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DERIVATIVES OF MAPPINGS WITH APPLICATIONS TO NONLINEAR DIFFERENTIAL EQUATIONS

BY

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ABSTRACT. We present a new definition of differentiation for mappings of sets in topological vector spaces. Complete flexibility is allowed in choosing the topology with which the derivative is taken. We determine the largest space on which the derivative can act. Our definition includes all others hitherto given, and the basic theorems of calculus hold for it. Applications are considered here and elsewhere.

1. The definition. Let X be a real vector space and let Q, Y be separated real topological vector spaces such that $Q \subset X$. Let G be a mapping from a subset $V \subset X$ to Y. We wish to define differentiation for the mapping G with respect to Q. We are not assuming that V is a linear manifold nor are we assuming that $V \subset Q$. Thus we cannot arbitrarily consider difference quotients of the form

$$t^{-1}[G(u+th)-G(u)] \tag{1.1}$$

since in general $u + th$ need not be in V. On the other hand if we restrict the derivative to apply only to those h for which $u + th$ is in V for t near 0, we may find that hardly any such h exist. For instance, if V is the surface of a sphere in $\mathbf{R}^n$ or in some Hilbert space, then $h = 0$ is the only element for which $u + th$ is in V for t near 0. Yet the only way one can define differentiation is to consider difference quotients of the form (1.1). This led us to examine those elements h which do not necessarily satisfy $u + th \in V$ but are the limits of elements which do. To this end, let $C(V, Q, u)$ denote the set consisting of those $q \in Q$ for which there exist sequences $\{t_n\} \subset \mathbf{R}$, $\{q_n\} \subset Q$ such that

$$q_n \to q \quad \text{in } Q, \qquad 0 \neq t_n \to 0, \qquad u + t_n q_n \in V. \tag{1.2}$$

We shall see that $C(V, Q, u)$ is a (double) cone. Next we let $E(V, Q, u)$ be the smallest linear manifold containing $C(V, Q, u)$, i.e. the set of all finite sums of elements in $C(V, Q, u)$. We are now ready to define the derivative of G with respect to Q.

DEFINITION. A linear operator A from X to Y is called the derivative of G at u with respect to Q and denoted by $G'_Q(u)$ if

(a) u is in V,

(b) $D(A) = E(V, Q, u)$,

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(c) for any sequences $\{q_n\}$, $\{t_n\}$ satisfying (1.2),

$$t_n^{-1}[G(u+t_nq_n)-G(u)] \to Aq \quad \text{in } Y \quad \text{as } n\to\infty. \tag{1.3}$$

Note that when it exists, $G_Q'(u)$ is uniquely determined on $E(V,Q,u)$. On the other hand, any larger domain will not determine it uniquely. We shall study the sets $C(V,Q,u)$ and $E(V,Q,u)$ in detail as well as the properties of the derivative. We shall also study one-sided derivatives as well.

Let us consider some examples. If V is the surface of the unit sphere in a Hilbert space H, then

$$E(V,H,u)=C(V,H,u)=\{v|v\perp u\}.$$

On the other hand, if V is the solid unit ball in H, then $E(V,H,u)=C(V,H,u)=H$. In the next section we discuss some basic properties of the derivative and in §3 we discuss the sets $C(V,Q,u)$ and $E(V,Q,u)$. In §§4, 5 we give some applications. Proofs will be given in §6.

2. Calculus theorems. We now present some of the basic theorems of calculus which are satisfied by our derivative. The spaces X, Y, Q and the mapping G are as described in the first section. The set of all continuous linear functionals on Y will be denoted by Y'.

THEOREM 2.1. *Let $q\in Q$ be such that $u+sq$ is in V for $0\leqslant s\leqslant 1$, and let y' be an element of Y'. Assume that $G_Q'(u+sq)$ exists for $0<s<1$ and that*

$$\begin{aligned} &y'G(u+sq)\to y'G(u) \quad \text{as } s\to 0,\\ &y'G(u+sq)\to y'G(u+q) \quad \text{as } s\to 1. \end{aligned} \tag{2.1}$$

Then there is a θ such that $0<\theta<1$ and

$$y'[G(u+q)-G(u)]=y'G_Q'(u+\theta q)q. \tag{2.2}$$

THEOREM 2.2. *Let F be a mapping of a subset D of Y to a topological vector space Z. Let u be an element of V and put $y=G(u)$. Assume that if $q\in Q$ and $u+q\in V$, then $G(u+q)\in D$ and $G(u+q)-y\in R$, a closed subspace of Y. Assume that $G_Q'(u)$, $F_R'(y)$ exist. Then if $H(v)=F(G(v))$, then $H_Q'(u)$ exists and equals $F_R'(y)G_Q'(u)$.*

THEOREM 2.3. *Let F be a mapping from $V\subset X$ to Y, and let N be a Banach space contained in X. Assume that there is a mapping $q(t)$ of $(0,m)$ into Q such that*

$$v=u+tq(t)+h\in V, \qquad 0<t<m,\ h\in N,\ \|h\|<m. \tag{2.3}$$

Assume further that Y is a Banach space and there are constants δ, C_0 and an operator $T\in B(N,Y)$ such that

$$\|T(h_1-h_2)-F(u+tq(t)+h_1)+F(u+tq(t)+h_2)\|\leqslant\delta\|h_1-h_2\|, \tag{2.4}$$
$$0<t<m,\ \|h_i\|<m,$$

and

$$d(h, N(T)) \equiv \inf_{w \in N(T)} \|h - w\| \leqslant C_0 \|Th\|, \qquad h \in N. \tag{2.5}$$

Finally, we assume that $R(T) = Y$ *and that* $\delta C_0 < 1$. *Then for any mapping* $L(t)$ *of* $(0, m)$ *into* Y *and any* $h \in N$ *there is a mapping* $h(t)$ *of interval* $(0, m')$ *into* N *such that*

$$F(u + tq(t) + th(t)) \equiv L(t), \qquad 0 < t < m' < m, \tag{2.6}$$

and

$$\|h(t) - h\| \leqslant Ct^{-1} \|F(u + tq(t) + th) - L(t)\| \tag{2.7}$$

provided $F(u + tq(t)) - L(t) \to 0$ *as* $t \to 0$.

COROLLARY 2.4. *Assume in addition that* $q(t) \to q$ *in* Q *as* $t \to 0$ *and* $F_Q'(u)$ *exists. Then there is a mapping* $h(t)$ *of* $(0, m)$ *into* N *such that*

$$F(u + tq(t) + th(t)) \equiv F(u), \qquad 0 < t < m, \tag{2.8}$$

and

$$\limsup_{t \to 0} \|h(t)\| \leqslant C \left\| F_Q'(u) q \right\|. \tag{2.9}$$

THEOREM 2.5. *Let* W *be a topological vector space continuously embedded in* Q. *If* $G_Q'(u)$ *exists, then* $G_W'(u)$ *exists, and is equal to the restriction of* $G_Q'(u)$ *to* $E(V, W, u)$.

THEOREM 2.6. *Let* F *be a mapping from* $V \subset X$ *to* Y *and let*

$$S = \{v \in V | F(v) = 0\}. \tag{2.10}$$

If $u \in S$ *and* $F_Q'(u)$ *exists, then*

$$C(S, Q, u) \subset \left\{q \in C(V, Q, u) \middle| F_Q'(u) q = 0\right\}. \tag{2.11}$$

If, in addition, for each $q \in C(V, Q, u)$ *the map* F *satisfies the hypotheses of Corollary* 2.4 *for either* q *or* $-q$ *with* N *continuously embedded in* Q, *then*

$$C(S, Q, u) = \left\{q \in C(V, Q, u) \middle| F_Q'(u) q = 0\right\}. \tag{2.12}$$

Theorem 2.6 generalizes theorems of Ljusternik [**1**] and Ioffe–Tihomirov [**2**].

THEOREM 2.7. *Under the same hypotheses let* G *be a mapping of* V *into* $\mathbf{R}$. *If*

$$G(u) = \min_S G(v) \tag{2.13}$$

and $G_Q'(u)$ *exists, then*

$$G_Q'(u) q = 0 \tag{2.14}$$

for all $q \in C(V, Q, u)$ *such that*

$$F_Q'(u) q = 0. \tag{2.15}$$

THEOREM 2.8. *Under the same hypotheses, $H = F'_N(u)$ maps N onto Y and*

$$G'_Q(u)\left[1 - H^{-1}F'_Q(u)\right]q = 0 \quad \forall q \in C(V, Q, u) \tag{2.16}$$

where $H^{-1}w$ is any element h of H such that $Hh = w$.

THEOREM 2.9. *Let F be a mapping of $V \subset X$ into Y and G a mapping of V into $\mathbf{R}$. Let S be given by* (2.10) *and assume that* (2.13) *holds for some $u \in S$. Assume that Q, Y are Banach spaces, that $V + Q \subset V$ and that $F'_Q(u + q)$ exists and is in $B(Q, Y)$ for $q \in Q$ small and*

$$\left\|F'_Q(u + q) - F'_Q(u)\right\| \to 0 \quad \text{as } q \to 0. \tag{2.17}$$

Assume further that the range of $F'_Q(u)$ is closed in Y and that $G'_Q(u)$ exists and is in Q'. Then there exist $\lambda \in \mathbf{R}$ and $y' \in Y'$ not both vanishing such that

$$\lambda G'_Q(u) + y'F'_Q(u) = 0. \tag{2.18}$$

Theorem 2.9 is a generalization of the Lagrange multiplier rule.

3. The tangent cones. Let Q be a topological vector space contained in a vector space X and let V be a subset of X. For a given element u of V we let $C_+(V, Q, u)$ denote the set of those $q \in Q$ for which there exist sequences $\{q_n\} \subset Q$ and $\{t_n\} \subset \mathbf{R}_+$ such that

$$q_n \to q \quad \text{in } Q, \qquad 0 < t_n \to 0, \qquad u + t_n q_n \in V. \tag{3.1}$$

We note

LEMMA 3.1. *If $q \in C_+(V, Q, u)$ and $\alpha > 0$ then $\alpha q \in C_+(V, Q, u)$.*

Thus $C_+(V, Q, u)$ is a cone. It is called the *tangent cone* to the set V at the point u with respect to Q. We also have

LEMMA 3.2. *If W is a topological vector space contained in Q with continuous injection, then*

$$C_+(V, W, u) \subset C_+(V, Q, u). \tag{3.2}$$

LEMMA 3.3. *q is in $C(V, Q, u)$ iff either q or $-q$ is in $C_+(V, Q, u)$.*

LEMMA 3.4. *$E(V, Q, u)$ is the set of all linear combinations of elements of $C_+(V, Q, u)$.*

Now we define a one-sided derivative corresponding to the definition of §1. Let X, Y, Q be as in that section. Let F map $V \subset X$ into Y. We shall write $A = F'_{+Q}(u)$ and call A the right-hand derivative of F at u with respect to Q if A is a linear operator from X to Y and

(a) u is in V,

(b) $D(A) = E(V, Q, u)$,

(c) for all sequences $\{q_n\}$, $\{t_n\}$ satisfying (3.1) we have

$$t_n^{-1}\left[F(u + t_n q_n) - F(u)\right] \to Aq \quad \text{in } Y \quad \text{as } n \to \infty. \tag{3.3}$$

Clearly we have

LEMMA 3.5. *If $F'_Q(u)$ exists, then $F'_{+Q}(u)$ exists and they are equal.*

We also have

THEOREM 3.6. *Let F be a mapping as above and assume that $F'_{+Q}(u)$ exists. Let B be any subset of Y', and put*

$$S = \{v \in V \mid y'F(v) \geqslant 0 \ \forall y' \in B\},$$
$$B_u = \{y' \in B \mid y'F(u) = 0\}. \tag{3.4}$$

Then

$$C_+(S,Q,u) \subset \{q \in C_+(V,Q,u) \mid y'F'_{+Q}(u)q \geqslant 0 \ \forall y' \in B_u\}.$$

THEOREM 3.7. *Under the same hypotheses, let*

$$R = \bigcap_{y' \in B_u} N(y')$$

and assume that $Y = R \oplus M$ with a bounded projection P of Y onto M. Assume that PF satisfies the hypotheses of Corollary 2.4 for each $q \in C_+(V,Q,u)$ with N continuously embedded in Q. Assume finally that the above hypotheses hold if we remove from B_u any of the $y' \in B$ for which no negative multiple of y' is in B. Then if u is in S we have

$$C_+(S,Q,u) = \{q \in C_+(V,Q,u) \mid y'F'_{+Q}(u)q \geqslant 0 \ \forall y' \in B_u\}. \tag{3.5}$$

THEOREM 3.8. *Under the same hypotheses let G be a mapping of V into $\mathbf{R}$. If*

$$G(u) = \min_S G(v) \tag{3.6}$$

and $G'_{+Q}(u)$ exists, then

$$G'_{+Q}(u)q \geqslant 0 \tag{3.7}$$

holds for all $q \in C_+(V,Q,u)$ such that

$$y'F'_{+Q}(u)q \geqslant 0 \quad \forall y' \in B_u. \tag{3.8}$$

THEOREM 3.9. *Under the same hypotheses, $H = PF'_{+N}(u)$ maps N onto M and*

$$G'_{+Q}(u)\left[1 - H^{-1}PF'_{+Q}(u)\right]q \geqslant 0 \quad \forall q \in C_+(V,Q,u) \tag{3.9}$$

where $H^{-1}w$ is any element h of N such that $Hh = w$.

THEOREM 3.10. *Let F be a mapping of $V \subset X$ into Y and G a mapping of V into $\mathbf{R}$. Let S be given by* (3.4) *and assume that* (3.6) *holds for some $u \in S$. Assume that Q, Y are Banach spaces and $V + Q \subset V$. Assume further that $F'_Q(u+q)$ exists and is in $B(Q,Y)S$ for $q \in Q$ small and* (2.17) *holds. Assume also that the range of $F'_Q(u)$ is closed in Y and that $G'_Q(u)$ exists and is in Q'. Then there exist $\lambda \geqslant 0$ and $y' \in M'$ not both vanishing such that*

$$\lambda G'_Q(u)q + y'PF'_Q(u)q \geqslant 0 \quad \forall q \in C_+(V,Q,u). \tag{3.10}$$

4. An application. Let Ω be an interval in $\mathbf{R}$ which is not necessarily bounded, and let $a(u,v)$ be a bilinear form on

$$W = W_0^{1,2}(\Omega) = \{v(r) \in L^2 \mid \dot{v} \in L^2,\ v(\partial\Omega) = 0\} \tag{4.1}$$

where $L^2 = L^2(\Omega)$, $\dot{v}(r) = dv(r)/dr$ and $\partial\Omega$ is the boundary of Ω. The operator A corresponding to $a(u, v)$ is defined as follows. For u, f in L^2 we say that u is in $D(A)$ and $Au = f$ if

$$a(u,v) = (f,v), \qquad v \in W, \tag{4.2}$$

where

$$(f,v) = \int_\Omega f(r)v(r)\,dr. \tag{4.3}$$

Let $g(v, r)$ be a continuous function on $\mathbf{R} \times \Omega$. We shall be looking for stationary points of the functional

$$G(v) = \alpha a(v) - I(v) \tag{4.4}$$

where

$$I(v) = \int_\Omega g(v(r), r)\,dr, \tag{4.5}$$

$\alpha \in \mathbf{R}$ and $a(v) = a(v, v)$. For Q we take the topological vector space of infinitely differentiable functions with compact supports in Ω. We note

LEMMA 4.1. *If $a(v)$ is continuous on W and $f(v, r) = \partial g(v, r)/\partial v$ is continuous on $\mathbf{R} \times \Omega$, then any stationary point u of* (4.4) *is a solution of*

$$2\alpha Au = f(u(r), r). \tag{4.6}$$

The proof of Lemma 4.1 is straightforward and is omitted. We shall examine some assumptions which will quarantee a stationary point of (4.4) and consequently a solution of (4.6). We shall assume

(I) There are positive constants c_0, C_0 such that

$$c_0 a(v) \leqslant \|v\|_1^2 \leqslant C_0 a(v) \tag{4.7}$$

where

$$\|v\|_1^2 = \|v\|^2 + \|\dot{v}\|^2, \qquad \|v\|^2 = \int_\Omega v(r)^2\,dr. \tag{4.8}$$

(II) $g(v, r) \leqslant B(v, r) = B_1(v, r) + B_2(v, r)$, where

$$B_1(v,r) = \sum_{k=1}^{k} y_k(r)v^2 m_k(v), \tag{4.9}$$

$$B_2(v,r) = \sum_{k=1}^{L} w_k(r)|v|^{s_k} h_k(v), \qquad 0 \leqslant s_k < 2, \tag{4.10}$$

the h_k and m_k are continuous functions, w_k is in L^{p_k} for some p_k satisfying $1 \leqslant p_k \leqslant 2/(2 - s_k)$, and y_k is in L^1_{loc} and satisfies

$$\int_{(x,x+1)\cap\Omega} |y_k(r)|\,dr \to 0 \quad \text{as } |x| \to \infty. \tag{4.11}$$

(III) $f(v, r) = \partial g(v, r)/\partial v$ is continuous on $\mathbf{R} \times \Omega$. Put

$$V = \{v \in W | g(v(r), r) \in L^1(\Omega)\}, \tag{4.12}$$

$$S_R = \{v \in V | a(v) \leqslant R^2\}, \tag{4.13}$$

$$\gamma_R = \sup_{S_R} I(v). \tag{4.14}$$

We have

THEOREM 4.2. *Under hypotheses* (I)–(III), γ_R *is attained for each* R *such that* $S_R \neq \varnothing$. *If* $S_m \neq \varnothing$, $R > m$ *and*

$$\alpha > \inf_{m \leqslant t \leqslant R} \frac{\gamma_R - \gamma_t}{R^2 - t^2}, \tag{4.15}$$

then (4.6) *has a solution in* S_R. *It is a stationary point of* (4.4).

Note that our hypotheses do not restrict $g(v, r)$ from below and that its restrictions from above are very mild. Theorem 4.2 will be proved in the next section. Now we consider some examples. In them we take

$$a(v) = \|\dot{v}\|^2 + 2\|v\|^2 \tag{4.16}$$

and make use of the inequality

$$|v(t)|^2 \leqslant \int_I \{|\dot{v}(r)|^2 + 2|v(r)|^2\} \, dr, \qquad t \in I, \tag{4.17}$$

which holds for any interval I of length one (cf. [**8**, p. 19]). First we consider

$$g(v, r) = w(r) e^v - h(v) \tag{4.18}$$

where

$$h(v),\ h'(v) \text{ are continuous, } h(v) \geqslant 0,\ h(0) = 0, \tag{4.19}$$

and

$$c_1 = \int_\Omega w(r) \, dr < \infty, \qquad w(r) \geqslant 0. \tag{4.20}$$

In this case $\gamma_R \leqslant c_1 e^R$ and $\gamma_0 = c_1$. Note that S_0 is not empty. Thus if

$$\alpha > \inf_{R \geqslant 0} \frac{\gamma_R - \gamma_0}{R^2}$$

we can apply Theorem 4.2 to conclude that (4.6) has a solution. Since

$$\min_{R \geqslant 0} \frac{e^R - 1}{R^2} < 1.5441383$$

we see that (4.6) has a solution if

$$\alpha \geqslant (1.5441383) c_1.$$

Moreover, the solution satisfies $|v| \leqslant 1.5936243$. Next consider

$$g(v, r) = c_1 v^2 e^v + v s(r) - h(v) \tag{4.21}$$

where $\int_\Omega |s(r)|^p \, dr = c_1^p < \infty$ for some p such that $1 \leqslant p \leqslant 2$ and h satisfies (4.19). Then

$$R^{-2}\gamma_R \leqslant c_1(e^R + R^{-1}), \qquad \gamma_0 = 0.$$

Again S_0 is not empty. Since

$$\min_{R \geqslant 0} (e^R + R^{-1}) < 3.442277$$

we see that (4.6) has a solution for $\alpha \geqslant (3.442277)c_1$ and the solution satisfies $|v| \leqslant 0.7034674$. Next let

$$g(v, r) = c_1 v^2 e^v - w(r)h(v) \tag{4.22}$$

where h, w satisfy (4.19), (4.20) with the exception that $h(0) = 1$. In this case

$$\gamma_R \leqslant c_1 R^2 e^R, \qquad \gamma_0 = -c_1.$$

Again S_0 is not empty. Thus

$$R^{-2}(\gamma_R - \gamma_0) \leqslant c_1(e^R + R^{-2}).$$

Since

$$\min_{R \geqslant 0} (e^R + R^{-2}) < 3.6906033$$

we know that (4.6) has a solution for $\alpha \geqslant (3.6906033)c_1$ and the solution satisfies $|v| \leqslant 0.92547893$.

In contrast with (4.21), let

$$g(v, r) = \left(cv^2 - |v|^{p+2}\right)e^v + vs(r) - h(v) \tag{4.23}$$

where $p > 0$, h satisfies (4.19) and $s(r)$ is as before. In this case the first term will be positive only if $|v|^p \leqslant c$. Thus we have

$$\gamma_R \leqslant cR^2 e^{c^{1/p}} + c_1 R, \qquad \gamma_0 = 0.$$

Thus

$$R^{-2}\gamma_R \leqslant ce^{c^{1/p}} + c_1 R^{-1} \to ce^{c^{1/p}} \quad \text{as } R \to \infty.$$

This implies that (4.6) has a solution for all $\alpha > \exp\{c^{1/p}\}$. The same conclusion holds when

$$g(v, r) = \left(cv^2 - |v|^{p+2}\right)e^v - w(r)h(v) \tag{4.24}$$

where h, w are as in (4.22). Next consider the case when

$$g(v, r) = cv(e^v - 1) - w(r)h(v) \tag{4.25}$$

where w satisfies (4.20) and $h(v) \geqslant h(0) \geqslant 0$. Thus

$$\gamma_R \leqslant cR(e^R - 1) + \gamma_0.$$

To obtain this we use the fact that $(e^R - 1)/R$ is an increasing function in R. Since

$$R^{-2}(\gamma_R - \gamma_0) \leqslant c(e^R - 1)/R$$

we see that (4.6) has a solution for $\alpha > c$. Next consider

$$g(v,r) = w(r)\cos v - h(v) \tag{4.26}$$

where h, w satisfy (4.19), (4.20). Now we have $\gamma_R \leqslant c_1$, $\gamma_0 = c_1$. Thus $R^{-2}(\gamma_R - \gamma_0) \leqslant 0$ and consequently (4.6) has a solution for any $\alpha > 0$. Finally consider

$$g(v,r) = cv^p + vs(r) - h(v)$$

where $p > 2$, h satisfies (4.19) and $s(r)$ is as before. Here we have

$$\gamma_R \leqslant cR^p + c_1 R, \qquad \gamma_0 = 0,$$

and

$$R^{-2}\gamma_R \leqslant cR^{p-2} + c_1 R^{-1}.$$

The minimum of the right-hand side is

$$c^{1(p-1)} c_1^{(p-2)/(p-1)}\left[(p-2)^{-(p-2)/(p-1)} + (p-2)^{1/(p-1)}\right].$$

Thus (4.6) has a solution for α greater than this value. The solution satisfies $|v|^{p-1} \leqslant [c_1/c(p-2)]$.

5. Finding stationary points. In this section we show how Theorem 4.2 is proved. First we give a few lemmas.

LEMMA 5.1. *If $a(v)$ is given by* (4.16) *and $a(v) \leqslant R^2$, then*

$$\int_\Omega w(r)v(r)^2 h(v(r))\,dr \leqslant R^2 M(w) \max_{v^2 \leqslant R^2} |h(v)| \tag{5.1}$$

and if $0 \leqslant s < 2$ and $1 \leqslant p \leqslant 2/(2-s)$, then

$$\int_\Omega w(r)|v(r)|^2 h(v(r))\,dr \leqslant R^s \max_{v^2 \leqslant R^2} |h(v)| \left(\int |w(r)|^p\,dr\right)^{1/p} \tag{5.2}$$

where

$$M(w) = \sup_x \int_{(x,x+1)\cap\Omega} |w(r)|\,dr. \tag{5.3}$$

PROOF. Let I be any interval of length one contained in Ω. By (4.17)

$$\int_I w(r)v^2 h(v)\,dr \leqslant M(w) \max_{v^2 \leqslant R^2} |h(v)| \int_I \left\{|\dot v(r)|^2 + 2|v(r)|^2\right\} dr.$$

Summing over a denumerable set of such intervals covering Ω, we obtain (5.1). To prove (5.2) we use Hölder's inequality and note that (4.17) implies that the L^p norm of v is $\leqslant R$ for $p \geqslant 2$. □

LEMMA 5.2. *Under the same hypothesis, suppose $a(v) \leqslant R^2$, $a(v_j) \leqslant R^2$ and v_j converges to v a.e. Then*

$$\int_\Omega B(v_j(r),r)\,dr \to \int_\Omega B(v(r),r)\,dr. \tag{5.4}$$

PROOF. Let $\varepsilon > 0$ be given, and take N so large that the left-hand sides of (4.11) are less than ε for $|x| > N - 1$. Put

$$J = \max_k \sum_{v^2 \leqslant R^2} |m_k(v)|.$$

Then by (5.1)

$$\int_{\Omega \setminus [-N,N]} |y_k(r)|v_j^2|m_k(v_j)|\,dr \leqslant R^2 J\varepsilon$$

with the same estimate for v. On the other hand

$$\int_{\Omega \cap [-N,N]} y_k(r)v_j(r)^2 m_k(v_j)\,dr \to \int_{\Omega \cap [-N,N]} y_k(r)v(r)^2 m_k(v)\,dr$$

since the integrand converges a.e. and is majorized by $R^2J|y_k(r)|$ which is integrable in $\Omega \cap [-N, N]$. This proves the lemma for $B_1(v, r)$. To prove the other part, put

$$J = \max_k \max_{v^2 \leqslant R^2} |h_k(v)|$$

and let $\varepsilon > 0$ be given and let N be so large that

$$\int_{\Omega \setminus [-N,N]} |w_k(r)|\,|v_j|^{s_k}|h_k(v_j)|\,dr \leqslant JR^2\left(\int_{\Omega \setminus [-N,N]} |w_k(r)|^{p_k}\,dr\right)^{1/p_k} < \varepsilon.$$

Since

$$\int_{\Omega \cap [-N,N]} w_k(r)|v_j|^{s_k} h_k(v_j)\,dr \to \int_{\Omega \cap [-N,N]} w_k(r)|v|^{s_k} h_k(v)\,dr,$$

the result follows. □

LEMMA 5.3. *If S_R is not empty for some $R > 0$, then there is a $t < R$ such that S_t is not empty.*

PROOF. Suppose $v \in V$ and $a(v) = R^2$. Since $v \neq 0$, there must be a q in Q such that $a(v, q) \neq 0$. By changing the sign of q we may arrange it such that $a(v, q) > 0$. Thus we have

$$a(v - tq) = a(v) - 2ta(v, q) + t^2 a(q).$$

By taking t sufficiently small we can make this less than R^2. Since $v - tq$ is in V, the lemma follows. □

LEMMA 5.4. *For $\alpha \geqslant 0$, $G(v)$ given by (4.4) has a minimum $\rho_{\alpha,R}$ on S_R.*

PROOF. By hypothesis (II) and Lemma 5.1, $G(v)$ is bounded from below on S_R. Put

$$\rho_{\alpha,R} = \inf_{S_R} G(v) \tag{5.5}$$

and let $\{v_j\} \subset S_R$ be a minimizing sequence. Since W is a Hilbert space, we can extract a weakly convergent subsequence. Moreover, since the sequence is uniformly bounded and equicontinuous, we can find a subsequence (also denoted by $\{v_j\}$)

which converges to the weak limit v uniformly on every bounded interval in Ω. Since

$$a(v, v_j) \leqslant a(v - v_j, v) + a(v_j),$$

we have

$$(5.6) \qquad a(v) \leqslant \liminf a(v_j) \leqslant R^2.$$

Thus (5.4) holds by Lemma 5.2. Since $B(v, r) - g(v, r) \geqslant 0$, we have by Fatou's lemma that

$$(5.7) \qquad \int_\Omega [B(v,r) - g(v,r)]\,dr \leqslant \liminf \int_\Omega [B(v_j,r) - g(v_j,r)]\,dr.$$

In particular, this shows that $g(v, r)$ is in L^1. Consequently v is in S_R. Combining (5.4), (5.6) and (5.7) we find $G(v) \leqslant \rho_{\alpha,R}$. This gives the desired result. □

COROLLARY 5.5 $\gamma_R = -\rho_{0,R}$.

Now we can give the

PROOF OF THEOREM 4.2. The first statement follows from Corollary 5.5. To prove the second, suppose there is no stationary point of (4.4) in the interior of S_R. This implies that for each t such that $m \leqslant t \leqslant R$, $\rho_{\alpha,t}$ is attained only on the boundary of S_t. For if it were attained for some v such that $a(v) < t^2$, we would have $v + q$ in S_t for $a(q)$ small. Thus v would be a stationary point of G. Hence we may assume that $G(v) = \rho_{\alpha,t}$, $v \in S_t$ implies $a(v) = t^2$. This also tells us that $I(v) = \gamma_t$. For if there were a w in S_t such that $I(w) > I(v)$, then we would have

$$G(w) = \alpha a(w) - I(w) < \alpha t^2 - I(v) = G(v),$$

contradicting the fact that $G(v) = \rho_{\alpha,t}$. Hence

$$(5.8) \qquad \rho_{\alpha,t} = G(v) = \alpha a(v) - I(v) = \alpha t^2 - \gamma_t.$$

The identity (5.8) must hold for every t such that $m \leqslant t \leqslant R$. Consequently,

$$\alpha t^2 - \gamma_t = \rho_{\alpha,t} \geqslant \rho_{\alpha,R} = \alpha R^2 - \gamma_R$$

or

$$\alpha(R^2 - t^2) \leqslant \gamma_R - \gamma_t.$$

This means that α is not greater than the right-hand side of (4.15). Consequently, if α is greater than this value there must be a value of t in the given interval for which $\rho_{\alpha,t}$ is achieved at an interior point of S_t. Such a point is a solution of (4.6). □

As another application, let G be given as in §1 with $Y = \mathbf{R}$ and $V = Q$ continuously embedded in X, a Hilbert space. Assume

(5.9) if $G(v_k) \leqslant C$, then $\{v_k\}$ has a subsequence converging in X, and

(5.10) if $G(v_k) \to \beta$ and $v_k \to v$ in X, then v is in Q and $G(v) = \beta$.

We shall prove

THEOREM 5.6. *Under hypotheses* (5.9) *and* (5.10) G'_Q *has a nonnegative eigenvalue. This means that there is a* $u \neq 0$ *in* Q *and a* $\lambda \geqslant 0$ *such that*

$$(5.11) \qquad G'_Q(u)q = \lambda(u, q), \qquad q \in Q.$$

PROOF. Pick $R > 0$ and put $F(v) = \|v\|^2 - R^2$, where the norm is that of X. Set

$$S = \{v \in Q \mid F(v) \geqslant 0\}, \qquad \mu = \inf_S G(v),$$

and let $\{v_k\} \subset S$ be such that $G(v_k) \to \mu$. Since $G(v_k) \leqslant C$, there is a subsequence (also denoted by $\{v_k\}$) converging to some v in X. This implies that v is in S and that $G(v) = \mu$. Note that $F_Q'(v)q = 2(v, q)$ and that its range is the whole of $\mathbf{R}$. Thus we may conclude that $G_Q'(v)q \geqslant 0$ whenever $(v, q) \geqslant 0$ (Theorem 3.8). But every q can be written in the form $q = tv + w$, where $(v, w) = 0$. Thus $G_Q'(v)w = 0$ and

$$G_Q'(v)q = tG_Q'(v)v = \frac{G_Q'(v)v}{\|v\|^2}(v, q) = \lambda(v, q). \quad \square$$

6. Some proofs. Now we turn to the proofs of the theorems of §§2, 3. First we prove the following result which is of interest in its own right.

THEOREM 6.1. *Let T be a closed operator from a Banach space N to a Banach space Y satisfying* (2.5). *Suppose $h_0 \in D(T)$ and f is a mapping of the ball $B = \{h \in N \mid \|h - h_0\| < m\}$ into Y such that*

$$\|f(h) - f(h')\| \leqslant \delta\|h - h'\|, \qquad h, h' \in B. \tag{6.1}$$

Assume that $R(T) = Y$ and $\delta C_0 < 1$. Let h_1 be any element of $D(T) \cap B$, and put $y_1 = Th_1 - f(h_1)$. Then for any $\varepsilon > 0$ and any $y \in Y$ such that

$$(C_0/(1 - \delta C_0))\|y - y_1\| + \|h_1 - h_0\| < m \tag{6.2}$$

there exists an $h \in D(T) \cap B$ such that

$$Th - f(h) = y$$

and

$$\|h - h_1\| \leqslant (C_0/(1 - \delta C_0) + \varepsilon)\|y - y_1\|. \tag{6.3}$$

PROOF. Let $\varepsilon > 0$ be given and take $C_1 > C_0$ such that (6.2) holds with C_0 replaced by C_1,

$$\frac{C_1}{1 - \delta C_1} < \frac{C_0}{1 - \delta C_0} + \varepsilon,$$

and $\rho = \delta C_1 < 1$. From (2.5) and the fact that $R(T) = Y$ we can find an element z_1 in $D(T)$ such that $Tz_1 = y - y_1$, $\|z_1\| \leqslant C_1\|y - y_1\|$. Define $\{h_k\}$, $\{z_k\}$ inductively by

$$h_{k+1} = h_1 + \sum_{j=1}^{k} z_j, \qquad Tz_k = f(h_k) - f(h_{k-1}), \qquad z_k = h_{k+1} - h_k,$$

$$\|z_k\| \leqslant C_1\|Tz_k\| = C_1\|f(h_k) - f(h_{k-1})\| \leqslant \rho\|z_{k-1}\|.$$

Thus

$$\|z_k\| \leqslant \rho^{k-1}\|z_1\|.$$

Since $\rho < 1$, we see that $h_k \to h$ in N. Moreover,

$$\|h - h_0\| \leqslant \|h_1 - h_0\| + \sum_{j=1}^{\infty} \|z_j\| \leqslant \|h_1 - h_0\| + \sum_{j=1}^{\infty} \rho^{j-1}\|z_1\|$$
$$\leqslant \|h_1 - h_0\| + \frac{C_1}{1-\rho}\|y - y_1\| < m.$$

Thus $h \in B$. Also

$$\|h - h_1\| \leqslant \frac{C_1}{1-\rho}\|y - y_1\|$$

showing that (6.3) holds. Finally, we note that

$$Th_{k+1} = Th_1 + \sum_{j=1}^{k} Tz_j = Th_1 + f(h_k) - f(h_1) + y - y_1$$
$$= f(h_k) + y \to f(h) + y \quad \text{as } k \to \infty.$$

Since T is a closed operator, we see that $Th - f(h) = y$ holds. □

Now we turn to the

Proof of Theorem 2.1. First we note that $q \in C(V, Q, u + sq)$ for $0 < s < 1$. For if $t_n \to 0$, then $u + sq + t_n q$ is in V for n large. If we put $q_n = q$, then (1.2) holds for $u + sq$. Put $f(s) = y'G(u + sq)$. Then for $0 < s < 1$, we have

$$t_n^{-1}[f(s + t_n) - f(s)] = t_n^{-1}[G(u + sq + t_n q) - G(u)]$$
$$\to G'_Q(u + sq)q \quad \text{as } t_n \to 0.$$

Thus $f(s)$ is differentiable in $(0,1)$. By (2.1) it is continuous in $[0,1]$. Thus there is a θ such that $0 < \theta < 1$ and $f(1) - f(0) = f'(\theta)$. This gives (2.2). □

Proof of Theorem 2.2. Note that $G'_Q(u)$ maps $C(V, Q, u)$ into $C(D, R, y)$. To see this let q be an element in $C(V, Q, u)$. Then there are sequences $\{q_n\}$, $\{t_n\}$ such that (1.2) holds. Therefore

$$r_n = t_n^{-1}[G(u + t_n q_n) - y] \to G'_Q(u)q \quad \text{in } Y. \tag{6.4}$$

Since r_n is in R and R is closed, this convergence is in R. Moreover,

$$y + t_n r_n = G(u + t_n q_n) \in D.$$

This shows that $G'_Q(u)q$ is in $C(D, R, y)$. We also have

$$t_n^{-1}[F(G(u + t_n q_n)) - F(G(u))] = t_n^{-1}[F(y + t_n r_n) - F(y)]$$
$$\to F'_R(y)G'_Q(u)q \quad \text{in } Y.$$

Thus

$$H'_Q(u)q = F'_R(y)G'_Q(u)q. \tag{6.5}$$

Finally, we note that $G'_Q(u)$ maps $E(V, Q, u)$ into $E(D, R, y)$. For if $q = \sum q_k$, where $q_k \in C(V, Q, u)$, then

$$G'_Q(u)q = \sum G'_Q(u)q_k \in E(D, R, y). \quad □$$

PROOF OF THEOREM 2.3. Put

$$f_t(h) = Th - F(v) + L(t), \qquad h \in N. \tag{6.6}$$

It maps the ball $\|h\| < m$ into Y. By (2.4)

$$\| f_t(h_1) - f_t(h_2) \| \leqslant \delta \| h_1 - h_2 \|, \qquad \| h_i \| < m, \qquad 0 < t < m.$$

We apply Theorem 6.1 with $h_0 = 0$, $h_1 = th$, $y = 0$, $f = f_t$. Then $\|y_1\| \to 0$ as $t \to 0$, where

$$y_1 = F(u + tq(t) + th) - L(t). \tag{6.7}$$

By that theorem we can find an element h_t in N such that $Th_t - f_t(h_t) = 0$ and

$$\| h_t - th \| \leqslant C \| y_1 \|.$$

If we now put $h(t) = t^{-1} h_t$, this gives (2.6) and (2.7). □

PROOF OF COROLLARY 2.4. Note that q is in $C(V, Q, u)$. Take $L(t) = F(u)$, $h = 0$ in Theorem 2.3. □

PROOF OF THEOREM 2.5. Suppose q is in $C(V, W, u)$. Then there are sequences $\{q_n\} \subset W$, $\{t_n\} \subset \mathbf{R}$ such that

$$q_n \to q \quad \text{in } W, \qquad 0 \neq t_n \to 0, \qquad u + t_n q_n \in V. \tag{6.8}$$

By hypothesis this implies (1.2). Thus $q \in C(V, Q, u)$ and (1.3) holds. This says that $G'_W(u)q$ exists and equals $G'_Q(u)q$. If q is in $E(V, W, u)$, then it is the sum of elements of $C(V, W, u)$, and the result follows. □

PROOF OF THEOREM 3.6. If $q \in C_+(S, Q, u)$, then there are sequences $\{q_n\}$, $\{t_n\}$ such that (3.1) holds and

$$y'F(u + t_n q_n) \geqslant 0, \qquad y' \in B. \tag{6.9}$$

If $y' \in B_u$, then $y'F(u) = 0$. Thus $y'F'_{+Q}(u)q \geqslant 0$. □

PROOF OF THEOREM 3.7. Suppose q is contained in the set on the right in (3.5). Remove from B_u all y' for which $y'F'_{+Q}(u)q > 0$. Clearly no such functional can have a negative multiple in B. For if it did, we would have $y'F(u) \geqslant 0$ and $-y'F(u) \geqslant 0$ and consequently $y'F(u) = 0$. Thus $-y'$ would also be in B_u and we would have $y'F'_{+Q}(u) > 0$. Thus we have

$$y'F'_{+Q}(u)q = 0, \qquad y' \in B_u. \tag{6.10}$$

By (3.5) this implies that $F'_{+Q}(u)q$ is in R. Thus $PF'_{+Q}(u)q = 0$. By Corollary 2.4 there is a mapping $h(t)$ of $(0, m)$ into N such that

$$PF(u + tq(t) + th(t)) = PF(u), \qquad 0 < t < m, \tag{6.11}$$

$q(t) \to q$ in Q and $h(t) \to 0$ in N. Thus $g_1(t) = q(t) + h(t) \to q$ in Q. But (6.11) is the same as

$$y'F(u + tq_1(t)) = y'F(u) = 0, \qquad y' \in B_u. \tag{6.12}$$

Next, let us examine those y' that were removed from B_u. For them we have

$$y'F(u) = 0, \qquad y'F'_{+Q}(u)q > 0. \tag{6.13}$$

Thus

$$y'F(u+tq_1(t)) = ty'F'_{+Q}(u)q + o(t) > 0 \tag{6.14}$$

for t sufficiently small. Finally, we examine those $y' \in B$ that were never placed in B_u. For them we have $y'F(u) > 0$. Thus

$$y'F(u+tq_1(t)) = y'F(u) + ty'F'_{+Q}(u)q + o(t) > 0 \tag{6.15}$$

for t small. If we combine (6.12), (6.14) and (6.15), we have

$$y'F(u+tq_1(t)) \geqslant 0, \qquad y' \in B. \tag{6.16}$$

This shows that $q \in C_+(S, Q, u)$. □

THEOREM 6.2. *Let N, Y be Banach spaces, and suppose T, H in $B(N,Y)$ are such that* (2.5) *holds and $\|T-H\| \leqslant \delta$ with $\delta C_0 < 1$. If $R(T) = Y$, then $R(H) = Y$.*

PROOF. Let y be any element of Y, and let $C_1 > C_0$ be such that $\rho = \delta C_1 < 1$. Then there is an element $z_0 \in N$ such that

$$Tz_0 = y, \qquad \|z_0\| \leqslant C_1\|y\|$$

and inductively, there are elements $z_k \in N$ such that

$$Tz_k = (T-H)z_{k-1}, \qquad \|z_k\| \leqslant C_1\|Tz_k\|, \qquad k = 1,2,\ldots.$$

Then

$$\|z_k\| \leqslant C_1\|(T-H)z_{k-1}\| \leqslant \rho\|z_{k-1}\| \leqslant \rho^k\|z_0\|.$$

Thus

$$h_k = \sum_0^k z_j \to h \quad \text{in } N$$

and

$$\begin{aligned} Th_k &= \sum_0^k Tz_j = \sum_1^k (T-H)z_{j-1} + y = (T-H)\sum_0^{k-1} z_j + y \\ &= (T-H)h_{k-1} + y \to (T-H)h + y. \end{aligned}$$

Thus $TH = (T-H)h + y$ and $Hh = y$. Thus $R(H) = Y$. □

THEOREM 6.3. *Under the hypotheses of Theorem* 2.3 *assume that N is continuously imbedded in Q and that $F'_{+Q}(u)$ exists. Then $R(F'_{+N}(u)) = Y$. If $q(t) \to q$ in Q and h is any element on N such that*

$$F'_{+N}(u)h = -F'_{+Q}(u)q, \tag{6.17}$$

then there is a mapping $h(t)$ of $(0, m)$ into N such that (2.8) *holds and*

$$h(t) \to h \quad \textit{in } N \quad \textit{as } t \to 0. \tag{6.18}$$

PROOF. By (2.4) for any $h \in N$, $t > 0$,

$$\|tTh - F(u+tq(t)+th) + F(u+tq(t))\| \leqslant \delta t\|h\|$$

and consequently,

$$\|Th - t^{-1}[F(u + tz(t) + th) - F(u)] + t^{-1}[F(u + tq(t)) - F(u)]\| \leqslant \delta\|h\|.$$

Letting $t \to 0$, we have

$$\|Th - F'_{+Q}(u)(q + h) + F'_{+Q}(u)q\| \leqslant \delta\|h\|.$$

It now follows from Theorem 6.2 that $R(F'_{+N}(u)) = Y$. Next note that q is in $C_+(V, Q, u)$. Take $L(t) = F(u)$ in Theorem 2.3. Then there is a mapping $h(t)$ of $(0, m)$ into N such that (2.6) and (2.7) hold. But

$$t^{-1}[F(u + tq(t) + th) - F(u)] \to F'_{+Q}(u)(q + h)$$

and this vanishes by (6.17). Thus (6.18) holds. □

PROOF OF THEOREM 3.8. If (3.8) holds, then $q \in C_+(S, Q, u)$ by Theorem 3.7. Thus there are sequences $\{q_n\}$, $\{t_n\}$ satisfying (3.1) such that $u + t_n q_n$ is in S. Thus

$$t_n^{-1}[G(u + t_n q_n) - G(u)] \geqslant 0 \quad \text{for } t > 0.$$

This gives (3.7). □

PROOF OF THEOREM 3.9. That H is onto follows from Theorem 6.3. Let q be any element of $C_+(V, Q, u)$ and let h be any element of N such that

$$Hh = -PF'_{+Q}(u)q. \tag{6.19}$$

Then $q_1 = q + h$ is in $C_+(V, Q, u)$ and

$$PF'_{+Q}(u)q_1 = 0. \tag{6.20}$$

Thus

$$F'_{+Q}(u)q_1 = 0, \qquad y' \in B_u. \tag{6.21}$$

If we now apply Theorem 3.8, we see that

$$G'_{+Q}(u)q_1 \geqslant 0. \tag{6.22}$$

This is precisely (3.9) □

PROOF OF THEOREM 2.6. Put $B = Y'$ in Theorem 3.6. Then $B_u = Y'$ as well. Thus

$$C_+(S, Q, u) \subset \{q \in C_+(V, Q, u) \mid F'_Q(u)q = 0\}. \tag{6.23}$$

If we now apply Lemma 3.3, we see that (6.23) implies (2.11). Next assume $q \in C_+(V, Q, u)$ and that F satisfies the hypotheses of Corollary 2.4. We apply Theorem 3.7. In our case $R = 0$ and $M = Y$ so that (3.5) becomes

$$C_+(S, Q, u) = \{q \in C_+(V, Q, u) \mid F'_Q(u)q = 0\}. \tag{6.24}$$

This and our hypotheses imply (2.12). □

PROOF OF THEOREM 2.7. By Theorem 2.6, the set of q described are in $C(S, Q, u)$. For such q (2.13) implies (2.14). □

PROOF OF THEOREM 2.9. If $R(F'_Q(u)) \neq Y$, then there is a $y' \in Y'$ such that $y' \neq 0$ and $y'F'_Q(u) = 0$. Thus we may take $\lambda = 0$ in (2.18). On the other hand, if $G'_Q(u)$ is onto, we may apply Theorem 2.8. All of the hypotheses of Theorem 2.3 are therefore satisfied, including (2.4). In fact we can take $T = F'_Q(u)$, $N = Q$. Then by Theorem 2.1

$$y'[T(q_1 - q_2) - F(u + q_1) + f(u + q_2)] = y'[T - F'_Q(u + q_\theta)](q_1 - q_2)$$

for any $y' \in Y'$. This implies (2.4) for the q_i sufficiently small. Note that the operator $G'_Q(u)H^{-1}$ is well defined and is in Y'. For if $Hq = 0$, then q satisfies (2.15) and consequently (2.14) holds. We now see that (2.16) implies (2.18) with $\lambda = 1$ and $y' = G'_Q(u)H^{-1}$. □

PROOF OF THEOREM 3.10. The proof is similar to that of Theorem 2.9. If $PF'_Q(u)$ is not onto M, then there is a nonzero $y' \in M'$ which annihilates it. Otherwise we apply Theorem 3.9. □

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