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A COMMUTATOR THEOREM AND WEIGHTED BMO

BY

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ABSTRACT. The main result of this paper is a commutator theorem: If μ and λ are A_p weights, then the commutator H, M_b is a bounded operator from $L^p(\mu)$ into $L^p(\lambda)$ if and only if $b \in \mathrm{BMO}_{(\mu\lambda^{-1})^{1/p}}$. The proof relies heavily on a weighted sharp function theorem. Along the way, several other applications of this theorem are derived, including a doubly-weighted L^p estimate for BMO. Finally, the commutator theorem is used to obtain vector-valued weighted norm inequalities for the Hilbert transform.

I. Introduction. In the last decade, there have been several major results involving weighted norm inequalities for the conjugate operator $f \to Hf$, given for trigonometric polynomials $f = \sum c_n r^{|n|} e^{in\theta}$ by

$$Hf(\theta) = i \sum_{n \leqslant -1} c_n r^{|n|} e^{in\theta} - i \sum_{n \geqslant 1} c_n r^n e^{in\theta}.$$

H may also be viewed as a convolution with conjugate Poisson kernel Q,

$$Q(x) = (\sin x)/1 - \cos x.$$

The major results involved the class of A_p weights and their logarithms.

DEFINITION 1.1. A nonnegative function w is in the *class* (A_p), for $1 < p < \infty$, if there exists a constant C so that, for $1/p + 1/q = 1$ and all intervals I contained in the boundary of the unit circle, we have

$$\frac{1}{|I|} \int_I w(x)\, dx \left(\frac{1}{|I|} \sum_I w^{-q/p}(x)\, dx \right)^{p-1} \leqslant C,$$

(where $|I|$ denotes the measure of the interval I).

Two useful properties of weights are the A_∞ condition: w is in the class (A_∞) if there exist constants C and $\delta > 0$ so that, for each interval I and measurable set $E \subseteq I$, we have

$$\left(\int_E w \right) \Big/ \left(\int_I w \right) \leqslant C \left(\frac{|E|}{|I|} \right)^{\delta},$$

and the *Reverse Hölder* condition: There exist constants C and $\delta > 0$ such that, for all intervals I,

$$\left(\frac{1}{|I|} \int_I w^{1+\delta}\, dx \right)^{1/(1+\delta)} \leqslant \frac{C}{|I|} \int_I w\, dx.$$

Muckenhoupt [**7, 8**] has shown that A_p for some $p > 1$, A_∞, and Reverse Hölder are all equivalent.

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The major weighted norm inequality for the conjugate operator was proven by Hunt, Muckenhoupt, and Wheeden [**5**].

THEOREM 1.2. *H is a bounded operator on the weighted L^p-space $L^p(w)$ if and only if $w \in (A_p)$.*

There is one more approach to the conjugate operator that has proved fruitful, via commutators and the class BMO. Let M_b denote multiplication by the function b. The commutator $[H, M_b]$ is the operator taking $f \to H(bf) - b(Hf)$. The bounded operators of this form comprise the dual space of H^1, so that, by Fefferman's Theorem, $[H, M_b]$ is a bounded operator of L^2 if and only if $b \in$ BMO, the class of functions of bounded mean oscillation:

$$\sup_I \frac{1}{|I|} \int_I \left| b(x) - \frac{1}{|I|} \int_I b(t)\, dt \right| dx < \infty.$$

(See Coifman, Rochberg, and Weiss [**3**].)

This theory is linked to the weighted norm inequalities in the following way: If $b \in$ BMO, by the John–Nirenberg Theorem, $\exp(tb) \in (A_p)$ for t sufficiently small and $p > 1$ fixed. Thus H is a bounded operator on $L^p(e^{tb})$.

Conversely, if H is a bounded operator on $L^2(e^b)$, then the operators $T_z = e^{zb}He^{-zb}$ are bounded operators on L^2 for $|z| \leq \frac{1}{2}$, as is $(d/dt)T_t|_{t=0} = [H, M_b]$, so that $b \in$ BMO.

In this paper, we extend this work to settings involving multiple weights. In §II, we present a weighted sharp function theorem, which plays a key role in the later analysis. In §III, we present some simple applications of this theorem. In §IV, we present the Commutator Theorem: If μ and λ are A_p weights, then the commutator $[H, M_b]$ is a bounded map from $L^p(\mu)$ into $L^p(\lambda)$ if and only if b is in an appropriate weighted BMO space. And in §V, we present a vector-valued version of Theorem 1.2.

II. The Sharp Function Theorem. A measure ν is a doubling measure if there exists a constant C such that, for any intervals I and J with $|J| = 2|I|$, we have $\nu(J) \leq C\nu(I)$.

For example, if $w \in (A_p)$, then the measure $w\,dx$ is a doubling measure.

The unit circle will be denoted by T.

Let ν be a doubling measure and u a nonnegative weight. Then u induces a measure, which we also call u, given by

$$u(E) = \int_E u\, d\nu.$$

u is in the class $A_\infty(d\nu)$ if there exist constants C and $\delta > 0$ for which

$$u(E)/u(I) \leq C(\nu(E)/\nu(I))^\delta$$

for all intervals I and measurable sets $E \subseteq I$. This measure u is also a doubling measure. The average of a function f over an interval I will be denoted by f_I,

$$f_I = \frac{I}{\nu(I)} \int_I f\, d\nu,$$

or sometimes $f_{I,d\nu}$ if the doubling measure is in doubt. The maximal function and the sharp function (relative to ν) are given by

$$f^*(x) = \sup\left\{ |f|_I \colon x \in I \right\}, \quad \text{and} \quad f^{\#}(x) = \sup\left\{ |f - f_I|_I \colon x \in I \right\}.$$

Finally, $L^p(u)$ will denote the L^p-space on the circle with norm

$$\|f\|_{L^p(u)} = \left(\frac{1}{u(T)} \int_T |f|^p u \, d\nu \right)^{1/p}.$$

THEOREM 2.1 (THE SHARP FUNCTION THEOREM). *Let $f \in L^1(d\nu)$, $1 < p < \infty$, and $f^{\#} \in L^p(u)$, for some $u \in A_\infty(d\nu)$. Then $f \in L^p(u)$, with*

$$\|f - f_T\|_{L^p(u)} \leqslant C_p \|f^{\#}\|_{L^p(u)}.$$

An unweighted version of this theorem, with $u = 1$ and ν Lebesgue measure, was given by Fefferman and Stein. Extending their proof to the present setting is straightforward and we omit the details (which can be found in [**1**]; see also [**11**]).

This theorem could have been proven for functions restricted to any interval $I \subseteq T$. Let $f^{\#,I}$ denote the sharp function restricted to I,

$$f^{\#,I}(x) = \sup\left\{ \frac{1}{\nu(J)} \int_J |f - f_J| \, d\nu \colon x \in J \subseteq I \right\}.$$

In this case, we would find

COROLLARY 2.2. *Let $f \in L^1(d\nu)$, $u \in A_\infty(d\nu)$, and $1 < p < \infty$. If $f^{\#} \in L^p(u \, d\nu)$, then so is f, and for any interval $I \subseteq T$, we have*

$$\int_I |f - f_I|^p u \, d\nu \leqslant C_p \int_I (f^{\#,I})^p u \, d\nu \leqslant C_p \int_I (f^{\#})^p u \, d\nu,$$

where C_p does not depend on I or f.

III. Applications of the Sharp Function Theorem. Our applications involve L^p estimates for some nonstandard weighted BMO classes. The first class we will consider is the doubly-weighted BMO class. Let u and v be weights, and suppose that

$$\inf_{c_I} \int_I |f - c_I| u \leqslant C \int_I v \quad \text{for } c_I \text{ constants.}$$

Is there an L^p version of this for any $p > 1$? When u and v are 1, this is standard BMO, and the John–Nirenberg Theorem gives the L^p estimate

$$\inf_{c_I} \int_I |f - c_I|^p \leqslant C|I|,$$

for any $1 \leqslant p < \infty$. The doubly-weighted version of this follows.

THEOREM 3.1 (THE DOUBLY-WEIGHTED BMO THEOREM). *Suppose that u^{-1} and v^{-1} are in (A_p) for some $p < 2$, and that $1/p + 1/q = 1$. If*

$$\text{(1)} \qquad \inf_{c_I} \int_I |f - c_I| u \leqslant C \int_I v \quad \textit{for all intervals } I,$$

then

$$\inf_{c_I} \frac{1}{|I|}\int_I |f - c_I|^{q/p} u^{q/p} \leqslant K\left(\frac{1}{|I|}\int_I v\right)^{q/p}. \tag{2}$$

PROOF. We will apply the Sharp Function Theorem with the measure $\nu = u\,dx$. Define $f^{\#}$, f^*, and f_I with respect to $u\,dx$. Suppose that (1) holds for some constant c_I. Then we can take $c_I = f_I$ without losing more than a factor of 2. Now fix x and let I contain x. Then

$$\frac{1}{u(I)}\int_I |f - f_I| u \leqslant \frac{c}{u(I)}\int_I v = \frac{c}{u(I)}\int_I vu^{-1}u \leqslant C(vu^{-1})^*(x),$$

and hence, $f^{\#}(x) \leqslant C(vu^{-1})^*(x)$. Now fix an interval I. By Corollary 2.2,

$$\begin{aligned}\int_I |f - f_I|^{q/p} u^{q/p}\,dx &= \int_I |f - f_I|^{q/p} u^{q/p-1} u\,dx \\ &\leqslant C_p \int_I (f^{\#})^{q/p} u^{q/p-1} u\,dx \\ &\leqslant C \cdot C_p \int_I (vu^{-1})^{*q/p} u^{q/p-1} u\,dx.\end{aligned}$$

We claim that

$$u^{q/p-1} \in A_{q/p}(u\,dx). \tag{3}$$

Given (3), Muckenhoupt's Theorem implies

$$\begin{aligned}\frac{1}{|I|}\int_I |f - f_I|^{q/p} u^{q/p}\,dx &\leqslant C' \frac{1}{|I|}\int_I (vu^{-1})^{q/p} u^{q/p}\,dx \\ &= C' \frac{1}{|I|}\int_I v^{q/p}\,dx \\ &\leqslant K\left(\frac{1}{|I|}\int_I v^{-1}\right)^{-q/p} \quad \text{as } v^{-1} \in (A_p), \\ &\leqslant K\left(\frac{1}{|I|}\int_I v\right)^{q/p}, \quad \text{by Cauchy–Schwarz.}\end{aligned}$$

So we must show (3). For this we must bound

$$\begin{aligned}&\frac{1}{u(I)}\int_I u^{q/p}\left[\frac{1}{u(I)}\int_I (u^{q/p-1})^{-1/(q/p-1)} u\right]^{q/(p-1)} \\ &= \frac{1}{u(I)}\int_I u^{q/p}\left(\frac{1}{u(I)}\int_I u^{-1}u\right)^{q/p-1} = \left(\frac{1}{u(I)}\right)^{q/p} |I|^{q/p} \frac{1}{|I|}\int_I u^{q/p} \\ &= \left(\frac{1}{|I|}\int_I u\right)^{-q/p} \frac{1}{|I|}\int_I u^{q/p} \leqslant \left(\frac{1}{|I|}\int_I u^{-1}\right)^{q/p}\left(\frac{1}{|I|}\int_I u^{q/p}\right)\end{aligned}$$

and this is bounded, since $u^{-1} \in (A_p)$.

For our other application, we will strengthen a Lipschitz type theorem of Lotkowski and Wheeden [**6**].

THEOREM 3.2. *Let F be a nonnegative function of sets, for which, if $I \subseteq J$, then $F(I) \leqslant CF(J)$. Let μ be a doubling measure, and let $g^{-1} \in A_p(g\,d\mu)$, $1 < p < \infty$. If for each interval I,*

$$\int_I |f - f_I| g\, d\mu \leqslant CF(I)\mu(I),$$

where $f_I = f_{I,g\,d\mu}$ is the average with respect to $g\,d\mu$, then

$$\int_I \left(|f - f_I| g\right)^q d\mu \leqslant CF(I)^q \mu(I), \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

(*Note*: *Here, and throughout, C will denote a universal constant, not necessarily the same at successive appearances.*)

Lotkowski and Wheeden also assumed the existence of constants $1 < \alpha < \beta$ for which $\alpha F(I) \leqslant F(2I) \leqslant \beta F(I)$. In particular, the restriction $\alpha > 1$ ruled out the function $F \equiv 1$.

PROOF OF THEOREM 3.2. Let ν be the measure $g\,d\mu$, and define f_I, f^*, and $f^\#$ with respect to ν. Fix I and let $x \in J \subseteq I$. Then

$$\int_J |f - f_J|\, d\nu \leqslant CF(J)\mu(J) \leqslant CF(I)\mu(J).$$

Hence,

$$\frac{1}{\nu(J)} \int_J |f - f_J|\, d\nu \leqslant CF(I) \frac{1}{\nu(J)} \int_J g^{-1} g\, d\mu \leqslant CF(I)(g^{-1})^*(x),$$

and so, taking supremums over $J \subseteq I$, we have $f^{\#,I}(x) \leqslant CF(I)(g^{-1})^*(x)$. By Corollary 2.2,

$$\begin{aligned} \int_I |f - f_I|^q g^{q-1}\, d\nu &\leqslant C \int_I (f^{\#,I})^q g^{q-1}\, d\nu \\ &\leqslant CF(I)^q \int_I (g^{-1})^{*q} g^{q-1}\, d\nu. \end{aligned}$$

But $g^{q-1} \in A_q(d\nu)$, as

$$\begin{aligned} &\frac{1}{\nu(I)} \int_I g^{q-1}\, d\nu \left(\frac{1}{\nu(I)} \int_I g^{-1}\, d\nu \right)^{q-1} \\ &\qquad = \frac{1}{\nu(I)} \int_I (g^{-1})^{1/(p-1)}\, d\nu \left(\frac{1}{\nu(I)} \int_I g^{-1}\, d\nu \right)^{q-1} \end{aligned}$$

which is bounded by the hypothesis $g^{-1} \in A_p(d\nu)$. Applying Muckenhoupt's Theorem, we conclude

$$\int_I |f - f_I|^q g^q\, d\mu \leqslant CF(I)^q \int_I (g^{-1})^q g^q\, d\mu = CF(I)^q \mu(I).$$

IV. The Commutator Theorem.

DEFINITION 4.1. Let w be an A_∞ weight and b an L^1 function. Then b is in the weighted BMO class BMO_w provided

$$\sup_I \frac{1}{w(I)} \int_I |b - b_I| < \infty \quad (\text{here } b_I = b_{I,dx}).$$

The main result of this paper is the following:

THEOREM 4.2 (THE COMMUTATOR THEOREM). *Let $\mu, \lambda \in (A_p)$ and put $\nu = (\mu\lambda^{-1})^{1/p}$, for some $1 < p < \infty$, and suppose that $b \in L^1$. Then*

(i) *If $b \in \mathrm{BMO}_\nu$, the commutator $[H, M_b]$ is a bounded map from $L^p(\mu)$ into $L^p(\lambda)$, with*

$$\int |[H, M_b]f|^p \lambda \leqslant C \int |f|^p \mu.$$

(ii) *Conversely, if $[H, M_b]$: $L^p(\mu) \to L^p(\lambda)$ is bounded, then $b \in \mathrm{BMO}_\nu$.*

To prove part (i), we will need a series of lemmas. Throughout, μ and λ will be in (A_p), $\nu = (\mu\lambda^{-1})^{1/p}$, and $b \in \mathrm{BMO}_\nu$. An exponent with a prime will denote the conjugate exponent, so $1/p + 1/p' = 1$.

LEMMA 4.3. *There exists an $\varepsilon > 0$ so that, for all $1 \leqslant r \leqslant p' + \varepsilon$,*

$$\frac{1}{|I|} \int_I |b - b_I|^r \mu^{-r/p} \leqslant C\left(\frac{1}{|I|} \int_I \lambda^{-1/p}\right)^r \quad \textit{for each interval } I.$$

PROOF. It will suffice to show this for some $r > p'$. Smaller values of r follow from Hölder's Inequality.

Choose r so that Reverse Hölder holds for the weights $\mu^{-p'/p}$ and $\lambda^{-p'/p}$ with exponent $1 + \delta = r/p'$. Fix I and let $x \in I$. If J contains x, then

$$\frac{1}{|I|} \int_J |b - b_J| \leqslant C \frac{1}{|J|} \int_J \nu \leqslant C\nu^*(x),$$

and hence $b^\#(x) \leqslant C\nu^*(x)$. By Corollary 2.2,

$$\int |b - b_I|^r \mu^{-r/p} \leqslant C \int_I (b^\#)^r \mu^{-r/p} \leqslant C \int_I (\nu^*)^r \mu^{-r/p}.$$

But

$$\frac{1}{|J|} \int_J \mu^{-r/p} \left(\frac{1}{|J|} \int_J \mu^{r'/p}\right)^{r/r'} \leqslant \frac{1}{|J|} \int_J \mu^{-r/p} \left(\frac{1}{|J|} \int_J \mu\right)^{r/p},$$

by Hölder's Inequality,

$$\leqslant C\left(\frac{1}{|J|} \int_J \mu^{-p'/p}\right)^{r/p'} \left(\frac{1}{|J|} \int_J \mu\right)^{r/p},$$

by Reverse Hölder,

$$\leqslant C \quad \text{by } (A_p).$$

Thus, $\mu^{-r/p} \in (A_r)$ and Muckenhoupt's Theorem applies. So

$$\int_I |b - b_I|^r \mu^{-r/p} \leqslant C \int_I \nu^r \mu^{-r/p} = C \int_I \lambda^{-r/p}.$$

Similarly, $\lambda^{-r/p} \in (A_r)$, and

$$\frac{1}{|I|} \int_I \lambda^{-r/p} \left(\frac{1}{|I|} \int_I \lambda \right)^{r/p} \leqslant C,$$

so that

$$\frac{1}{|I|} \int_I \lambda^{-r/p} \leqslant C \left(\frac{1}{|I|} \int_I \lambda \right)^{-r/p}.$$

But by Cauchy–Schwartz,

$$1 \leqslant \frac{1}{|I|} \int_I \lambda^{1/p} \frac{1}{|I|} \int_I \lambda^{-1/p}$$

so that

$$\frac{1}{|I|} \int |b - b_I|^r \mu^{-r/p} \leqslant C \left(\frac{1}{|I|} \int_I \lambda \right)^{-r/p} \leqslant C \left(\frac{1}{|I|} \int_I \lambda^{1/p} \right)^{-r}, \quad \text{by Hölder's,}$$

$$\leqslant C \left(\frac{1}{|I|} \int_I \lambda^{-1/p} \right)^{r}.$$

We will need some further notation. q will be a number near p but less than p. Let $r \geqslant 1$ and w a weight. Define

$$S_r(b; w, I) = \left(\frac{1}{|I|} \int_I |b - b_I|^r w^r \, dx \right)^{1/r},$$

$$\Lambda_r(f; w, I) = \left(\frac{1}{|I|} \int_I |fw|^r \right)^{1/r}, \quad \text{and}$$

$$K_r^*(b, f, w)(x) = \sup_{1 \ni x} S_{rq'}(b, w, I) \Lambda_{rq} l(f; w^{-1}, I).$$

Also put $K^* = K_1^*$, and let M_λ^* denote the weighted maximal function

$$M_\lambda^* g(x) = \sup \left\{ \frac{1}{\lambda(I)} \int_I |g| \lambda \, dy \colon x \in I \right\}.$$

LEMMA 4.4. *For an appropriate choice of $q < p$, and for any r with $1 \leqslant r < p/q$, there exists a weight w depending on r such that*

(i) $w^{rq'} \in (A_{q'})$, *and*

(ii) $\int [K_r^*(b, f, w)(x)]^p \lambda(x) \, dx \leqslant C \int |f|^p \mu(x) \, dx$.

PROOF. We will choose w as $w = \mu^{1/p}\lambda^{1/p-1/rq}$. To show (i), it will suffice to show that $w^{p'} \in (A_{p'})$. For then (i) will hold by Reverse Hölder if q is chosen sufficiently near p. For the $A_{p'}$ condition, let $t > 1$. Then

$$\frac{1}{|I|}\int_I w^{p'}\left(\frac{1}{|I|}\int_I w^{-p}\right)^{p'/p} = \left(\frac{1}{|I|}\int_I \mu^{-p'/p}\lambda^{p'/p-p'/rq}\right)\left(\frac{1}{|I|}\int_I \mu\lambda^{p/rq-1}\right)^{p'/p}$$
$$\leqslant \left[\frac{1}{|I|}\int_I \mu^{-tp'/p}\left(\frac{1}{|I|}\int_I \mu^{t}\right)^{p'/p}\right]^{1/t}$$
$$\cdot\left[\frac{1}{|I|}\int_I \lambda^{t'(p'/p-p'/rq)}\left(\frac{1}{|I|}\int_I \lambda^{t'(p/rq-1)}\right)^{p'/p'}\right]^{1/t'}.$$

The first term

$$\left[\frac{1}{|I|}\int_I \mu^{-tp'/p}\left(\frac{1}{|I|}\int_I \mu^{t}\right)^{p'/p}\right]^{1/t}$$

is bounded by Reverse Hölder and $\mu \in (A_p)$ for t near one, say $t \leqslant t_0$. For the second term, consider the exponent $t'(p/rq - 1)$. As $q \to p$, $r \to 1$, and $p/rq \to 1$. So we can choose q sufficiently near p so that choosing t' with $t'(p/rq - 1) = 1$ still keeps $t \leqslant t_0$ for $r = 1$, and hence for $r > 1$ as well. Then the second term is

$$\left[\frac{1}{|I|}\int_I \bar{\lambda}^{p'/p}\left(\frac{1}{|I|}\int_I \lambda\right)^{p'/p}\right]^{1/t'},$$

which is bounded, since $\lambda \in (A_p)$. Hence $w^{p'} \in (A_{p'})$ and (i) follows.

We will show that

$$\left(\frac{1}{|I|}\int_I \lambda\right)^{1/rq} S_{rq'}(b; w, I) \leqslant C \quad \text{for all } I. \tag{4}$$

Assume (4) for the moment. For each x, there exist intervals I_x containing x which approximate K_r^*, that is

$$\int \left[K_r^*(b, f, w)(x)\right]^p \lambda(x)\, dx \leqslant 2\int \left[S_{rq'}(b; w, I_x)\Lambda_{rq}(f; w^{-1}, I_x)\right]^p \lambda.$$

Now

$$\Lambda_{rq}(f; w^{-1}, I_x) = \left(\frac{1}{|I|}\int_{I_x} |f|^{rq} w^{-rq}\lambda^{-1}\lambda\right)^{1/rq}$$
$$= (\lambda_{I_x})^{1/rq}\left(\frac{1}{\lambda(I_x)}|fw^{-1}|^{rq}\lambda^{-1}\lambda\right)^{1/rq}$$
$$\leqslant (\lambda_{I_x})^{1/rq}\left[M_\lambda^*\left(|fw^{-1}|^{rq}\lambda^{-1}\right)(x)\right]^{1/rq},$$

so that, by (4),

$$\int K_r^*(b, f, w)^p \lambda \leqslant C \int \left[M_\lambda^*\left(|fw^{-1}|^{rq} \lambda^{-1} \right) \right]^{p/rq} \lambda$$

$$\leqslant C \int |fw^{-1}|^p \lambda^{-p/rq} \lambda,$$

by the boundedness of the Hardy–Littlewood maximal function,

$$= C \int |f|^p \mu.$$

So to show (ii), we must only verify (4). First,

$$S_{rq'}^{rq'}(b; w, I) = \frac{1}{|I|} \int_I |b - b_I|^{rq'} \mu^{-rq'/p} \lambda^{-q'(1/q - r/p)}.$$

Choose s so that $sq'(1/q - r/p) = p'/p$. q near p means that s is large, so that $rq's' \leqslant p' + \varepsilon$, the exponent in Lemma 4.3. Hence,

$$S_{rq'}^{rq'}(b; w, I) \leqslant \left(\frac{1}{|I|} \int_I |b - b_I|^{rq's'} \mu^{rq's'/p} \right)^{1/s'} \left(\frac{1}{|I|} \int_I \lambda^{-p'/p} \right)^{1/s}$$

$$\leqslant C \left(\frac{1}{|I|} \int_I \lambda^{-1/p} \right)^{rq'} \left(\frac{1}{|I|} \int_I \lambda^{-p'/p} \right)^{1/s}, \quad \text{by Lemma 4.3,}$$

$$\leqslant C \left(\frac{1}{|I|} \int_I \lambda^{-p'/p} \right)^{rq'/p' + 1/s} = C \left(\frac{1}{|I|} \int_I \lambda^{-p'/p} \right)^{pq'/p'q},$$

or

$$S_{rq'}(b; w, I) \leqslant C \left[\left(\frac{1}{|I|} \int_I \lambda^{-p'/p} \right)^{p/p'} \right]^{1/rq}$$

and (4) holds by the A_p condition.

The main ingredient of the proof is an estimate of the sharp function of $[H, M_b]f$, set out in the lemma below.

LEMMA 4.5. *Let w and $\tilde{w}$ be weights with $w^{q'}, \tilde{w}^{rq'} \in (A_{q'})$ for some $r > 1$. Then*

$$([H, M_b]f)^{\#}(x) \leqslant C \Big[K^*(b, f, w)(x) + K^*(b, Hf, w)(x)$$

$$+ K_r^*(b, f, \tilde{w})(x) + \left(M_\lambda^*\left(|f\nu|^q \right)(x) \right)^{1/q} \Big].$$

PROOF. Let $g = [H, M_b]f$. We must estimate $g^{\#}$. So fix x and I containing x. Let x_0 be the center of I. Define $f_1 = f\chi_{2I}$, and $f_2 = f - f_1$. For any constant c,

$$\frac{1}{|I|} \int_I |g - g_I| \leqslant \frac{2}{|I|} \int_I |g - c|.$$

In particular,

$$\frac{1}{|I|} \int_I |g - g_I| \leqslant \frac{2}{|I|} \int_I |g - H(b - b_I) f_2(x_0)|.$$

Now

$$g = [H, M_b]f = \left[H, M_{b-b_I}\right]f$$
$$= H(b - b_I)f_1 + H(b - b_I)f_2 - (b - b_I)Hf$$

so that

$$\frac{1}{|I|}\int_I |g - g_I| \leqslant \frac{2}{|I|}\int |b - b_I||Hf| + \frac{2}{|I|}\int_I |H(b - b_I)f_1|$$
$$+ \frac{2}{|I|}\int_I |H(b - b_I)f_2(t) - H(b - b_I)f_2(x_0)|\, dt$$
$$= 2(K_1 + K_2 + K_3).$$

For these terms, first

$$K_1 = \frac{1}{|I|}\int_I |b - b_I|w|Hf|w^{-1}$$
$$\leqslant \left(\frac{1}{|I|}\int_I |b - b_I|^{q'} w^{q'}\right)^{1/q'} \left(\frac{1}{|I|}\int_I |Hf|^q w^{-q}\right)^{1/q}$$
$$= S_{q'}(b; w, I)\Lambda_q(Hf; w^{-1}, I) \leqslant K^*(b, Hf, w)(x).$$

For the second piece,

$$K_2 \leqslant \left(\frac{1}{|I|}\int_I |H(b - b_I)f_1|^r\right)^{1/r}$$
$$\leqslant |I|^{-1/r}\left(\int_0^{2\pi} |H(b - b_I)f_1|\right)^{1/r}$$
$$\leqslant C|I|^{-1/r}\left(\int |b - b_I|^r |f_1|^r\right)^{1/r}, \quad \text{by the Theorem of M. Riesz,}$$
$$= 2^{1/r}C\left(\frac{1}{|2I|}\int_{2I} |b - b_I|^r |f|^r\right)^{1/r}$$
$$\leqslant 2^{1/r}C\left[\left(\frac{1}{|2I|}\int_{2I} |b - b_{2I}|^r |f|^r\right)^{1/r} + |b_I - b_{2I}|\left(\frac{1}{|2I|}\int_{2I} |f|^r\right)^{1/r}\right]$$
$$= 2^{1/r}C[A + B].$$

Here,

$$A = \left(\frac{1}{|2I|}\int_{2I} |b - b_I|^r \tilde{w}^r |f|^r \tilde{w}^{-r}\right)^{1/r}$$
$$\leqslant S_{rq'}(b; \tilde{w}, 2I)\Lambda_{rq}(f; \tilde{w}^{-1}, 2I) \leqslant K_r^*(b, f, \tilde{w})(x).$$

To estimate B,

$$\begin{aligned}|b_I - b_{2I}| &\leqslant \frac{1}{|I|}\int_I |b - b_{2I}| \leqslant \frac{2}{|2I|}\int_{2I} |b - b_{2I}| \\ &\leqslant 2S_{q'}(b;\tilde{w},2I)\left(\frac{1}{|2I|}\int_{2I}\tilde{w}^{-q}\right)^{1/q} \\ &\leqslant 2S_{rq'}(b;\tilde{w},2I)\left(\frac{1}{|2I|}\int_{2I}\tilde{w}^{-rq}\right)^{1/rq}\end{aligned}$$

so that

$$\begin{aligned}B &\leqslant 2S_{rq'}(b;\tilde{w},2I)\left(\frac{1}{|2I|}\int_{2I}\tilde{w}^{-rq}\right)^{1/rq}\left(\frac{1}{|2I|}\int_{2I}|f|^r\tilde{w}^{-r}\tilde{w}^{r}\right)^{1/r} \\ &\leqslant 2S_{rq'}(b;\tilde{w},2I)\Lambda_{rq}(f;\tilde{w}^{-1},2I)\left(\frac{1}{|2I|}\int_{2I}\tilde{w}^{-rq}\right)^{1/rq}\left(\frac{1}{|2I|}\int_{2I}\tilde{w}^{rq'}\right)^{1/rq'} \\ &\leqslant CK_r^*(b,f,\tilde{w})(x)\end{aligned}$$

since $w^{rq'} \in (A_{q'})$. Hence $K_2 \leqslant CK_r^*(b,f,\tilde{w})(x)$. For K_3, let $t \in I$. Then

$$\begin{aligned}&|H(b-b_I)f_2(t) - H(b-b_I)f_2(x_0)| \\ &\quad\leqslant \int |Q(t-y) - Q(x_0-y)||b-b_I||f_2|\,dy \\ &\quad\leqslant C\int \frac{|t-x_0|}{|t-y||x_0-y|}|f_2(y)||b-b_I|\,dy \\ &\quad= C\int_{|x_0-y|>\delta}\frac{t-x_0}{|t-y||x_0-y|}|f||b-b_I|\end{aligned}$$

where $\delta = |I|$.

Since $t \in I$,

$$|t-x_0| \leqslant \delta/2 \leqslant \tfrac{1}{2}|x_0-y|,$$

and

$$|t-y| \geqslant |x_0-y| - |t-x_0| \geqslant \tfrac{1}{2}|x_0-y|.$$

Hence,

$$|H(b-b_I)f_2(t) - H(b-b_I)f_2(x_0)| \leqslant C\delta\int_{|x_0-y|>\delta}\frac{1}{|x_0-y|^2}|f||b-b_I|\,dy,$$

and since this holds for all $t \in I$, the same bound must hold for K_3. Therefore,

$$\begin{aligned}K_3 &\leqslant C\delta\sum_k\int_{2^{k-1}\delta<|x_0-y|\leqslant 2^k\delta}\frac{1}{|x_0-y|^2}|f||b-b_I|\,dy \\ &\leqslant C\sum_k 2^{2-2k}\delta^{-1}\int_{|x_0-y|\leqslant 2^k\delta}|f||b-b_I|\,dy.\end{aligned}$$

Let $I_k = 2^k I$. Then

$$\begin{aligned}
K_3 &\leqslant 8C\sum_k 2^{-k}\frac{1}{|I_{k+1}|}\int_{I_{k+1}} |b - b_I||f| \\
&\leqslant 16C\sum_k 2^{-k}\left(\frac{1}{|I_k|}\int_{I_k} |b - b_{I_k}||f| + \frac{1}{|I_k|}\int_{I_k} |b_I - b_{I_k}||f|\right) \\
&= 16C\sum_k 2^{-k}(L_k + M_k).
\end{aligned}$$

But

$$L_k \leqslant S_{q'}(b; w, I_k)\Lambda_q(f; w^{-1}, I_k) \leqslant K^*(b, f, w)(x),$$

so that

$$K_3 \leqslant 16C\left[K^*(b, f, w)(x) + \sum_k 2^{-k}M_k\right].$$

We must show

$$\sum_k 2^{-k}M_k \leqslant C\left[M_\lambda^*\left(|f\nu|^q\right)(x)\right]^{1/q}. \tag{5}$$

To prove (5), we will use two facts. First, since $b \in \mathrm{BMO}_\nu$,

$$\int_J |b - b_J| \leqslant C\nu(J) \quad \text{for each interval } J, \tag{6}$$

and second, since $\nu \in (A_\infty)$, there exists a $\delta > 0$ such that

$$\frac{\nu(E)}{\nu(J)} \leqslant C\left(\frac{|E|}{|J|}\right)^\delta \quad \text{for all measurable sets } E \subseteq J. \tag{7}$$

Thus,

$$\begin{aligned}
|b_I - b_{I_k}| &\leqslant \sum_{n=0}^{k-1} |b_{I_n} - b_{I_{n+1}}| \leqslant \sum_{n=0}^{k-1} \frac{1}{|I_n|}\int_{I_n} |b - b_{I_{n+1}}| \\
&\leqslant 2\sum_{n=0}^{k-1} \frac{1}{|I_{n+1}|}\int_{I_{n+1}} |b - b_{I_{n+1}}| \\
&\leqslant C\sum_{n=0}^{k-1} \frac{\nu(I_{n+1})}{|I_{n+1}|} \quad \text{by (2)} \\
&\leqslant C\nu_{I_k}\sum_{n=0}^{k-1} \frac{\nu(I_{n+1})}{\nu(I_k)}\cdot\frac{|I_k|}{|I_{n+1}|} \\
&\leqslant C\nu_{I_k}\sum_{n=0}^{k-1}\left(\frac{|I_k|}{|I_{n+1}|}\right)^{1-\delta} \quad \text{by (3)}, \\
&= C\nu_{I_k}\sum_{n=0}^{k-1} 2^{(k-n-1)(1-\delta)} \leqslant C\nu_{I_k}2^{k(1-\delta)},
\end{aligned}$$

and hence

$$\begin{aligned}\sum_k 2^{-k} M_k &\leqslant C \sum_k 2^{-k\delta} \nu_{I_k} \frac{1}{|I_k|} \int_{I_k} |f| \\ &\leqslant C \sum_k 2^{-k\delta} \nu_{I_k} \left(\frac{1}{|I_k|} \int_{I_k} |f\nu|^q \lambda \right)^{1/q} \left(\frac{1}{|I_k|} \int_{I_k} \nu^{-q'} \lambda^{-q'/q} \right)^{1/q'} \\ &= C \sum_k 2^{-k\delta} \nu_{I_k} (\lambda_{I_k})^{1/q} \left(\frac{1}{\lambda(I_k)} \int_{I_k} |f\nu|^q \lambda \right)^{1/q} \left(\frac{1}{|I_k|} \int_{I_k} \nu^{-q'} \lambda^{-q'/q} \right)^{1/q'} \\ &\leqslant C \left[M_\lambda^* (|f\nu|^q)(x) \right]^{1/q} \sum_k 2^{-k\delta} \nu_{I_k} (\lambda_{I_k})^{1/q} \left(\frac{1}{|I_k|} \int_{I_k} \nu^{-q'} \lambda^{-q'/q} \right)^{1/q'}.\end{aligned}$$

So (5) will hold provided

$$\nu_I (\lambda_I)^{1/q} \left(\frac{1}{|I|} \int_I \nu^{-q'} \lambda^{-q'/q} \right)^{1/q'} \leqslant C \quad \text{for all } I. \tag{8}$$

To show (8),

$$\nu^{-q'} \lambda^{-q'/q} = \mu^{-q'/p} \lambda^{-q'(1/q - 1/p)}.$$

Choose s so that $sq'(1/q - 1/p) = p'/p$. So s is large, and Reverse Hölder will apply to $\mu^{-p'/p}$ with exponent $q's'/p'$ for q near p. So

$$\begin{aligned}(8) &\leqslant (\lambda_I)^{1/q} (\mu_I)^{1/p} \left(\lambda_I^{-p'/p} \right)^{1/p'} \left(\frac{1}{|I|} \int_I \mu^{\frac{-q's'}{p}} \right)^{1/s'q'} \left(\frac{1}{|I|} \int_I \lambda^{-p'/p} \right)^{1/sq'} \\ &\leqslant C (\lambda_I)^{1/q} (\mu_I)^{1/p} \left(\lambda_I^{-p'/p} \right)^{1/p' + 1/sq'} \left(\mu^{-p'/p} \right)^{1/p'}, \quad \text{by Reverse Hölder,} \\ &= C (\lambda_I)^{1/q} \left(\lambda_I^{-p'/p} \right)^{p/p'q} (\mu_I)^{1/p} \left(\mu_I^{-p'/p} \right)^{1/p'}\end{aligned}$$

which is bounded, since μ and λ are in (A_p).

PROOF OF THEOREM 4.2, PART (i). By Lemma 4.5,

$$\int ([H, M_b] f)^{\#p} \lambda \leqslant C \left[\int K^*(b, f, w)^p \lambda + \int K^*(b, Hf, w)^p \lambda + \int K_r^*(b, f, \tilde{w})^p \lambda + \int \left(M_\lambda^* (|f\nu|^q) \right)^{p/q} \lambda \right],$$

for w and $\tilde{w}$ satisfying $w^{q'}$ and $\tilde{w}^{rq'} \in (A_{q'})$. By Lemma 4.4, we can choose an $r > 1$ and such weights w and $\tilde{w}$ so that

$$\int K^*(b, f, w)^p \lambda \leqslant C \int |f|^p \mu,$$

and

$$\int K_r^*(b, f, \tilde{w})^p \lambda \leqslant \int |f|^p \mu.$$

Therefore,

$$\int \left[([H, M_b]f)^{\#}(x)\right]^p \lambda(x) \leqslant C\left[\int |f|^p\mu + \int |Hf|^p\mu + \int M_\lambda\left(|f\mu|^q\right)^{p/q}\lambda\right].$$

By the Theorem of Hunt, Muckenhoupt, and Wheeden,

$$\int |Hf|^p\mu \leqslant C\int |f|^p\mu,$$

and by Hardy and Littlewood's Theorem,

$$\int M_\lambda^*\left(|f\nu|^q\right)^{p/q}\lambda \leqslant C\int |f\nu|^p\lambda = C\int |f|^p\mu.$$

So we conclude

$$\int ([H, M_b]f)^{\#p}\lambda \leqslant C\int |f|^p\mu.$$

Now let

$$k = \frac{1}{2\pi}\int_0^{2\pi} [H, M_b]f.$$

By the Sharp Function Theorem,

$$\int |[H, M_b]f - k|\lambda \leqslant C\int |f|^p\mu.$$

Thus,

$$\left(\int |[H, M_b]f|^p\lambda\right)^{1/p} \leqslant \left(\int |[H, M_b]f - k|^p\lambda\right)^{1/p} + k\left(\int \lambda\right)^{1/p}$$
$$\leqslant C\left(\int |f|^p\mu\right)^{1/p} + k\left(\int \lambda\right)^{1/p}.$$

Finally, $|k| = (1/2\pi)\int_T [H, M_{b-b_T}]f\,|$, and estimating this just as in Lemma 4.5 for K_1 and K_2 gives

$$|k| \leqslant \frac{I}{2\pi}\int_T |b - b_T||Hf| + \frac{1}{2\pi}\int_T |H(b - b_T)f|$$
$$\leqslant [K^*(b, Hf, \omega)(x) + K^*(b, f, \tilde{\omega})(x)]$$

for any $x \in T$, so that

$$\left(\int_T |k|^\rho\lambda\right)^{1/\rho} \leqslant \left[\left(\int_T K^*(b, Hf, \omega)^\rho\lambda\right)^{1/\rho} + \left(\int_T K^*(b, f, \tilde{\omega})^\rho\lambda\right)^{1/\rho}\right]$$
$$\leqslant \left(\|Hf\|_{L^\rho(\mu)} + \|f\|_{L^\rho(\mu)}\right) \leqslant \|f\|_{L^\rho(\mu)}$$

by Lemma 4.4 with appropriate choices of $\omega, \tilde{\omega}$ and $r > 1$, and by the Hunt, Muckenhoupt, and Wheeden Theorem, and part (i) follows.

PROOF OF THEOREM 4.2, PART (ii). We may assume that

$$\int |[H, M_b]f|^p \lambda \leqslant \int |f|^p \mu,$$

and that b is real. Fix I, centered at x_0. Since $[H, M_b] = [H, M_{b-b_I}]$, we may further assume that $b_I = 0$. Put $M = (1/\nu(I))\int_I |b|$. We must bound M, independent of I. So we can assume that M is quite large. Let $E = \{x \in I\colon b(x) \geqslant 0\}$. We may assume without loss of generality that $|E| \geqslant \frac{1}{2}|I|$. Let $E' \subseteq E$ have measure $|E'| = |I \sim E|$. Define ψ by $\psi = \chi_{E'} - \chi_{I\sim E}$. Then $\psi b \geqslant 0$, and $\int \psi = 0$. Also, since $\int_I b = 0$, $\int_{I\sim E}(-b) = \int_E b$. Hence,

$$\int \psi b \geqslant \int_{I\sim E}(-b) = \frac{1}{2}\left(\int_{i\sim E}(-b) + \int_E b\right) = \frac{1}{2}\int_I |b| = \frac{1}{2}M\nu(I).$$

Now let $x \in 2I$. Then

$$|[H, M_b]\psi(x)| \geqslant |H(b\psi)(x)| - |b(x)||H\psi(x)|.$$

To analyze this,

$$\begin{aligned} |H(b\psi)(x)| &= \int_I |Q(x-y)|(b\psi)(y)\,dy \\ &\geqslant \frac{2C_1}{|x-x_0|}\int_I (b\psi) \geqslant \frac{C_1}{|x-x_0|}M\nu(I), \end{aligned}$$

while

$$\begin{aligned} |H\psi(x)| &= \left|\int Q(x-y)\psi(y)\,dy\right| \\ &= \left|\int [Q(x-y) - Q(x-x_0)]\psi(y)\,dy\right|, \quad \text{since} \int \psi = 0, \\ &\leqslant C_2\int_I \frac{|y-x_0|}{|x-y||x-x_0|}\,dy \leqslant C_2|I|^2|x-x_0|^{-2}, \end{aligned}$$

and we conclude,

$$|[H, M_b]\psi(x)| \geqslant \frac{C_1 M\nu(I)}{|x-x_0|} - \frac{C_2|I|^2|b(x)|}{|x-x_0|^2}, \qquad x \in 2I. \tag{9}$$

Choose $\alpha < C_1$ and β a small universal constant. We can think of the unit circle as $(x_0 - \pi, x_0 + \pi)$. β will be chosen so small that $2\beta M^{1/p}|I| < \pi$. For otherwise,

$$\frac{\pi}{2\beta M^{1/p}} \leqslant |I| = \int_I \nu^{1/p}\nu^{-1/p} \leqslant \nu(I)^{1/p}\left(\int_0^{2\pi} \nu^{-q/p}\right)^{1/q},$$

or

$$\nu(I) \geqslant \left(\frac{\pi}{2\beta M^{1/p}}\right)^p \left(\int \nu^{-q/p}\right)^{-p/q} = \frac{1}{C\beta^p M},$$

and so

$$M = \frac{1}{\nu(I)}\int_I |b| \leqslant C\beta^p M\|b\|_1.$$

We can choose β sufficiently small so that $C\beta^p\|b\|_1 < 1$, leading to a contradiction. Now put

$$J = \left\{x\colon |I| < (x - x_0) < \beta M^{1/p}|I|\right\},$$

$$F = \left\{x \in J\colon C_2|b(x)||I|^2 < (c_1 - \alpha)M\nu(I)(x - x_0)\right\}, \quad \text{and}$$

$$G = J \sim F.$$

By the argument above, $J \subseteq (x_0, x_0 + \pi/2)$, and so $2J \subseteq T$, the unit circle. We can assume that M is large enough so that $I \subseteq 2J$. Then

$$\begin{aligned}
\mu(I) &\geqslant \int |\psi|^p\mu \geqslant \int |[H, M_b]\psi|^p\lambda \geqslant \int_F |[H, M_b]\psi|^p\lambda \\
&\geqslant \int_F \left|\frac{c_1 M\nu(I)}{x - x_0} - \frac{c_2|I|^2 b(x)}{(x - x_0)^2}\right|^p \lambda, \quad \text{by (1)}, \\
&\geqslant \int_F \left[\frac{\alpha M\nu(I)}{x - x_0}\right]^p \lambda \geqslant \int_F \left[\frac{\alpha M\nu(I)}{\beta M^{1/p}|I|}\right]^p \lambda \\
&= \lambda(F)\left(\frac{\alpha}{\beta}\nu_I\right)^p M^{p-1} \geqslant \lambda(F)\left(\frac{\alpha}{\beta}\nu_I\right)^p,
\end{aligned}$$

so that

$$\begin{aligned}
\lambda(F) &\leqslant \left(\frac{\beta}{\alpha}\right)^p (\nu_I)^{-p}\mu(I) \leqslant \left(\frac{\beta}{\alpha}\right)^p \left(\frac{1}{|I|}\int_I \nu^{-1}\right)^p \mu(I) \\
&= \left(\frac{\beta}{\alpha}\right)^p \left(\frac{1}{|I|}\int_I \lambda^{1/p}\mu^{-1/p}\right)^p \mu(I) \\
&\leqslant \left(\frac{\beta}{\alpha}\right)^p \lambda_I \left(\frac{1}{|I|}\int_I \mu^{-q/p}\mu(I)\right), \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1, \\
&= \left(\frac{\beta}{\alpha}\right)^p \lambda(I)\mu_I(\mu^{-q/p})_I^{p/q} \leqslant C\left(\frac{\beta}{\alpha}\right)^p \lambda(I), \quad \text{by the } A_p \text{ condition}, \\
&\leqslant C(\beta/\alpha)^p\lambda(2J).
\end{aligned}$$

Now

$$\begin{aligned}
\frac{|F|}{|2J|} &= \frac{1}{|2J|}\int_F \lambda^{1/p}\lambda^{-1/p} \leqslant \left(\frac{1}{|2J|}\int_F \lambda\right)^{1/p} \left(\frac{1}{|2J|}\int_{2J} \lambda^{-q/p}\right)^{1/q} \\
&= \left[\frac{\lambda(F)}{\lambda(J)}\right]^{1/p} \left(\frac{1}{|2J|}\int_{2J} \lambda\right)^{1/p} \left(\frac{1}{|2J|}\int_{2J} \lambda^{-q/p}\right)^{1/q} \\
&\leqslant C\left[\frac{\lambda(F)}{\lambda(2J)}\right]^{1/p}, \quad \text{by } (A_p), \\
&\leqslant C\frac{\beta}{\alpha},
\end{aligned}$$

and thus $|F| \leqslant 2C\beta/\alpha \cdot \beta M^{1/p}|I|$. We will also require β to be so small that $2C\beta/\alpha \leqslant \frac{1}{3}$. Then for M large,

$$|G| \geqslant \tfrac{1}{2}\beta M^{1/p}|I|. \tag{10}$$

Notice that β does not depend on M or $|I|$. Next let

$$H^*f(x) = \int Q(x-y)f(y)\,dy.$$

Then the adjoint of the commutator is $[H, M_b]^* = -[H^*, M_b]$. Also, if T is any operator satisfying $\int |Tf|^p\lambda \leqslant \int |f|^p\mu$, then any easy argument shows that its adjoint satisfies

$$\left(|T^*f|^q\mu^{-q/p}\right) \leqslant \left(\int |f|^q\lambda^{-q/p}\right).$$

In particular, for $T = [H, M_b]$, we have

$$\int |[H^*, M_b]g|^q\mu^{-q/p} \leqslant \int |g|^q\lambda^{-q/p}. \tag{11}$$

Now let $g = (\operatorname{sgn} b)\chi_G$. Then for $x \in I$,

$$\begin{aligned}
|[H^*, M_b]g(x)| &\geqslant \int_G Q(y-x)(bg)(y)\,dy - |b(x)||H^*g(x)| \\
&\geqslant 2c_1\int \frac{|b(y)|}{|y-x_0|} - |b(x)|\int_G Q(y-x)\,dy \\
&\geqslant \frac{2c_1}{c_2}(c_1-\alpha)\frac{M\nu(I)}{|I|^2}|G| - c_3|b(x)|\int_1^{\beta M^{1/p}|I|}\frac{dy}{y} \\
&\geqslant CM^{1+1/p}\nu_I - C'|b(x)|\log M, \quad \text{by (2)}.
\end{aligned}$$

Let $D = \{x \in I\colon |b(x)| \leqslant 2M\nu_I\}$. Then for $x \in D$,

$$\begin{aligned}
|[H^*, M_b]g(x)| &\geqslant \nu_I\left[CM^{1+1/p} - 2C'M\log M\right] \\
&\geqslant C\nu_I M^{1+1/p} \quad \text{for large } M.
\end{aligned}$$

Next,

$$2M|I \sim D| = \int_{I\sim D} 2M \leqslant |I|\int_I \frac{|b|}{\nu(I)} \leqslant M|I|,$$

so that $|I \sim D| \leqslant \frac{1}{2}|I|$, or $|D| \geqslant \frac{1}{2}|I|$. By (11),

$$\int |[H^*, M_b]g|^q\mu^{-q/p} \leqslant \int |g|^q\lambda^{-q/p} \leqslant \int_G \lambda^{-q/p} \leqslant \int_{2J}\lambda^{-q/p}.$$

Thus,

$$\int_{2J}\lambda^{-q/p} \geqslant \int_D |[H^*, M_b]g|^q\mu^{-q/p} \geqslant C(\nu_I)^q M^{q+q/p}\int_D \mu^{-q/p}.$$

But

$$\frac{1}{2} \leqslant \frac{|D|}{|I|} = \frac{1}{|I|}\int_D \mu^{1/p}\mu^{-1/p} \leqslant (\mu_I)^{1/p}\left(\frac{1}{|I|}\int_D \mu^{-q/p}\right)^{1/q},$$

so that $(1/|I|)\int_D \mu^{-q/p} \geqslant 2^{-q}(\mu_I)^{-q/p}$, and thus

$$\frac{1}{|I|}\int_{2J} \lambda^{-q/p} \geqslant C(\nu_1)^q M^{q+q/p} 2^{-q}(\mu_I)^{-q/p},$$

or

$$\begin{aligned}
M^{q+q/p} &\leqslant C\frac{1}{|I|}\int_{2J} \lambda^{-q/p}(\mu_I)^{q/p}(\nu_I)^{-q} \\
&\leqslant C\frac{1}{|I|}\int_{2J} \lambda^{-q/p}(\mu_I)^{q/p}\left(\frac{1}{|I|}\int_I \nu^{-1}\right)^q \\
&= C\frac{|2J|}{|I|}(\lambda^{-q/p})_{2J}(\mu_I)^{q/p}\left(\frac{1}{|I|}\int_I \lambda^{1/p}\mu^{-1/p}\right)^q \\
&\leqslant C\frac{|2J|}{|I|}(\lambda^{-q/p})_{2J}(\mu_I)^{q/p}(\mu^{-q/p})_I\left(\frac{1}{|I|}\int_I \lambda\right)^{q/p} \\
&\leqslant C\left(\frac{|2J|}{|I|}\right)^{1+q/p}(\mu_I)^{q/p}(\mu^{-q/p})_I(\lambda_{2J})^{q/p}(\lambda^{-q/p})_{2J} \\
&\leqslant C\left(\frac{|2J|}{|I|}\right)^{1+q/p}, \quad \text{as } \lambda, \mu \in (A_p), \\
&\leqslant C(2\beta M^{1/p})^{1+q/p}.
\end{aligned}$$

So we have

$$C \geqslant M^{q+(q/p)(1-1/p)-1/p} = M^q,$$

and we have an upper bound on M.

V. A weighted norm inequality for vectors. Let W be a symmetric, positive definite, $n \times n$ matrix-valued function on the unit circle T. $W(x)$ induces a pointwise inner product on the vector space C^n given by $(f, g)_{W(x)} = (W(x)f, g)$ where the latter is the standard dot product on C^n. This extends to vector-valued functions as

$$(f, g)_W = \frac{1}{2\pi}\int_T (W(x)f(x), g(x))\, dx.$$

This inner product in turn induces a Hilbert space $L^2(W)$ of vector-valued functions whose W-norm is finite.

We wish to extend Theorem 1.2 to this setting. For what weights W is the conjugate operator H a bounded operator on $L^2(W)$? Nonconstructive necessary and sufficient conditions have been found by Pousson [**9**] and Rabindranathan [**10**] using the Hilbert space arguments of Helson and Szegö [**4**]. We will present a sufficient condition which is constructive, and which can be generalized to appropriately defined $L^p(W)$ spaces [**1**].

THEOREM 5.1. *Let $W = U^*\Lambda U$, where U is unitary, Λ diagonal, and the diagonal entries λ_k of Λ are A_2 weights. If for each r and j,*

$$u_{rj} \in \mathrm{BMO}_{(\lambda_r\lambda_k^{-1})^{1/2}} \quad \textit{for } k = 1, 2, \ldots, n,$$

then H is a bounded operator on $L^2(W)$.

PROOF. The W-norm of Hf is given by

$$\|Hf\| = \sum_k \frac{1}{2\pi} \int_T |(UHf)_k(x)|^2 \lambda_k(x)\, dx.$$

We will bound each $|(UHf)_k|$.

$$\begin{aligned} UHf &= H(Uf) + UHf - H(Uf) \\ &= H(Uf) + UHU^*(Uf) - UU^*HUf, \end{aligned}$$

so that, with $U = (u_{rj})$,

$$(UHf)_k = H(Uf)_k + \sum_j u_{kj}\big(HU^*(Uf)_j - U^*H(Uf)_j\big).$$

Now

$$\begin{aligned} HU^*(Uf)_j - U^*H(Uf)_j &= \sum_r \bar{u}_{rj}(Uf)_r - \bar{u}_{rj}H(Uf)_r \\ &= \sum_r \left[H, M_{\bar{u}_{rj}}\right](Uf)_r. \end{aligned}$$

Since U is unitary, each $|u_{kj}| \leqslant 1$. Thus

$$|(UHf)_k| \leqslant |H(Uf)_k| + \sum_{r,j}\sum \left|\left[H, M_{\bar{u}_{rj}}\right](Uf)_r\right|,$$

and so

$$\|Hf\|_W^2 \leqslant c\left(\sum_k \int |H(Uf)_k|^2 \lambda_k + \sum_{r,j,k} \int \left|\left[H, M_{\bar{u}_{rj}}\right](Uf)_r\right|^2 \lambda_k\right)$$

where c depends only on the dimension n.

By the Hunt, Muckenhoupt and Wheeden Theorem 1.2,

$$\int |H(Uf)_k|^2 \lambda_k \leqslant C \int |(Uf)_k|^2 \lambda_k,$$

and by the Commutator Theorem 4.2,

$$\int \left|\left[H, M_{\bar{u}_{rj}}\right](Uf)_r\right|^2 \lambda_k \leqslant C \int |(Uf)_r|^2 \lambda_r,$$

since each $\bar{u}_{rj} \in \mathrm{BMO}_{(\lambda_r\lambda_k^{-1})^{1/2}}$ by assumption. So

$$\|Hf\|_w^2 \leqslant C \sum_r \int |(Uf)_r|^2 \lambda_r = C\|f\|_w^2.$$

We close with some remarks on the converse of Theorem 5.1. The requirement that the λ_k be A_2 weights causes no great pain. There are examples of good weights with a diagonalization for which the diagonal entries are not in A_2, but these examples reflect a choice in diagonalization rather than the structure of the weight. In particular, if $U^*\Lambda U$ is a good weight with U continuous, the λ_k's must be in A_2 [**1**].

In any converse to Theorem 5.1, very little can be said about the arguments of the unitary entries. For if $U^*\Lambda U$ is a good weight, and if J is any diagonal, unitary matrix, then $U^*J^*\Lambda JU = U^*\Lambda U$, so that necessary conditions must apply to JU as well as U. Multiplication by J smears the arguments of each row.

The condition that we suspect is necessary is the following

CONJECTURE 5.2 Let H be a bounded operator on $L^2(U^*\Lambda U)$, where

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

is diagonal, and $U = (u_{ij})$ is unitary. Then each $|u_{rj}| \in \mathrm{BMO}_{(\lambda_r \lambda_k^{-1})^{1/2}}$, $k = 1, 2, \ldots, n$.

This author has also studied the simpler moving average operator

$$A_h f(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t)\, dt.$$

A weight W is said to be a good weight for the moving average if A_h is bounded on $L^2(W)$, with bound independent of h.

Conjecture 5.2 holds for the moving average in two dimensions, but our proof breaks down in higher dimensions [**1**].

Similarly, one can ask these questions about the Hardy–Littlewood maximal function, defined in the vector setting to maximize the W-norm.

One example motivated much of these ideas. Let α, $\beta > 0$, $\alpha < 1$. Put

$$\Lambda = \begin{bmatrix} |x|^{\alpha} & 0 \\ 0 & |x|^{-\alpha} \end{bmatrix},$$

and

$$U = \begin{bmatrix} \cos|x|^{\beta} & \sin|x|^{\beta} \\ -\sin|x|^{\beta} & \cos|x|^{\beta} \end{bmatrix}.$$

Then $U^*\Lambda U$ is a good weight for any of the operators discussed if and only if $\beta \geqslant \alpha$, which is exactly the condition called for by Theorem 5.1 of [**1**].

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