \sigma_4 f \\ 0 & 1 & 0 \\ -\sigma_4 f\sigma_1^2 & 0 & 1 + \sigma_4 f\sigma_1 \end{pmatrix}.$$

(4.8)
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + \sigma_4 fg\sigma_1 & -\sigma_4 fg^2\sigma_1^2 \\ 0 & \sigma_4 f & 1 - \sigma_4 fg\sigma_1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & -\sigma_2 g \\ 0 & 1 & \sigma_1 g \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \sigma_4 f & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & \sigma_2 g \\ 0 & 1 & -\sigma_1 g \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \sigma_4 fg\sigma_2 & -\sigma_4 fg^2\sigma_1\sigma_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is certainly induced, and similarly

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \sigma_4 fg\sigma_1 & \sigma_4 f \\ 0 & -\sigma_4 fg^2\sigma_1^2 & 1 + \sigma_4 fg\sigma_1 \end{pmatrix}$$

is also induced.

From (4.7), we see that

$$\begin{pmatrix} 1-\sigma_4 f\sigma_1 & \sigma_4 f & 0 \\ -\sigma_4 f\sigma_1^2 & 1+\sigma_4 f\sigma_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\sigma_4 f\sigma_1 & \sigma_4 f & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -\sigma_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sigma_4 f\sigma_1 & -\sigma_4 f & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \sigma_1 \\ 0 & 0 & 1 \end{pmatrix}$$

is induced as is the first factor in the product. Therefore,

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & -\sigma_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sigma_4 f\sigma_1 & -\sigma_4 f & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \sigma_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\sigma_4 f\sigma_1 & 0 & -\sigma_4 f\sigma_1 \\ 0 & 1 & 0 \\ \sigma_4 f\sigma_1 & 0 & 1+\sigma_4 f\sigma_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\sigma_4 f\sigma_1^2 & 1 & -\sigma_4 f\sigma_1^2 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\sigma_4 f\sigma_1 & \sigma_4 f\sigma_1^2 \\ 0 & -\sigma_4 f & 1-\sigma_4 f\sigma_1 \end{pmatrix} \begin{pmatrix} 1 & \sigma_4 f & \sigma_4 f\sigma_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is induced as are the last three factors of the last product, the second of these factors by virtue of (4.8). We conclude that

$$\begin{pmatrix} 1-\sigma_4 f\sigma_1 & 0 & -\sigma_4 f\sigma_1 \\ 0 & 1 & 0 \\ \sigma_4 f\sigma_1 & 0 & 1+\sigma_4 f\sigma_1 \end{pmatrix} \tag{4.9}$$

is induced.

In a similar manner, we can prove that

$$\begin{pmatrix} 1-\sigma_4 f\sigma_1 & 0 & \sigma_4 f\sigma_1 \\ 0 & 1 & 0 \\ -\sigma_4 f\sigma_1 & 0 & 1+\sigma_4 f\sigma_1 \end{pmatrix}, \begin{pmatrix} 1-\sigma_4 f\sigma_1 & -\sigma_4 f\sigma_1 & 0 \\ \sigma_4 f\sigma_1 & 1+\sigma_4 f\sigma_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1-\sigma_4 f\sigma_1 & \sigma_4 f\sigma_1 & 0 \\ -\sigma_4 f\sigma_1 & 1+\sigma_4 f\sigma_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are all induced.

$$\begin{pmatrix} 1-\sigma_4 f\sigma_1 & -\sigma_4 f & 0 \\ \sigma_4 f\sigma_1^2 & 1+\sigma_4 f\sigma_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sigma_4 f\sigma_1 & \sigma_4 f & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -\sigma_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\sigma_4 f\sigma_1 & -\sigma_4 f & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \sigma_1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sigma_4 f\sigma_1 & \sigma_4 f & 1 \end{pmatrix} \begin{pmatrix} 1-\sigma_4 f\sigma_1 & 0 & \sigma_4 f\sigma_1 \\ 0 & 1 & 0 \\ -\sigma_4 f\sigma_1 & 0 & 1+\sigma_4 f\sigma_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \sigma_4 f\sigma_1^2 & 1 & -\sigma_4 f\sigma_1^2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\sigma_4 f\sigma_1 & \sigma_4 f\sigma_1^2 \\ 0 & -\sigma_4 f & 1-\sigma_4 f\sigma_1 \end{pmatrix} \begin{pmatrix} 1 & -\sigma_4 f & -\sigma_4 f\sigma_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

All five factors of this last product are induced (use (4.9) and (4.8)), whence

$$\begin{pmatrix} 1-\sigma_4 f\sigma_1 & -\sigma_4 f & 0 \\ \sigma_4 f\sigma_1^2 & 1+\sigma_4 f\sigma_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is induced.

The proofs that

$$\begin{pmatrix} 1-\sigma_4 f\sigma_1 & \sigma_4 f\sigma_1^2 & 0 \\ -\sigma_4 f & 1+\sigma_4 f\sigma_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1+\sigma_4 f\sigma_1 & \sigma_4 f\sigma_1^2 & 0 \\ -\sigma_4 f & 1-\sigma_4 f\sigma_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1-\sigma_4 f\sigma_1 & 0 & -\sigma_4 f \\ 0 & 1 & 0 \\ \sigma_4 f\sigma_1^2 & 0 & 1+\sigma_4 f\sigma_1 \end{pmatrix}$$

are induced are done in a like fashion, and then these results are used to prove that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\sigma_4 f\sigma_1 & \sigma_4 f\sigma_1 \\ 0 & -\sigma_4 f\sigma_1 & 1-\sigma_4 f\sigma_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\sigma_4 f\sigma_1 & \sigma_4 f\sigma_1 \\ 0 & -\sigma_4 f\sigma_1 & 1+\sigma_4 f\sigma_1 \end{pmatrix}$$

are induced.

We are now ready to prove that matrices of form (4.6) of a more general type are induced.

Replacing f by fg in (4.9), we have the induced matrix

$$\begin{pmatrix} 1-\sigma_4 fg\sigma_1 & 0 & -\sigma_4 fg\sigma_1 \\ 0 & 1 & 0 \\ \sigma_4 fg\sigma_1 & 0 & 1+\sigma_4 fg\sigma_1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ g\sigma_1 & 1 & g\sigma_1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \sigma_4 f & 0 \\ 0 & 1 & 0 \\ 0 & -\sigma_4 f & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -g\sigma_1 & 1 & -g\sigma_1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\sigma_4 f & 0 \\ 0 & 1 & 0 \\ 0 & \sigma_4 f & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1-\sigma_4 fg\sigma_1 & \sigma_4 f & 0 \\ -\sigma_4 fg^2\sigma_1^2 & 1+\sigma_4 fg\sigma_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -\sigma_4 fg\sigma_1 \\ 0 & 1 & -\sigma_4 fg^2\sigma_1^2 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\sigma_4 fg\sigma_1 & \sigma_4 fg^2\sigma_1^2 \\ 0 & -\sigma_4 f & 1+\sigma_4 fg\sigma_1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \sigma_4 fg^2\sigma_1^2 & 1 & 0 \\ \sigma_4 fg\sigma_1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\sigma_4 f & 0 \\ 0 & 1 & 0 \\ 0 & \sigma_4 f & 1 \end{pmatrix},$$

and the last four factors of this last product are induced by (4.8). Therefore the first factor of the product

$$\begin{pmatrix} 1-\sigma_4 fg\sigma_1 & \sigma_4 f & 0 \\ -\sigma_4 fg^2\sigma_1^2 & 1+\sigma_4 fg\sigma_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is induced.

Likewise, we can prove that

$$\begin{pmatrix} 1-\sigma_4 fg\sigma_1 & \sigma_4 fg^2\sigma_1^2 & 0 \\ -\sigma_4 f & 1+\sigma_4 fg\sigma_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1-\sigma_4 fg\sigma_1 & 0 & \sigma_4 fg^2\sigma_1^2 \\ 0 & 1 & 0 \\ -\sigma_4 f & 0 & 1+\sigma_4 fg\sigma_1 \end{pmatrix},$$

and

$$\begin{pmatrix} 1+\sigma_4 fg\sigma_1 & 0 & -\sigma_4 f \\ 0 & 1 & 0 \\ \sigma_4 fg^2\sigma_1^2 & 0 & 1-\sigma_4 fg\sigma_1 \end{pmatrix}$$

are induced.

Using the preceding methods, we can also easily verify that the following matrices are induced:

$$\left[I_3 + \sigma_4^2 f(\vec{e}_i - \vec{e}_j)^t \cdot (\vec{e}_i + \vec{e}_j)\right]$$

and

$$\left[I_3 + \sigma_4(f\vec{e}_i + fg\sigma_4\vec{e}_j)^t \cdot (-g\sigma_4\vec{e}_i + \vec{e}_j)\right],$$

where $i \neq j$ and $f, g \in \mathbf{Z}[A(4)]$, and

$$\text{(4.10)} \qquad \left[I_3 + \sigma_4 m(\vec{e}_i \pm \vec{e}_j)^t \cdot (\mp\vec{e}_i + \vec{e}_j)\right] = \left[I_3 + \sigma_4(\vec{e}_i \pm \vec{e}_j)^t \cdot (\mp\vec{e}_i + \vec{e}_j)\right]^m,$$

where $m \in \mathbf{Z}$ and $i \neq j$.

Combining the above results and using the symmetry of s_1, s_2, and s_3, we have that, for $1 \leqslant k \leqslant 4$,

$$(4.11)\qquad \left[I_3 + \sigma_4\left(f\vec{e}_i + fg\sigma_k\vec{e}_j\right)^t \cdot \left(-g\sigma_k\vec{e}_i + \vec{e}_j\right)\right]$$

and

$$\left[I_3 + \sigma_4 f\sigma_k\left(\vec{e}_i \pm \vec{e}_j\right)^t \cdot \left(\mp\vec{e}_i + \vec{e}_j\right)\right],$$

where $i \neq j$ and $f, g \in \mathbf{Z}[A(4)]$, are all induced.

Using (4.10) and (4.11) and the additivity of

$$(4.12)\qquad \left[I_3 + \sigma_4 f\left(\vec{e}_i \pm \vec{e}_j\right)^t \cdot \left(\mp\vec{e}_i + \vec{e}_j\right)\right], \qquad i \neq j,$$

on f, we can conclude that (4.12) is induced for all $f \in \mathbf{Z}[A(4)]$.

Up to this point, we have demonstrated that special cases of the matrices (4.6), namely (4.11) and (4.12), are induced. These cases may be considered as the beginning of an induction procedure. We are now ready to prove that the matrices

$$(4.6)\qquad \left[I_3 + \sigma_4\left(f\vec{e}_i + fg\vec{e}_j\right)^t\left(-g\vec{e}_i + \vec{e}_j\right)\right], \qquad i \neq j,$$

are induced for arbitrary f, g in $\mathbf{Z}[A(4)]$. The process used will be referred to as the "induction procedure".

Let $g = g_1 + g_2$. Then, for $i = 1, j = 2$, (4.6) becomes

$$(4.13)\qquad \begin{aligned}
&\begin{pmatrix} 1-\sigma_4 fg & \sigma_4 f & 0 \\ -\sigma_4 fg^2 & 1+\sigma_4 fg & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -g & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -\sigma_4 f \\ 0 & 1 & -\sigma_4 f(g_1+g_2) \\ 0 & 0 & 1 \end{pmatrix} \\
&\quad\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & \sigma_4 f \\ 0 & 1 & \sigma_4 f(g_1+g_2) \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -g & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -\sigma_4 f \\ 0 & 1 & -\sigma_4 fg_1 \\ 0 & 0 & 1 \end{pmatrix} \\
&\quad\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -g & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -\sigma_4 f \\ 0 & 1 & -\sigma_4 fg_2 \\ 0 & 0 & 1 \end{pmatrix} \\
&\quad\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & \sigma_4 f \\ 0 & 1 & \sigma_4 f(g_1+g_2) \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

Since the last factor in the last product of (4.13) is induced, in order to prove that (4.13) is induced, we must demonstrate that

(4.14)
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -(g_1+g_2) & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -\sigma_4 f \\ 0 & 1 & -\sigma_4 f g_k \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (g_1+g_2) & -1 & 1 \end{pmatrix}, \qquad k = 1, 2,$$

are induced. Both cases are handled in the same manner. For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -g_1-g_2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -\sigma_4 f \\ 0 & 1 & -\sigma_4 f g_1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g_1+g_2 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -g_2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -g_1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -\sigma_4 f \\ 0 & 1 & -\sigma_4 f g_1 \\ 0 & 0 & 1 \end{pmatrix}$$

(4.15)
$$\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g_1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & \sigma_4 f \\ 0 & 1 & \sigma_4 f g_1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g_2 & 0 & 1 \end{pmatrix}$$

$$\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -g_2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -\sigma_4 f \\ 0 & 1 & -\sigma_4 f g_1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g_2 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -g_2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1-\sigma_4 f_1 & \sigma_4 f & 0 \\ -\sigma_4 f g_1^2 & 1+\sigma_4 f g_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g_2 & 0 & 1 \end{pmatrix}$$

$$\cdot\begin{pmatrix} 1-\sigma_4 f g_2 & 0 & -\sigma_4 f \\ 0 & 1 & 0 \\ \sigma_4 f g_2^2 & 0 & 1+\sigma_4 f g_2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -\sigma_4 f g_1 g_2 & 1 & -\sigma_4 f g_1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\sigma_4 f g_1 & \sigma_4 f & 0 \\ -\sigma_4 f g_1^2 & 1+\sigma_4 f g_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sigma_4 f g_1 g_2 & -\sigma_4 f g_2 & 1 \end{pmatrix}$$

$$\cdot\begin{pmatrix} 1-\sigma_4 f g_2 & 0 & -\sigma_4 f \\ 0 & 1 & 0 \\ \sigma_4 f g_2^2 & 0 & 1+\sigma_4 f g_2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -\sigma_4 f g_1 g_2 & 1 & -\sigma_4 f g_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The second and fourth factors in this last product are induced. Thus, if we know that the first and third factors of this product are induced, (4.15) would be induced, whence (4.14) would be induced, whence (4.13) would be induced. A similar procedure holds for any i and j such that $i \neq j$ in (4.6).

Therefore, the completion of the proof that (4.6) is induced is now clear. We prove (4.6) is induced when g is an integer by induction on g, beginning with (4.12). Then, we prove (4.6) is induced successively for $g = r + h_1\sigma_1$, $g = r + h_1\sigma_1 + h_2\sigma_2$, $g = r + h_1\sigma_1 + h_2\sigma_2 + h_3\sigma_3$, and $g = r + h_1\sigma_1 + h_2\sigma_2 + h_3\sigma_3 + h_4\sigma_4$ where $r \in \mathbf{Z}$ and

$h_i \in \mathbf{Z}[A(4)]$, using the induction procedure and the induction hypothesis that (4.11) and (4.12) are induced. Since any element g of $\mathbf{Z}[A(4)]$ can be written as $g = r + h_1\sigma_1 + h_2\sigma_2 + h_3\sigma_3 + h_4\sigma_4$, we are done. □

5. Proof of Theorem B. Let

$$\alpha = \left[I_n + (a_{ij})\right] \in \mathscr{A}(n) \cap \mathrm{GL}_n\left(\mathbf{Z}[A(n)], \sum_{j=\nu}^{n} \sigma_j \mathbf{Z}[A(n)]\right).$$

By condition (1.1),

$$\sigma_1 a_{1j} + \sigma_2 a_{2j} + \cdots + \sigma_n a_{nj} = 0, \qquad 1 \leqslant j \leqslant n. \tag{5.1}$$

Let $\bar{\alpha} = [I_n + (\bar{a}_{ij})] = [I_n + (\sigma_n \bar{b}_{ij})]$ be the result of setting $\sigma_\nu = \cdots = \sigma_{n-1} = 0$ in α. Thus $\bar{\alpha} \in \mathrm{GL}_n(\mathbf{Z}[A(\nu - 1)][s_n, s_n^{-1}], \sigma_n \mathbf{Z}[A(\nu - 1)][s_n, s_n^{-1}])$. (*Note.* When $\nu = 1$, it is understood that $\mathbf{Z}[A(\nu - 1)] = \mathbf{Z}$. When $\nu = n$, $\alpha = \bar{\alpha}$.) Equation (5.1) takes on the equivalent forms

$$\sigma_1 \bar{a}_{1j} + \sigma_2 \bar{a}_{2j} + \cdots + \sigma_{\nu-1} \bar{a}_{(\nu-1)j} + \sigma_n \bar{a}_{nj} = 0, \qquad 1 \leqslant j \leqslant n, \tag{5.2}$$

$$\sigma_1 \bar{b}_{1j} + \sigma_2 \bar{b}_{2j} + \cdots + \sigma_{\nu-1} \bar{b}_{(\nu-1)j} + \sigma_n \bar{b}_{nj} = 0, \qquad 1 \leqslant j \leqslant n. \tag{5.3}$$

If we set $\sigma_n = 0$ in $(\bar{b}_{1n}, \dots, \bar{b}_{(n-1)n})$, we obtain $(\bar{c}_{1n}, \dots, \bar{c}_{(n-1)n})$ where

$$\sigma_1 \bar{c}_{1n} + \cdots + \bar{\sigma}_{(\nu-1)n} = 0. \tag{5.4}$$

Proposition 5.1. *In the preceding setting, there is*

$$\gamma = (c_{ij}) \in \mathscr{B}(n) \cap \mathrm{GL}_n\left(\mathbf{Z}[A(n)], \sum_{j=\nu}^{n} \sigma_j \mathbf{Z}[A(n)]\right),$$

such that when one sets $\sigma_\nu = \cdots = \sigma_{n-1} = 0$ in γ,

$$\begin{pmatrix} I_{n-1} & \begin{matrix} \bar{c}_{1n} \\ \vdots \\ \bar{c}_{(n-1)n} \end{matrix} \\ 0 \cdots 0 & 1 \end{pmatrix}$$

is obtained.

Proof.

$$\begin{pmatrix} I_{n-1} & \begin{matrix} \bar{c}_{1n} \\ \vdots \\ \bar{c}_{(n-1)n} \end{matrix} \\ 0 \cdots 0 & 1 \end{pmatrix} = \begin{pmatrix} I_{n-1} & \begin{matrix} \bar{c}_{1n} \\ \vdots \\ \bar{c}_{(\nu-1)n} \\ 0 \\ \vdots \\ 0 \end{matrix} \\ 0 \cdots 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} I_{n-1} & \begin{matrix} 0 \\ \vdots \\ 0 \\ \bar{c}_{\nu n} \\ \vdots \\ \bar{c}_{(n-1)n} \end{matrix} \\ 0 \cdots 0 & 1 \end{pmatrix}.$$

Since by virtue of (5.2) the first factor of the product satisfies condition (1.1) and has at most one nontrivial column, it is in $\mathscr{B}(n)$ by Proposition 3.3. Moreover, the second factor lies in $\mathrm{GL}_n(\mathbf{Z}[A(n)], \sigma_n \mathbf{Z}[A(n)])$ and breaks up into the following product:

$$\begin{pmatrix} & & 0 \\ & & \vdots \\ & & 0 \\ & I_{n-1} & \bar{c}_{\nu n} \\ & & \vdots \\ & & \bar{c}_{(n-1)n} \\ 0 \cdots 0 & & 1 \end{pmatrix} = \prod_{k=\nu}^{n-1} (I_n + \bar{c}_{kn} E_{kn}).$$

We shall prove that each $(I_n + \bar{c}_{kn} E_{kn})$ arises in the manner described in Proposition 5.1.

$$\bar{c}_{kn} = \sigma_1 \sigma_n f_1 + \cdots + \sigma_{\nu-1} \sigma_n f_{\nu-1} + \sigma_n^2 f_n + \sigma_n r$$

where $r \in \mathbf{Z}$ and $f_1, \ldots, f_n \in \mathbf{Z}[A(\nu - 1)][s_n, s_n^{-1}]$. Let

$$\gamma_k = \begin{pmatrix} & & 0 \\ & & \vdots \\ & & 0 \\ & I_{n-1} & s_k^{-1}\sigma_n \\ & & 0 \\ & & \vdots \\ & & 0 \\ 0 \cdots 0 & & s_k^{-1} \end{pmatrix} \begin{pmatrix} & & -\sigma_k \sigma_n f_1 \\ & & \vdots \\ & & -\sigma_k \sigma_n f_{\nu-1} \\ & I_{n-1} & 0 \\ & & \sigma_1 \sigma_n f_1 + \cdots + \sigma_{\nu-1} \sigma_n f_{\nu-1} \\ & & 0 \\ 0 \cdots 0 & & 1 \end{pmatrix}$$

$$\cdot \begin{pmatrix} I_{k-1} & & 0 & \\ & 1 + \sigma_k \sigma_n f_n & 0 & \sigma_n^2 f_n \\ 0 & 0 & I_{n-k-1} & 0 \\ & -\sigma_k^2 f_n & 0 & 1 - \sigma_k \sigma_n f_n \end{pmatrix}.$$

By Proposition 3.3 and Proposition 3.4,

$$\gamma_k \in \mathscr{B}(n) \cap \mathrm{GL}_n\left(\mathbf{Z}[A(n)], \sum_{j=\nu}^{n} \sigma_j \mathbf{Z}[A(n)]\right)$$

and when we set $\sigma_\nu = \cdots = \sigma_{n-1} = 0$ in γ_k, we obtain $(I_n + \bar{c}_{kn} E_{kn})$.

It is clear that

$$\gamma = \begin{pmatrix} & & \bar{c}_{1n} \\ & & \vdots \\ & I_{n-1} & \bar{c}_{(\nu-1)n} \\ & & 0 \\ 0 \cdots 0 & & 1 \end{pmatrix} \cdot \gamma_\nu \cdots \gamma_{n-1}$$

satisfies the demands of Proposition 5.1. □

By multiplying α on the left by γ^{-1} where γ is as in Proposition 5.1, if necessary, we may suppose that when we set $\sigma_\nu = \cdots = \sigma_{n-1} = 0$ in α,

$$\bar{a}_{in} \in \sigma_n^2 \mathbf{Z}[A(n-1)][s_n, s_n^{-1}], \qquad 1 \leqslant i \leqslant n-1.$$

Let

$$M = \begin{pmatrix} & & 0 \\ & I_{n-1} & \vdots \\ & & 0 \\ \sigma_1 & \cdots & \sigma_n \end{pmatrix},$$

and let

$$\overline{M} = \begin{pmatrix} & I_{n-1} & & & \begin{matrix} 0 \\ \vdots \end{matrix} \\ \sigma_1 \cdots \sigma_{\nu-1} & & 0 \cdots 0 & & \sigma_n \end{pmatrix}$$

be the result of setting $\sigma_\nu = \cdots = \sigma_{n-1} = 0$ in M. Then,

$$M\alpha M^{-1} = I_n + \begin{pmatrix} & \sigma_n^{-1}a_{1n} \\ \left(a_{ij} - \sigma_n^{-1}\sigma_j a_{in}\right) & \vdots \\ & \sigma_n^{-1}a_{(n-1)n} \\ 0 \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} & \sigma_n^{-1}a_{1n} \\ I_{n-1} & \vdots \\ & \sigma_n^{-1}a_{(n-1)n} \\ 0 \cdots 0 & 1 \end{pmatrix} \left[I_n + \begin{pmatrix} \left(a_{ij} - \sigma_n^{-1}\sigma_j a_{in}\right) & 0 \\ & \vdots \\ 0 \cdots & 0 \end{pmatrix} \right],$$

$$\overline{M}\bar{\alpha}\overline{M}^{-1} = \begin{pmatrix} & \sigma_n^{-1}\bar{a}_{1n} \\ I_{n-1} & \vdots \\ & \sigma_n^{-1}\bar{a}_{(n-1)n} \\ 0 \cdots 0 & 1 \end{pmatrix} \left[I_n + \begin{pmatrix} & 0 \\ \left(\bar{a}_{ij} - \sigma_n^{-1}\sigma_j \bar{a}_{in}\right) & \vdots \\ 0 \cdots & 0 \end{pmatrix} \right]$$

$$= \begin{pmatrix} & \sigma_n \bar{d}_{1n} \\ I_{n-1} & \vdots \\ & \sigma_n \bar{d}_{(n-1)n} \\ 0 \cdots 0 & 1 \end{pmatrix} \left[I_n + \begin{pmatrix} & 0 \\ \left(\sigma_n \bar{d}_{ij}\right) & \vdots \\ 0 \cdots & 0 \end{pmatrix} \right],$$

where $\bar{d}_{ij} \in \mathbf{Z}[A(\nu-1)][s_n, s_n^{-1}]$, and the diagram

$$\begin{array}{ccc} \alpha & \to & M\alpha M^{-1} \\ \downarrow & & \downarrow \\ \bar{\alpha} & \to & \overline{M}\bar{\alpha}\overline{M}^{-1} \end{array}$$

commutes.

For $1 \leqslant k \leqslant n-1$, let

$$\delta_k = I_n + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ \sigma_1\sigma_n\bar{d}_{kn} & \sigma_2\sigma_n\bar{d}_{kn} & \cdots & \sigma_n^2\bar{d}_{kn} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ -\sigma_1\sigma_k\bar{d}_{kn} & -\sigma_2\sigma_k\bar{d}_{kn} & \cdots & -\sigma_n\sigma_k\bar{d}_{kn} \end{pmatrix}$$
$$= \prod_{i \neq k, n} \left(I_n + \sigma_i\bar{d}_{kn}F_{kni}\right) \cdot \left(I_n + \sigma_k\bar{d}_{kn}F_{knk} - \sigma_n\bar{d}_{kn}F_{knn}\right),$$

which is in $\mathscr{B}(n)$ by Propositions 3.2 and 3.4.

Let $\bar{\delta}_k$ be the result of setting $\sigma_\nu = \cdots = \sigma_{n-1} = 0$ in δ_k. Then,

$$\overline{M}\bar{\delta}_1 \cdots \bar{\delta}_{n-1}\overline{M}^{-1} = \begin{pmatrix} & & \sigma_n\bar{d}_{1n} \\ & I_{n-1} & \vdots \\ & & \sigma_n\bar{d}_{(n-1)n} \\ 0 & \cdots\ 0 & 1 \end{pmatrix}.$$

Thus, by multiplying α on the left by $(\delta_1 \cdots \delta_{n-1})^{-1}$ if necessary, we may assume that

$$\overline{M}\bar{\alpha}\overline{M}^{-1} = \left[I_n + \begin{pmatrix} \left(\sigma_n\bar{d}_{ij}\right) & \begin{matrix} 0 \\ \vdots \end{matrix} \\ 0\ \cdots & 0 \end{pmatrix} \right].$$

Let

$$\varepsilon = I_n + \begin{pmatrix} \sigma_n\bar{d}_{11} & \cdots & \sigma_n\bar{d}_{1(n-1)} & 0 \\ \vdots & & \vdots & \vdots \\ \sigma_n\bar{d}_{(n-1)1} & \cdots & \sigma_n\bar{d}_{(n-1)(n-1)} & 0 \\ -\sum_{j=1}^{n-1}\sigma_j\bar{d}_{j1} & \cdots & -\sum_{j=1}^{n-1}\sigma_j\bar{d}_{j(n-1)} & 0 \end{pmatrix}.$$

By the Main Lemma, $\varepsilon \in \mathscr{B}(n) \cap \mathrm{GL}_n(\mathbf{Z}[A(n)], \sum_{j=\nu}^{n}\sigma_j\mathbf{Z}[A(n)])$, and

$$M\varepsilon M^{-1} = \overline{M}\bar{\varepsilon}\overline{M}^{-1} = \left[I_n + \begin{pmatrix} \left(\sigma_n\bar{d}_{ij}\right) & \begin{matrix} 0 \\ \vdots \end{matrix} \\ 0\ \cdots & 0 \end{pmatrix} \right].$$

By multiplying α by ε^{-1} on the left if necessary, we may assume that $\overline{M}\bar{\alpha}\overline{M}^{-1} = I_n$, whence

$$\alpha \in \mathscr{A}(n) \cap \mathrm{GL}_n\left(\mathbf{Z}[A(n)], \sum_{j=\nu}^{n-1}\sigma_j\mathbf{Z}[A(n)]\right).$$

In effect what we have proved is

(*) If $\alpha \in \mathscr{A}(n) \cap \mathrm{GL}_n(\mathbf{Z}[A(n)], \sum_{j=\nu}^{n} \sigma_j \mathbf{Z}[A(n)])$, then there are γ, δ, $\varepsilon \in \mathscr{B}(n)$ such that

$$\varepsilon^{-1}\delta^{-1}\gamma^{-1}\alpha = \beta^{-1}\alpha \in \mathscr{A}(n) \cap \mathrm{GL}_n\left(\mathbf{Z}[A(n)], \sum_{j=\nu}^{n-1} \sigma_j \mathbf{Z}[A(n)]\right).$$

To prove Theorem B, a modified version of (*) is needed, as was seen in the outline of the proof of Theorem B in §2.

PROPOSITION 5.2. *Let* $\alpha \in \mathscr{A}(n) \cap \mathrm{GL}_n(\mathbf{Z}[A(n)], \sum_{j=\nu}^{n} \sigma_j \mathbf{Z}[A(n)])$. *Then, there exists* $\beta_\nu \in \mathscr{B}(n) \cap \mathrm{GL}_n(\mathbf{Z}[A(n)], \sum_{j=\nu}^{n} \sigma_j \mathbf{Z}[A(n)])$ *such that*

$$\beta_\nu^{-1}\alpha \in \mathrm{GL}_n\left(\mathbf{Z}[A(n)], \sum_{j=\nu+1}^{n} \sigma_j \mathbf{Z}[A(n)]\right).$$

PROOF. All the indeterminates $s_1, \ldots, s_n$ play equivalent roles. Thus, when we set $\sigma_{\nu+1} = \cdots = \sigma_n = 0$ in α, we obtain $\bar{\alpha} \in \mathrm{GL}_n(\mathbf{Z}[A(\nu)], \sigma_\nu \mathbf{Z}[A(\nu)])$, and from (*) we have the existence of an element $\beta_\nu \in \mathscr{B}(n)$ such that

$$\beta_\nu^{-1}\alpha \in \mathrm{GL}_n\left(\mathbf{Z}[A(n)], \sum_{j=\nu+1}^{n} \sigma_j \mathbf{Z}[A(n)]\right). \qquad \square$$

We are finally prepared to prove

THEOREM B. *If* $n \geqslant 4$, *then the natural map* IA-$\mathrm{Aut}(F(n)) \to IA$-$\mathrm{Aut}(\Phi(n))$ *is surjective, i.e. each* IA*-automorphism of* $\Phi(n)$ *is induced by an* IA*-automorphism of* $F(n)$.

PROOF. In our notation, we need to prove that $\mathscr{A}(n) = \mathscr{B}(n)$. To this end, let $\alpha \in \mathscr{A}(n)$. By applying Proposition 5.2 successively, we have the existence of elements $\beta_1, \ldots, \beta_n \in \mathscr{B}(n)$ such that

$$\beta_\nu^{-1} \cdots \beta_2^{-1}\beta_1^{-1}\alpha \in \mathrm{GL}_n\left(\mathbf{Z}[A(n)], \sum_{j=\nu+1}^{n} \sigma_j \mathbf{Z}[A(n)]\right),$$

$\nu = 1, 2, \ldots, n$. In particular, $\beta_n^{-1} \cdots \beta_1^{-1}\alpha = I_n$, whence $\alpha \in \mathscr{B}(n)$. $\square$

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