BROWNIAN MOTION WITH POLAR DRIFT

BY

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ABSTRACT. Consider a strong Markov process X^0 that has continuous sample paths in R^d ($d \geqslant 2$) and the following two properties.

(1) Away from the origin X^0 behaves like Brownian motion with a *polar* drift given in spherical polar coordinates by $\mu(\theta)/2r$. Here μ is a bounded Borel measurable function on the unit sphere in R^d, with average value $\bar{\mu}$.

(2) X^0 is absorbed at the origin.

It is shown that X^0 reaches the origin with probability zero or one as $\bar{\mu} \geqslant 2 - d$ or $< 2 - d$. Indeed, X^0 is transient to $+\infty$ if $\bar{\mu} > 2 - d$ and null recurrent if $\bar{\mu} = 2 - d$. Furthermore, if $\bar{\mu} < 2 - d$ (i.e., X^0 reaches the origin), then X^0 does not approach the origin in any particular direction. Indeed, there is a single Martin boundary point for X^0 at the origin. The question of the existence and uniqueness of a strong Markov process with continuous sample paths in R^d that behaves like X^0 away from the origin, but spends zero time there (in the sense of Lebesgue measure), is also resolved here.

1. Introduction. Let X^0 be a strong Markov process with continuous sample paths in R^d ($d \geqslant 2$) such that the following two properties hold.

(i) In $R^d \setminus \{0\}$, X^0 behaves like a diffusion process with (local) generator

$$(1.1) \qquad L \equiv \frac{1}{2}\Delta + (2r)^{-1}\mu(\theta)\frac{\partial}{\partial r}.$$

(ii) X^0 is absorbed at the origin.

Here Δ is the usual Laplace operator in R^d, μ is a bounded Borel measurable function on the unit sphere S^{d-1} in R^d, and $(r, \theta) \in (0, \infty) \times S^{d-1}$ denote polar coordinates in $R^d \setminus \{0\}$. Paraphrasing (i) we say that X^0 behaves like Brownian motion with *polar drift* $\mu(\theta)/2r$ in $R^d \setminus \{0\}$. The following questions are addressed in this paper. Does X^0 reach the origin (i.e. the pole)? If so, does it have an asymptotic direction of approach there? What is the nature of the Martin boundary for X^0 at the origin? Does there exist a strong Markov process with continuous sample paths in R^d that behaves like X^0 away from the origin, but spends zero time there (in the sense of Lebesgue measure)?

There are two basic motivations for the study of this particular process X^0. The first is the desire to increase the detailed knowledge of the behavior of multidimensional diffusions near singular points of the coefficients. The second is the fact that X^0 is a basic "model" for comparison with processes having more general drift

and/or diffusion coefficients, but for which direct computations are usually not possible.

This paper is divided into five sections. In §2, an alternative representation is given for X^0 as a "twisted product". This is similar to the skew product constructions of Itô and McKean [6, §7.15–§7.17], which cover the special case $\mu \equiv$ constant. The difference here is that the radial part of the process can depend in a nontrivial way on the spherical part, via $\mu(\theta)$. The twisted product representation provides intuition for the results obtained in §3 and §4 on the behavior of X^0 near the origin. In §3 it is shown that starting from $x \neq 0$, X^0 hits the origin with probability zero or one, respectively, as

$$\bar{\mu} \equiv \int_{S^{d-1}} \mu(\theta)\, d\bar{\sigma}(\theta)$$

is $\geq (2 - d)$ or $< (2 - d)$. Here $\bar{\sigma}$ denotes surface area on S^{d-1}, normalized so that $\bar{\sigma}(S^{d-1}) = 1$. Furthermore, X^0 is transient to $+\infty$ if $\bar{\mu} > 2 - d$ and null recurrent if $\bar{\mu} = 2 - d$. In §4, the asymptotic behavior of the resolvent of X^0 near the origin is obtained. This reflects the properties that X^0 does not approach the origin in any particular direction and that the Martin boundary is a single point there. The question of whether X^0 can be modified to a continuous strong Markov process that spends zero time at the origin is answered in §5.

For the special case $\mu \equiv$ constant, the results obtained here are consistent with those that can be deduced by one-dimensional analysis using the skew product constructions of Itô and McKean [6]. The interested reader might like to compare this paper with that of Kendall [7] on "Pole seeking Brownian motion and bird navigation". The drift part of the two-dimensional diffusion studied there includes a singular term which has components in the radial and tangential directions (see [7, p. 381]).

For each domain $D \subset R^d$ and $n \geq 0$, $C^n(D)$ denotes the set of functions f: $D \to R$ that are n-times continuously differentiable in D. The set of functions in $C^n(D)$ whose partial derivatives up to and including those of order n are bounded on D is denoted by $C_b^n(D)$. The symbol $C_c^n(D)$ denotes the set of functions in $C^n(D)$ that have compact support in D. If $n = 0$, the superscript n is usually omitted. The same notations will be used with $R^1 \times S^{d-1}$ in place of R^d. The abbreviation "a.e." is used for "almost everywhere" with respect to Lebesgue measure on R^d.

The Laplace operator on $R^d \setminus \{0\}$ is given in polar coordinates by

$$(1.2) \qquad \Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{d-1}}$$

where $\Delta_{S^{d-1}}$ denotes the Laplace–Beltrami operator on S^{d-1}. The set S^{d-1} is endowed with the topology (and hence Borel σ-field) induced from R^d and σ denotes the surface measure there. The normalized surface measure $\sigma/\sigma(S^{d-1})$ is denoted by $\bar{\sigma}$.

2. Twisted product representation. Given μ as described in §1, the existence and uniqueness (in law) of the process X^0 follows by applying the theory of Stroock and

Varadhan [**13**, **14**] to the diffusion with generator L that is absorbed at the boundary of $R^d \setminus \{x: |x| \leq \varepsilon\}$, and then letting $\varepsilon \downarrow 0$. Indeed, X^0 is characterized (in law) as the unique process that has continuous paths in R^d and associated probability measures $\{P_x^0\}$ (one for each starting point $x \in R^d$) satisfying the following three properties.

(i) $P_x^0(X^0(0) = x) = 1$.

(ii) For each $f \in C_c^2(R^d)$ with compact support in $R^d \setminus \{0\}$,

(2.1) $$\left\{ f(X^0(t \wedge \tau_0)) - \int_0^{t \wedge \tau_0} (Lf)(X^0(s)) \, ds, \ \mathscr{F}_{t \wedge \tau_0}^0, \ t \geq 0 \right\}$$

is a P_x^0-martingale, where

$$\mathscr{F}_t^0 = \sigma\{X^0(s): 0 \leq s \leq t\} \quad \text{and} \quad \tau_0 = \inf\{t \geq 0: X^0(t) = 0\}.$$

(iii) $P_x^0(X^0(t) = 0 \text{ for all } t \geq \tau_0) = 1$.

In this section we give a representation for X^0 which provides useful intuition for the results given in later sections on the behavior of X^0 near the origin.

Suppose $(\Omega, \mathscr{G})$ is a fixed measurable space on which is defined a standard one-dimensional Brownian motion B, and an independent spherical Brownian motion Z with paths in S^{d-1}. For each $t \in [0, \infty)$, define

(2.2) $$Y(t) = B_1(t) + \frac{1}{2}(d-2)t + \frac{1}{2}\int_0^t \mu(Z(s)) \, ds.$$

Then (Y, Z) is a diffusion process with continuous paths in $R^1 \times S^{d-1}$ whose generator acting on functions in $C_c^2(R^1 \times S^{d-1})$ is given by

(2.3) $$L_{YZ} = \frac{1}{2}\left(\frac{\partial^2}{\partial y^2} + (d-2)\frac{\partial}{\partial y} + \mu(z)\frac{\partial}{\partial y} + \Delta_{S^{d-1}} \right)$$

for $(y, z) \in R^1 \times S^{d-1}$. In particular, for each $g \in C_c^2(R^1 \times S^{d-1})$,

(2.4) $$\left\{ g(Y(t), Z(t)) - \int_0^t (L_{YZ}g)(Y(s), Z(s)) \, ds, \ \mathscr{G}_t, \ t \geq 0 \right\}$$

is a $Q_{y,z}$-martingale, where $\{\mathscr{G}_t = \sigma\{(Y(s), Z(s)): 0 \leq s \leq t\}, \ t \geq 0\}$ is the natural filtration generated by (Y, Z), and $Q_{y,z}$ is the probability measure on $(\Omega, \mathscr{G})$ associated with (Y, Z) starting from (y, z).

Next define

(2.5) $$A(t) = \int_0^t e^{2Y(s)} \, ds \quad \text{for all } t \geq 0,$$

and

(2.6) $$A_\infty = \int_0^\infty e^{2Y(s)} \, ds.$$

Let A^{-1} denote the (continuous) functional inverse of A, with $A^{-1}(t) \equiv \infty$ if $t \geq A_\infty$. For each $t \in [0, A_\infty)$, define $X^*(t)$ in polar coordinates by

(2.7) $$X^*(t) = (R(t), \Theta(t))$$

where

(2.8) $$R(t) = e^{Y(A^{-1}(t))} \quad \text{and} \quad \Theta(t) = Z(A^{-1}(t)).$$

Further define

(2.9) $X^*(t) = 0 \in R^d$ for all $t \geq A_\infty$.

Then we have the following

THEOREM 2.1. X^* *is a representation for* X^0, *i.e.,* X^* *is equivalent in law to* X^0.

PROOF. By construction, X^* has continuous paths in R^d and is absorbed at the origin. Thus, it suffices to verify that property (ii) of the characterization of X^0 holds for X^*, for each starting point $x \in R^d$. For this it is enough to verify that, for each $x \in R^d \setminus \{0\}$, $0 < \varepsilon < 1$ and $f \in C_c^2(R^d \setminus \{0\})$,

(2.10) $\left\{ f\big(X^*(t \wedge \tau_\varepsilon^*)\big) - \int_0^{t \wedge \tau_\varepsilon^*} (Lf)(X^*(s)) \, ds, \mathscr{F}_{t \wedge \tau_\varepsilon^*}^*, \, t \geq 0 \right\}$

is a $Q_{y,z}$-martingale, where $(y, z) = (\ln|x|, x/|x|) \in R^1 \times S^{d-1}$,

$$\tau_\varepsilon^* = \inf\left\{ s \geq 0 : |X^*(s)| \leq \varepsilon \text{ or } |X^*(s)| \geq \varepsilon^{-1} \right\},$$

and $\mathscr{F}_t^* = \sigma\{ X^*(s) : 0 \leq s \leq t \}$. Given $0 < \varepsilon < 1$ and $f(r, \theta) \in C_c^2(R^d \setminus \{0\})$, let

$$\tau_\varepsilon^Y = \inf\left\{ s \geq 0 : Y(s) \leq \ln \varepsilon \text{ or } Y(s) \geq \ln(\varepsilon^{-1}) \right\}$$

and define $g \in C_c^2(R^1 \times S^{d-1})$ by

$$g(y, z) = f(e^y, z) \quad \text{for all } (y, z) \in R^1 \times S^{d-1}.$$

Then by the martingale property (2.4) of (Y, Z) and Doob's stopping theorem, since $E^{Q_{y,z}}(\tau_\varepsilon^Y) < \infty$, we have

(2.11) $\Big\{ g\big(Y(A^{-1}(t) \wedge \tau_\varepsilon^Y), Z(A^{-1}(t) \wedge \tau_\varepsilon^Y)\big)$

$$- \int_0^{A^{-1}(t) \wedge \tau_\varepsilon^Y} (L_{YZ}g)(Y(s), Z(s)) \, ds, \mathscr{G}_{A^{-1}(t) \wedge \tau_\varepsilon^Y}, \, t \geq 0 \Big\}$$

is a $Q_{y,z}$-martingale for each $(y, z) \in R^1 \times S^{d-1}$. Now, by (2.3),

$$(L_{YZ}g)(y, z) = \left\{ \frac{1}{2}\left(e^{2y} \frac{\partial^2}{\partial r^2} + (d - 1 + \mu(z)) e^y \frac{\partial}{\partial r} + \Delta_{S^d} \right) f \right\}(e^y, z).$$

Also, $A^{-1}(t) \wedge \tau_\varepsilon^Y = A^{-1}(t \wedge \tau_\varepsilon^*)$. By substituting the above in (2.11) and changing the variable of integration there to $u = A(s)$ (so that $du = e^{2Y(s)} ds$), we obtain

$\Big\{ f\big(\exp(Y(A^{-1}(t \wedge \tau_\varepsilon^*))), Z(A^{-1}(t \wedge \tau_\varepsilon^*))\big)$

$$- \int_0^{t \wedge \tau_\varepsilon^*} (Lf)\big(\exp(Y(A^{-1}(u))), Z(A^{-1}(u))\big) \, du, \mathscr{G}_{A^{-1}(t \wedge \tau_\varepsilon^*)}, \, t \geq 0 \Big\}$$

is a $Q_{y,z}$-martingale. Here (cf. (1.1)–(1.2)),

(2.12) $L = \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + r^{-1}(d - 1 + \mu(\theta)) \frac{\partial}{\partial r} + r^{-2}\Delta_{S^{d-1}} \right).$

Recalling the definition (2.7)–(2.8) of X^* and noting that $\mathscr{F}_{t \wedge \tau_\varepsilon^*}^* \subset \mathscr{G}_{A^{-1}(t \wedge \tau_\varepsilon^*)}$, we see that the desired martingale property (2.10) holds.   □

Henceforth we shall use X^* as our representation for X^0.

By dividing (2.2) by t and letting $t \to \infty$, we obtain, for $(y, z) \in R^1 \times S^{d-1}$

(2.13) $\lim_{t \to \infty} \frac{Y(t)}{t} = \frac{1}{2}\left\{ d - 2 + \int_{S^{d-1}} \mu(\theta) \, d\bar\sigma(\theta) \right\}$ $Q_{y,z}$-a.s.,

since the steady-state distribution for Z on S^{d-1} is uniform. Thus, if

$$\bar\mu \equiv \int_{S^{d-1}} \mu(\theta)\, d\bar\sigma(\theta) < 2 - d,$$

then $Q_{y,z}$-a.s. there is a constant $c > 0$ and time t_0 (depending on $\omega \in \Omega$) such that $Y(t) \leqslant -ct$ for all $t \geqslant t_0$. This implies $A_\infty = \int_0^\infty e^{2Y(s)}\, ds < \infty\ Q_{y,z}$-a.s. and, consequently (cf. (2.9)), $\tau_0 \equiv \inf\{t \geqslant 0:\ X^0(t) = 0\} < \infty\ P_x^0$-a.s. for all $x \in R^d$. On the other hand, if $\bar\mu > 2 - d$, it follows that $A_\infty = \infty\ Q_{y,z}$-a.s., and for $x \neq 0$, P_x^0-a.s., $\tau_0 = \infty$ and $|X^0(t)| = R(t) = e^{Y(A^{-1}(t))} \to \infty$ as $t \to \infty$. In the next section these results will be strengthened to show that if $x \neq 0$ and $\bar\mu = 2 - d$, then P_x^0-a.s., $\tau_0 = \infty$. However, X^0 is (finely) null recurrent then.

In case $\bar\mu < 2 - d$, $A^{-1}(t)\uparrow\infty$ as $t\uparrow\tau_0 < \infty\ P_x^0$-a.s. $(x \neq 0)$. Since $\Theta(t) = Z(A^{-1}(t))$ and Z is positive recurrent on S^{d-1}, it then follows that X^0 does not have an asymptotic direction of approach to the origin. This property is reflected in the asymptotic behavior obtained in §4 for the resolvent of X^0 near the origin.

3. Is the pole reached? To determine whether the origin (the pole of $\mu(\theta)/2r$) is reached by the process X^0, we seek a nonconstant function u such that $u(X^0(\cdot))$ is a local martingale under P_x^0 for each $x \neq 0$. For this, since L is the generator of X^0, we seek a function u such that (at least in a weak sense)

$$(3.1) \qquad\qquad Lu = 0 \quad \text{in } R^d\backslash\{0\}$$

and u has either a minimum (of zero) or a singularity at the origin. Because L is homogeneous in r, it is natural to look for a u of the following form for some $\alpha \in R^1$:

$$(3.2) \qquad\qquad u(r,\theta) = \begin{cases} r^\alpha \phi_\alpha(\theta) & \text{if } \alpha \in R^1\backslash\{0\}, \\ \ln r + \chi(\theta) & \text{if } \alpha = 0, \end{cases}$$

where $\phi_\alpha > 0$ on S^{d-1}.

Since μ was only assumed to be bounded and measurable, in general the most regularity we can expect for ϕ_α (or χ) is that it lie in some Sobolev space contained in $C^1(S^{d-1})$ such that (3.1) holds in the sense of (Schwartz) distributions. Let p be a fixed real number such that $p > d - 1$, and let $W^{2,p}(S^{d-1})$ denote the Sobolev space of functions on S^{d-1} that have derivatives through order 2 in $L^p(S^{d-1})$. By the Sobolev embedding theorem (cf. [5, §7.7]), $W^{2,p}(S^{d-1}) \subset C^1(S^{d-1})$. It is shown below that there is an α (depending on μ and d) and an associated $\phi_\alpha > 0$ (or χ if $\alpha = 0$) in $W^{2,p}(S^{d-1})$ such that u given by (3.2) satisfies (3.1) in the sense of distributions. An extended version of the martingale property (2.1) is then applied to u to show that, starting from $x \neq 0$, X^0 reaches the origin with P_x^0-probability zero or one as $\alpha \leqslant 0$ or $\alpha > 0$. Although a closed form expression is not known for α (except when $\mu \equiv$ constant), the following criterion is given for determining the sign of α: $\alpha < 0$, $= 0$, or > 0 as

$$(3.3) \qquad\qquad \bar\mu \equiv \int_{S^d{}_1} \mu(\theta)\, d\bar\sigma(\theta)$$

is $> 2 - d$, $= 2 - d$, or $< 2 - d$ (note the reversal of the inequality signs here).

To prove the above, we first determine whether there is a u of the form (3.2) with $\alpha \neq 0$. By the definition (2.12) of L, there is such a u if and only if there is $\alpha \neq 0$ and $\phi_\alpha \in W^{2,p}(S^{d-1})$ satisfying $\phi_\alpha > 0$ on S^{d-1} and

$$(3.4) \qquad \Delta_{S^{d-1}}\phi_\alpha + \{\alpha\mu + \alpha(\alpha + d - 2)\}\phi_\alpha = 0 \quad \text{on } S^{d-1}.$$

Here and henceforth, an equation involving functions in Sobolev spaces is interpreted in the weak sense of distributions. If $\mu \equiv$ constant $\neq 2 - d$, then $\alpha = 2 - d - \mu$ and $\phi_\alpha \equiv 1$ determine a suitable u. But in general, α and ϕ_α cannot be determined explicitly. This prompts the following restatement (Q) of the question of the existence of α and ϕ_α, in terms of an eigenvalue problem.

For each $\alpha \in R^1$, for all $\lambda \in R^1$ sufficiently negative, $(-\Delta_{S^{d-1}} - \alpha\mu - \lambda)^{-1}$ is a bounded positive linear operator from $L^p(S^{d-1})$ to $W^{2,p}(S^{d-1})$. Since $W^{2,p}(S^{d-1})$ is compactly embedded in $L^p(S^{d-1})$, it then follows by the Krein–Rutman [8, Theorem 6.3] theory of positive linear operators that, for each $\alpha \in R^1$, there is a smallest eigenvalue $\lambda(\alpha)$ and associated (strictly) positive eigenfunction $\psi_\alpha \in W^{2,p}(S^{d-1})$ satisfying $\int_{S^{d-1}} \psi_\alpha^2 \, d\bar\sigma = 1$ and

$$(3.5) \qquad \Delta_{S^{d-1}}\psi_\alpha + \{\alpha\mu + \lambda(\alpha)\}\psi_\alpha = 0 \quad \text{on } S^{d-1}.$$

The question of the existence of $\alpha \neq 0$ and $\phi_\alpha > 0$ satisfying (3.4) can then be rephrased as follows.

(Q)   Is there an $\alpha \neq 0$ such that $\lambda(\alpha) = \alpha(\alpha + d - 2)$?

To answer this, the following properties of $\lambda(\alpha)$ and ψ_α are used (cf. [5]). For $\alpha = 0$, $\lambda(0) = 0$ and $\psi_0 = 1/(\bar\sigma(S^{d-1}))^{1/2}$. For each $\alpha \in R^1$,

$$(3.6) \quad \lambda(\alpha) = \inf_{0 \neq \psi \in W^{2,p}(S^{d-1})} \left\{ \int_{S^{d-1}} \psi(-\Delta_{S^{d-1}}\psi - \alpha\mu\psi) \, d\bar\sigma \Big/ \int_{S^{d-1}} \psi^2 \, d\bar\sigma \right\}.$$

It follows that $\lambda(\alpha)$ is a concave function of α, and by setting $\psi \equiv 1$ in (3.6) we see that $\lambda(\alpha) \leqslant |\alpha| \, \|\mu\|_\infty$ where $\|\mu\|_\infty = \sup\{|\mu|(\theta)\colon \theta \in S^{d-1}\}$. By the perturbation theory of $(\Delta_{S^{d-1}} + \alpha\mu + \lambda)^{-1}$ (cf. [3, Lemma 1.3]), $\alpha \to \lambda(\alpha) \in R^1$ and $\alpha \to \psi_\alpha \in W^{2,p}(S^{d-1})$ are real analytic functions of α. By integrating (3.5) over S^{d-1} and using Green's identity: $\int_{S^{d-1}} \Delta_{S^{d-1}}\psi_\alpha \, d\bar\sigma = 0$, we conclude that

$$(3.7) \qquad \lambda(\alpha) = -\left\{ \alpha\int_{S^{d-1}} \mu\psi_\alpha \, d\bar\sigma \right\} \Big/ \left\{ \int_{S^{d-1}} \psi_\alpha \, d\bar\sigma \right\}.$$

Then, by differentiating (3.7) with respect to α and setting $\alpha = 0$, we obtain

$$(3.8) \qquad \lambda'(0) = -\int_{S^{d-1}} \mu(\theta) \, d\bar\sigma(\theta) \equiv -\bar\mu.$$

Thus, $\eta(\alpha) \equiv \lambda(\alpha) - \alpha(\alpha + d - 2)$ is a strictly concave function of α, $\eta(0) = 0$ and $\eta(\alpha) \to -\infty$ as $\alpha \to \pm\infty$. It follows that there is (a unique) $\alpha > 0$ (resp. $\alpha < 0$) such that $\lambda(\alpha) = \alpha(\alpha + d - 2)$ if and only if $\eta'(0) > 0$ (resp. $\eta'(0) < 0$). By combining this with (3.8) and giving a separate argument for the exceptional case $\eta'(0) = 0$, we obtain the following.

THEOREM 3.1. *If $\bar{\mu} < 2 - d$ (resp. $> 2 - d$), there is (a unique) $\alpha > 0$ (resp. $\alpha < 0$) such that (3.5) holds with $\lambda(\alpha) = \alpha(\alpha + d - 2)$. Then by setting $\phi_\alpha = \psi_\alpha$ for this α, we obtain a solution $u(r, \theta) = r^\alpha \phi_\alpha(\theta)$ of (3.1) with $\phi_\alpha \in W^{2,p}(S^{d-1}) \subset C^1(S^{d-1})$ and $\phi_\alpha > 0$ on S^{d-1}. If $\bar{\mu} = 2 - d$, there is $\chi \in W^{2,p}(S^{d-1}) \subset C^1(S^{d-1})$ such that*

$$(3.9) \qquad\qquad u(r, \theta) = \ln r + \chi(\theta)$$

is a solution of (3.1).

PROOF. The cases $\bar{\mu} < 2 - d$ and $\bar{\mu} > 2 - d$ follow from the discussion above and the observation that $\eta'(0)$ is > 0 (resp. < 0) as $\bar{\mu} \equiv -\lambda'(0)$ is $< 2 - d$ (resp. $> 2 - d$).

So suppose $\bar{\mu} = 2 - d$. By the definition (2.12) of L, there is a u of the form (3.9) satisfying (3.1) if and only if there is $\chi \in W^{2,p}(S^{d-1})$ satisfying

$$(3.10) \qquad\qquad \Delta_{S^{d-1}} \chi = -(d - 2 + \mu) \quad \text{on } S^{d-1}.$$

Now $\lambda = 0 \in R^1$ is the smallest eigenvalue of $(\Delta_{S^{d-1}} + \lambda)^{-1}$ and the constant function is the solution of the corresponding homogeneous adjoint problem. It then follows by the Fredholm alternative (cf. [5, §5.4]) that there is a solution $\chi \in W^{2,p}(S^{d-1})$ satisfying (3.10) if and only if

$$(3.11) \qquad\qquad \int_{S^{d-1}} (d - 2 + \mu) \, d\bar{\sigma} = 0.$$

But (3.11) is precisely the condition that $\bar{\mu} = 2 - d$. $\quad\square$

Henceforth, α will be used to denote the distinguished real number such that there is a solution u of the form (3.2) satisfying (3.1). In particular, $\alpha = 0$ if $\bar{\mu} = 2 - d$.

The function u will now be used to determine whether the origin is reached by X^0. But first we must show that X^0 escapes from any bounded, punctured neighborhood of the origin in finite (expected) time. For this it is expeditious to use the times at which X^0 first reaches a level set of the function v defined on $R^d \setminus \{0\}$ by

$$(3.12) \qquad\qquad v = \begin{cases} u & \text{if } \alpha > 0, \\ e^u & \text{if } \alpha = 0, \\ 1/u & \text{if } \alpha < 0, \end{cases}$$

and defined at the origin by $v(0) = 0$. The function v is continuous on R^d and $v(r, \theta)$ increases with $r \in (0, \infty)$ for each fixed $\theta \in S^{d-1}$. For each $r \geq 0$, define

$$(3.13) \qquad\qquad \tau_r = \inf\{t \geq 0: v(X^0(t)) = r\},$$

and for $0 \leq r < K < \infty$, let $\tau_{rK} = \tau_r \wedge \tau_K$.

LEMMA 3.1. *Let $x \in R^d \setminus \{0\}$. Then for any $K \geq v(x)$, we have*

$$(3.14) \qquad\qquad E^{P_x^0}\left[\int_0^{\tau_{0K}} h(X^0(s)) \, ds\right] \leq g(u(x)),$$

where for $(r, \theta) \in R^d \setminus \{0\}$,

$$(3.15) \qquad h(r, \theta) = \begin{cases} \phi_\alpha^{(2/\alpha) - 2}(\theta)\left(\alpha^2 \phi_\alpha^2 + |\nabla_{S^{d-1}} \phi_\alpha|^2\right)(\theta) & \text{if } \alpha \neq 0, \\ e^{2\chi(\theta)}\left(1 + |\nabla_{S^{d-1}} \chi|^2\right)(\theta) & \text{if } \alpha = 0, \end{cases}$$

and if $\alpha \in (0, \infty) \setminus \{2\}$,

$$(3.16) \quad g(y) = \begin{cases} 0 & \text{for } y = 0, \\ \alpha^2 y \left(K^{(2/\alpha)-1} - y^{(2/\alpha)-1} \right)/(2 - \alpha) & \text{for } 0 \leqslant y \leqslant K, \\ \alpha K^{2/\alpha}(1 - y/K) & \text{for } y \geqslant K, \end{cases}$$

if $\alpha = 2$,

$$(3.17) \quad g(y) = \begin{cases} 0 & \text{for } y = 0, \\ 2y(\ln K - \ln y) & \text{for } 0 < y \leqslant K, \\ 2K(1 - (y/K)) & \text{for } y \geqslant K, \end{cases}$$

if $\alpha = 0$,

$$(3.18) \quad g(y) = \begin{cases} (K^2 - e^{2y})/2 & \text{for } -\infty < y < \ln K, \\ K^2(\ln K - y) & \text{for } y \geqslant \ln K, \end{cases}$$

if $\alpha < 0$,

$$(3.19) \quad g(y) = \begin{cases} -2\alpha K^{1-(2/\alpha)}(y - K^{-1})/(2 - \alpha) & \text{for } 0 \leqslant y < K^{-1}, \\ \alpha^2 \left(K^{-2/\alpha} - y^{2/\alpha} \right)/(2 - \alpha) & \text{for } y \geqslant K^{-1}. \end{cases}$$

REMARKS. (1) Since ϕ_α (or χ if $\alpha = 0$) is in $C^1(S^{d-1})$ and $\phi_\alpha > 0$, it follows that $h \in C(R^d \setminus \{0\})$ and there are constants c_1, c_2 such that

$$(3.20) \qquad\qquad 0 < c_1 \leqslant h \leqslant c_2 \quad \text{on } R^d \setminus \{0\}.$$

(2) The relation (3.14) actually holds with equality in place of inequality there. This is proved in Theorem 3.3, since it requires the result of Theorem 3.2, which in turn depends on the above lemma.

PROOF. On $R^d \setminus \{0\}$, define $f = g(u)$. Then $f \in C^1(R^d \setminus \{0\}) \cap W^{2,p}_{\mathrm{loc}}(R^d \setminus \{0\})$. Here $W^{2,p}_{\mathrm{loc}}(R^d \setminus \{0\})$ denotes the space of functions that are in the Sobolev space $W^{2,p}(D)$ for each domain $D \subset R^d \setminus \{0\}$ such that $\bar{D}$ is a compact subset of $R^d \setminus \{0\}$. Now,

$$(3.21) \qquad Lf = \tfrac{1}{2} g''(u) |\nabla u|^2 + g'(u) Lu = \tfrac{1}{2} g''(u) |\nabla u|^2,$$

where

$$(3.22) \quad |\nabla u|^2 = \begin{cases} r^{2\alpha-2} \left(\alpha^2 \phi_\alpha^2 + |\nabla_{S^{d-1}} \phi_\alpha|^2 \right) = u^{2-(2/\alpha)} h & \text{if } \alpha \neq 0, \\ r^{-2} \left(1 + |\nabla_{S^{d-1}} \chi|^2 \right) = e^{-2u} h & \text{if } \alpha = 0. \end{cases}$$

The function g was chosen such that, if $\alpha > 0$,

$$g''(y) = \begin{cases} -2y^{(2/\alpha)-2} & \text{for } 0 < y < K, \\ 0 & \text{for } y > K, \end{cases}$$

if $\alpha = 0$,

$$g''(y) = \begin{cases} -2e^{2y} & \text{for } -\infty < y < \ln K, \\ 0 & \text{for } y > \ln K, \end{cases}$$

and if $\alpha < 0$,

$$g''(y) = \begin{cases} 0 & \text{for } 0 < y < K^{-1}, \\ -2y^{(2/\alpha)-2} & \text{for } y > K^{-1}. \end{cases}$$

By combining these with (3.21)–(3.22), we see that

(3.23) $$Lf(z) = \begin{cases} -h(z) & \text{for } 0 < v(z) < K, \\ 0 & \text{for } v(z) > K. \end{cases}$$

It is readily verified that f is nonnegative and bounded on $\{ z \in R^d \colon 0 < v(z) < K \}$, that $f(z) = 0$ on $v^{-1}(K)$ and $\lim_{z \to 0} f(z) = 0$ if $\alpha > 0$. Consistent with the last property, we define $f(0) = 0$ if $\alpha > 0$.

For $0 < r < v(x)$, let $D_{rK} = \{ z \in R^d \colon r < v(z) < K \}$. It follows by the same kind of approximation argument as in Stroock and Varadhan [**12**, II, Theorem 11.2] that the martingale property (2.1) extends to $f \in C^1(R^d \setminus \{0\}) \cap W^{2,p}_{\text{loc}}(R^d \setminus \{0\})$ with τ_{rK} in place of τ_0. Hence, by (3.23), for all $t \geqslant 0$:

(3.24) $$E^{P^0_x}\left[f\left(X^0(t \wedge \tau_{rK})\right)\right] + E^{P^0_x}\left[\int_0^{t \wedge \tau_{rK}} h\left(X^0(s)\right) ds \right] = f(x).$$

By letting $t \to \infty$ and then $r \to 0$ in the above, and using the nonnegativity of f and Fatou's lemma, we obtain

(3.25) $$E^{P^0_x}\left[\int_0^{\tau_{0K}} h\left(X^0(s)\right) ds \right] \leqslant f(x).$$

Since $f(x) = g(u(x))$, the proof is complete. $\square$

COROLLARY 3.1. *For each* $x \in R^d \setminus \{0\}$ *and* $K \geqslant v(x)$,

(3.26) $$E^{P^0_x}[\tau_{0K}] < \infty.$$

PROOF. Since $h \geqslant c_1 > 0$ on $R^d \setminus \{0\}$, this follows immediately from (3.14). $\square$

THEOREM 3.2. *Let* $x \in R^d \setminus \{0\}$. *Then*,

(3.27) $$P^0_x(\tau_0 < \infty) = \begin{cases} 1 & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha \leqslant 0, \end{cases}$$

where $\alpha > 0$ *or* $\alpha \leqslant 0$ *as* $\bar{\mu} < 2 - d$ *or* $\bar{\mu} \geqslant 2 - d$.

PROOF. Let $0 < \varepsilon < v(x) < K < \infty$ and recall that

$$\tau_{\varepsilon K} = \inf\{ t \geqslant 0 \colon v\left(X^0(t)\right) = \varepsilon \text{ or } v\left(X^0(t)\right) = K \}.$$

As in the proof of Lemma 3.1 above, the martingale property (2.1), with $\tau_{\varepsilon K}$ in place of τ_0, can be extended to

$$u \in C^1\left(R^d \setminus \{0\}\right) \cap W^{2,p}_{\text{loc}}\left(R^d \setminus \{0\}\right).$$

Thus $\{ u(X^0(t \wedge \tau_{\varepsilon K})),\ \mathscr{F}^0_{t \wedge \tau_{\varepsilon K}},\ t \geqslant 0\}$ is a P^0_x-martingale. Since u is bounded on $\overline{D}_{\varepsilon K} = \{ z \colon \varepsilon \leqslant v(z) \leqslant K \}$ and $P^0_x(\tau_{\varepsilon K} < \infty) = 1$, by (3.26), after taking the expected value of $u(X^0(t \wedge \tau_{\varepsilon K}))$ and letting $t \to \infty$, we conclude that

(3.28) $$E^{P^0_x}\left[u\left(X^0(\tau_{\varepsilon K})\right)\right] = u(x).$$

If $\alpha > 0$,

$$u\left(X^0(\tau_{\epsilon K})\right) = \epsilon 1_{\{\tau_\epsilon < \tau_K\}} + K1_{\{\tau_K < \tau_\epsilon\}} \quad P_x^0\text{-a.s.}$$

By combining this with (3.28) we obtain

$$P_x^0(\tau_\epsilon < \tau_K) = (K - u(x))/(K - \epsilon).$$

Since $\tau_{0K} < \infty$ P_x^0-a.s. by (3.26), we can let $\epsilon \downarrow 0$ in the above to conclude that

(3.29) $$P_x^0(\tau_0 < \tau_K) = 1 - \{u(x)/K\},$$

and then let $K \to \infty$ to obtain $P_x^0(\tau_0 < \infty) = 1$ if $\alpha > 0$. On the other hand, for $\alpha \leqslant 0$, by using the definitions of $\tau_{\epsilon K}$ and v (see (3.12)), we see that (3.28) implies

(3.30) $$P_x^0(\tau_\epsilon < \tau_K) = \begin{cases} (u(x) - K^{-1})/(\epsilon^{-1} - K^{-1}) & \text{if } \alpha < 0, \\ (\ln K - u(x))/(\ln K - \ln \epsilon) & \text{if } \alpha = 0. \end{cases}$$

Then letting $\epsilon \downarrow 0$ and using (3.26) we conclude that

$$P_x^0(\tau_0 < \tau_K) = 0 \quad \text{if } \alpha \leqslant 0. \quad \square$$

By letting $t \to \infty$, and then $r \to 0$ in (3.24), and using (3.27) to conclude that $f(X^0(\tau_{0K})) = 0$ P_x^0-a.s., we obtain the following strengthened version of Lemma 3.1. To emphasize its importance for work in later sections, it is labelled as a theorem.

THEOREM 3.3. *Let* $x \in R^d \setminus \{0\}$, $K \geqslant v(x)$, *and* g, h *be defined as in Lemma* 3.1. *Then*,

(3.31) $$E^{P_x^0}\left[\int_0^{\tau_{0K}} h\left(X^0(s)\right) ds \right] = g(u(x)).$$

By using (3.30), the following result on transience/recurrence of X^0 can be proved in a similar manner to that employed in Williams [16]. For details and terminology, the interested reader is referred to that paper.

COROLLARY 3.2. *If* $\bar{\mu} > 2 - d$,

$$P_x^0\left(\liminf_{t \to \infty} |X^0(t)| = +\infty \right) = 1$$

for all $x \in R^d \setminus \{0\}$. *If* $\bar{\mu} = 2 - d$, *starting from* $x \neq 0$, X^0 *is (finely) null recurrent on* R^d.

4. An ergodic property. For each $\gamma > 0$ and $k \in C_b(R^d)$, let $R_\gamma^0 k$ denote the stopped resolvent defined by

$$R_\gamma^0 k(x) = E^{P_x^0}\left[\int_0^{\tau_0} e^{-\gamma t} k\left(X^0(t)\right) dt \right] \quad \text{for all } x \in R^d.$$

The following result plays a key role in determining whether there is a strong Markov process with continuous paths in R^d that behaves like X^0 in $R^d \setminus \{0\}$, but spends zero time at the origin. It also implies that there is a single point of the Martin boundary for X^0 at the origin (cf. Rogers [11, §4]). In the sequel, uniformly as $|x| \to 0$ means uniformly in the spherical coordinate $\theta \equiv x/|x|$.

THEOREM 4.1. *For each $\gamma > 0$, $\beta > 0$ and $k \in C_b(R^d)$ having compact support in $R^d \setminus \{0\}$,*

$$(4.1) \qquad \frac{R_\gamma^0 k(x)}{R_\beta^0 1(x)} \to \begin{cases} c(\gamma, \beta, k) & \text{if } \alpha < 2, \\ 0 & \text{if } \alpha \geqslant 2 \end{cases}$$

uniformly as $|x| \to 0$ for $x \in R^d \setminus \{0\}$, where $c(\gamma, \beta, k)$ is a constant depending on γ, β, k such that if $0 \not\equiv k \geqslant 0$, then $c(\gamma, \beta, k) > 0$.

The proof of Theorem 4.1 relies on the following two theorems. Specifically, in Theorem 4.2 we establish the asymptotic behavior as $|x| \to 0$ of $R_\gamma^0 k(x)$ for $k \in C_b(R^d)$ and $\alpha < 2$, and vital estimates on $E^{P_x^0}[e^{-\lambda \tau_{0\epsilon}}; \tau_\epsilon < \tau_0]$ and $E^{P_x^0}[e^{-\beta \tau_0}]$ follow from Theorem 4.3.

THEOREM 4.2. *Suppose $\alpha < 2$. For each $\gamma > 0$ and $k \in C_b(R^d)$ there is a constant $c = c(\gamma, k) < \infty$ such that*

$$(4.2) \qquad R_\gamma^0 k(x)/u(x) \to c \quad \text{if } \alpha > 0,$$

$$(4.3) \qquad R_\gamma^0 k(x) \to c \quad \text{if } \alpha \leqslant 0$$

uniformly as $|x| \to 0$ for $x \in R^d \setminus \{0\}$.

The following lemmas are fundamental to the proof of Theorem 4.2. The first lemma follows immediately from Theorem 3.3 and (3.20). Here the relation $\sim$ is defined such that, for two real-valued functions f and g with the same domain, $f \sim g$ if and only if there are constants $a_1 > 0$ and $a_2 > 0$ such that $a_1 f \leqslant g \leqslant a_2 f$.

LEMMA 4.1. *For each $\epsilon > 0$ and $x \in \{z \in R^d : 0 < v(z) < \epsilon\}$,*

$$(4.4)$$

$$E^{P_x^0}[\tau_{0\epsilon}] \sim \begin{cases} \alpha^2 u(x)\left(\epsilon^{(2/\alpha)-1} - u(x)^{(2/\alpha)-1}\right)/(2 - \alpha) & \text{if } \alpha \in (0, \infty) \setminus \{2\}, \\ 2u(x)\ln(\epsilon/u(x)) & \text{if } \alpha = 2, \\ \frac{1}{2}(\epsilon^2 - e^{2u(x)}) & \text{if } \alpha = 0, \\ \alpha^2\left(\epsilon^{-(2/\alpha)} - (u(x))^{2/\alpha}\right)/(2 - \alpha) & \text{if } \alpha < 0. \end{cases}$$

The next lemma gives a Riesz-type decomposition for the stopped resolvent.

LEMMA 4.2. *Let $x \in R^d$ and τ be a stopping time (with respect to $\{\mathscr{F}_t^0\}$) such that $\tau \leqslant \tau_0$ P_x^0-a.s. and $E^{P_x^0}[\tau] < \infty$. Then for each $\gamma > 0$ and $k \in C_b(R^d)$, $f = R_\gamma^0 k$ satisfies*

$$(4.5) \quad f(x) = E^{P_x^0}\left[\int_0^\tau \{k(X^0(t)) - \gamma f(X^0(t))\}\, dt\right] + E^{P_x^0}[f(X^0(\tau))].$$

PROOF. The first term in the right member of (4.5) is finite because f and k are bounded and we assumed that $E^{P_x^0}[\tau] < \infty$. The proof of this lemma involves standard manipulations using the strong Markov property and Fubini's theorem (cf. Chung [2, p. 203]). The details are left to the reader. □

The following "scaling" lemma derives from the homogeneity of L with respect to the radial coordinate r.

LEMMA 4.3. *For $\lambda > 0$ and $x \in R^d$, $X^0(\cdot)$ under P_x^0 is equivalent in law to $\lambda^{-1} X^0(\lambda^2 \cdot)$ under $P_{\lambda x}^0$.*

REMARK. Here "equivalent in law" means that the probability measures induced on the space of continuous paths in R^d by the two processes are the same.

PROOF. By the characterization given for X^0 in §2, it suffices to verify that properties (i)–(iii) listed there hold with $\lambda^{-1} X^0(\lambda^2 \cdot)$ and $P_{\lambda x}^0$, respectively, in place of $X^0(\cdot)$ and P_x^0, for each $x \in R^d$. Indeed, property (i) is readily verified and so too is (iii) with the aid of the identity

$$(4.6) \qquad \tau_0(X^0(\cdot)) = \lambda^2 \tau_0(\lambda^{-1} X^0(\lambda^2 \cdot))$$

where $\tau_0(w) = \inf\{t \geq 0: w(t) = 0\}$, for $w = X^0(\cdot)$ or $\lambda^{-1} X^0(\lambda^2 \cdot)$. To verify (ii), let $f \in C_c^2(R^d)$ with compact support in $R^d \setminus \{0\}$. By applying property (ii) of $P_{\lambda x}^0$ to $f(\lambda^{-1} \cdot)$ and using the homogeneity of L with respect to r, we obtain

$$\left\{ f\left(\lambda^{-1} X^0\left(\lambda^2 t \wedge \tau_0(X^0)\right)\right) \right.$$
$$\left. - \int_0^{\lambda^2 t \wedge \tau_0(X^0)} \lambda^{-2} (Lf)(\lambda^{-1} X^0(s))\, ds, \ \mathscr{F}_{\lambda^2 t \wedge \tau_0(X^0)}, \ t \geq 0 \right\}$$

is a $P_{\lambda x}^0$-martingale. Then by changing the variable of time integration from s to $\lambda^{-2} s$ and using (4.6), we conclude that

$$\left\{ f\left(\lambda^{-1} X^0\left(\lambda^2 (t \wedge \tau_0^\lambda)\right)\right) - \int_0^{t \wedge \tau_0^\lambda} (Lf)(\lambda^{-1} X^0(\lambda^2 s))\, ds, \ \mathscr{F}_{t \wedge \tau_0^\lambda}^\lambda, \ t \geq 0 \right\}$$

is a $P_{\lambda x}^0$-martingale, where

$$\tau_0^\lambda \equiv \tau_0(\lambda^{-1} X^0(\lambda^2 \cdot)) \quad \text{and} \quad \mathscr{F}_t^\lambda = \sigma\{\lambda^{-1} X^0(\lambda^2 s): 0 \leq s \leq t\}. \quad \square$$

As in §2, $R(t)$ and $\Theta(t)$ are used below to denote the radial and spherical parts, respectively, of $X^0(t)$ for all $t < \tau_0$.

Fix $\varepsilon > 0$. For each $x = (r, \theta) \in v^{-1}(\frac{1}{2}\varepsilon)$ and each Borel measurable set $F \subset S^{d-1}$, let $\Pi(\theta, F)$ denote the conditional probability defined by

$$(4.7) \qquad \Pi(\theta, F) = \frac{P_x^0(\tau_\varepsilon < \tau_0; \ \Theta(\tau_\varepsilon) \in F)}{P_x^0(\tau_\varepsilon < \tau_0)}.$$

Here, by (3.27) and (3.29) we have

$$(4.8) \qquad P_x^0(\tau_\varepsilon < \tau_0) = \begin{cases} 1 & \text{if } \alpha \leq 0, \\ \frac{1}{2} & \text{if } \alpha > 0. \end{cases}$$

For each $x = (r, \theta)$ and F, $\Pi(\theta, \cdot)$ is a (Borel) probability measure on S^{d-1} and $\Pi(\cdot, F)$ is Borel measurable on S^{d-1}. The latter follows from the continuity of $\Phi_{\varepsilon/2}(\theta) = (r_{\varepsilon/2}(\theta), \theta)$, where $r_{\varepsilon/2}(\theta)$ is the unique value in $(0, \infty)$ such that $v(r_{\varepsilon/2}(\theta), \theta) = \frac{1}{2}\varepsilon$, and the Borel measurability of $P_x^0(\tau_\varepsilon < \tau_0; \ \Theta(\tau_\varepsilon) \in F)$ on $\{x: 0 < v(x) < \varepsilon\}$ (cf. Chung [2, p. 75; p. 153, ex. 6]). Thus, Π is the transition probability function for a Markov chain with state space S^{d-1}. For each integer $n \geq 1$, the n-step transition probabilities of this chain are given by

$$(4.9) \quad \Pi^n(\xi, F) = \int_{S^{d-1}} \Pi(\xi, d\xi_1) \cdots \int_{S^{d-1}} \Pi(\xi_{n-2}, d\xi_{n-1}) \Pi(\xi_{n-1}, F)$$

for any $\xi \in S^{d-1}$ and Borel set $F \subset S^{d-1}$.

Next, an ergodic result is proved for this Markov chain. Following Revuz [10, p. 201], we say that $\{\Pi^n\}_{n=1}^\infty$ converges uniformly to a (Borel) probability measure κ on S^{d-1} if and only if

$$(4.10) \quad \lim_{\substack{n \to \infty}} \sup_{\substack{f \in L^\infty(S^{d-1}) \\ \|f\|_\infty \leq 1}} \sup_{x \in S^{d-1}} \left| \int_{S^{d-1}} \Pi^n(x, dy) f(y) - \int_{S^{d-1}} \kappa(dy) f(y) \right| = 0$$

where $\| \cdot \|_\infty$ denotes the essential supremum norm on $L^\infty(S^{d-1})$.

LEMMA 4.4. $\{\Pi^n\}_{n=1}^\infty$ converges uniformly to a (Borel) probability measure κ on S^{d-1}.

PROOF. It is shown below that there are constants $c > 0$ and $\delta > 0$, and a (Borel) probability measure ν on S^{d-1} such that

$$(4.11) \qquad \Pi(\xi, \cdot) \geq c\nu(\cdot)^\delta \text{ for all } \xi \in S^{d-1}.$$

This implies that the Markov chain associated with Π is an aperiodic Harris chain (see Revuz [10, pp. 92, 194–196]) and that it satisfies Doeblin's condition (see [10, p. 210]). The desired result then follows immediately from Revuz [10, pp. 203, 211].

For the verification of (4.11), by (4.7)–(4.8), it suffices to show that there are constants $c > 0$ and $\delta > 0$, and a (Borel) probability measure ρ on $v^{-1}(\varepsilon)$ such that

$$(4.12) \qquad \inf_{x \in v^{-1}(\varepsilon/2)} P_x^0\left(X^0(\tau_\varepsilon \wedge \tau_{\varepsilon/4}) \in F \right) \geq c\rho(F)^\delta$$

for all Borel sets $F \subset v^{-1}(\varepsilon)$. Indeed, since $\theta \to \Phi_\varepsilon(\theta) = (r_\varepsilon(\theta), \theta)$ is one-to-one and continuous, $\nu \equiv \rho(\Phi_\varepsilon(\cdot))$ would then satisfy (4.11).

For each $x = (r, \theta) \in R^d \setminus \{0\}$, let $\vec{\mu}(x) = \mu(\theta)\vec{e}_r$, where $\vec{e}_r$ denotes the unit vector in the *outward* radial direction at x. Let B_0 be a d-dimensional Brownian motion starting from the origin, defined on a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P}_0)$. For each $(\omega, \tilde{\omega}) \in \Omega \times \tilde{\Omega}$, let

$$B(t; \omega, \tilde{\omega}) = \begin{cases} X^0(t, \omega) - \int_0^t \dfrac{\vec{\mu}(X^0(s, \omega))}{2|X^0(s, \omega)|} ds & \text{for } 0 \leq t \leq \tau(\omega), \\ B(\tau(\omega)) + B_0(t - \tau(\omega); \tilde{\omega}) & \text{for } t > \tau(\omega), \end{cases}$$

where $\tau(\omega) \equiv (\tau_\varepsilon \wedge \tau_{\varepsilon/4})(\omega)$. Then for each $x \in G_\varepsilon \equiv \{z \in R^d : \varepsilon/4 \leq v(z) \leq \varepsilon\}$, on $(\overline{\Omega}, \overline{\mathscr{F}}, \overline{P}_x) \equiv (\Omega \times \tilde{\Omega}, \mathscr{F} \times \tilde{\mathscr{F}}, P_x^0 \times \tilde{P}_0)$, B is a d-dimensional Brownian motion starting from x. This follows from the facts that under P_x^0, X^0 behaves in $R^d \setminus \{0\}$ like Brownian motion starting from x with a polar drift of the form $\mu(\theta)/2r$, and $\tau(\omega)$ is independent of $B_0(\cdot; \tilde{\omega})$. For all $(t, \omega, \tilde{\omega}) \in [0, \infty) \times \overline{\Omega}$, let

$$\overline{X}(t; \omega, \tilde{\omega}) = B(t; \omega, \tilde{\omega}) + \int_0^{t \wedge \tau(\omega)} \dfrac{\vec{\mu}(X^0(s, \omega))}{2|X^0(s, \omega)|} ds$$

so that $\overline{X}(t; \omega, \tilde{\omega}) = X^0(t, \omega)$ for all $t \leq \tau(\omega)$. On $\overline{\Omega}$, define $\tau(\omega, \tilde{\omega}) = \tau(\omega)$,

$$\eta_\varepsilon = \inf\{ t \geq 0 : v(B(t)) \leq \varepsilon/4 \text{ or } \geq \varepsilon \},$$

and

$$\chi_\epsilon = \exp\left(\int_0^{\eta_\epsilon} \frac{\bar{\mu}(B(s))}{2|B(s)|} \, dB(s) - \frac{1}{8} \int_0^{\eta_\epsilon} \frac{|\bar{\mu}(B(s))|^2}{|B(s)|^2} \, ds \right).$$

Now,

$$m_\epsilon \equiv \sup_{x \in G_\epsilon} \{ |\bar{\mu}(x)|/2|x| \} < \infty, \qquad P_x(\eta_\epsilon < \infty) = 1,$$

and by (3.26), $P_x^0(\tau_\epsilon \wedge \tau_{\epsilon/4} < \infty) = 1$. It then follows from a version of Girsanov's change of measure formula (cf. Lipster and Shiryayev [9, Theorem 7.10]) that, for any Borel set $F \subset v^{-1}(\epsilon)$ and $x \in G_\epsilon$,

$$(4.13) \quad P_x^0\left(X^0(\tau_\epsilon \wedge \tau_{\epsilon/4}) \in F \right) = \bar{P}_x\left(\bar{X}(\tau) \in F \right) = E^{\bar{P}_x}[\chi_\epsilon 1_F(B(\eta_\epsilon))].$$

Now by Chung [2, (X), p. 148], there is $a_\epsilon > 0$ such that, for all $0 < a < a_\epsilon$,

$$(4.14) \qquad\qquad \sup_{x \in G_\epsilon} E^{\bar{P}_x}[\exp(a\eta_\epsilon)] < \infty.$$

Moreover, by the Cauchy–Schwarz inequality, for $b > 0$ and $x \in G_\epsilon$,
(4.15)

$$\left(E^{\bar{P}_x}[\chi_\epsilon^{-b}] \right)^2 \leqslant E^{\bar{P}_x}\left[\exp\left\{ -\int_0^{\eta_\epsilon} \frac{b\bar{\mu}(B(s))}{|B(s)|} \, dB(s) - \frac{1}{2}\int_0^{\eta_\epsilon} \left(\frac{b|\bar{\mu}(B(s))|}{|B(s)|^2} \right)^2 ds \right\} \right]$$

$$\cdot E^{\bar{P}_x}\left[\exp\left\{ \int_0^{\eta_\epsilon} \frac{(2b^2 + b)|\bar{\mu}(B(s))|^2}{4|B(s)|} \, ds \right\} \right].$$

Since $E^{\bar{P}_x}[\chi_\epsilon] = 1$, with $-2b\bar{\mu}$ in place of $\bar{\mu}$, the first factor of the product in (4.15) is identically one. By combining this with the definition of m_ϵ, we see that the right member of (4.15) is dominated above by

$$E^{\bar{P}_x}\left[\exp\{ (2b^2 + b)m_\epsilon^2\eta_\epsilon \} \right].$$

Then using (4.14) we conclude that there is $b_\epsilon > 0$ such that, for $0 < b \leqslant b_\epsilon$,

$$(4.16) \qquad\qquad \sup_{x \in G_\epsilon} E^{\bar{P}_x}[\chi_\epsilon^{-b}] < \infty.$$

Choose real numbers p_ϵ and q_ϵ such that $1 < q_\epsilon < p_\epsilon < \infty$, $1/p_\epsilon + 1/q_\epsilon = 1$ and $q_\epsilon/p_\epsilon \leqslant b_\epsilon$. Then by Hölder's inequality, for $x \in G_\epsilon$ we have

$$(4.17) \quad E^{\bar{P}_x}[\chi_\epsilon 1_F(B(\eta_\epsilon))] \geqslant \left(E^{\bar{P}_x}[1_F(B(\eta_\epsilon))] \right)^{p_\epsilon} / \left(E^{\bar{P}_x}[\chi_\epsilon^{-q_\epsilon/p_\epsilon}] \right)^{p_\epsilon/q_\epsilon}$$

$$\geqslant \left(E^{\bar{P}_x}[1_F(B(\eta_\epsilon))] \right)^{p_\epsilon} / c_{1,\epsilon}$$

where

$$c_{1,\epsilon} \equiv \sup_{x \in G_\epsilon} \left(E^{\bar{P}_x}[\chi_\epsilon^{-q_\epsilon/p_\epsilon}] \right)^{p_\epsilon/q_\epsilon} < \infty,$$

by (4.16).

Recall (cf. (3.12)) that $v(r, \theta) = r^\beta \psi(\theta)$ where $\beta = |\alpha|$ if $\alpha \neq 0$ or 1 if $\alpha = 0$, and $\psi(\theta) = \phi_\alpha(\theta)$ if $\alpha > 0$, $(\phi_\alpha(\theta))^{-1}$ if $\alpha < 0$, or $e^{\chi(\theta)}$ if $\alpha = 0$. Since ψ is strictly positive and C^1 on S^{d-1}, it follows that, for any $a > 0$, $v^{-1}(a) = \{ (r_a(\theta), \theta): v(r_a(\theta), \theta) = a, \ \theta \in S^{d-1} \}$ is the graph of a C^1 function on S^{d-1}. Hence the boundary of the domain $\{ x \in R^d: \epsilon/4 < v(x) < \epsilon \}$ is C^1.

Now by the properties of harmonic measure for domains with C^1 (hence Lipschitz) boundary (see Dahlberg [4, p. 276]), and Harnack's inequality (see Gilbarg and Trudinger [5, Theorem 2.5]), it follows that there are constants $c_{2,\varepsilon} > 0$ and $\delta_\varepsilon > 0$ such that

$$(4.18) \qquad \inf_{x \in v^{-1}(\varepsilon/2)} E^{\bar{P}_x}\big[1_F(B(\eta_\varepsilon))\big] \geq c_{2,\varepsilon}\bar{\sigma}(F)^{\delta_\varepsilon}$$

where $\bar{\sigma}$ denotes the surface measure on $v^{-1}(\varepsilon)$, normalized so that $\bar{\sigma}(v^{-1}(\varepsilon)) = 1$. By combining this with (4.13) and (4.17), we conclude that (4.12) holds with $c = c_{2,\varepsilon}^{p_\varepsilon}/c_{1,\varepsilon}$, $\delta = p_\varepsilon\delta_\varepsilon$ and $\rho = \bar{\sigma}$.   □

The following lemma establishes a link between the ergodic behavior of the Markov chain on S^{d-1} (determined by Π) and the asymptotic behavior of P_x^0 as $|x| \to 0$.

LEMMA 4.5. *For each integer $n \geq 1$, each $x_n = (r_n, \theta_n) \in v^{-1}(\varepsilon 2^{-n})$ and each Borel set $F \subset S^{d-1}$, we have*

$$P_{x_n}^0\big(\tau_\varepsilon < \tau_0; \Theta(\tau_\varepsilon) \in F\big) = \overline{\Pi}^n(\theta_n, F)$$

where

$$\overline{\Pi}^n = \begin{cases} 2^{-n}\Pi^n & \text{if } \alpha > 0, \\ \Pi^n & \text{if } \alpha \leq 0. \end{cases}$$

PROOF. The proof is by induction on n. The result is true for $n = 1$ by definition (see (4.7)–(4.8)). Suppose it is true for some $n \geq 1$. Then,

$$(4.19) \quad P_{x_{n+1}}^0\big(\tau_\varepsilon < \tau_0; \Theta(\tau_\varepsilon) \in F\big)$$

$$= P_{x_{n+1}}^0\Big(\tau_{\varepsilon 2^{-n}} < \tau_0; P_{X^0(\tau_{\varepsilon 2^{-n}})}^0\big(\tau_\varepsilon < \tau_0; \Theta(\tau_\varepsilon) \in F\big)\Big)$$

$$= \int_{S^{d-1}} P_{x_{n+1}}^0\big(\tau_{\varepsilon 2^{-n}} < \tau_0; \Theta(\tau_{\varepsilon 2^{-n}}) \in d\xi\big)\overline{\Pi}^n(\xi, F).$$

Here we have used the strong Markov property to obtain the second line and the induction hypothesis for the last line. Let $\lambda_n = 2^{n/|\alpha|}$ if $\alpha \neq 0$ or $\lambda_n = 2^n$ if $\alpha = 0$. Then $\lambda_n x_{n+1} \in v^{-1}(\varepsilon/2)$ and by the scaling lemma 4.3, the last line of (4.19) is equal to

$$\int_{S^{d-1}} P_{\lambda_n x_{n+1}}^0\big(\tau_\varepsilon < \tau_0; \Theta(\tau_\varepsilon) \in d\xi\big)\overline{\Pi}^n(\xi, F) = \int_{S^{d-1}} \overline{\Pi}(\theta_{n+1}, d\xi)\overline{\Pi}^n(\xi, F).$$

The result now follows for $n + 1$ from the definitions of $\overline{\Pi}^{n+1}$ and Π^{n+1}.   □

In the following, κ_ε is defined to be equal to κ/ε if $\alpha > 0$ or κ if $\alpha \leq 0$, where κ is the probability measure of Lemma 4.4, which may itself depend on ε.

LEMMA 4.6. *For each continuous function f on S^{d-1} and for q defined on $R^d \setminus \{0\}$ by*

$$q = \begin{cases} u & \text{if } \alpha > 0, \\ 1 & \text{if } \alpha \leq 0, \end{cases}$$

we have

$$(4.20) \quad \left\{\int_{S^{d-1}} P_x^0\big(\tau_\varepsilon < \tau_0; \Theta(\tau_\varepsilon) \in d\xi\big)f(\xi)\,d\xi\right\}/q(x) \to \int_{S^{d-1}} \kappa_\varepsilon(d\xi)f(\xi)$$

uniformly in θ as $x = (r, \theta) \in R^d \setminus \{0\}$ tends to 0.

PROOF. Since $v(x) \sim |x|^{|\alpha|}$ if $\alpha \neq 0$ or $v(x) \sim |x|$ if $\alpha = 0$, it suffices to prove that (4.20) holds as $v(x) \to 0$.

Consider $n \geq 1$ and $x \in R^d \setminus \{0\}$ such that $v(x) < \varepsilon 2^{-n}$. Then by Lemma 4.5, the left member of (4.20) is equal to

$$(4.21) \quad \int_{S^{d-1}} P_x^0(\tau_{\varepsilon 2^{-n}} < \tau_0; \Theta(\tau_{\varepsilon 2^{-n}}) \in d\xi) \int_{S^{d-1}} \overline{\Pi}^n(\xi, d\xi_1) f(\xi_1)/q(x).$$

By (3.27) and (3.29),

$$P_x^0(\tau_{\varepsilon 2^{-n}} < \tau_0) = \begin{cases} 2^n u(x)/\varepsilon & \text{if } \alpha > 0, \\ 1 & \text{if } \alpha \leq 0. \end{cases}$$

Thus, $a_n P_x^0(\tau_{\varepsilon 2^{-n}} < \tau_0; \Theta(\tau_{\varepsilon 2^{-n}}) \in d\xi)/q(x)$, where $a_n = \varepsilon 2^{-n}$ if $\alpha > 0$ or $a_n = 1$ if $\alpha \leq 0$, defines a (Borel) probability measure on S^{d-1}. By combining this with the definitions of $\overline{\Pi}^n$ and κ_ε, and the uniform convergence in Lemma 4.4, it follows that (4.21) converges to the right member of (4.20) as $n \to \infty$, i.e. as $v(x) \to 0$. $\square$

PROOF OF THEOREM 4.2. Suppose $\alpha < 2$ and γ, k are defined as in the statement of the theorem. Then by the Feller continuity of P_x^0 on $R^d \setminus \{0\}$ (cf. Stroock and Varadhan [13, 14]), $f(x) = R_\gamma^0 k(x)$ is continuous there and f_ε defined by

$$f_\varepsilon(\theta) = f(r_\varepsilon(\theta), \theta) \quad \text{for all } \theta \in S^{d-1}$$

is continuous on S^{d-1}. Here r_ε is defined in the same manner as $r_{\varepsilon/2}$ (see the paragraph following (4.8)). Then for q as defined in Lemma 4.6, we have

$$E^{P_x^0}[f(X^0(\tau_{0\varepsilon}))]/q(x) = \left\{ \int_{S^{d-1}} P_x^0(\tau_\varepsilon < \tau_0; \Theta(\tau_\varepsilon) \in d\xi) f_\varepsilon(\xi) \right\}/q(x)$$

$$\to \int_{S^{d-1}} \kappa_\varepsilon(d\xi) f_\varepsilon(\xi) \equiv c_\varepsilon$$

uniformly as $|x| \to 0$ in $R^d \setminus \{0\}$.

By combining this with Lemmas 4.1 and 4.2 (setting $\tau = \tau_{0\varepsilon}$ in the latter), since $\alpha < 2$, we obtain

$$(4.22) \quad \lim \binom{\sup}{\inf}_{|x| \to 0} f(x)/q(x) = c_\varepsilon + O(\beta(\varepsilon))$$

where

$$\beta(\varepsilon) = \begin{cases} \varepsilon^{(2/\alpha) - 1} & \text{if } 0 < \alpha < 2, \\ \varepsilon^2 & \text{if } \alpha = 0, \\ \varepsilon^{2/|\alpha|} & \text{if } \alpha < 0. \end{cases}$$

To complete the proof of Theorem 4.2 it suffices to show that $\{c_\varepsilon\}$ is Cauchy as $\varepsilon \to 0$. Indeed, for $0 < \varepsilon' < \varepsilon$, by Lemmas 4.1 and 4.2 we have

$$c_\varepsilon - c_{\varepsilon'} \equiv \lim_{|x| \to 0} E^{P_x^0}[f(X^0(\tau_{0\varepsilon})) - f(X^0(\tau_{0\varepsilon'}))]/q(x)$$

$$= \lim_{|x| \to 0} E^{P_x^0}\left[\tau_{\varepsilon'} < \tau_0; \int_{\tau_{\varepsilon'}}^{\tau_{0\varepsilon}} \{k(X^0(t)) - \gamma f(X^0(t))\} dt\right] / q(x)$$

$$\leq C \limsup_{|x| \to 0} E^{P_x^0}[\tau_{0\varepsilon}]/q(x), \quad \text{where } C \equiv \sup_{x \in R^d} \{(|k| + \gamma |f|)(x)\} < \infty$$

$$= O(\beta(\varepsilon)).$$

Now $\beta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and so $c_\varepsilon \to c$ for some constant c as $\varepsilon \to 0$. Since all of the above limits and estimates have been uniform in $x/|x|$, Theorem 4.2 follows from letting $\varepsilon \to 0$ in (4.22). $\square$

The next theorem enables us to obtain estimates for $R_\gamma^0 1$ and $R_\gamma^0 k$ when $k \in C_b(R^d)$ has compact support in $R^d \setminus \{0\}$. For this theorem, let K_ν, I_ν denote the modified Bessel functions of order $\nu = |\alpha|/2$ which satisfy

(4.23) $\qquad w^2 g''(w) + w g'(w) - (w^2 + \nu^2) g(w) = 0 \quad \text{for } w \in (0, \infty)$

with $g(w) = K_\nu(w)$ or $I_\nu(w)$. Further details on the properties of K_ν and I_ν used below can be found in Abramowitz and Stegun [1, pp. 374–378].

THEOREM 4.3. *Let $\lambda > 0$, $\varepsilon > 0$, and let h be defined by (3.15). For $y \in [0, \infty)$, define*

$$
g_1(y) = \begin{cases}
\dfrac{\sqrt{y}\, I_\nu\!\left(\sqrt{2\lambda\alpha^2}\, y^{1/\alpha}\right)}{\sqrt{\varepsilon}\, I_\nu\!\left(\sqrt{2\lambda\alpha^2}\, \varepsilon^{1/\alpha}\right)} & \text{if } \alpha > 0, \\[3ex]
\dfrac{I_0\!\left(\sqrt{2\lambda}\, y\right)}{I_0\!\left(\sqrt{2\lambda}\, \varepsilon\right)} & \text{if } \alpha = 0, \\[3ex]
\dfrac{\sqrt{y}\, I_\nu\!\left(\sqrt{2\lambda\alpha^2}\, y^{1/\alpha}\right)}{\sqrt{\varepsilon^{-1}}\, I_\nu\!\left(\sqrt{2\lambda\alpha^2}\, \varepsilon^{-1/\alpha}\right)} & \text{if } \alpha < 0,\ y \neq 0,
\end{cases}
$$

and

$$
g_2(y) = c\sqrt{y}\, K_\nu\!\left(\sqrt{2\lambda\alpha^2}\, y^{1/\alpha}\right) \quad \text{if } \alpha > 0,
$$

where

$$
c \equiv 2^{1 - (\alpha/4)} \alpha^{\alpha/2} \lambda^{\alpha/4} \left(\Gamma(\alpha/2)\right)^{-1}
$$

is such that $g_2(0) \equiv \lim_{y \to 0} g_2(y) = 1$ for $\alpha > 0$.

For $\alpha \neq 0$ define $f_1 = g_1(u)$, for $\alpha = 0$ define $f_1 = g_1(e^u)$, and for $\alpha > 0$ define $f_2 = g_2(u)$. Then for each $x \in R^d \setminus \{0\}$ such that $0 < v(x) \leq \varepsilon$ we have

(4.24) $\qquad E^{P_x^0}\!\left[\exp\!\left(-\lambda \int_0^{\tau_{0\varepsilon}} h(X^0(s))\, ds\right); \tau_\varepsilon < \tau_0\right] = f_1(x),$

and for each $x \in R^d$ and $\alpha > 0$ we have

(4.25) $\qquad E^{P_x^0}\!\left[\exp\!\left(-\lambda \int_0^{\tau_0} h(X^0(s))\, ds\right)\right] = f_2(x).$

PROOF. It follows from (4.23) and the properties of I_ν, K_ν, including

$$
I_\nu(w) = \left(\tfrac{1}{2}w\right)^\nu / \Gamma(\nu + 1) + o(w^\nu) \quad \text{as } w \to 0,
$$

$K_\nu(w) \sim w^{-\nu}$ as $w \to 0$ for $\nu > 0$, $K_\nu(w) \sim w^{-1/2} e^{-w}$ as $w \to \infty$, that for $\alpha > 0$,

(4.26) $\qquad \begin{cases} \tfrac{1}{2} g_1''(y) y^{2 - (2/\alpha)} - \lambda g_1(y) = 0 & \text{for } y \in (0, \infty), \\ g_1(0) = 0, \quad g_1(\varepsilon) = 1, \end{cases}$

(4.27) $\qquad \begin{cases} \tfrac{1}{2} g_2''(y) y^{2 - (2/\alpha)} - \lambda g_2(y) = 0 & \text{for } y \in (0, \infty), \\ g_2(0) = 1, \quad |g_2| \leq 1, \quad g_2(y) \to 0 \quad \text{as } y \to \infty; \end{cases}$

for $\alpha = 0$,

(4.28)
$$\begin{cases} \frac{1}{2}\{g_1''(y) + y^{-1}g_1'(y)\} - \lambda g_1(y) = 0 & \text{for } y \in (0, \infty), \\ g_1(\varepsilon) = 1, \quad 0 < g_1(y) < 1 & \text{for } 0 \leqslant y < \varepsilon; \end{cases}$$

for $\alpha < 0$,

(4.29)
$$\begin{cases} \frac{1}{2}g_1''(y)y^{2-(2/\alpha)} - \lambda g_1(y) = 0 & \text{for } y \in (0, \infty), \\ g_1(\varepsilon^{-1}) = 1, \quad 0 < \inf_{y \in [\varepsilon^{-1}, \infty)} g_1(y) \leqslant \sup_{y \in [\varepsilon^{-1}, \infty)} g_1(y) < \infty. \end{cases}$$

Then f_1, f_2 are functions in $C^1(R^d \setminus \{0\}) \cap W^{2,p}_{\text{loc}}(R^d \setminus \{0\})$, where $p > d - 1$ is as in §3. By a similar derivation to that of (3.23), using (4.26)–(4.29), we have on $R^d \setminus \{0\}$

(4.30)
$$Lf_1 = \begin{cases} \frac{1}{2}g_1''(u)|\nabla u|^2 & \text{if } \alpha \neq 0, \\ \frac{1}{2}\{g_1''(e^u)e^{2u} + g_1'(u)e^u\}|\nabla u|^2 & \text{if } \alpha = 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2}g_1''(u)u^{2-(2/\alpha)}h & \text{if } \alpha \neq 0, \\ \frac{1}{2}\{g_1''(e^u) + g_1'(u)e^{-u}\}h & \text{if } \alpha = 0 \end{cases}$$

$$= \lambda f_1 h,$$

and

(4.31)
$$Lf_2 = \lambda f_2 h \quad \text{if } \alpha > 0.$$

Furthermore, f_1 is bounded on $\{x: 0 < v(x) \leqslant \varepsilon\}$, $f_1|_{v^{-1}(\varepsilon)} = 1$, and if $\alpha > 0$, $f_1(0) = 0$, $f_2(0) = 1$, $f_2(x) \to 0$ as $|x| \to \infty$; if $\alpha \leqslant 0$,

$$0 < \inf_{0 < v(x) \leqslant \varepsilon} f_1(x) \leqslant \sup_{0 < v(x) \leqslant \varepsilon} f_1(x) < \infty.$$

Let $0 < r < \varepsilon$ and $x \in \{z: r < v(z) \leqslant \varepsilon\}$. As in Lemma 3.1, the martingale property (2.1) with $\tau_{r\varepsilon}$ in place of τ_0 extends to f_1 and f_2, so that by (4.30)–(4.31),

$$\left\{ M_t^i = f_i(X^0(t \wedge \tau_{r\varepsilon})) - \int_0^{t \wedge \tau_{r\varepsilon}} (\lambda f_i h)(X^0(s)) \, ds, \ \mathscr{F}^0_{t \wedge \tau_{r\varepsilon}}, \ t \geqslant 0 \right\}$$

is a P_x^0-martingale for $i = 1, 2$.

Since

$$V_t \equiv \exp\left(-\lambda \int_0^{t \wedge \tau_{r\varepsilon}} h(X^0(s)) \, ds\right) \quad \text{and} \quad U_t^i \equiv \int_0^{t \wedge \tau_{r\varepsilon}} (\lambda f_i h)(X^0(s)) \, ds$$

are locally of bounded variation, by the product rule of stochastic calculus we have, P_x^0-a.s.,

(4.32)
$$d(V_t(M_t^i + U_t^i)) = V_t dM_t^i + V_t dU_t^i + (M_t^i + U_t^i) dV_t.$$

The first term on the right of (4.32) integrates to a P_x^0-martingale since M_t^i is such a martingale and $|V_t| \leqslant 1$. The last two terms in (4.32) sum to zero. Hence we may conclude that

(4.33)
$$\left\{ V_t(M_t^i + U_t^i) \equiv \exp\left(-\lambda \int_0^{t \wedge \tau_{r\varepsilon}} h(X^0(s)) \, ds\right) f_i(X^0(t \wedge \tau_{r\varepsilon})), \ \mathscr{F}^0_{t \wedge \tau_{r\varepsilon}}, \ t \geqslant 0 \right\}$$

is a P_x^0-martingale for $i = 1, 2$. Then taking expectations for $i = 1$ and letting $t \to \infty$ and then $r \to 0$, we obtain by bounded convergence (recall $\tau_{0\varepsilon} < \infty$ P_x^0-a.s.)

$$(4.34) \qquad E^{P_x^0}\left[\equiv \exp\left(-\lambda \int_0^{\tau_{0\varepsilon}} h(X^0(s))\, ds\right) f_1(X^0(\tau_{0\varepsilon}))\right] = f_1(x).$$

Note that for $0 < v(x) \leqslant \varepsilon$, if $\alpha \leqslant 0$, $\tau_{0\varepsilon} = \tau_\varepsilon$ P_x^0-a.s., and if $\alpha > 0$, $f_1(0) = 0$, and for all α, $f_1|_{v^{-1}(\varepsilon)} = 1$. Consequently, for any α, $f_1(X^0(\tau_{0\varepsilon})) = 1_{\{\tau_\varepsilon < \tau_0\}}$ P_x^0-a.s. and (4.24) follows from (4.34).

When $\alpha > 0$, by taking the expectation of the martingale (4.33) with $i = 2$ and letting $t \to \infty$, then $r \to 0$, and then $\varepsilon \to \infty$, it follows, since f_2 is bounded and $\tau_0 < \infty$ P_x^0-a.s., that

$$E^{P_x^0}\left[\exp\left(-\lambda \int_0^{\tau_0} h(X^0(s))\, ds\right) f_2(X^0(\tau_0))\right] = f_2(x)$$

for all $x \in R^d \setminus \{0\}$. Since $f_2(0) = 1$, (4.25) follows immediately, even for $x = 0$. $\square$

PROOF OF THEOREM 4.1. Let γ, β and k satisfy the hypotheses of Theorem 4.1 and let $\varepsilon > 0$ be so small that $\{x: 0 \leqslant v(x) \leqslant \varepsilon\}$ does not meet the support of k.

Consider $x \in R^d \setminus \{0\}$. First suppose $\alpha \leqslant 0$. Then, $P_x^0(\tau_0 = \infty) = 1$ and so by Theorem 4.2

$$(4.35) \qquad R_\gamma^0 k(x)/R_\beta^0 1(x) = \beta R_\gamma^0 k(x) \to \beta c(\gamma, k) \equiv c(\gamma, \beta, k)$$

uniformly as $|x| \to 0$. If $0 \neq k \geqslant 0$, then by the choice of ε, for $0 < v(x) < \varepsilon$,

$$(4.36) \qquad R_\gamma^0 k(x) = E^{P_x^0}\left[e^{-\gamma \tau_\varepsilon} R_\gamma^0 k(X^0(\tau_\varepsilon))\right]$$

$$\geqslant E^{P_x^0}\left[e^{-\gamma \tau_\varepsilon}\right] \|R_\gamma^0 k\|_\varepsilon$$

where $\|R_\gamma^0 k\|_\varepsilon = \inf\{R_\gamma^0 k(z): v(z) = \varepsilon\} > 0$. Moreover, by Theorem 4.3, for c_1 as defined in (3.20),

$$E^{P_x^0}\left[e^{-\gamma \tau_\varepsilon}\right] \geqslant E^{P_x^0}\left[\exp\left(-\gamma c_1^{-1} \int_0^{\tau_{0\varepsilon}} h(X^0(s))\, ds\right); \tau_\varepsilon < \tau_0\right]$$

$$= f_1(x) \quad \text{with } \lambda = \gamma c_1^{-1}.$$

Now as $|x| \to 0$, $u(x)^{1/\alpha} \to 0$ if $\alpha < 0$ and $e^{u(x)} \to 0$ if $\alpha = 0$, and so

$$(4.37) \qquad f_1(x) = \begin{cases} \dfrac{\sqrt{u(x)}\, I_\nu\left(\sqrt{2\lambda\alpha^2}\, u(x)^{1/\alpha}\right)}{\sqrt{\varepsilon^{-1}}\, I_\nu\left(\sqrt{2\lambda\alpha^2}\, \varepsilon^{-1/\alpha}\right)} & \text{if } \alpha < 0, \\[3ex] \dfrac{I_0\left(\sqrt{2\lambda}\, e^{u(x)}\right)}{I_0\left(\sqrt{2\lambda}\, \varepsilon\right)} & \text{if } \alpha = 0 \end{cases}$$

$$\sim \begin{cases} \sqrt{u(x)}\, u(x)^{\nu/\alpha} = 1 & \text{if } \alpha < 0, \\ I_0(0) > 0 & \text{if } \alpha = 0. \end{cases}$$

Here we have used the facts that $I_\nu(w) \sim w^\nu$ as $w \to 0$ and $\nu \equiv |\alpha|/2 = -\alpha/2$ if $\alpha < 0$. By combining (4.36)–(4.37), we conclude that if $0 \neq k \geqslant 0$,

$$c(\gamma, k) = \lim_{|x| \to 0} R_\gamma^0 k(x) > 0,$$

and hence so too is $c(\gamma, \beta, k) \equiv \beta c(\gamma, k)$. Thus, Theorem 4.1 holds if $\alpha \leqslant 0$.

Now suppose $\alpha > 0$. For c_2 as defined in (3.20), by Theorem 4.3 we have

(4.38) $R_\beta^0 1(x) = \beta^{-1}\left(1 - E^{P_x^0}[e^{-\beta\tau_0}]\right)$

$$\geqslant \beta^{-1}\left(1 - E^{P_x^0}\left[\exp\left(-\beta c_2^{-1}\int_0^{\tau_0} h(X^0(s))\,ds\right)\right]\right)$$

$$= \beta^{-1}(1 - f_2(x)) \quad \text{with } \lambda = \beta c_2^{-1}.$$

By [1, 9.6.2 and 9.6.10], as $w \to 0$,

$$w^\nu K_\nu(w) = \begin{cases} 2^{\nu-1}\Gamma(\nu) - a_\nu w^{2 \wedge (2\nu)} + o(w^{2 \wedge (2\nu)}) & \text{if } \nu \neq 1, \\ 2^{\nu-1}\Gamma(\nu) + a_\nu w^2 \ln w + O(w^2) & \text{if } \nu = 1, \end{cases}$$

where the a_ν are strictly positive constants.

It follows from this and the definition of g_2 that

$$1 - f_2(x) = 1 - g_2(u(x))$$

$$= \begin{cases} a_\nu' u(x)^{(2/\alpha) \wedge 1} + o\left(u(x)^{(2/\alpha) \wedge 1}\right) & \text{if } \alpha \neq 2, \\ -a_\nu' u(x)^{2/\alpha} \ln u(x)^{1/\alpha} + O\left(u(x)^{2/\alpha}\right) & \text{if } \alpha = 2, \end{cases}$$

where the a_ν' are strictly positive constants depending on λ and ε as well as $\nu = |\alpha|/2$. Then, by dividing the above by $u(x)$, letting $|x| \to 0$, and combining the result with (4.38) we obtain

(4.39) $\displaystyle\liminf_{|x|\to 0} \frac{R_\beta^0 1(x)}{u(x)} \geqslant \begin{cases} \beta^{-1}a_\nu' & \text{if } 0 < \alpha < 2, \\ +\infty & \text{if } \alpha \geqslant 2. \end{cases}$

By replacing $\geqslant$ and c_2, respectively, by $\leqslant$ and c_1 in the above argument, we conclude that there is a constant $a_\nu'' > 0$ such that

(4.40) $\displaystyle\limsup_{|x|\to 0} \frac{R_\beta^0 1(x)}{u(x)} \leqslant \begin{cases} \beta^{-1}a_\nu'' & \text{if } 0 < \alpha < 2, \\ +\infty & \text{if } \alpha \geqslant 2. \end{cases}$

On the other hand, for $0 < v(x) < \varepsilon$,

(4.41) $R_\gamma^0 k(x) = E^{P_x^0}\left[e^{-\gamma\tau_{0\varepsilon}} R_\gamma^0 k(X^0(\tau_{0\varepsilon}))\right]$

$$\leqslant E^{P_x^0}\left[\exp\left(-\gamma c_2^{-1}\int_0^{\tau_{0\varepsilon}} h(X^0(s))\,ds\right); \ \tau_\varepsilon < \tau_0\right]\|k\|_{\gamma,\infty}$$

$$= f_1(x)\|k\|_{\gamma,\infty} \quad \text{with } \lambda = \gamma c_2^{-1},$$

where $\|k\|_{\gamma,\infty} = \gamma^{-1}\sup_{z \in R^d}|k(z)| < \infty$. Now as $w \to 0$,

$$I_\nu(w) = \left(\tfrac{1}{2}w\right)^\nu / \Gamma(\nu + 1) + o(w^\nu)$$

and consequently, as $|x| \to 0$,

$$f_1(x) = g_1(u(x)) = \frac{\sqrt{u(x)}\, I_\nu\left(\sqrt{2\lambda\alpha^2}\, u(x)^{1/\alpha}\right)}{\sqrt{\varepsilon}\, I_\varepsilon\left(\sqrt{2\lambda\alpha^2}\, \varepsilon^{1/\alpha}\right)} = b_\nu u(x) + O(u(x)),$$

where $b_\nu > 0$ is a constant depending on ν, ε and $\lambda = \gamma c_2^{-1}$. By combining this with (4.41) we obtain

(4.42) $\displaystyle\limsup_{|x|\to 0} \frac{R_\gamma^0 k(x)}{u(x)} \leqslant b_\nu\|k\|_{\gamma,\infty}.$

By replacing $\leqslant$, c_2 and $\|k\|_{\gamma,\infty}$, respectively, by $\geqslant$, c_1 and $\|R_\gamma^0 k\|_\epsilon = \inf_{z \,\in\, v^{-1}(\epsilon)} R_\gamma^0 k(z)$, from (4.41) onwards, we conclude that, for $0 \neq k \geqslant 0$,

$$(4.43) \qquad \liminf_{|x| \to 0} \frac{R_\gamma^0 k(x)}{u(x)} \geqslant b_\nu' \|R_\gamma^0 k\|_\epsilon > 0$$

for some constant $b_\nu' > 0$ depending on ν, ϵ, γc_1^{-1}. Note that the limits in (4.39)–(4.40), (4.42)–(4.43) are uniform in $x/|x|$.

It follows from (4.39)–(4.40) that, for $0 < \alpha < 2$, the constant $c(\beta, 1)$ defined via Theorem 4.2 satisfies $0 < c(\beta, 1) < \infty$. Consequently, for $0 < \alpha < 2$,

$$\frac{R_\gamma^0 k(x)}{R_\beta^0 1(x)} = \frac{R_\gamma^0 k(x)/u(x)}{R_\beta^0 1(x)/u(x)} \to \frac{c(\gamma, k)}{c(\beta, 1)} \equiv c(\gamma, \beta, k)$$

uniformly as $|x| \to 0$, where $c(\gamma, \beta, k) < \infty$, and if $0 \neq k \geqslant 0$, it follows from (4.43) that $c(\gamma, k) > 0$ and hence $c(\gamma, \beta, k) > 0$. Thus Theorem 4.1 holds if $0 < \alpha < 2$.

Finally, if $\alpha \geqslant 2$, by (4.39) and (4.42)

$$\limsup_{|x| \to 0} \frac{|R_\gamma^0 k(x)|}{R_\beta^0 1(x)} \leqslant \limsup_{|x| \to 0} \frac{R_\gamma^0 |k|(x)/u(x)}{R_\beta^0 1(x)/u(x)} \leqslant \frac{b_\nu \|k\|_{\gamma,\infty}}{\liminf_{|x| \to 0} R_\beta^0 1(x)/u(x)} = 0,$$

uniformly in $x/|x|$. $\square$

5. Continuation from the pole.

It follows from Theorem 4.1 and the work of Rogers [11], particularly the discussion of the Martin compactification on pp. 249–251 of [11], that there is a single Martin boundary point for X^0 at the origin so that the Martin kernel $K(x, y)$ defined for $(x, y) \in (R^d \setminus \{0\}) \times (R^d \setminus \{0\})$ where

$$\int_{R^d \setminus \{0\}} K(x, y) k(y) \, dy = R_\gamma^0 k(x)/R_\gamma^0 1(x)$$

for all $k \in C_c(R^d \setminus \{0\})$, can be extended continuously to $x = 0$:

$$\int_{R^d \setminus \{0\}} K(0, y) k(y) \, dy = \lim_{x \to 0} R_\gamma^0 k(x)/R_\gamma^0 1(x) = \begin{cases} c(\gamma, \gamma, k) & \text{if } \alpha < 2, \\ 0 & \text{if } \alpha \geqslant 2. \end{cases}$$

It further follows that, for $\alpha < 2$, there is a unique strong Markov process with continuous paths in R^d that behaves like X^0 away from the origin, but the amount of time that it spends at the origin has Lebesgue measure zero. Indeed, for $\gamma > 0$ and $k \in C_c(R^d)$ with compact support in $R^d \setminus \{0\}$, the resolvent of X is given by (cf. [11, (2), (33)])

$$R_\gamma k(x) \equiv E^{P_x} \left[\int_0^\infty e^{-\gamma t} k(X(t)) \, dt \right] = R_\gamma^0 k(x) + E^{P_x^0} [e^{-\gamma \tau_0}] \gamma^{-1} c(\gamma, \gamma, k).$$

Conversely, if $\alpha \geqslant 2$, $K(0, y) = 0$ for all $y \in R^d \setminus \{0\}$ and it follows from [11] that there is no such process X in this case.

Following the approach in Varadhan and Williams [15], for $\alpha < 2$, X can be characterized in law as follows using families of martingales. We can use martingales here, rather than submartingales as in [15], because there are no boundary points besides the origin. The process X is uniquely characterized in law as the process with continuous sample paths in R^d and associated probability measures $\{P_x, \, x \in R^d\}$

(one for each starting state x) satisfying the following three properties:

(i) $P_x(X(0) = x) = 1$.

(ii) For each $f \in C_c^2(R^d)$ with compact support in $R^d \setminus \{0\}$,

$$\left\{ f(X(t)) - \int_0^t Lf(X(s))\, ds,\ \mathscr{F}_t,\ t \geqslant 0 \right\}$$

is a P_x-martingale, where $\mathscr{F}_t \equiv \sigma\{X(s): 0 \leqslant s \leqslant t\}$.

(iii)

$$E^{P_x}\left[\int_0^\infty 1_{\{0\}}(X(t))\, dt \right] = 0.$$

One question that remains unanswered here and provides an area for further research is the following: Is there a simple criterion, expressed in terms of μ and d, for determining whether $\alpha \geqslant 2$? When $\mu \equiv$ constant, the answer is known: $\alpha = 2 - d - \mu$ and so $\alpha \geqslant 2$ is equivalent to $\mu \leqslant -d$.

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