TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 292, Number 1, November 1985

GROUP-GRADED RINGS AND DUALITY

BY

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ABSTRACT. We give an alternative construction of the duality between finite group actions and group gradings on rings which was shown by Cohen and Montgomery in [1]. This duality is then used to extend known results on skew group rings to corresponding results for large classes of group-graded rings. Finally we modify the construction slightly to handle infinite groups.

Introduction. In the first section we give an alternate construction of the duality between finite group actions and group gradings on rings which was shown by Cohen and Montgomery in [1]. This duality is then used to extend results on skew group rings and their modules to corresponding results for large classes of group-graded rings.

In the second section we modify the construction slightly to handle infinite groups and apply it to a theorem on prime and semiprime infinite crossed products.

I would like to thank my thesis advisor, D. S. Passman, for his guidance and encouragement throughout the writing of this paper.

1. Finite group-graded rings and the smash product. Let G be a multiplicative group. An associative ring with identity is said to be G-graded if

$$R = \oplus \sum_{x \in G} R(x) \tag{1}$$

is a direct sum of additive subgroups $R(x)$, with $R(x)R(y) \subseteq R(xy)$. It follows that necessarily $1_R \in R(1)$ so that each $R(x)$ is a unitary $R(1)$-bimodule. If $r \in R$, we write $r(x)$ for the component of r in $R(x)$ so that $r = \sum_{x \in G} r(x)$. R is said to be strongly G-graded if $R(x)R(y) = R(xy)$ for all $x, y \in G$.

Throughout this section R is assumed to be G-graded, where G is a finite group with $|G| = n$.

Let $M_G(R)$ denote the set of $n \times n$ matrices over R with the rows and columns indexed by the elements of G. If $\alpha \in M_G(R)$, we write $\alpha_{x,y}$ for the entry in the $[x, y]$-position of α. Then if $\alpha, \beta \in M_G(R)$, the matrix product $\alpha\beta$ is given by

$$(\alpha\beta)_{x,y} = \sum_{z \in G} \alpha_{x,z}\beta_{z,y}. \tag{2}$$

If $U \subseteq G$ is any subset of G, let $R(U) = \sum_{x \in U} R(x)$. In particular, $R = R(G)$. Now suppose $H \subseteq G$ is a subgroup of G. We define $R\{H\} \subseteq M_G(R)$ by

$$R\{H\} = \left\{ \alpha \in M_G(R) | \alpha_{x,y} \in R(xHy^{-1}) \right\},$$

that is, $R\{H\} = \sum_{x,y \in G} R(xHy^{-1})e(x, y)$, where $e(x, y)$ is the matrix unit with 1_R in the $[x, y]$-position and zeroes elsewhere. Note that $R\{G\} = M_G(R)$.

Received by the editors September 27, 1984.
1980 *Mathematics Subject Classification*. Primary 16A03.

Since $R(xHy^{-1})R(yHz^{-1}) \subseteq R(xHz^{-1})$, $R\{H\}$ is a subring of $M_G(R)$. Furthermore, $1_R \in R(1) \subseteq R(xHx^{-1})$ for each $x \in G$, so that $I \in R\{H\}$, where I is the identity matrix.

$R\{H\}$ can be viewed as a generalized conjugate of the matrix ring $M_G(R(H))$, where $M_G(R(H)) \subseteq M_G(R)$ denotes the $n \times n$ matrices over the ring $R(H)$. Let D and D^{-1} be the diagonal subsets of $M_G(R)$ given by

(3) $$D = \left\{ \alpha \in M_G(R) | \alpha_{x,x} \in R(xH) \text{ and } \alpha_{x,y} = 0 \text{ if } x \neq y \right\},$$
$$D^{-1} = \left\{ \beta \in M_G(R) | \beta_{x,x} \in R(Hx^{-1}) \text{ and } \beta_{x,y} = 0 \text{ if } x \neq y \right\}.$$

Then $DM_G(R(H))D^{-1} \subseteq R\{H\}$ and $D^{-1}R\{H\}D \subseteq M_G(R(H))$. If R is strongly G-graded these inclusions are both equalities.

The ideals of $R(H)$ correspond to the ideals of $M_G(R(H))$ and we can extend this, using conjugation, to a correspondence between the ideals of $R(H)$ and those of $R\{H\}$ (see Lemma 1.4.).

Another description of $R\{H\}$, as a skew group ring over $R\{1\}$, is given in Lemma 1.2. We begin by studying the ring $R\{1\}$.

First we observe that R can be embedded in $R\{1\}$ by a ring monomorphism η, with $\eta(1_R) = I$. We define η by $\eta(r) = \sum_{x,y \in G} r(xy^{-1})e(x, y)$ and denote $\eta(r)$ by $\tilde{r}$. If $x, y \in G$ and $r, s \in R$ then using equation (2) we find

$$(\tilde{r}\tilde{s})_{x,y} = \sum_{z \in G} \tilde{r}_{x,z}\tilde{s}_{z,y} = \sum_{z \in G} r(xz^{-1})s(zy^{-1}) = (rs)(xy^{-1}) = (\widetilde{rs})_{x,y}.$$

Thus $\tilde{r}\tilde{s} = \widetilde{rs}$. Similarly, $\eta(r + s) = \eta(r) + \eta(s)$ and so η is a ring homomorphism. Clearly $\eta(1_R) = I$, η is injective and $\tilde{R} = \eta(R) \subseteq R\{1\}$.

We call $R\{1\}$ the smash product of $\tilde{R}$ with G and denote it by $\tilde{R}\#G$. The following lemma shows that this definition coincides with the definition of the smash product given in [**1**]. Let $p(x) = e(x, x)$.

Lemma 1.1. *$\tilde{R}\#G = \underline{\oplus \sum_{x \in G}} \tilde{R}p(x)$ is a free $\tilde{R}$-module with basis $\{p(x) | x \in G\}$. Also, $\tilde{r}p(x)\tilde{s}p(y) = \tilde{r}\widetilde{s(xy^{-1})}p(y)$, where $r, s \in R$ and $x, y \in G$.*

Proof. Notice that $\widetilde{r(x)}p(y) = r(x)e(xy, y)$, so $\tilde{r}p(y) = \sum_{x \in G} r(x)e(xy, y)$ and $\tilde{R}p(y) = \sum_{x \in G} R(x)e(xy, y)$ is a free $\tilde{R}$-module with generator $p(y)$. Now, $\sum_{y \in G}\tilde{R}p(y) = \sum_{y \in G}\sum_{x \in G} R(x)e(xy, y) = \tilde{R}\#G$ and the sum $\sum_{y \in G}\tilde{R}p(y)$ is direct since $I = \sum_{y \in G} p(y)$ is a decomposition of I into orthogonal idempotents. The final formula follows easily. □

The group G can be embedded in the matrix ring $M_G(R)$. Indeed, if $g \in G$, we let $\bar{g} = \sum_{x \in G} e(x, xg)$. Now $\bar{g}$ is a unit of $M_G(R)$ and G is isomorphic to $\overline{G}$, where $\overline{G} = \{\bar{g} | g \in G\}$, via the map taking g to $\bar{g}$. We now describe $R\{H\}$ as a skew group ring over $\tilde{R}\#G$.

Lemma 1.2. *Let H be a subgroup of G. Then*

(i) *$R\{H\} = \oplus \sum_{h \in H} (\tilde{R}\#G)\bar{h}$, as a direct sum of abelian groups.*

(ii) *$\bar{g}^{-1}(\tilde{R}\#G)\bar{g} = \tilde{R}\#G$ for all $g \in G$.*

(iii) *$R\{G\} = (\tilde{R}\#G)\overline{G}$ is a skew group ring of the group $\overline{G}$ over the ring $\tilde{R}\#G$. Furthermore, $R\{H\}$ is the naturally embedded sub skew group ring.*

PROOF.

(i) $$(\tilde{R}\#G)\bar{h} = \sum_{x,y\in G} R(xy^{-1})e(x,y)\sum_{z\in G} e(z,zh)$$

$$= \sum_{x,y\in G} R(xy^{-1})e(x,yh) = \sum_{x,w\in G} R(xhw^{-1})e(x,w).$$

Since for fixed x, w the elements xhw^{-1} are distinct for distinct h, it follows from equation (1) that $\Sigma_{h\in H}(\tilde{R}\#G)\bar{h}$ is direct. Also,

$$\sum_{h\in H}(\tilde{R}\#G)\bar{h} = \sum_{h\in H}\sum_{x,w\in G} R(xhw^{-1})e(x,w)$$

$$= \sum_{x,w\in G}\sum_{h\in H} R(xhw^{-1})e(x,w)$$

$$= \sum_{x,w\in G} R(xHw^{-1})e(x,w) = R\{H\}.$$

(ii) $$\bar{g}^{-1}(\tilde{R}\#G)\bar{g} = \sum_{u\in G} e(u,ug^{-1})\sum_{x,y\in G} R(xy^{-1})e(x,y)\sum_{v\in G} e(v,vg)$$

$$= \sum_{x,y\in G} R(xy^{-1})e(xg,yg)$$

$$= \tilde{R}\#G \quad \text{since } (xg)(yg)^{-1} = xy^{-1}.$$

(iii) Follows directly from (i) and (ii). □

The duality result of Cohen and Montgomery [**1**, Theorem 3.5] now follows. Observe that $G \simeq \bar{G}$ acts by conjugation on $\tilde{R}\#G$ and we let $(\tilde{R}\#G)^G$ denote the set of fixed points.

THEOREM 1.3. *$M_G(R) = (\tilde{R}\#G)\bar{G}$ is a skew group ring of $\bar{G} \simeq G$ over the ring $\tilde{R}\#G$. Furthermore, $(\tilde{R}\#G)^G = \tilde{R}$.*

PROOF. Since $M_G(R) = R\{G\}$, the first statement is a special case of Lemma 1.1(iii).

If $\alpha \in \tilde{R}\#G$ note that $(\bar{g}^{-1}\alpha\bar{g})_{x,y} = \alpha_{xg^{-1},yg^{-1}}$. So $\alpha \in (\tilde{R}\#G)^G$ if and only if $\alpha_{x,y} = \alpha_{xg^{-1},yg^{-1}}$ for all $x, y, g \in G$ and hence if and only if $\alpha = \eta(\Sigma_{x\in G}\alpha_{x,1}) \in \tilde{R}$. □

A group-graded ring R is said to be left component regular if $l_R(R(x)) = 0$ for all $x \in G$, where $l_R(R(x))$ is the left annihilator of $R(x)$ in R. Right component regular is defined similarly, and the term component regular is used to imply that both these conditions hold. If the left annihilator of $R(x)$ in $R(x^{-1})$, written $l_{R(x^{-1})}(R(x))$, is zero for all $x \in G$, then we say R is left nondegenerate. It was shown in [**2**] that if $R(1)$ is semiprime then left nondegenerate is equivalent to right nondegenerate. It was also shown that if G is finite, then semiprime implies nondegenerate.

We now use the generalized conjugation to give the ideal correspondence between the rings $R(H)$ and $R\{H\}$. If T is any ring, $\mathscr{I}(T)$ denotes the set of two-sided ideals of T.

LEMMA 1.4. *Let R be a G-graded ring with G finite and suppose H is subgroup of G. Then there exist maps*

$$\phi: \mathcal{I}(R(H)) \to \mathcal{I}(R\{H\}), \qquad \psi: \mathcal{I}(R\{H\}) \to \mathcal{I}(R(H))$$

such that

(i) *ϕ is injective and if $A, B \triangleleft R(H)$ then $\phi(A)\phi(B) \subseteq \phi(AB)$,*

(ii) *if $J, K \triangleleft R\{H\}$ then $\psi(J)\psi(K) \subseteq \psi(JK)$,*

(iii) *$\psi \circ \phi$ is the identity on $\mathcal{I}(R(H))$.*

Furthermore, let R be component regular and let $I \triangleleft R\{H\}$. Then $\psi(I) = 0$ if and only if $I = 0$.

PROOF. Let $A \triangleleft R(H)$ and recall from equation (3) that

$$D = \left\{ \alpha \in M_G(R) | \alpha_{x,x} \in R(xH), \alpha_{x,y} = 0 \text{ if } x \neq y \right\}$$

and

$$D^{-1} = \left\{ \beta \in M_G(R) | \beta_{x,x} \in R(Hx^{-1}), \beta_{x,y} = 0 \text{ if } x \neq y \right\}.$$

Take $A^0 = DM_G(A)D^{-1}$, where $M_G(A)$ is the set of $n \times n$ matrices over A. $A^0 \subseteq DM_G(R(H))D^{-1} \subseteq R\{H\}$. We claim that A^0 is an ideal of $R\{H\}$. First we show that $R\{H\}D \subseteq DM_G(R(H))$. To see this consider the $[x, y]$-position of each side. For the left-hand side we get $R(xHy^{-1})R(yH) \subseteq R(xH)$, while on the right we have $R(xH)R(H) = R(xH)$ since $1 \in R(H)$. Thus,

$$\begin{aligned} R\{H\}A^0 &= R\{H\}DM_G(A)D^{-1} \\ &\subseteq DM_G(R(H))M_G(A)D^{-1} = DM_G(A)D^{-1} = A^0. \end{aligned}$$

Hence A^0 is a left ideal of $R\{H\}$. Similarly, it is also a right ideal.

Now suppose $I \triangleleft R\{H\}$. Take $I_{1,1} = \{\alpha_{1,1} | \alpha \in I\} \subseteq R(H)$. Clearly, $I_{1,1}$ is an ideal of $R(H)$ since $R\{H\} \supseteq R(H)e(1,1)$.

We define ϕ and ψ as follows: If $A \triangleleft R(H)$, let $\phi(A) = A^0$ and, for $I \triangleleft R\{H\}$, take $\psi(I) = I_{1,1}$.

Note that $A^0_{1,1} = R(H)AR(H) = A$. That is, $\psi \circ \phi(A) = A$, so (iii) is proved and also ϕ is injective. To complete the proof of (i) note that $D^{-1}D \subseteq \sum_{x \in G} R(H)e(x,x) \subseteq M_G(R(H))$. Thus if $A, B \triangleleft R(H)$ then

$$\begin{aligned} \phi(A)\phi(B) &= DM_G(A)D^{-1}DM_G(B)D^{-1} \\ &\subseteq DM_G(A)M_G(R(H))M_G(B)D^{-1} = DM_G(AB)D^{-1} = \phi(AB). \end{aligned}$$

To prove (ii) note that $e(1,1) \in R\{H\}$ and so if J, $K \triangleleft R\{H\}$ then $e(1,1)Je(1,1)e(1,1)Ke(1,1) \subseteq JK$. Evaluating both sides at the $[1,1]$-position gives $J_{1,1}K_{1,1} \subseteq (JK)_{1,1}$ or equivalently $\psi(J)\psi(K) \subseteq \psi(JK)$.

Finally, to prove the last statement, suppose R is component regular and let $J \triangleleft R\{H\}$ with $J \neq 0$. Since $J \neq 0$, choose $\alpha \in J \setminus 0$ and say $\alpha_{x,y} \neq 0$. Note that $R(Hx^{-1})e(1,x) \subseteq R\{H\}$ and $e(y,1)R(yH) \subseteq R\{H\}$, so $R(Hx^{-1})e(1,x) \cdot \alpha \cdot e(y,1)R(yH) \subseteq J$. Thus, $\psi(J) = J_{1,1} \supseteq R(Hx^{-1})\alpha_{x,y}R(yH) \neq 0$ since R is component regular. $\square$

Now suppose H is a fixed subgroup of G and let $x \in G$. Then the conjugate xHx^{-1} is also a subgroup of G and we let ϕ_x and ψ_x be the maps between $\mathscr{I}(R(xHx^{-1}))$ and $\mathscr{I}(R\{xHx^{-1}\})$. Note that $\bar{x}R\{H\}\bar{x}^{-1} = R\{xHx^{-1}\}$ since

$$\bar{x}R\{H\}\bar{x}^{-1} = \bar{x}(\tilde{R}\#G)\overline{H}\bar{x}^{-1} = (\tilde{R}\#G)\bar{x}\overline{H}\bar{x}^{-1} = R\{xHx^{-1}\}.$$

Thus if $A \triangleleft R\{H\}$ then $\bar{x}A\bar{x}^{-1} \triangleleft R\{xHx^{-1}\}$.

LEMMA 1.5. *Let R be a left nondegenerate group-graded ring and let $A \triangleleft R\{H\}$ be a nonzero ideal. Then there exists $x \in G$ such that $\psi_x(\bar{x}A\bar{x}^{-1}) \triangleleft R(xHx^{-1})$ is nonzero.*

PROOF. Since $A \neq 0$, there exists $x, y \in G$ such that $A_{x,y} \neq 0$. Now $A_{x,y}e(x, y) \subseteq A$ and $R(yHx^{-1})e(y, x) \subseteq R\{H\}$ so that

$$A_{x,y}R(yHx^{-1})e(x, x) = A_{x,y}e(x, y)R(yHx^{-1})e(y, x) \subseteq A.$$

Thus $A_{x,y}R(yHx^{-1}) \subseteq A_{x,x}$. Now since $0 \neq A_{x,y} \subseteq R(xHy^{-1})$, R is left nondegenerate, $yHx^{-1} = (xHy^{-1})^{-1}$, we can conclude that $A_{x,x} \neq 0$. But $A_{x,x} = (\bar{x}A\bar{x}^{-1})_{1,1} = \psi_x(\bar{x}A\bar{x}^{-1})$. □

A crossed product $R * G$, where R is an arbitrary ring, is said to be weakly semiprime if whenever P is an elementary abelian p-subgroup of G where R has p-torsion, or $P = 1$, then $R * P$ is semiprime. In [**4**, Theorems 1.8 and 1.9] Passman showed that if $R * G$ is a crossed product with G finite then

(A) the semiprime condition on $R * G$ is inherited by each subring $R * H$ where H is a subgroup of G;

(B) if $R * G$ is weakly semiprime then it is semiprime.

We can now extend (A) and (B) to corresponding results for more general group-graded rings.

THEOREM 1.6. *Let R be a G-graded ring with G finite and suppose H is a subgroup of G. If R is semiprime then so is $R(H)$.*

PROOF. Since R is semiprime, so is the matrix ring $M_G(R)$. By Lemma 1.2(iii), $M_G(R) = (\tilde{R}\#G)\overline{G}$ is a skew group ring, so (A) applies to give that $R\{H\} = (\tilde{R}\#G)\overline{H}$ is also semiprime.

Now suppose $I \triangleleft R(H)$ with $I^2 = 0$. Then by 1.4(i), $\phi(I)^2 \subseteq \phi(I^2) = 0$. But $\phi(I) \triangleleft R\{H\}$ and $R\{H\}$ is semiprime. Thus $\phi(I) = 0$, and since ϕ is injective, $I = 0$. □

By analogy with the definition for crossed products, we say a G-graded ring R is weakly semiprime if $R(P)$ is semiprime for $P = 1$ or P an elementary abelian p-subgroup of G, where the ring $R(1)$ has p-torsion.

THEOREM 1.7. *Let R be a G-graded ring with G finite. Then R is semiprime if and only if R is weakly semiprime and left nondegenerate.*

PROOF. Suppose R is semiprime. Then Theorem 1.6 implies R is weakly semiprime and [**2**, Proposition 1.2] implies R is nondegenerate.

Conversely, suppose R is weakly semiprime and left nondegenerate. To show R is semiprime it suffices to show that $M_G(R)$ is semiprime. In turn, using (B), we need only show $M_G(R) = (\tilde{R}\#G)\overline{G}$ is weakly semiprime.

Suppose $\tilde{R}\#G$ has p-torsion. Then R has p-torsion, so there exists $r \in R$, with $r \neq 0$, such that $pr = 0$. Choose $x \in G$ with $r(x) \neq 0$ and note that $pr(x) = (pr)(x) = 0$. Hence $p(r(x)R(x^{-1})) = 0$. But $0 \neq r(x)R(x^{-1}) \subseteq R(1)$ since R is left nondegenerate. Thus if $\tilde{R}\#G$ has p-torsion then so also has $R(1)$.

Now consider $P \subseteq G$, an elementary abelian p-subgroup of G. We need to show that $R\{P\} = (\tilde{R}\#G)\overline{P}$ is semiprime. Suppose $I \triangleleft R\{P\}$ is a nonzero ideal with $I^2 = 0$. Then by Lemma 1.5 there exists $x \in G$ such that $\psi_x(\bar{x}I\bar{x}^{-1}) \neq 0$. Now by 1.4(ii), $\psi_x(\bar{x}I\bar{x}^{-1})^2 \subseteq \psi_x(\bar{x}I^2\bar{x}^{-1}) = \psi_x(0) = 0$. But $\psi_x(\bar{x}I\bar{x}^{-1}) \triangleleft R(xHx^{-1})$ and $R(x^{-1}Hx)$ is semiprime since $x^{-1}Hx$ is an elementary abelian p-subgroup of G. This is a contradiction and completes the proof. □

We now turn our attention to modules over finite group-graded rings. Let V be a left R-module and let $\operatorname{Col}_G V$ denote the set of $n \times 1$ column vectors over V with the positions indexed by the elements of G. If $v \in \operatorname{Col}_G V$ we write v_x for the element of V in the $[x]$-position of v. $\operatorname{Col}_G V$ is naturally a left $M_G(R)$-module under matrix multiplication. The action of $\alpha \in M_G(R)$ on $v \in \operatorname{Col}_G V$ is given by

$$(\alpha v)_x = \sum_{y \in G} \alpha_{x,y} v_y. \tag{4}$$

It can be shown that up to isomorphism all left $M_G(R)$-modules are of the form $\operatorname{Col}_G V$ for some left R-module V.

If $v \in V$ then we let $vf(x)$ denote the element of $\operatorname{Col}_G V$ with v in the $[x]$-position and zeroes elsewhere. Thus if $u \in \operatorname{Col}_G V$ then $u = \sum_{x \in G} u_x f(x)$. If ${}_R W \subseteq {}_R V$ are R-modules we write $W \operatorname{ess}_R V$ to indicate the W is an essential R-submodule of V. Also $\operatorname{rank}_R V$ is used to denote the Goldie rank of V as an R-module.

The following two results are known for modules over crossed products. (See [**3**].) Let R be an arbitrary ring and let $R * G$ be a crossed product with G finite. Then

(C) if V is $|G|$-torsion free then $W \operatorname{ess}_R V$ if and only if $W \operatorname{ess}_{R*G} V$.

(D) $\operatorname{rank}_{R*G} V \leqslant \operatorname{rank}_R V \leqslant |G| \cdot \operatorname{rank}_{R*G} V$.

As before we can find analogs of these results for the group-graded rings. Suppose V is a module over a G-graded ring R. We say V is component regular if whenever $v \in V$, with $v \neq 0$, then $R(x)v \neq 0$ for all $x \in G$.

THEOREM 1.8. *Let R be a G-graded ring with G-finite and let ${}_R W \subseteq {}_R V$ be R-modules where V is component regular with no $|G|$-torsion. Then $W \operatorname{ess}_R V$ if and only if $W \operatorname{ess}_{R(1)} V$.*

Note. Two proofs of this result are given. The first proof uses the smash product $\tilde{R}\#G$ to obtain this result directly from (C). In Theorem 1.9 an equivalent result is proved by adapting the proof of (C) given in [**3**].

PROOF. It is clear that $W \operatorname{ess}_{R(1)} V$ implies $W \operatorname{ess}_R V$.

Conversely suppose $W \operatorname{ess}_R V$. We can form the $M_G(R)$-modules $\operatorname{Col}_G W$ and $\operatorname{Col}_G V$. Note that $\operatorname{Col}_G W \subseteq \operatorname{Col}_G V$, and by considering the diagonal of $M_G(R)$ acting on $\operatorname{Col}_G V$ we have that $\operatorname{Col}_G W$ is an essential $M_G(R)$-submodule of $\operatorname{Col}_G V$. Since V has no $|G|$-torsion, neither has $\operatorname{Col}_G V$. Thus we can apply (C) to $\operatorname{Col}_G W \subseteq \operatorname{Col}_G V$ as modules over the skew group ring $M_G(R) = (\tilde{R}\#G)\overline{G}$ to give that $\operatorname{Col}_G W$ is an essential $\tilde{R}\#G$-submodule of $\operatorname{Col}_G V$.

To prove $W \operatorname{ess}_{R(1)} V$ we need to show that, if $v \in V$ with $v \neq 0$, then $R(1)v \cap W \neq 0$. Consider $vf(1) \in \operatorname{Col}_G V$. Since $\operatorname{Col}_G W$ is an essential $\tilde{R}\#G$-submodule of $\operatorname{Col}_G V$, there exists $\alpha \in \tilde{R}\#G$ such that $\alpha vf(1) \neq 0$ and $\alpha vf(1) \in \operatorname{Col}_G W$. Using equation (4) we get $\alpha vf(1) = \sum_{x \in G} \alpha_{x,1} vf(x)$. So $\alpha_{x,1} v \in W$ for each $x \in G$ and since $\alpha vf(1) \neq 0$ we can find $x \in G$ such that $\alpha_{x,1} v \neq 0$. But $\alpha_{x,1} \in R(x)$ since $\alpha \in \tilde{R}\#G$, and W is an R-module. Thus $R(x^{-1})\alpha_{x,1} v \subseteq R(1)v \cap W$ and $R(x^{-1})\alpha_{x,1} v \neq 0$ because V is component regular. □

THEOREM 1.9. *Let $W \subseteq V$ be left R-modules and suppose V is component regular with no $|G|$-torsion. Then there exists $K \subseteq V$, an R-submodule, such that $W \cap K = 0$ and $W \oplus K$ is an essential $R(1)$-submodule of V.*

PROOF. Using Zorn's Lemma there exists an $R(1)$-submodule $U \subseteq V$ such that $W \oplus U$ is an essential $R(1)$-submodule of V. If $x \in G$ it is easily seen, since V is component regular, that $W \oplus R(x)U$ is an essential $R(1)$-submodule of V. Thus, since G is finite, we have that $E = \bigcap_{x \in G}(W + R(x)U)$ is again an essential $R(1)$-submodule of V. Note that

$$R(y)E \subseteq \bigcap_{x \in G} [R(y)W + R(y)R(x)U] \subseteq \bigcap_{x \in G} (W + R(yx)U) = E.$$

Hence E is an R-submodule of V.

Since $W \subseteq E \subseteq W \oplus R(x)U$, E can be written $E = W \oplus U_x$, where $U_x = R(x)U \cap E$. Let π_x be the projection from E to W relative to this decomposition. Now let θ: $E \to W$ be given by $\theta(t) = \sum_{x \in G} \pi_x(t)$, where $t \in E$.

Let $t \in E$ and write $t = w + s$ where $w \in W$ and $s \in R(x)U$, so that $\pi_x(t) = w$. Now, if $r(y) \in R(y)$, we get $r(y)t = r(y)w + r(y)s$. Note that $r(y)w \in W$ and $r(y)s \in R(y)R(x)U \subseteq R(yx)U$. Thus, $\pi_{yx}(r(y)t) = r(y)w = r(y)\pi_x(t)$. Summing over all $x \in G$, we obtain $\sum_{x \in G} \pi_{yx}(r(y)t) = r(y) \cdot \sum_{x \in G} \pi_x(t)$, that is $\theta(r(y)t) = r(y)\theta(t)$. Since θ is additive, we conclude that θ is R-linear.

Let $K = \ker \theta$, so K is an R-submodule of E. If $w \in W$ then $\pi_x(w) = w$, so $\theta(w) = nw$. But V is $|G|$-torsion free, so $K \cap W = 0$. Also if $t \in E$ and $\theta(t) = w$, we note that $nt - w \in \ker \theta = K$. Thus $n \cdot E \subseteq W \oplus K$ and so $W \oplus K$ is an essential $R(1)$-submodule of E by the torsion free assumption. Finally, since E is an essential $R(1)$-submodule of V we conclude that $W \oplus K$ is essential as an $R(1)$-submodule of V. □

In Theorem 1.11 we prove a generalization of (D) but first we require a lemma. A collection $W_1, \ldots, W_k$ of submodules of a module V is said to be independent if $W_1 + \cdots + W_k$ is a direct sum.

LEMMA 1.10. *Let V be an R-module. Then*

(i) $\operatorname{rank}_R V = \operatorname{rank}_{M_G(R)} \operatorname{Col}_G V$.

Furthermore if V is component regular then

(ii) $\operatorname{rank}_{R(1)} V = \operatorname{rank}_{\tilde{R}\#G} \operatorname{Col}_G V$.

PROOF. (i) It is easy to see that $M_G(R)$-submodules of $\operatorname{Col}_G V$ are of the form $\operatorname{Col}_G W$ where $W \subseteq V$ is an R-submodule. Note that $W_1, \ldots, W_k$ is an independent set of R-submodules of V if and only if $\operatorname{Col}_G W_1, \ldots, \operatorname{Col}_G W_k$ forms an independent set of $M_G(R)$-submodules of $\operatorname{Col}_G V$. This proves (i).

(ii) Suppose $V_1, \ldots, V_S$ form an independent set of nonzero $R(1)$-submodules of V. Then we claim $V_1^0, \ldots, V_S^0$ is an independent collection of nonzero $(\tilde{R}\#G)$-submodules of $\mathrm{Col}_G V$ where

$$V_i^0 = \sum_{x \in G} R(x) V_i f(x) = (\tilde{R}\#G) V_i f(1).$$

Clearly, V_i^0 is nonzero for each i. Also, it is easily checked, since V is component regular, that $R(x)V_1, \ldots, R(x)V_s$ is an independent set of subgroups of V. Hence $V_1^0 + \cdots + V_s^0$ is direct, and so $\mathrm{rank}_{\tilde{R}\#G} \mathrm{Col}_G V \geqslant \mathrm{rank}_{R(1)} V$.

To obtain the reverse inequality, suppose that $U_1, \ldots, U_t$ is an independent collection of nonzero $\tilde{R}\#G$-submodules of $\mathrm{Col}_G V$. Let $U_i' = \{u \in V \mid uf(1) \in U_i\}$ and note that $U_1' + \cdots + U_t'$ is a direct sum. Thus, we need only show that each U_i' is nonzero.

Since $U_i \neq 0$ and $U_i = \sum_{x \in G} e(x, x) U_i$, we have that $e(x, x)U_i \neq 0$ for some $x \in G$. Thus $R(x^{-1})e(1, x)U_i = R(x^{-1})e(1, x) \cdot e(x, x)U_i \neq 0$ since V is component regular. Also, since $R(x^{-1})e(1, x) \subseteq \tilde{R}\#G$, we have that $R(x^{-1})e(1, x)U_i \subseteq U_i$. But $R(x^{-1})e(1, x)U_i \subseteq Vf(1)$, which implies $U_i' \neq 0$. Thus $\mathrm{rank}_{\tilde{R}\#G} \mathrm{Col}_G V = \mathrm{rank}_{R(1)} V$. □

THEOREM 1.11. *Let V be a component regular R-module. Then*

$$\mathrm{rank}_R V \leqslant \mathrm{rank}_{R(1)} V \leqslant |G| \cdot \mathrm{rank}_R V.$$

PROOF. Using that $M_G(G) = (\tilde{R}\#G)\overline{G}$ is a skew group ring, we can apply (D) to conclude

$$\mathrm{rank}_{M_G(R)} \mathrm{Col}_G V \leqslant \mathrm{rank}_{\tilde{R}\#G} \mathrm{Col}_G V \leqslant |G| \cdot \mathrm{rank}_{M_G(R)} \mathrm{Col}_G V.$$

Now applying Lemma 1.10 gives the result. □

We conclude this section with a note on graded modules. A left R-module V is said to be G-graded if $V = \oplus \sum_{x \in G} V(x)$ is a direct sum of additive groups such that $R(x)V(y) \subseteq V(xy)$. If $v \in V$ we write $v(x)$ for the component of v in $V(x)$. Clearly, R is graded as a left module over itself.

If V is a G-graded R-module then we can embed V in $\mathrm{Col}_G V$ by a map ρ, where $\rho(v) = \sum_{x \in G} v(x) f(x)$ for all $v \in V$. We write $\tilde{v}$ for $\rho(v)$ and denote the image of ρ by $\tilde{V}$. It is easily checked that $\tilde{V}$ is an $\tilde{R}\#G$-submodule of $\mathrm{Col}_G V$. If we let $r \cdot \tilde{v} = \tilde{r}\tilde{v}$ then $\tilde{V}$ becomes an R-module and ρ is an R-module isomorphism.

In [1] it was noted that the $\tilde{R}\#G$-modules correspond to the graded R-modules. The map taking V to $\tilde{V}$ gives one direction. Conversely, if W is an $\tilde{R}\#G$-module one can check that taking $W(x) = e(x, x)W$ and $r \cdot w = \tilde{r}w$ for all $w \in W$ gives W a graded R-module structure.

The modules $\tilde{V}$ and $\mathrm{Col}_G V$ are related as follows:

LEMMA 1.12. *Let R be a G-graded ring with G finite and let V be a graded left R-module. Then* $\mathrm{Col}_G V = \oplus \sum_{g \in G} \bar{g}\tilde{V}$ *and each $\bar{g}\tilde{V}$ is an $\tilde{R}\#G$-submodule of* $\mathrm{Col}_G V$. *Thus* $\mathrm{Col}_G V$ *is the $M_G(R)$-module induced from the $\tilde{R}\#G$-module $\tilde{V}$.*

PROOF. Note that $e(x, y)V(z)f(z) = V(y)f(x)$ if $y = z$ and is zero otherwise. Hence

$$\bar{g}\tilde{V} = \sum_{x\in G} e(x, xg) \sum_{y\in G} V(y)f(y) = \sum_{x\in G} V(xg)f(x),$$

and since $\Sigma_{x\in G}V(x)$ is a direct sum, it follows that $\Sigma_{g\in G}\bar{g}\tilde{V}$ is also direct. Also,

$$\sum_{g\in G} \bar{g}\tilde{V} = \sum_{g\in G}\sum_{x\in G} V(xg)f(x) = \sum_{x\in G} Vf(x) = \mathrm{Col}_G V.$$

Finally, from Lemma 1.2(ii), we have that $(\tilde{R}\#G)\bar{g} = \bar{g}(\tilde{R}\#G)$, and so $(\tilde{R}\#G)\bar{g}\tilde{V} = \bar{g}(\tilde{R}\#G)\tilde{V} = \bar{g}\tilde{V}$. Thus $\bar{g}V$ is an $\tilde{R}\#G$-submodule of $\mathrm{Col}_G V$. □

2. Infinite group graded rings. The constructions of §1 can be repeated, with some minor changes, when R is G-graded with G infinite. Here we work inside $M_G(R)$—the set of row and column finite matrices over R with the rows and columns again indexed by the elements of G. We use $M_G^*(R)$ to denote the matrices with finitely many nonzero entries.

As before, R can be embedded in $M_G(R)$ by a ring monomorphism η, taking $r = \Sigma_{x\in G}r(x)$ to $\tilde{r} \in M_G(R)$, where $\tilde{r}_{x,y} = r(xy^{-1})$. Now $\tilde{R}\#G$ is defined to be the subring of $M_G(R)$ generated by $\tilde{R}$ and $\{p(x)\}_{x\in G}$, where $p(x) = e(x, x) \in M_G(R)$.

LEMMA 2.1. *Let R be a G-graded ring with G infinite. If $r, s \in R$, then $\tilde{r}p(x)\tilde{s}p(y) = \tilde{r}\widetilde{s(xy^{-1})}p(y)$ and $\tilde{R}\#G = \tilde{R} \oplus [\oplus\Sigma_{x\in G}\tilde{R}p(x)]$ is a free $\tilde{R}$-module with basis $\{I\} \cup \{p(x)\}_{x\in G}$. Furthermore, $\Sigma_{x\in G}\tilde{R}p(x) = \Sigma_{x,y\in G}R(xy^{-1})e(x, y)$ is an ideal of $\tilde{R}\#G$ which is essential both as a left ideal and as a right ideal.*

PROOF. The argument of Lemma 1.1 applies again to give that $\tilde{r}p(x)\tilde{s}p(y) = \tilde{r}\widetilde{s(xy^{-1})}p(y)$ and that $\tilde{R}p(x)$ is a free $\tilde{R}$-module with generator $p(x)$. The sum $\Sigma_{x\in G}\tilde{R}p(x)$ is direct since $\{p(x)\}_{x\in G}$ is a set of orthogonal idempotents. It is easily checked that $\Sigma_{x\in G}\tilde{R}p(x) = \Sigma_{x,y\in G}R(xy^{-1})e(x, y)$ and so $\Sigma_{x\in G}\tilde{R}p(x) \cap \tilde{R} = 0$, since nonzero elements of $\tilde{R}$ have infinitely many nonzero entries.

It is now clear that $\tilde{R}\#G = \tilde{R} + \Sigma_{x\in G}\tilde{R}p(x)$. Note that

$$\tilde{R}\#G \subseteq \left\{ \alpha \in M_G(R) | \alpha_{x,y} \in R(xy^{-1})\right\}$$

and

$$\tilde{R}\#G \supseteq \left\{ \alpha \in M_G^*(R) | \alpha_{x,y} \in R(xy^{-1})\right\} = \sum_{x,y\in G} R(xy^{-1})e(x, y).$$

Thus $\Sigma_{x\in G}\tilde{R}p(x) = (\tilde{R}\#G) \cap M_G^*(R)$, and since $M_G^*(R) \triangleleft M_G(R)$, this implies $\Sigma_{x\in G}\tilde{R}p(x) \triangleleft \tilde{R}\#G$. To show that it is essential as a left ideal, let $\alpha \in \tilde{R}\#G$ with $\alpha \neq 0$. We can find $x \in G$ such that $p(x)\alpha \neq 0$. Note that $p(x)\alpha$ has finitely many nonzero entries, and so $p(x)\alpha \in \Sigma_{x\in G}\tilde{R}p(x)$. Thus, $\Sigma_{x\in G}\tilde{R}p(x)$ is essential as a left ideal and similarly it is essential as a right ideal. □

Now G is embedded in $M_G(R)$ by the map taking g to $\bar{g}$, where $\bar{g} = \Sigma_{x\in G}e(x, xg)$. As in §1 this map is a group isomorphism from G to $\bar{G} = \{\bar{g} | g \in G\}$ and $\bar{G}$ is a subgroup of the units of $M_G(R)$.

LEMMA 2.2. *Let R be a G-graded ring with G infinite. Then*
(i) $\bar{g}^{-1}(\tilde{R}\#G)\bar{g} = \tilde{R}\#G$ *and* $(\tilde{R}\#G)^G = \tilde{R}$,
(ii) $\Sigma_{g\in G}(\tilde{R}\#G)\bar{g}$ *is a direct sum of additive groups,*
(iii) $(\tilde{R}\#G)\overline{G}$ *is a skew group ring of the group* $\overline{G}$ *over the ring* $\tilde{R}\#G$,
(iv) $(\tilde{R}\#G)\overline{G} = (\oplus\Sigma_{g\in G}\tilde{R}\bar{g}) \oplus M_G^*(R)$ *as additive groups and* $M_G^*(R)$ *is an ideal of* $(\tilde{R}\#G)\overline{G}$ *which is essential both as a left ideal and as a right ideal.*

PROOF. Parts (i), (ii) and (iii) are proved as in the case of G finite.

(iv)
$$\begin{aligned}(\tilde{R}\#G)\overline{G} &= \sum_{g\in G}(\tilde{R}\#G)\bar{g}\\ &= \sum_{g\in G}\left[\tilde{R} + \sum_{x,y\in G} R(xy^{-1})e(x,y)\right]\bar{g} \quad \text{by Lemma 2.1}\\ &= \sum_{g\in G}\tilde{R}\bar{g} + \sum_{x,y,g\in G} R(xy^{-1})e(x,y)\bar{g}.\end{aligned}$$

Note that
$$\begin{aligned}\sum_{x,y,g\in G} R(xy^{-1})e(x,y)\bar{g} &= \sum_{x,y,g\in G} R(xy^{-1})e(x,yg)\\ &= \sum_{x,w,g\in G} R(xgw^{-1})e(x,w) = \sum_{x,w\in G} Re(x,w) = M_G^*(R).\end{aligned}$$

Thus $(\tilde{R}\#G)\overline{G} = \Sigma_{g\in G}\tilde{R}\bar{g} + M_G^*(R)$.

$\Sigma_{g\in G}\tilde{R}\bar{g}$ is a direct sum from (ii), and since nonzero elements of $\Sigma_{g\in G}\tilde{R}\bar{g}$ have infinitely many nonzero entries, $\Sigma_{g\in G}\tilde{R}\bar{g} \cap M_G^*(R) = 0$.

Clearly, $M_G^*(R)$ is an ideal of $(\tilde{R}\#G)\overline{G}$. Now if $\alpha \in (\tilde{R}\#G)\overline{G}$ with $\alpha \neq 0$, we can find $x \in G$ such that $p(x)\alpha \neq 0$. Note that $p(x)\alpha \in M_G^*(R)$, and so $M_G^*(R)$ is essential as a left ideal. Similarly, it is essential as a right ideal. □

We now define $R\{H\}$ to be $(\tilde{R}\#G)\overline{H} = \Sigma_{h\in H}(\tilde{R}\#G)\bar{h}$. Note that $R\{H\} \subseteq \{\alpha \in M_G(R) | \alpha_{x,y} \in R(xHy^{-1})\}$. Now let
$$R^*\{H\} = \left\{\alpha \in M_G^*(R) | \alpha_{x,y} \in R(xHy^{-1})\right\}.$$

The following lemma is proved in the same way as Lemma 2.2(iv):

LEMMA 2.3. *With the above notation,* $R\{H\} = \Sigma_{h\in H}\tilde{R}\bar{h} \oplus R^*\{H\}$. *Furthermore,* $R^*\{H\}$ *is a two-sided ideal which is essential on either side.* □

LEMMA 2.4. *Let R be a G-graded ring with G infinite and suppose H is a subgroup of G. Then there exist maps*
$$\phi: \mathscr{I}(R(H)) \to \mathscr{I}(R\{H\}), \qquad \psi: \mathscr{I}(R\{H\}) \to \mathscr{I}(R(H)),$$
such that
(i) ϕ *is injective and if* $A, B \triangleleft R(H)$ *then* $\phi(A)\phi(B) \subseteq \phi(AB)$,
(ii) *if* $J, K \triangleleft R\{H\}$ *then* $\psi(J)\psi(K) \subseteq \psi(JK)$,
(iii) $\psi \circ \phi$ *is the identity on* $\mathscr{I}(R(H))$.

Furthermore, let R be component regular, and let $I \triangleleft R\{H\}$. *Then* $\psi(I) = 0$ *if and only if* $I = 0$.

PROOF. Let $D = \Sigma_{x \in G} R(xH)e(x, x)$ and let $D^{-1} = \Sigma_{x \in G} R(Hx^{-1})e(x, x)$. If $A \lhd R(H)$, we take $\phi(A)$ to be $DM_G^*(A)D^{-1}$ where $M_G^*(A)$ is the set of matrices over A with finitely many nonzero entries. Next, if $I \lhd R\{H\}$ we define $\psi(I)$ to be $I_{1,1} = \{\alpha_{1,1} | \alpha \in I\}$. The remainder of the proof is identical to that of Lemma 1.4. □

As in the case where G was finite, results on infinite skew group rings can be translated into results on infinite group-graded rings. In Theorems 2.7 and 2.8 we apply the above construction to a result of Passman [5, Theorem 1.3]. While the theorem was proved for strongly group-graded rings, we state it for skew group rings, since that is all we require here.

(E) THEOREM. *Let $R * G$ be a skew group ring with G infinite. Then $R * G$ contains nonzero ideals A and B with $AB = 0$ if and only if there exist*

(i) *subgroups $N \lhd H \subseteq G$ with N finite,*

(ii) *an H-invariant ideal I of R with $I^x I = 0$ for $x \in G \setminus H$,*

(iii) *nonzero H-invariant ideals $\tilde{A}, \tilde{B} \lhd R * N$ with $\tilde{A}, \tilde{B} \subseteq I * N$ and $\tilde{A}\tilde{B} = 0$.*

Furthermore, $A = B$ if and only if $\tilde{A} = \tilde{B}$. □

Let R be a G-graded ring and suppose $N \lhd H \subseteq G$ are subgroups. Then if $I \lhd R(N)$ and $x \in H$, we let $I^x = R(x^{-1}N)IR(Nx)$. I^x is again an ideal of $R(N)$. This may not be a group action on the ideals of $R(N)$ since it is not true in general that $(I^x)^y = I^{xy}$, but we do have $I^1 = I$ and $(I^x)^y \subseteq I^{xy}$. We say $I \lhd R(N)$ is H-stable if $I^h \subseteq I$ for all $h \in H$. If $I \lhd R(N)$ is H-stable then $\bar{I} = R(H)IR(H) \lhd R(H)$ and $I = \bar{I} \cap R(N)$. In particular, an ideal $I \lhd R(N)$ is H-stable if and only if $I = J \cap R(N)$ for some ideal $J \lhd R(H)$. An ideal $K \lhd R(N)$ is said to be a graded ideal if $K = \oplus \Sigma_{x \in N} K(x)$, where $K(x) = K \cap R(x)$ for each $x \in N$.

LEMMA 2.5. *Let R be a G-graded ring and let $N \lhd H \subseteq G$ be subgroups of G. Also suppose A is an $\overline{H}$-stable ideal of $R\{N\}$. Then $\psi(A) \lhd R(N)$ is an H-stable ideal of $R(N)$. Furthermore, let $I \lhd \tilde{R}\#G$ be an $\overline{H}$-stable ideal of $\tilde{R}\#G$ such that $I^{\bar{t}}I = 0$ for all $t \in G \setminus H$. Then $I\overline{N} \lhd R\{N\}$ and $\psi(I\overline{N})$ is a graded H-stable ideal of $R(N)$ with the property that $\psi(I\overline{N}) \cdot R(t) \cdot \psi(I\overline{N}) = 0$ for all $t \in G \setminus H$.*

PROOF. Let $A \lhd R\{N\}$ and let $x \in H$. Notice that

$$R(x^{-1}N)e(1,1)\bar{x}^{-1} = R(x^{-1}N)e(1, x^{-1}) \subseteq R\{N\}$$

since $N \lhd H$ implies $x^{-1}N = Nx^{-1}$. Similarly, $\bar{x}e(1,1)R(Nx) \subseteq R\{N\}$. Since $A \lhd R\{N\}$, this yields that

$$R(x^{-1}N)e(1,1)\bar{x}^{-1} \cdot A \cdot \bar{x}e(1,1)R(Nx) \subseteq A.$$

But A is $\overline{H}$-stable, so that $\bar{x}^{-1}A\bar{x} = A$ and $e(1,1)Ae(1,1) = \psi(A)e(1,1)$. Thus $R(x^{-1}N)\psi(A)R(Nx)e(1,1) \subseteq A$ or, equivalently, $R(x^{-1}N)\psi(A)R(Nx) \subseteq \psi(A)$, and so $\psi(A)$ is H-stable.

Next let $I \lhd \tilde{R}\#G$ be $\overline{H}$-stable so that $J = I\overline{N}$ is an $\overline{H}$-stable ideal of $R\{N\}$. From $I\bar{t}I = 0$ for all $t \in G \setminus H$ we get that $J\bar{t}J = 0$. This follows since $H \supseteq N$ implies $Nt \subseteq G \setminus H$. Notice that $e(1,1)R(t)\bar{t}^{-1} \subseteq \tilde{R}\#G \subseteq R\{N\}$, and so since $J \lhd R\{N\}$, we have that $Je(1,1)R(t)J = Je(1,1)R(t)\bar{t}^{-1} \cdot \bar{t}J \subseteq J\bar{t}J = 0$. But $e(1,1)Je(1,1) = J_{1,1}e(1,1) = \psi(J)e(1,1)$, so that $J_{1,1}R(t)J_{1,1} = 0$ or $\psi(J)R(t)\psi(J) = 0$. Finally,

$$\psi(J) = (I\bar{N})_{1,1} = \Big(\sum_{h \in N} I\bar{h}\Big)_{1,1} = \sum_{h \in N} (I\bar{h})_{1,1} = \sum_{h \in N} I_{1,h^{-1}}$$

and, since $I \subseteq \tilde{R}\#G$, $I_{1,h^{-1}} \subseteq R(h)$, so that $\psi(J)$ is a graded ideal. □

LEMMA 2.6. *Let R be a left component regular G-graded ring with G finite and suppose A is a nonzero ideal of R. Then $A^0 = \bigcap_{x \in G} AR(x)$ is a nonzero ideal of R. Furthermore, if $A \subseteq I$, where I is a graded ideal of R, then $A^0 \subseteq I(1)R$.*

PROOF. It is clear that A^0 is an ideal of R. To show $A^0 \neq 0$, let $x_1, \ldots, x_n$ be a listing of the elements of G where $n = |G|$. Now let

$$B = AR(x_n x_{n-1}^{-1})R(x_{n-1}x_{n-2}^{-1}) \cdots R(x_2 x_1^{-1})R(x_1).$$

Note that $B \neq 0$ since R is left component regular. Also,

$$\begin{aligned} B &= [AR(x_n x_{n-1}^{-1}) \cdots R(x_{i+1}x_i^{-1})][R(x_i x_{i-1}^{-1}) \cdots R(x_1)] \\ &\subseteq AR(x_i x_{i-1}^{-1} x_{i-1} x_{i-2}^{-1} \cdots x_2 x_1^{-1} x_1) = AR(x_i) \quad \text{for } i = 1, 2, \ldots, n. \end{aligned}$$

Thus $0 \neq B \subseteq \bigcap_{x \in G} AR(x) = A^0$.

Finally note that if $A \subseteq I$ then $A^0 \subseteq I^0$ and, since I is graded,

$$I^0 = \sum_{x \in G}\left[\bigcap_{y \in G} I(xy^{-1})R(y)\right] \subseteq \sum_{x \in G} I(1)R(x) = I(1)R. \quad \square$$

If $I \lhd R(1)$ is an H-stable ideal then $R(H)IR(H) \lhd R(H)$, and $\hat{I} = R(H)IR(H) \cap R(N)$ is an H-stable graded ideal of $R(N)$ with $\hat{I} \cap R(1) = I$.

THEOREM 2.7. *Let R be a left component regular G-graded ring with G infinite. Then there exists a nonzero ideal $A \lhd R$ with $A^2 = 0$ if and only if there exist*

(i) *subgroups $N \lhd H \subseteq G$, with N finite,*

(ii) *an H-stable ideal $I \lhd R(1)$ with $IR(t)I = 0$ for all $t \in G \setminus H$,*

(iii) *a nonzero H-stable ideal $\hat{A} \lhd R(N)$ with $\hat{A} \subseteq \hat{I}$ and $\hat{A}^2 = 0$, where $\hat{I} = R(H)IR(H) \cap R(N)$.*

PROOF. Assume R has a nonzero ideal A with $A^2 = 0$. Then $M_G^*(A)$ is a nonzero ideal of $(\tilde{R}\#G)\bar{G}$ which has square zero. We can apply (E) to give

(a) subgroups $N \lhd H \subseteq G$ with N finite,

(b) I' an $\bar{H}$-stable ideal of $\tilde{R}\#G$ such that $I'\bar{t}I' = 0$ for all $t \in G \setminus H$,

(c) a nonzero $\bar{H}$-stable ideal $A' \lhd (\tilde{R}\#G)\bar{H}$ with $A' \subseteq I'\bar{N}$ and $(A')^2 = 0$.

Assume $\psi(A') \neq 0$ and let $B = \psi(A')$ and $J = \psi(I'N)$. By Lemma 2.5, B and J are H-stable ideals of $R(N)$ and J is graded. Also $B \subseteq J$, since $A' \subseteq I'\bar{N}$, and by Lemma 2.4(ii), $B^2 = \psi(A')^2 \subseteq \psi(0) = 0$. Now let $I = J \cap R(1)$ so that I is an H-stable ideal of $R(1)$. Also $\hat{I} = R(H)IR(H) \cap R(N)$ is an H-stable ideal of $R(N)$, and so if we take $A = \hat{I} \cap B$ then $A \lhd R(N)$ is again H-stable and $A^2 \subseteq B^2 = 0$. Note that $A \neq 0$ since applying Lemma 2.6 to the group N gives that $B \cap IR(N) \neq 0$ and $A = B \cap \hat{I} \supseteq B \cap IR(N)$.

To complete the proof in this direction it remains to show that we can assume $\psi(A') \neq 0$. Note that since $A' \neq 0$, there exist $x, y \in G$ such that $A'_{x,y} \neq 0$. $A'_{x,y}e(x, y) \subseteq A' \triangleleft R\{N\}$ and $R(yNx^{-1})e(y, x) \subseteq R\{N\}$ so that $A'_{x,y}R(yNx^{-1})e(x, x) = A'_{x,y}e(x, y)R(yNx^{-1})e(y, x) \subseteq A'$.

Since $A'_{x,y}$ is nonzero and R is left component regular, it follows that $A'_{x,y}R(yNx^{-1}) \neq 0$ and hence $A'_{x,x} \neq 0$. Now

$$\bar{x}A'\bar{x}^{-1} \triangleleft \bar{x}R\{N\}\bar{x}^{-1} = R\{xNx^{-1}\} \quad \text{and} \quad \psi_x(\bar{x}A'\bar{x}^{-1}) = A'_{x,x} \neq 0.$$

Thus replacing I', A', N and H by $\bar{x}I'\bar{x}^{-1}$, $\bar{x}A'\bar{x}^{-1}$, xNx^{-1} and xHx^{-1}, respectively, gives the desired result.

Conversely, suppose conditions (i), (ii) and (iii) are satisfied. Let $A = R\hat{A}R \triangleleft R$. Since $\hat{A} \neq 0$, we have that $A \neq 0$. If $t \in G \setminus H$ then $HtH \subseteq G \setminus H$ and thus

$$\begin{aligned} \hat{I}R(t)\hat{I} &\subseteq R(H)IR(H) \cdot R(t) \cdot R(H)IR(H) \\ &\subseteq R(H)IR(HtH)IR(H) = 0 \quad \text{by (ii)}. \end{aligned}$$

Hence, $\hat{A}R(t)\hat{A} \subseteq \hat{I}R(t)\hat{I} = 0$. Next let $x \in H$. Since $\hat{A}$ is H-stable, $R(xN)\hat{A}R(Nx^{-1}) \subseteq \hat{A}$. Thus $\hat{A}R(xN)\hat{A}R(Nx^{-1}) \subseteq \hat{A}^2 = 0$. But since R is left component regular, this implies that $\hat{A}R(xN)\hat{A} = 0$. In particular, $\hat{A}R(x)\hat{A} = 0$. Thus we have that $\hat{A}R\hat{A} = 0$, and so $A^2 = R\hat{A}R\hat{A}R = 0$. □

The proof of the final result is left to the reader.

THEOREM 2.8. *Let R be a component regular G-graded ring with G infinite. Then there exist nonzero ideals $A, B \triangleleft R$ with $AB = 0$ if and only if there exist*

(i) *subgroups $N \triangleleft H \subseteq G$ with N finite,*

(ii) *an H-stable ideal $I \triangleleft R(1)$ with $IR(t)I = 0$ for all $t \in G \setminus H$,*

(iii) *nonzero H-stable ideals $\hat{A}$, $\hat{B} \triangleleft R(N)$ with $\hat{A}$, $\hat{B} \subseteq \hat{I}$ and $\hat{A}\hat{B} = 0$, where $\hat{I} = R(H)IR(H) \cap R(N)$.* □

If M is a graded left R-module, where R is G-graded, then M is said to be graded Noetherian if M satisfies the ascending condition on graded submodules. We write $\text{K-dim}_R M$ for the Krull dimension of M and $\text{gr K-dim}_R M$ for the graded Krull dimension of M. The methods of this paper can be used to obtain the following theorem which extends a result of C. Nastasescu.

THEOREM. *Let R be a G-graded ring where G is a polycyclic-by-finite group. If M is a graded Noetherian R-module, then M is Noetherian as an R-module. Furthermore*

$$\text{K-dim}_R M \leqslant \text{gr K-dim}_R M + h(G), \quad \textit{where } h(G) \textit{ is the Hirsch number of } G.$$

This result will be included in a subsequent paper.

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