SELF-MAPS OF LOOP SPACES. II

BY

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ABSTRACT. We study under what conditions on a finite CW complex X is $Q(X)$ atomic.

1. Introduction. Let $Q(X) = \Omega^\infty \Sigma^\infty(X)$. In [3] we showed that $Q(S^r)$ is atomic at p if $r \geqslant 2$ and that the universal covering space of $Q(S^1)$ is atomic at p. In this paper we study the question of what conditions one must have on a finite complex X so that $Q(X)$ is atomic at 2. For example, we show in Theorem 2.5 that $Q(\Sigma CP^n)$ is atomic at 2 for all n. We conjecture that $Q(\Sigma^s X)$ is atomic at 2 if $s \geqslant 1$ and X is "stably atomic". We can prove this only when $s \geqslant 2$ and X is restricted to a special kind of CW-complex.

The proofs of our results depend heavily on the methods of [3], especially those used to prove the stable case. We use algebraic properties of the algebra of Dickson invariants in a crucial way in order to use an induction on the skeleta of X. This is done in §3. We also note that something different happens when X is no longer finite dimensional by showing that $Q(\Sigma CP^\infty)$ is not atomic at 2 (Theorem 2.7). Finally, we note the finite cases proved in [3] do not generalize by showing that $\Omega^n \Sigma^n X$ is not atomic at p unless X is a sphere or a point (Theorem 2.8), although [2] gives a different finite analogue.

A detailed statement of our results is given in §2 and the proofs are given in §§3–7.

2. Statement of results. We first define the term atomic and various related notions (see also [2, 3]).

DEFINITION. An $(r - 1)$-connected space Y is called *atomic* if given any self-map $f\colon Y \to Y$ such that $f_*\colon H_r(Y) \to H_r(Y)$ is an isomorphism, then $f_*\colon H_*(Y) \to H_*(Y)$ is an isomorphism. Y is called *H-atomic* if Y is an H-space and we are given only self-H-maps f. Y is called *atomic at p*, where p is a prime, if Y has the same property with all homology groups having Z/pZ coefficients. Y is called *H-atomic at p* if further, Y is an H-space and we are given only self-H-maps f.

DEFINITION. A connected space X is called stably atomic (stably atomic at p) if $\Sigma^s(X)$ is atomic (atomic at p) for all $s \geqslant 0$.

We first note the following simple proposition.

PROPOSITION 2.1. *If Y is 1-connected and ΩY is H-atomic (H-atomic at p), then Y is atomic (atomic at p).*

We will be interested in finding conditions on X that imply that $Q(\Sigma^s X)$ is atomic. Many of our results will be stated in the form: $Q(X)$ is H-atomic at p under certain conditions on X. Since $\Omega Q(\Sigma X)$ is homotopy equivalent to $Q(X)$, Proposition 2.1 then implies that $Q(\Sigma X)$ is atomic at p.

The following proposition gives a necessary condition on X. We prove it in §5.

PROPOSITION 2.2. *If $Q(\Sigma^s X)$ is atomic at p for some s, then X is stably atomic at p.*

We now assume that $p = 2$; we presume there are similar theorems for p odd. We also assume all homology and cohomology groups have coefficients $Z/2Z$. Before stating our main results, we recall $H_*(Q(X))$. Let Q_i be the Dyer-Lashof (Kudo-Araki) operations defined for infinite loop spaces. Let $I = (i_1, \ldots, i_k)$ be a sequence of integers with $i_j \leqslant i_{j+1}$ and $i_1 > 0$. Let $Q_I = Q_{i_1} \cdots Q_{i_k}$, where $Q_I = \mathrm{id}$ if $k = 0$. Let $\{x_i\}$ be a basis for $H_*(X)$. Then $H_*(Q(X)) = Z/2Z[Q_I(x_i)]$. Let i_*: $H_*(X) \to H_*(Q(X))$ and π: $H_*(Q(X)) \to H_*(X)$ be the natural maps. Let f: $Q(X) \to Q(X)$ be a self-map which induces an isomorphism on the lowest nonvanishing homology group. Let $x_i' = \pi f_* i_*(x_i)$. If X is stably atomic at 2, then $\{x_i'\}$ is also a basis for $H_*(X)$.

We need to put a condition on our CW-complexes X. A CW-complex is called *special* (at 2) if it is 2-homotopy equivalent to a CW-complex X with $C_*(X) = H_*(X)$, that is, if there is a CW-structure with zero boundaries. (Using homology decompositions (see Moore [17]) and the fact that Moore spaces for finitely generated groups are special, it is not hard to see that a simply-connected CW-complex of finite type is special.)

In §3 we study the Dickson algebra and prove the following general theorem.

THEOREM 2.3. *Let X be a finite CW-complex, which is special. Assume that X is stably atomic at 2. Let f: $Q(X) \to Q(X)$ be an H-map. Assume that $f_*(Q_1 x_i) = Q_1 x_i' +$ other terms and that $f_*(Q_1 Q_1 x_i) = Q_1 Q_1 x_i' +$ other tems. (Here x_i and x_i' are as above, and other terms mean terms of another filtration in the given basis.) Then f_* is an isomorphism.*

Since double loop maps commute with Q_1, we get the following corollary.

COROLLARY 2.4. *Let X be a finite CW-complex, which is special. Assume X is stably atomic at 2. Then $Q(\Sigma^s X)$ is atomic at 2 if $s \geqslant 2$.*

In §4, we apply Theorem 2.3 to projective spaces and prove the following theorem.

THEOREM 2.5. *$Q(CP^n)$ is H-atomic at 2 for all n. $Q(RP^n)$ is H-atomic at 2 for $n \neq 3$ or 7.*

In §5, we apply Theorem 2.3 to prove a general theorem for finite CW-complexes X.

THEOREM 2.6. *Let X be a finite CW-complex which is special and which is stably atomic at 2. Assume that all elements of $H_*(X)$ are primitive. Furthermore, assume $H^*(X)$ is a trivial A-module. Then $Q(X)$ is H-atomic at 2.*

On the basis of our results, we offer a conjecture.

CONJECTURE. Let X be a connected finite CW-complex. Assume X is stably atomic at 2. Then $Q(\Sigma^s X)$ is atomic at 2 if $s \geq 1$.

This conjecture is not true when X is an infinite dimensional CW-complex. In §6 we use the results of [3] to study RP^∞ and CP^∞.

THEOREM 2.7. $Q(\Sigma RP^\infty)$ *and* $Q(\Sigma CP^\infty)$ *are not atomic at 2.*

The proof of this theorem is a corollary of our discussion of the Kahn-Priddy theorem [15] in §6.

We have studied some finite versions of $Q(X)$ to see if they could be atomic at 2. [2] gives our results for $\mathrm{Map}_*(P^3(2), S^n)$, and in §7, we prove the following theorem.

THEOREM 2.8. *If* $\Omega^n \Sigma^n X$ *is atomic at p for $n \geq 2$, then* $H_*(X; Z/pZ) = H_*(pt; Z/pZ)$ *or* $H_*(S^q; Z/pZ)$ *for some q.*

3. The Dickson algebra and the proof of 2.3. Let $f\colon Q(X) \to Q(X)$ be a self-H-map. To show that $f_*\colon H_*(QX) \to H_*(QX)$ is an isomorphism it is enough to show that the induced map on $QH_*(QX)$, the module of indecomposables, is an isomorphism. (This is because f_* is multiplicative.)

Let $Q_k(X) \subset QH_*(QX)$ be the subspace generated by Dyer-Lashof operations of length k on $H_*(X)$ modulo those starting with Q_0. $R[k]$ denotes the Dyer-Lashof operations of length k. Recall [3, §1] that $R[k]^* = Z/2Z[a_1, \ldots, a_k]$, the algebra of Dickson invariants. The following two propositions relate $Q_k(S^r)$ to the Dickson algebra. Here $R[k]/(Q_0, \ldots, Q_r)$ denotes the Dyer-Lashof operations of length k all of whose subscripts are $> r$.

PROPOSITION 3.1. $Q_k(S^r) \to \Sigma^r(R[k]/(Q_0, \ldots, Q_r))$ *given by* $Q_{i_1} \cdots Q_{i_k}(x_r) \mapsto Q_{i_1 + r} \cdots Q_{i_k + r}$ *is an isomorphism of A-modules.*

PROPOSITION 3.2. $(R[k]/(Q_0, \ldots, Q_r))^* = (a_k^{r+1}) \subset Z/2Z[a_1, \ldots, a_k]$.

3.1 is an easy application of the Nishida formula and 3.2 is proved in the same way as Theorem 1.7 of [3].

DEFINITION. An $\mathscr{A}$-module is $\mathscr{A}$-atomic if given an $\mathscr{A}$-map $f\colon M \to M$ which is an isomorphism in the lowest nontrivial dimension, then f is an isomorphism.

A key fact is the following proposition, which is proved below.

PROPOSITION 3.3. *Let* $M = \prod_k \langle a_k^{r+1} \rangle$ *and let $f\colon M \to M$ be an $\mathscr{A}$-map such that* $f(a_k^{r+1}) = a_k^{r+1} +$ *others for all k. Then f is an isomorphism. In particular, $\langle a_k^{r+1} \rangle$ is $\mathscr{A}$-atomic.*

We now start the proof of Theorem 2.3. Let X^r be the r-skeleton of X. Let $P(X)$ denote $PH^*(QX) = QH_*(QX)^* = \prod_k Q_k(X)^*$. We want to show that $f^*\colon P(X) \to P(X)$ is an isomorphism. We first want to prove by downward induction that f^* induces a map $f_r\colon P(X^r) \to P(X^r)$. We will then prove by upward induction that f_r is an isomorphism. Let $Q_1^k = Q_1 \cdots Q_1 \in R[k]$.

LEMMA 3.4. *Under the hypotheses of 2.3, $f_*(Q_1^k x_i) = Q_1^k x_i' + others$.*

PROOF. The hypotheses of 2.3 include this condition for $k \leqslant 2$. Assume $k > 2$, and prove this by induction on k. We note that "others" must be primitive. We have $Sq_*^1 Q_1^k x_i = Q_0 Q_1^{k-1} x_i$. Since $k > 2$, no other basis element has this property. Hence, $f_*(Q_1^k x_i) = Q_1^k x_i' + $ primitives.

LEMMA 3.5. *Let $x \in P(X^r/X^{r-1})$, $|x| = t$. Then $x \in \bigcap_{i > t-r+2} \operatorname{Ker} Sq^i$.*

PROOF. $P(X^r/X^{r-1}) = \oplus P(S^r) = \oplus \prod_k \Sigma^r(a_k^{r+1})$ by Propositions 3.1 and 3.2. Thus $x = \sigma^r y$ with $|y| = t - r$. Since (a_k^{r+1}) is an unstable $\mathscr{A}$-module, $y \in \bigcap_{i > t-r+2} \operatorname{Ker} Sq^i$.

LEMMA 3.6. $0 \rightarrow P(X^r/X^{r-1}) \rightarrow P(X^r) \rightarrow P(X^{r-1}) \rightarrow 0$ *is exact.*

PROOF. The cofibration $X^{r-1} \rightarrow X^r \rightarrow X^r/X^{r-1}$ induces a fibration $Q(X^{r-1}) \rightarrow Q(X^r) \rightarrow Q(X^r/X^{r-1})$ and it is easy to see that the Serre spectral sequence of this fibration collapses. The lemma now follows.

LEMMA 3.7. *Let $0 \neq x \in P(X^{r-1})$, $|x| = t$. Then $Sq^i(x) \neq 0$ for some $i > t - r + 2$.*

PROOF. By Lemma 3.6, either $x \in P(X^{r-1}/X^{r-2})$ or $0 \neq x \in P(X^{r-2})$. Continue in this manner and find $s < r$ such that x projects to a nonzero element in $P(X^s/X^{s-1})$. Then $0 \neq Sq^{t-s}(x) \in P(X^s/X^{s-1})$ and hence in $P(X^{r-1})$. This is because $P(X^s/X^{s-1}) = \oplus \prod_k \Sigma^s(a_k^{s+1})$ by Propositions 3.1 and 3.2, and Sq^{t-s} is a monomorphism in degree $t - s$ of the Dickson algebra [3]. Since $s < r$, we have $t - s > t - r$.

We return to the proof of 2.3. We assume that we have $f_r: P(X^r) \rightarrow P(X^r)$ which is a map of $\mathscr{A}$-modules. This certainly exists when $r = \dim X$. We want to fill in the following diagram by maps of $\mathscr{A}$-modules:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & P(X^r/X^{r-1}) & \overset{\theta}{\rightarrow} & P(X^r) & \rightarrow & P(X^{r-1}) & \rightarrow & 0 \\
 & & & & \downarrow & & & & \\
0 & \rightarrow & P(X^r/X^{r-1}) & \overset{\theta}{\rightarrow} & P(X^r) & \rightarrow & P(X^{r-1}) & \rightarrow & 0
\end{array}
$$

By Lemmas 3.5 and 3.7, $\operatorname{Im} \theta = \bigcap_{i > t-r} \operatorname{Ker} Sq^i$ and hence it is preserved by f_r. This defines f_{r-1} on $P(X^{r-1})$ and makes a commutative ladder. By induction we know that f_{r-1} is an isomorphism and thus so is f_r if $f_r|P(X^r/X^{r-1})$ is an isomorphism. (We write $f_r|$ for $f_r|P(X^r/X^{r-1})$.) The bottom classes of $P(X^r/X^{r-1})$ are dual to $Q_1^k x_i$ and thus by Lemma 3.4, $f_r|$ preserves the classes a_k^{r+1} (at this point we use the fact that X is special). By Proposition 3.3, $f_r|$ is an isomorphism and the five lemma shows that $f_{\dim X}$ is an isomorphism, and the proof of 2.3 is complete.

We now prove Proposition 3.3. The idea of the proof is similar to that of Corollary 4.2 of [3]. Let $x \in \prod_k \langle a_k^{r+1} \rangle$ be in $\operatorname{Ker} f$. Using the techniques of Corollary 4.2 of [3], we can find a Steenrod operation α such that $\alpha(x) = a_k^{2^s}$ for some s and some k, and $\alpha(x) \in \operatorname{Ker} f$. However, there is a Steenrod operation ψ such that $\psi(a_k^{r+1}) = a_k^{2^s}$ and so $a_k^{2^s}$ is not in $\operatorname{Ker} f$.

4. $Q(CP^n)$ and $Q(RP^n)$. We first study $Q(CP^n)$. Let $x_{2i} \in H_{2i}(CP^n)$ be the generators. To apply Theorem 2.3 we must show that $f_*(x_{2i}) = x_{2i} + \text{of}$, $f_*(Q_1 x_{2i}) = Q_1 x_{2i} + \text{of}$, and $f_*(Q_1 Q_1 x_{2i}) = Q_1 Q_1 x_{2i} + \text{of}$ for $i \leqslant n$. ("of" stands for other filtrations.) These facts will be proved in Lemmas 4.1–4.5.

LEMMA 4.1. *If $f_*(x_{2i}) = x_{2i} + \text{of}$, then $f_*(Q_2 x_{2n}) = Q_2 x_{2n} + \text{of}$. Further, if n is odd, then $f_*(Q_2 x_{2n-2}) = Q_2 x_{2n-2} + \text{of}$.*

PROOF. *Case 1, n even.* $|Q_2 x_{2n}| = 4n + 2$, so there are no decomposable elements of filtration 2 in that dimension. Hence $f_*(Q_2 x_{2n}) = \sum a_{2i} Q_{4n+2-4i} x_{2i} + \text{of}$. Apply Sq_*^2 to get $0 = \sum i a_{2i} Q_{4n-4i} x_{2i} + \text{of}$. Hence, $a_{2i} = 0$ if i is odd. We now apply Sq_*^4 to the same equation to get

$$f_*(Q_2 x_{2n-2}) = a_{2n} Q_2 x_{2n-2} + \sum a_{4j}\left(j Q_{4n-2-8j} x_{4j} + Q_{4n+2-8j} x_{4j-2}\right) + \text{of}.$$

Apply Sq_*^2 to this last equation to get

$$f_*(Q_0 x_{2n-2}) = a_{2n} Q_0 x_{2n-2} + \sum a_{4j} Q_{4n-8j} x_{4j-2} + \text{of}.$$

Since $f_*(x_{2n-2}) = x_{2n-2} + \text{of}$ and f_* is multiplicative, we see that $a_{2n} = 1$ and $a_{4j} = 0$, $j < (n/2) - 1$.

Case 2, n odd. As before, apply Sq_*^2 to $f_*(Q_2 x_{2n})$ to get $f_*(Q_0 x_{2n}) = \sum i a_{2i} Q_{4n-4i} x_{2i} + \text{of}$. Hence $a_{2n} = 1$ and $a_{2i} = 0$ if i is odd, $i < n$. As before, apply $\text{Sq}_*^2 \text{Sq}_*^4$ to $f_*(Q_2 x_{2n})$ to get $0 = \sum a_{4j} Q_{4n-8j} x_{4j-2} + \text{of}$. Hence $a_{4j} = 0$, $j \leqslant (n-1)/2$, and we have shown Case 2.

Finally, consider

$$f_*(Q_2 x_{2n-2}) = \sum a_{2i} Q_{4n-4i-2} x_{2i} + \text{of} + b x_{2n-2} \cdot x_{2n}.$$

Apply Sq_*^2 to get $0 = \sum i a_{2i} Q_{4n-4i-4} x_{2i} + b x_{2n-4} \cdot x_{2n} + \text{of}$. Hence $b = 0$ and $a_{2i} = 0$ if i is odd. Now continue as in Case 1.

LEMMA 4.2. *If $f_*(x_{2i}) = x_{2i} + \text{of}$, all i, then $f_*(Q_1 x_{2i}) = Q_1 x_{2i} + \text{of}$, all i.*

PROOF. By Lemma 4.1, $f_*(Q_2 x_{2n}) = Q_2 x_{2n} + \text{of}$. Apply Sq_*^1 to see that $f_*(Q_1 x_{2n}) = Q_1 x_{2n} + \text{of}$. We now use downward induction on i. Assume the lemma is true for $i > k$.

Case 1, k odd. Then $\text{Sq}_*^4 Q_1 x_{2(k+1)} = Q_1 x_{2k}$ and this case follows easily by induction.

Case 2, k even. $f_*(Q_2 x_{2k}) = \sum b_{2j} Q_{4k+2-4j} x_{2j} + \text{decomp.} + \text{of}$. Apply Sq_*^2 to deduce that $b_{2j} = 0$ if j is odd. Now apply $\text{Sq}_*^4 \text{Sq}_*^1$ to get

$$f_*(Q_1 x_{2k-2}) = \sum b_{4j}\left[(1+j) Q_{4k-3-8j} x_{4j} + Q_{4k+1-8j} x_{4j-2}\right] + b_{2k} Q_1 x_{2k-2} + \text{of}.$$

Case 2a, $k \equiv 0 \mod 4$. Since $\text{Sq}_*^4 x_{2k+2} = x_{2k-2}$, by induction we have

$$f_*\left(\text{Sq}_*^8 Q_1 x_{2k+2}\right) = \text{Sq}_*^8\left(Q_1 x_{2k+2} + \text{of}\right) = Q_1 x_{2k-2} + \text{of} = f_*(Q_1 x_{2k-2}).$$

Thus $b_{2k} = 1$ and $b_{4j} = 0$ if $j < k/2$. Note that x_{2k+2} exists because $k < n$.

Case 2b, $k \equiv 2 \mod 4$. Then $\text{Sq}_*^4 x_{2k+4} = x_{2k}$. Hence the induction hypothesis gives $f_*(Q_1 x_{2k}) = f_*(\text{Sq}_*^8 Q_1 x_{2k+4}) = Q_1 x_{2k} + \text{of}$. Note that x_{2k+4} exists as $k \leqslant n - 2$ if n is even and $k \leqslant n - 3$ if n is odd by Lemma 4.1.

LEMMA 4.3. *If* $f_*(x_{2i}) = x_{2i} + of$, *all* i, *then* $f_*(Q_1Q_1x_{2i}) = Q_1Q_1x_{2i} + decomp. + of.$

PROOF.

$$f_*(Q_2Q_2x_{2i}) = \sum a_{kjr}Q_{2k}Q_{2j}x_{2r} + \sum b_{krjs}Q_{2k}x_{2r} \cdot Q_{2j}x_{2s}$$
$$+ \sum c_{krjs}Q_{2k+1}x_{2r} \cdot Q_{2j-1}x_{2s} + \sum d_{kjr}Q_{2k}Q_{2j+1}x_{2s}$$
$$+ \text{squares} + (\text{products of length} \geq 3) + of.$$

Apply $Sq_*^1 Sq_*^2 Sq_*^1$ to get

$$f_*(Q_0Q_1x_{2k}) = \sum a_{kjr}Q_{2k-2}Q_{2j-1}x_{2r}$$
$$+ \sum b_{krjs}\big[(k + r)Q_{2k-3}x_{2r} \cdot Q_{2j-1}x_{2s} + (j + s + 1)Q_{2k-1}x_{2r}$$
$$\cdot Q_{2j-3}x_{2s} + (k + s + 1)Q_{2k-3}x_{2r} \cdot Q_{2j-1}x_{2j}$$
$$+ (j + s)Q_{2k-1}x_{2r} \cdot Q_{2j-3}x_{2s}\big] + (\text{length} \geq 3) + of.$$

By Lemma 4.2, $f_*(Q_0Q_1x_{2i}) = Q_0Q_1x_{2i} + of.$ Comparing coefficients gives $a_{11i} = 1$ and $a_{kjr} = 0$ otherwise.

LEMMA 4.4. *Let* $n \neq 2^t - 1$. *If* $f_*(x_2) = x_2$, *then* $f_*(x_{2i}) = x_{2i} + of$, *all* i.

PROOF. $H_*(CP^n) \to H_*(QCP^n) \to H_*(QCP^n) \to H_*(CP^n)$ is a map of A-modules. Since $H_*(CP^n)$ is A-atomic, the lemma follows.

LEMMA 4.5. *Let* $n = 2^t - 1$. *If* $f_*(x_2) = x_2$, *then* $f_*(x_{2i}) = x_{2i} + of$, *all* i.

PROOF. $H_*(CP^{n-1}) \to H_*(CP^n) \to H_*(QCP^n) \to H_*(QCP^n) \to H_*(CP^{n-1})$ is a map of A-modules. Hence, by Lemma 4.4, this lemma is true for $i < n$. $\psi(x_{2i}) = \sum x_{2i} \otimes x_{2n-2i}$. We have that

$$\psi f_*(x_{2n}) = \sum f_*(x_{2i}) \otimes f_*(x_{2n-2i}) = \sum x_{2i} \otimes x_{2n-2i} + \text{others}.$$

The only elements having this diagonal are $x_{2n} + of$ or $\sum x_{2i} \cdot x_{2n-2i} + \text{others}$. If the latter, then apply Sq_*^2 to get $0 = Q_0x_{2^t-2} + \text{others}$. Since only $Q_2x_{2^t-2}$ goes to $Q_0x_{2^t-2}$ by Sq_*^2 besides $x_{2^t-2} \cdot x_{2^t}$, we have $f_*(x_{2n}) = \sum x_{2i} \cdot x_{2n-2i} + Q_2x_{2^t-2} + \text{others}$. Apply Sq_*^1 to get $0 = 0 + Q_1x_{2^t-2} + \text{others}$ and others cannot cancel $Q_1x_{2^t-2}$. Hence the lemma is proved.

We now study $Q(RP^n)$. The proof is very analogous to that for $Q(CP^n)$ and we state the appropriate lemmas and give a short outline of the proof of each.

LEMMA 4.6. *If* $f_*(x_i) = x_i + of$, *then* $f_*(Q_1x_n) = Q_1x_n + of$. *Further, if* n *is odd, then* $f_*(Q_1x_{n-1}) = Q_1x_{n-1} + of.$

PROOF. *Case 1,* n *even.* $f_*(Q_1x_n) = \sum a_iQ_{2n-2i+1}x_i + of$. Apply Sq_*^1 to deduce $a_i = 0$ if i is odd. Then apply $Sq_*^1Sq_*^2$ to deduce that $a_n = 1$ and $a_{2j} = 0$ if $2j < n$.

Case 2, n *odd.* Apply Sq_*^1 to deduce $a_i = 0$ for i odd, $i < n$, and $a_n = 1$. Apply Sq_*^2 to deduce that $a_{2j} = 0$, $2j \leq n - 1$. Finally, consider $f_*(Q_1x_{n-1})$. Apply Sq_*^1 to deduce that $b = 0$ and $a_i = 0$ for i odd, and then continue as in Case 1.

LEMMA 4.7. *If* $f_*(x_i) = x_i + of$, *then* $f_*(Q_1 x_i) = Q_1 x_i + of + decomp$.

PROOF. By Lemma 4.6, this is true for $i = n$. Assume true for $i > k$.

Case 1, k *odd*. Since $Sq^2_* Q_1 x_{k+1} = Q_1 x_k$, this case follows immediately by downward induction.

Case 2, k *even*. $f_*(Q_1 x_k) = \Sigma b_j Q_{2k-2j+1} x_j + decomp. + of$. Apply Sq^1_* to deduce that $b_j = 0$ if j is odd. Now apply $Sq^3_* + Sq^1_* Sq^2_*$ to deduce $b_{2j} = 0$ but we cannot conclude that $b_k = 1$ because $(Sq^3_* + Sq^1_* Sq^2_*)(x_{k-1} \cdot x_{k+2}) = Q_0 x_{k-1}$.

Case 2a, $k \equiv 2 \bmod 4$. Then $Sq^4_* Q_1 x_{k+2} = Q_1 x_k$ and we use "further" in Lemma 4.6 and induction.

Case 2b, $k \equiv 0 \bmod 4$. Apply Sq^2_* to $f_*(Q_1 x_k)$ and Sq^4_* to $f_*(Q_1 x_{k+1})$ to show no decomp. appear and hence $b_k = 1$.

LEMMA 4.8. *If* $f_*(Q_1 x_i) = Q_1 x_i + decomp. + of$, *then* $f_*(Q_1 Q_1 x_i) = Q_1 Q_1 x_i + decomp. + of$.

PROOF. *Case* 1, i *odd*. $f_*(Q_1 Q_1 x_i) = a Q_1 Q_1 x_i + others$. Apply Sq^1_* to deduce that $a = 1$ because $Sq^1_*(Q_1 x_i \cdot Q_2 x_i) = Q_0 x_i \cdot Q_2 x_i$.

Case 2, i *even*. $f_*(Q_1 Q_2 x_i) = \Sigma a_{i-j} Q_1 Q_{2+2j} x_{i-j} + b Q_3 Q_3 x_{i-1} + \Sigma c Q_I x_j + decomp. + of$. Apply $Sq^4_* Sq^2_*$ to deduce $a_i = 1$ by Case 1.

LEMMA 4.9. *Let* $n \neq 2^t - 1$. *If* $f_*(x_1) = x_1$, *then* $f_*(x_i) = x_i + of$, *all* i.

PROOF. The proof is the same as for Lemma 4.4.

LEMMA 4.10. *Let* $n = 2^t - 1$, $t \geq 4$. *If* $f_*(x_1) = x_1$, *then* $f_*(x_i) = x_i + of$, *all* i.

PROOF. The proof of 4.5 does not generalize to the real case. For example, if $n = 3$, then $x_3 + x_1 \cdot x_2 + (x_1)^3 + Q_1 x_1$ could be in $Ker f_*$. (Indeed, there is a homotopy equivalence $g: \Sigma^2 RP^3 \to \Sigma^2(RP^2 \cup S^3)$ and thus $Q(RP^3)$ is homotopy equivalent to $Q(RP^2 \cup S^3)$ by a map $\Omega^2 Q(g)$.) We first prove that f_* is an isomorphism in dimensions $< n$. The first step in the proof of Lemma 4.5 works, so we know that $f_*(x_i) = x_i + of$ for $i < n$. Then we note that the proof of Lemma 4.6 shows that if $f_*(x_i) = x_i + of$ for $i < n$, then $f_*(Q_1 x_{n-1}) = Q_1 x_{n-1} + of$. Using this, we note that the induction in the proof of Lemma 4.7 still works in dimensions $< n$, as does the proof of 4.8. Next, we examine the proof of 2.3 in §3. The induction which constructs f_r is correct in our case and the only step that fails is the induction from X^{n-1} to X^n, and it fails for $Q_0(X^n)$, i.e. it fails first in dimension n.

If $Ker f_* = 0$ in dimension n, Lemma 4.10 is proved. If $Ker f_* \neq 0$ in dimension n, then an element of the form $y = x_n + \Sigma x_i \cdot x_{n-i} + others \in Ker f_*$. As in the proof of 3.2 in [2], we let M be the mapping telescope of f and let $f': Q(RP^n) \to M$. Let F be the fibre of f'. F is $(n-1)$-connected and hence $y \in Ker f_*$ is a spherical element. By taking an adjoint of this spherical class we see that the top cell of RP^n is stably spherical and this can only happen in Hopf invariant 1 cases, i.e. when $t \leq 3$ (see [13]).

5. Proofs of Proposition 2.2 and Theorem 2.6. We first prove Proposition 2.2. It is enough to prove that $\Sigma^N X$ is atomic for large N. Let $g: \Sigma^N X \to \Sigma^N X$ be such that g_* is an isomorphism on the bottom dimension, but $g_*: H_*(\Sigma^N X) \to H_*(\Sigma^N X)$ is

not an isomorphism. Let $x \in H_*(X)$, $x \neq 0$, $g_*(\sigma^N x) = 0$. Choose $N > s$ and $N > |x|$. Then g induces $f = Q(g)$: $Q\Sigma^N X \to Q\Sigma^N X$ such that f_* is an isomorphism on the bottom dimension, but f_* is not an isomorphism because $H_*(Q\Sigma^N X) = H_*(\Sigma^N X)$ if $* \leq |\sigma^N x|$. Hence f is not a homotopy equivalence at p and thus $\Omega^{N-s} f$ is also not a homotopy equivalence at p.

We now examine an attempt to prove the conjecture in §2. If we assume that every element of $H_*(X)$ is primitive, i.e. that $H^*(X)$ has no nontrivial cup products, then the proof of Lemma 3.4 shows that we only need assume that $f_*(Q_1 x_i) = Q_1 x_i' +$ others terms to apply the argument for Theorem 2.3. We now show that the assumptions of Theorem 2.6 are enough to prove this. Let $|x_i| = 2^s \mod 2^{s+1}$. Then $\mathrm{Sq}^{2^s} Q_{2^s} x_i = Q_0 x_i$. Since x_i' is not in the image of any Steenrod operation, we get $f_*(Q_{2^s} x_i) = Q_{2^s} x_i' +$ others. Apply $\mathrm{Sq}^1_* \mathrm{Sq}^2_* \cdots \mathrm{Sq}^{2^{s-1}}_*$ to this equation to get $f_*(Q_1 x_i) = Q_1 x_i' +$ others. This proves Theorem 2.6.

A more delicate analysis would show that we can weaken the assumptions of Theorem 2.6, but the results are fragmentary.

6. Analogues of the Kahn-Priddy theorem. Let X have the homotopy type of a connected CW-complex of finite type. If X is not 1-connected, we assume that X is an H-space. The following lemma is easily proved using the Whitehead theorem.

LEMMA 6.1. *If f: $X \to Y$ and g: $Y \to X$, with $(gf)_*$ an isomorphism in the bottom dimension and X atomic at p, then X is a retract of Y at p.*

Theorem 2.7 and the Kahn-Priddy theorem follow from the following corollary. ($Q^\wedge(S^1)$ denotes the universal covering space of $Q(S^1)$.)

COROLLARY 6.2. *Let* (1) g_1: $Q(\Sigma RP^\infty) \to Q^\wedge(S^1)$, (2) g_2: $Q(\Sigma CP^\infty) \to SU$, *and* (3) g_3: $\mathrm{Map}_*(P^4(2), S^5) \to \Omega_0^3 S^3$ *be maps which induce isomorphisms on the first nonvanishing homotopy group of the domain. Then there exist maps f_i such that $g_i f_i$ is a homotopy equivalence at 2.*

PROOF. Since all three ranges of g_i are atomic at 2, it suffices to exhibit maps f_i and then apply Lemma 6.1. ($Q^\wedge(S^1)$ is atomic at 2 by [3], $\Omega_0^3 S^3$ is atomic at 2 by [2], and it is easily checked that SU is atomic at 2.) To define f_1, consider the second stable Hopf invariant h_2: $Q(S^1) \to Q(D_2(S^1))$ [14]. $D_2(S^1)$ is homotopy equivalent to ΣRP^∞ and $Q(S^1)$ is homotopy equivalent to $S^1 \times Q^\wedge(S^1)$. Then f_1 is the composite $Q^\wedge(S^1) \to Q(S^1) \to Q(D_2)$. It is immediate that $g_1 f_1$ induces an isomorphism on π_2. Note that f_1 is not an H-map.

To define f_2, use the adjoint of James' stable retraction in [12]. Then $g_2 f_2$ induces an isomorphism on π_3. Note that f_2 is not an H-map.

To define f_3, use the lift of the second Hopf invariant given in Lemma 3.1 of [6]. Again $g_3 f_3$ induces an isomorphism on π_1.

To apply Corollary 6.2, we need to know that the g_i exist. To obtain g_1, use the extension of $\Sigma RP^\infty \to Q(S^1)$ obtained by taking the adjoint of J. H. C. Whitehead's map $RP^\infty \to Q_0(S^0)$. To obtain g_2, use the extension of $\Sigma CP^\infty \to SU$ given in [13]. The construction of g_3 is the main result of [7].

We remark that one point of Corollary 6.2 is that the maps g_i (and f_i) are essentially arbitrary. In particular, g_i may not necessarily be an H-map. Kahn and Priddy [14, 15], J. F. Adams [1], and P. Loeffler and N. Ray [16] use a map g_1 which is an infinite loop map. The map g_2 can be traced to the work of I. M. James [12]. Later, G. Segal [18] proved that BU is a retract of $Q(CP^\infty)$, and this follows by looping g_2. Since g_2 induces an isomorphism on rational homotopy, it follows that the other factor of $Q(\Sigma CP^\infty)$ has finite homotopy groups.

7. Theorem 2.8 and product decompositions. In this section, all spaces X are localized at p and are assumed to be of finite type over $Z_{(p)}$. Our main result is the following theorem.

THEOREM 7.1. *If $\Sigma^2 X$ is not homotopy equivalent to a sphere or a point, then $\Omega \Sigma^2 X$ admits a nontrivial product decomposition up to homotopy.*

This theorem was known to Cohen, Moore, and Neisendorfer and they gave a proof in an unpublished computation [9]. Some analogous decompositions and related applications are given in [5 and 10]. We note that Theorem 2.8 is an immediate corollary.

We first set up some notation and prove two lemmas. We require maps β_k: $(\Sigma X)^{[k]} \to (\Sigma X)^{[k]}$, where $Y^{[k]}$ denotes the k-fold smash product of the space Y. Let σ_k: $Y^{[k]} \to Y^{[k]}$ denote the cyclic permutation $(1, k, k-1, \ldots, 2)$ and recall that $[(\Sigma X)^{[k]}; Z]$ is a group for $k \geq 1$. Define β_k inductively by setting $\beta_2 = 1 - \sigma_2$ and $\beta_{k+1} = (1 \wedge \beta_k) - (\beta_k \wedge 1) \cdot \sigma_k$.

LEMMA 7.2 (DYNKIN-SPECHT-WEVER RELATION). $\beta_{k*}\beta_{k*} = k\beta_{k*}$.

PROOF. Let $M = \overline{H}_*(\Sigma X)^{[k]}$ and $[a, b] = a \otimes b - (-1)^{|a||b|}b \otimes a$ in $M \otimes M$. By definition, we have

$$\beta_{k*}(m_1 \otimes \cdots \otimes m_k) = \big[m_1\big[m_2\big[\ldots[m_{k-1}, m_k]\ldots\big],\big],\big].$$

The lemma now follows from the computation in the proof of Theorem 8 of [11, p. 169].

Let $L_k(\Sigma X)$ and $M_k(\Sigma X)$ denote the mapping telescopes of β_k and $k - \beta_k$ respectively. Define ϕ_k: $(\Sigma X)^{[k]} \to L_k(\Sigma X) \vee M_k(\Sigma X)$ to be the composite $(\Sigma X)^{[k]} \to (\Sigma X)^{[k]} \vee (\Sigma X)^{[k]} \to L_k(\Sigma X) \vee M_k(\Sigma X)$. By Lemma 7.2, ϕ_{k*} induces an isomorphism on $H_*(\ ; Z[1/k])$. Thus as in [4], we have proved the following lemma.

LEMMA 7.3. *If $(k, p) = 1$, then ϕ_k is a homotopy equivalence.*

PROOF OF THEOREM 7.1. We shall exhibit a loop map $\Omega(w)$: $\Omega \Sigma L_k(\Sigma X) \to \Omega \Sigma^2 X$ and a map h: $\Omega \Sigma^2 X \to \Omega \Sigma L_k(\Sigma X)$ such that $h \cdot \Omega(w)$ induces an isomorphism on $H_*( )$ if $(k, p) = 1$. Thus $\Omega \Sigma L_k(\Sigma X)$ is a p-local retract of $\Omega \Sigma^2 X$. We will show that $\Omega \Sigma^2 X$ is homotopy equivalent to $\Omega \Sigma L_k(\Sigma X) \times Z$ for some Z. To prove Theorem 7.1, we must check that $L_k(\Sigma X)$ is more highly connected than ΣX and that $\overline{H}_* L_k(\Sigma X)$ is nontrivial. But $L_k(\Sigma X)$ is a retract of $(\Sigma X)^{[k]}$ and so it is more

highly connected than ΣX. Also, if ΣX is not a sphere or a point, there exist at least two elements u and v in $\overline{H}_*\Sigma X$. Then $\mathrm{ad}^{k-1}(v)(u)$ is a nonzero element of $\overline{H}_* L_k(\Sigma X)$.

Next we show that $\Omega\Sigma^2 X$ splits. Let F be the homotopy fibre of w and let π: $\Omega\Sigma^2 X \to F$ be the natural map obtained from the Barratt-Puppe sequence. There is a morphism of fibrations:

$$
\begin{array}{ccc}
\Omega\Sigma L_k(\Sigma X) & \overset{h\cdot\Omega(w)}{\to} & \Omega\Sigma L_k(\Sigma X) \\
\Omega(w)\downarrow & & \downarrow \\
\Omega\Sigma^2 X & \overset{\pi\times h}{\to} & F \times \Omega\Sigma L_k(\Sigma X) \\
\pi\downarrow & & \downarrow \\
F & \to & F
\end{array}
$$

and so $\pi \times h$ induces a homology isomorphism.

Now we exhibit $\Omega(w)$ and h. Consider the Freudenthal suspension E: $\Sigma X \to \Omega\Sigma^2 X$ and the k-fold Samelson product $\mathrm{ad}^{k-1}(E)(E)$: $(\Sigma X)^{[k]} \to \Omega\Sigma^2 X$. Notice that this map induces the map β_{k*} in homology. Thus if $(k, p) = 1$, then $\mathrm{ad}^{k-1}(E)(E)$ restricted to $L_k(\Sigma X)$ induces an isomorphism from $H_* L_k(\Sigma X)$ to the Lie tensors of length k. Let $\Omega(w)$: $\Omega\Sigma L_k(\Sigma X) \to \Omega\Sigma^2 X$ be the canonical multiplicative extension of the induced map $\mathrm{ad}^{k-1}(E)(E)$: $L_k(\Sigma X) \to \Omega\Sigma^2 X$. Finally, we construct g: $\Sigma\Omega\Sigma^2 X \to \Sigma\Omega\Sigma L_k(\Sigma X)$ such that $g \cdot \Sigma\Omega(w)$ induces a homology isomorphism. It follows from Lemma 1.6 in [8] that there is a map h such that $h \cdot \Omega(w)$ is an equivalence. To construct g, recall that $\Sigma\Omega\Sigma^2 X$ is homotopy equivalent to $\Sigma(\bigvee_{j\geq 1}(\Sigma X)^{[j]})$. Pinch the spaces $(\Sigma X)^{[j]}$ to a point if $j \neq 0 \bmod k$ to obtain a map $\Sigma\Omega\Sigma^2 X \to \Sigma(\bigvee_{j\geq 1}(\Sigma X)^{[jk]})$. Next, observe that by Lemma 7.3, $(\Sigma X)^{[jk]}$ is homotopy equivalent to $(L_k(\Sigma X) \vee M_k(\Sigma X))^{[j]}$. Project $(\Sigma X)^{[jk]}$ to $(L_k(\Sigma X))^{[j]}$. The composite of these maps gives g: $\Sigma\Omega\Sigma^2 X \to \Sigma\Omega\Sigma L_k(\Sigma X)$ which by inspection has the property that $g \cdot \Sigma\Omega(w)$ induces a homology isomorphism.

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