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REGULARIZATION FOR nTH-ORDER LINEAR BOUNDARY VALUE PROBLEMS USING mTH-ORDER DIFFERENTIAL OPERATORS

BY

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ABSTRACT. Let X and Y denote real Hilbert spaces, and let $L: X \to Y$ be a closed densely-defined linear operator having closed range. Given an element $y \in Y$, we determine least squares solutions of the linear equation $Lx = y$ by using the method of regularization.

Let Z be a third Hilbert space, and let $T: X \to Z$ be a linear operator with $\mathscr{D}(L) \subseteq \mathscr{D}(T)$. Under suitable conditions on L and T and for each $\alpha \neq 0$, we show that there exists a unique element $x_\alpha \in \mathscr{D}(L)$ which minimizes the functional $G_\alpha(x) = \|Lx - y\|^2 + \alpha^2\|Tx\|^2$, and the x_α converge to a least squares solution x_0 of $Lx = y$ as $\alpha \to 0$.

We apply our results to the special case where L is an nth-order differential operator in $X = L^2[a, b]$, and we regularize using for T an mth-order differential operator in $L^2[a, b]$ with $m \leqslant n$. Using an approximating space of Hermite splines, we construct numerical solutions to $Lx = y$ by the method of continuous least squares and the method of discrete least squares.

I. Introduction and general theory.

(A) *Introduction*. The method of regularization we introduced by Phillips [**24**] and Tikhonov [**28**, **29**] as a means of overcoming instabilities of numerical computations for first kind integral equations. More recently, Locker and Prenter [**19**, **20**] have examined regularization in a general Hilbert space setting, applying it to first kind integral equations while regularizing with differential operators. Other work of Locker and Prenter [**21**] involves regularizing ill-posed two-point linear boundary value problems using the identity operator.

Throughout this paper we let X and Y denote real Hilbert spaces with inner products $(\ ,\ )$ and norms $\|\ \|$, and we let $L: X \to Y$ be a closed densely-defined linear operator having closed range. Given an element $y \in Y$, we want to determine least squares solutions of the linear equation

$$(1.1) \qquad Lx = y$$

by using the method of regularization.

To introduce regularization, let Z be a third Hilbert space and let $T: X \to Z$ be a linear operator with $\mathscr{D}(L) \subseteq \mathscr{D}(T)$. For each real number $\alpha \neq 0$ let G_α be the functional defined on $\mathscr{D}(L)$ by

$$G_\alpha(x) = \|Lx - y\|^2 + \alpha^2\|Tx\|^2.$$

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Under suitable conditions on L and T there exists a unique element $x_\alpha \in \mathscr{D}(L)$ satisfying

$$(1.2) \qquad G_\alpha(x_\alpha) = \inf G_\alpha(x), \qquad x \in \mathscr{D}(L),$$

and the x_α converge to a least squares solution of (1.1) as $\alpha \to 0$.

In the following exposition many well-known results regarding regularization are given. Most are included in §I, where we establish the general theory of regularization. In §II we study regularizing linear boundary value problems using similar differential operators, and in §III we examine regularization in a discrete setting. These two sections represent the principal contributions of this paper.

§I follows the work of Locker and Prenter [**19**, **21**] by making appropriate assumptions on L and T, the regularizing operator, and by viewing regularization as a least squares process, an idea introduced and outlined by Nashed [**23**]. For each real number $\alpha \neq 0$ we establish the existence and uniqueness of the regularized approximate solution x_α, and show that the x_α converge to a least squares solution of (1.1) as $\alpha \to 0$. The technique employed to establish error estimates is the *alternative method*, surveys of which have been given by Cesari [**4**, **5**] and Hale [**10**]. The special form of the alternative method used in this paper was developed by Kannan and Locker [**13**]. The rate of convergence will be shown to be of order α^2, which is in contrast to the rate of convergence of order $|\alpha|$ obtained by Ivanov [**12**] with $T = I$, the identity operator.

In §§II and III we let L be an nth-order differential operator in $X = L^2[a, b]$, and we regularize using for T an mth-order differential operator in $L^2[a, b]$ with $m \leqslant n$. Using an approximating space of Hermite splines, we construct numerical solutions to (1.1) by the method of continuous least squares in §II, and by the method of discrete least squares in §III.

In both sections we establish error estimates by first showing superconvergence at the knots of the partition associated with the spline approximating space. This is known to occur with Galerkin approximates to particular two-point boundary value problems [**9**]. Developments in both sections extend work of Sammon [**26**] and Locker and Prenter [**18**] in a well-posed setting. The discrete least squares analysis in [**18**] is an alternate analysis of the method of collocation applied to well-posed ordinary differential equations, which was done by deBoor and Swartz [**8**]. Ascher [**1**] has studied discrete least squares together with Gaussian quadrature in a well-posed setting. The analysis is relatively complicated, but it allows for a more general spline approximating space than the Hermite splines used here.

An outline of §II follows: Part (A) describes the mathematical setting and introduces $\tilde{x}_\alpha$, the (continuous) *least squares approximate* to x_α which is selected from a spline subspace. In (F) we establish the convergence of $\tilde{x}_\alpha$ to $x_0 = L^\mu y$, a particular least squares solution of (1.1).

To develop the error estimates, we introduce in (B) the generalized inverse L^+ and list properties of the associated generalized Green's function $G(t, s)$, which has been characterized by Locker in [**16**, **17**]. This information along with appropriate adjoints of L and $A = T|\mathscr{D}(L)$, which are examined in (C), is used to establish smoothness and uniform boundedness of the x_α in (D). In (E) we examine the linear operator $T_\alpha x = (Lx, \alpha Ax)$, explicitly characterizing the local and global smoothness of the associated representors for its generalized inverse. This parallels the representor theory of Locker and Prenter [**19**, **22**].

Because of the exact integration involved in the computation of $\tilde{x}_\alpha$, the least squares approximate to x_α, the method in §II is in general impractical. This is readily overcome by the implementation of discrete least squares and numerical quadrature in §III.

We begin in (A) of the third section by introducing $\hat{x}_\alpha$, the *discrete least squares approximate* to x_α, proving its existence and uniqueness in (B) and (C). In (D) we introduce $\overset{\triangle}{x}_\alpha$, the *quasi-discrete least squares approximate* to x_α, using it to establish error estimates for $\hat{x}_\alpha - x_0$ in (E).

(B) *The method of regularization.* To analyze the method of regularization, it is necessary to work with $\mathscr{D}(L)$ under two different structures. The first is the graph norm structure of L:

$$(x, y)_L = (x, y) + (Lx, Ly), \qquad \|x\|_L = (x, x)_L^{1/2}.$$

Because L is a closed operator, it follows that $\mathscr{D}(L)$ is a Hilbert space under this structure. We now make the following assumptions on L and T:

(I) $\mathscr{N}(L) \cap \mathscr{N}(T) = \{0\}$.

(II) The restriction $A = T|\mathscr{D}(L)$ is continuous from $\mathscr{D}(L)$ under the graph norm structure of L into Z under its standard structure.

(III) There exists a constant $\beta > 0$ such that $\|Tx\| \geqslant \beta\|x\|$ for all $x \in \mathscr{N}(L)$.

The second structure on $\mathscr{D}(L)$ is the $*$*-structure*:

$$(x, y)_* = (Lx, Ly) + (Ax, Ay), \qquad \|x\|_* = (x, x)_*^{1/2}.$$

From [**19**, Lemma 3.3, p. 509] it can be shown that $(\ ,\ )_*$ is an inner product on $\mathscr{D}(L)$, $\mathscr{D}(L)$ is a Hilbert space under the inner product $(\ ,\ )_*$ and $\|\ \|_L$ and $\|\ \|_*$ are equivalent norms on $\mathscr{D}(L)$. It follows that both L and A are bounded linear operators from $\mathscr{D}(L)$ under the $*$-structure into Y and Z under their standard structures, resp. Thus, there exist bounded everywhere-defined adjoint operators

$$L^\#: Y \to \mathscr{D}(L) \quad \text{and} \quad A^\#: Z \to \mathscr{D}(L)$$

given by the equations

$$(Lx, y) = (x, L^\# y)_* \quad \text{and} \quad (Ax, z) = (x, A^\# z)_*$$

for all $x \in \mathscr{D}(L)$, $y \in Y$, and $z \in Z$.

We get the main existence and uniqueness theorem for the method of regularization from [**19**, Theorem 3.1, p. 508].

THEOREM 1.1. *For each $y \in Y$ and each $\alpha \neq 0$, the equation*

$$L^\# Lx + \alpha^2 A^\# Ax = L^\# y$$

has a unique solution $x_\alpha \in \mathscr{D}(L)$. Moreover, the element x_α is also characterized as the unique element in $\mathscr{D}(L)$ satisfying $G_\alpha(x_\alpha) = \inf_{x \in \mathscr{D}(L)} G_\alpha(x)$.

It is possible to view regularization as a least squares process, an approach introduced by Nashed [23]. Let us use the standard product structure on $Y \oplus Z$:

$$\langle (u,v),(\xi,\eta)\rangle = (u,\xi) + (v,\eta), \qquad |(u,v)| = \left[\|u\|^2 + \|v\|^2\right]^{1/2}$$

for (u, v), $(\xi, \eta) \in Y \oplus Z$. For each real $\alpha \neq 0$ let T_α be the linear operator from $\mathscr{D}(L)$ into $Y \oplus Z$ defined by $T_\alpha x = (Lx, \alpha Ax)$. From assumption (I) we see that T_α is a 1-1 mapping, and clearly T_α is bounded from $\mathscr{D}(L)$ under the $*$-structure into $Y \oplus Z$ under the product structure. Thus, T_α has a bounded, everywhere-defined adjoint $T_\alpha^\#$: $Y \oplus Z \to \mathscr{D}(L)$ determined by the equation

$$\langle T_\alpha x, z\rangle = \left(x, T_\alpha^\# z\right)_* \quad \text{for all } x \in \mathscr{D}(L),\ z \in Y \oplus Z.$$

It follows from [19, Lemma 3.3, p. 509] that the range $\mathscr{R}(T_\alpha)$ is closed in $Y \oplus Z$ under the product structure. Let Q_α be the orthogonal projection from $Y \oplus Z$ onto $\mathscr{N}(T_\alpha^\#)$ in terms of the product structure, so that $I - Q_\alpha$ is the orthogonal projection onto $\mathscr{R}(T_\alpha)$. Let T_α^+ be the generalized inverse of T_α given by $T_\alpha^+ = T_\alpha^{-1}(I - Q_\alpha)$. From the Open Mapping Theorem the operator T_α^+: $Y \oplus Z \to \mathscr{D}(L)$ is bounded from the product structure into the $*$-structure. More precisely, we have

$$\|T_\alpha^+\|_* = \|T_\alpha^{-1}\|_* \leqslant 1/|\alpha| \quad \text{for } 0 < |\alpha| \leqslant 1, \tag{1.3}$$

where the operator norm is taken from the product structure into the $*$-structure on $\mathscr{D}(T_\alpha) = \mathscr{D}(L)$.

If $y \in Y$ and if we set $\hat{y} = (y, 0) \in Y \oplus Z$, then $|T_\alpha x - \hat{y}|^2 = \|Lx - y\|^2 + \alpha^2\|Ax\|^2 = G_\alpha(x)$. We immediately get

LEMMA 1.2. *The element x_α in Theorem* 1.1 *is characterized as the unique least squares solution of the equation $T_\alpha x = \hat{y}$, i.e. $x_\alpha = T_\alpha^+ \hat{y}$.*

(C) *Error estimates in the $*$-norm for $x_\alpha - L^\mu y$.* Next, we establish the convergence of the x_α to a least squares solution of (1.1) as $\alpha \to 0$ and develop the associated error estimates. Let $N^{\perp *}$ denote the orthogonal complement of a subspace N in $\mathscr{D}(L)$ with respect to the inner product $(\ ,\ )_*$, and let P_* and $I - P_*$ be the orthogonal projections from $\mathscr{D}(L)$ onto $\mathscr{N}(L)$ and $\mathscr{N}(L)^{\perp *} = \mathscr{R}(L^\#)$, resp. In terms of the standard inner product on Y, let Q and $I - Q$ be the orthogonal projections from Y onto $\mathscr{N}(L^\#) = \mathscr{R}(L)^\perp$ and $\mathscr{N}(L^\#)^\perp = \mathscr{R}(L)$, resp. Let L^μ: $Y \to \mathscr{D}(L)$, be the generalized inverse of L, where we work with $\mathscr{D}(L)$ under the $*$-structure and with Y under the standard structure:

$$L^\mu = \left[L|\mathscr{D}(L) \cap \mathscr{N}(L)^{\perp *}\right]^{-1}(I - Q).$$

Since we are assuming $\mathscr{R}(L)$ is closed in Y, the linear operator L^μ is continuous from Y under its standard structure into $\mathscr{D}(L)$ under the $*$-structure. The range $\mathscr{R}(L)$ being closed in Y also implies that the ranges $\mathscr{R}(L^\#)$ and $\mathscr{R}(L^\# L)$ are closed in $\mathscr{D}(L)$ under the $*$-structure, and $\mathscr{R}(L^\# L) = \mathscr{R}(L^\#)$ and $\mathscr{N}(L^\# L) = \mathscr{N}(L)$ [3, pp. 481–482]. Form the generalized inverse $(L^\#)^\mu$: $\mathscr{D}(L) \to Y$, which is continuous from $\mathscr{D}(L)$ under the $*$-structure into Y under the standard structure

$$(L^\#)^\mu = \left[L^\# | \mathscr{N}(L^\#)^\perp\right]^{-1}(I - P_*).$$

Also, the operator $L^\sharp L\colon \mathscr{D}(L) \to \mathscr{D}(L)$ is continuous under the $*$-structure, and it is easy to check that $L^\sharp L$ is symmetric, and hence, is selfadjoint. Form its generalized inverse $K = (L^\sharp L)^\mu\colon \mathscr{D}(L) \to \mathscr{D}(L)$, which is given by

$$K = (L^\sharp L)^\mu = \left[L^\sharp L | \mathscr{D}(L) \cap \mathscr{N}(L)^{\perp *} \right]^{-1}(I - P_*),$$

and which is continuous under the $*$-structure. The operator $K = (L^\sharp L)^\mu = L^\mu (L^\sharp)^\mu$ plays a key role in establishing the following uniform boundedness and convergence of the x_α **[21]**.

THEOREM 1.3. *The element $x_\alpha \in \mathscr{D}(L) \cap \mathscr{N}(L)^{\perp *}$, for each $\alpha \neq 0$, and the x_α are uniformly bounded and converge to $L^\mu y$ with rate of convergence of order α^2 as $\alpha \to 0$, with estimates*

$$\|x_\alpha\|_* \leqslant \tfrac{8}{3}\|L^\mu y\|_* \tag{1.4}$$

and

$$\|x_\alpha - L^\mu y\|_* \leqslant \left[\tfrac{32}{9}\|K\| + \tfrac{4}{3}\right]\|L^\mu y\|_* \alpha^2 \tag{1.5}$$

for $0 < |\alpha| \leqslant \min\{\tfrac{1}{2}, \sqrt{3}/2\sqrt{2\|K\|}\}$.

II. Regularization and nth-order linear boundary value problems with T an mth-order differential operator.

(A) *Mathematical preliminaries and the least squares approximate $\tilde{x}_\alpha$.* In this section the regularization method is applied to the numerical solution of ill-posed two-point linear boundary value problems. This extends earlier work **[18]** for well-posed problems. We will work in the real Hilbert space $X = Y = Z = L^2[a, b]$ under the standard inner product and norm, and in the subspace

$$H^n[a, b] = \left\{ x \in C^{n-1}[a, b] | x^{n-1} \text{ is absolutely continuous on } [a, b] \text{ and } x^{(n)} \in L^2[a, b] \right\}.$$

Let L be an nth-order differential operator in $L^2[a, b]$ determined by a formal differential operator $\tau = \sum_{i=0}^n a_i(t)(d/dt)^i$ with coefficients $a_i(t) \in C^\infty[a, b]$ and $a_n(t) \neq 0$ on $[a, b]$, and by a set k_0 $(0 \leqslant k_0 \leqslant 2n)$ linearly independent boundary values $B_1, \ldots, B_{k_0}$:

$$\mathscr{D}(L) = \left\{ x \in H^n[a, b] | B_i(x) = 0, i = 1, \ldots, k_0 \right\}, \qquad Lx = \tau x.$$

Let T be an mth-order differential operator in $L^2[a, b]$ with $m \leqslant n$ which is determined by a formal differential operator $\sigma = \sum_{i=0}^m b_i(t)(d/dt)^i$ with coefficients $b_i(t) \in C^\infty[a, b]$ and $b_m(t) \neq 0$ on $[a, b]$ and by a set of k_1 $(0 \leqslant k_1 \leqslant 2m)$ linearly independent boundary values $C_1, \ldots, C_{k_1}$:

$$\mathscr{D}(T) = \left\{ x \in H^m[a, b] | C_i(x) = 0, i = 1, \ldots, k_1 \right\}, \qquad Tx = \sigma x.$$

It is well known that L and T are closed densely-defined linear operators in $L^2[a, b]$ with $\mathscr{R}(L)$ and $\mathscr{R}(T)$ closed subspaces in $L^2[a, b]$.

For a given function $y \in L^2[a, b]$ we want to numerically construct approximate least squares solutions of the linear boundary value problem $Lx = y$, using T as the regularizing operator. In the sequel we assume that $\mathscr{D}(L) \subseteq \mathscr{D}(T)$ with L and T satisfying conditions (I)–(III) of §I.

In our analysis we will frequently use the norm

$$\|x\|_{H^n} = \sum_{i=0}^{n-1} \|x^{(i)}\|_\infty + \|x^{(n)}\|$$

on $\mathscr{D}(L)$, where $\| \|_\infty$ is the standard L^∞-norm. It is well known that $\mathscr{D}(L)$ is a Banach space under the norm $\| \|_{H^n}$, and $\| \|_{H^n}$ is equivalent to the graph norm $\| \|_L$ of L. We will refer to $\| \|_{H^n}$ as the H^n-norm and to the associated Banach space structure as the *H^n-structure* for $\mathscr{D}(L)$.

Note that $A = T|\mathscr{D}(L)$ is continuous from $\mathscr{D}(L)$ under the H^n-structure into $L^2[a, b]$ under its standard structure, and hence, condition (II) is automatically satisfied. Also, if condition (I) is satisfied, then T is 1-1 on the finite-dimensional subspace $\mathscr{N}(L)$, implying that condition (III) is satisfied. Thus, conditions (I)–(III) hold whenever condition (I) is satisfied.

Treating L and T in the setting of unbounded linear operators, the adjoints L^* and T^* exist as closely densely-defined operators given by $(Lx, z) = (x, L^*z)$ for all $x \in \mathscr{D}(L)$ and $z \in \mathscr{D}(L^*)$, and $(Tx, z) = (x, T^*z)$ for all $x \in \mathscr{D}(T)$ and $z \in \mathscr{D}(T^*)$. We know that L^* is an nth-order differential operator in $L^2[a, b]$ determined by the formal adjoint τ^* and by a set of $2n - k_0$ adjoint boundary values $B_1^*, \ldots, B_{2n-k_0}^*$, and T^* is an mth-order differential operator in $L^2[a, b]$ determined by σ^* and a set of $2m - k_1$ adjoint boundary values $C_1^*, \ldots, C_{2m-k_1}^*$. Also, the product operator L^*L is a $2n$th-order differential operator in $L^2[a, b]$ determined by the formal differential operator $\tau^*\tau$ and by the set of $2n$ boundary values $B_1, \ldots, B_{k_0}, B_1^*\tau, \ldots, B_{2n-k_0}^*\tau$, and L^*L is selfadjoint [**17**, Chapter IV].

Let P denote the family of all partitions Δ of $[a, b]$. For $\Delta \in P$ given by Δ: $a = t_0 < t_1 < \cdots < t_N = b$, let

$$h = \max_{1 \leqslant i \leqslant N} (t_i - t_{i-1}) \quad \text{and} \quad \underline{h} = \min_{1 \leqslant i \leqslant N} (t_i - t_{i-1}).$$

In terms of a partition Δ we introduce the subspace

$$C_\Delta^{n,k}[a, b] = \left\{ x \in H^n[a, b] | x \in C^{n+k}[t_{i-1}, t_i] \text{ for } i = 1, \ldots, N \right\}$$

for $0 \leqslant k \leqslant \infty$, which is a Banach space under the norm

$$\|x\|C_\Delta^{n,k} = \sum_{j=0}^{n+k} \|x^{(j)}\|_\infty \quad \text{for } 0 \leqslant k < \infty.$$

Let $\mathrm{Sp}(2n - 1, \Delta, n - 1)$ denote the space of Hermite splines of degree $2n - 1$ having C^{n-1} global continuity. The cardinal basis

$$\Phi_N = \left\{ \varphi_{ij} | 0 \leqslant i \leqslant N, 0 \leqslant j \leqslant n - 1 \right\}$$

for $\mathrm{Sp}(2n - 1, \Delta, n - 1)$ consists of the unique splines from $\mathrm{Sp}(2n - 1, \Delta, n - 1)$ solving the interpolation problems

$$\varphi_{ij}^{(m)}(t_l) = \delta_{il}\delta_{jm}, \qquad 0 \leqslant i, l \leqslant N; 0 \leqslant j, m \leqslant n - 1.$$

Each $x \in \mathrm{Sp}(2n - 1, \Delta, n - 1)$ has the unique representation

$$x(t) = \sum_{i=0}^{N} \sum_{j=0}^{n-1} x^{(j)}(t_i)\varphi_{ij}(t), \tag{2.1}$$

which will be used later to establish certain error estimates.

Assume that the regularization parameter $\alpha \neq 0$ has been fixed so that the corresponding x_α is a good approximation to the least squares solution $x_0 = L^\mu y$ (see Theorem 1.3). We are going to use the method of least squares to approximate x_α by means of Hermite splines. Let $S_N = \mathrm{Sp}(2n-1, \Delta, n-1) \cap \mathscr{D}(L)$ and let $\varphi_1, \dots, \varphi_M$, where $M = n(N+1) - k_0$, be a basis for S_N.

By Lemma 1.2 we know that

$$|T_\alpha x_\alpha - \hat{y}| = \inf_{x \in \mathscr{D}(L)} |T_\alpha x - \hat{y}|, \tag{2.2}$$

or equivalently,

$$\langle T_\alpha x_\alpha - \hat{y}, T_\alpha x \rangle \quad \text{for all } x \in \mathscr{D}(T_\alpha) = \mathscr{D}(L). \tag{2.3}$$

Define the (continuous) least squares approximate $\tilde{x}_\alpha \in S_N$ to x_α by the condition

$$|T_\alpha \tilde{x}_\alpha - \hat{y}| = \inf_{\varphi \in S_N} |T_\alpha \varphi - \hat{y}|, \tag{2.4}$$

or equivalently,

$$\langle T_\alpha \tilde{x}_\alpha - \hat{y}, T_\alpha \varphi \rangle = 0 \quad \text{for all } \varphi \in S_N. \tag{2.5}$$

Combining (2.3) and (2.5) we have

$$\langle T_\alpha x_\alpha - T_\alpha \tilde{x}_\alpha, T_\alpha \varphi \rangle = 0 \quad \text{for all } \varphi \in S_N, \tag{2.6}$$

or equivalently,

$$|T_\alpha x_\alpha - T_\alpha \tilde{x}_\alpha| \leqslant |T_\alpha x_\alpha - T_\alpha \varphi| \quad \text{for all } \varphi \in S_N. \tag{2.7}$$

If we write $\tilde{x}_\alpha = \sum_{j=1}^{M} c_j \varphi_j$, then from (2.6) the coefficients $c_1, \dots, c_M$ satisfy the linear system

$$\sum_{j=1}^{M} \left[(L\varphi_i, L\varphi_j) + \alpha^2 (A\varphi_i, A\varphi_j) \right] c_j = (y, L\varphi_i), \qquad 1 \leqslant i \leqslant M. \tag{2.8}$$

(B) *The generalized inverse L^+ and the generalized Green's function $G(t,s)$.* The remainder of this section is devoted to establishing error estimates for $\tilde{x}_\alpha^{(j)} - x_0^{(j)}$, where $x_0 = L^\mu Y$ and $j = 0, 1, \dots, n-1$. This will require establishing smoothness and uniform boundedness of the x_α, and investigating the operators T_α and their generalized inverses and associated representors. Toward this end we introduce the generalized inverse L^+ and the generalized Green's function $G(t,s)$ for L, and consider various adjoints of the operators L and $A = T|\mathscr{D}(L)$.

Let L^+ be the generalized inverse of L given by

$$L^+ = \left[L|\mathscr{D}(L) \cap \mathscr{N}(L)^\perp \right]^{-1} (I - Q) \colon L^2[a,b] \to \mathscr{D}(L),$$

where Q and $I - Q$ are the L^2-orthogonal projections from $L^2[a,b]$ onto $\mathscr{N}(L^*)$ and $\mathscr{R}(L)$, resp., and let G be the generalized Green's function associated with L, which satisfies

$$L^+ z(t) = \int_a^b G(t,s) z(s)\, ds \quad \text{for all } z \in L^2[a,b],\ a \leqslant t \leqslant b$$

(see [**16** or **17**]). For each $t \in [a,b]$ we have $G_t(\cdot) = G(t, \cdot) \in \mathscr{R}(L)$ and

$$\|G_t\| \leqslant \|L^+\|, \tag{2.9}$$

where we work with L^+ from $L^2[a,b]$ under the standard structure into $\mathscr{D}(L)$ under the H^n-structure.

We have the following result concerning the boundedness of the operators L and L^+. See [**17**, Chapter IV].

LEMMA 2.1. *For $\Delta \in P$, $0 \leqslant j < \infty$, $0 \leqslant k < \infty$, there exist positive constants α_{jk} and β_{jk} independent of Δ such that*

(i) $$\|Lx\|_{C_\Delta^{j,k}} \leqslant \alpha_{jk}\|x\|_{C_\Delta^{n+j,k}} \quad \text{for all } x \in \mathscr{D}(L) \cap C_\Delta^{n+j,k}[a,b],$$

and

(ii) $$\|L^+ z\|_{C_\Delta^{n+j,k}} \leqslant \beta_{jk}\|z\|_{C_\Delta^{j,k}} \quad \text{for all } z \in C_\Delta^{j,k}[a,b].$$

(C) *The adjoints L_* and A_* and their relationships with the adjoints $L^\sharp$ and $A^\sharp$.* We will now examine the operator L and A on $\mathscr{D}(L)$ with an emphasis on the graph norm structure of L. We know that L and A are both bounded linear operators from the graph norm structure of L into the L^2-structure, and hence, there exist adjoint operators L_*: $L^2[a,b] \to \mathscr{D}(L)$ and A_*: $L^2[a,b] \to \mathscr{D}(L)$ determined by the equations

$$(Lx, z) = (x, L_* z)_L \tag{2.10}$$

and

$$(Ax, z) = (x, A_* z)_L \tag{2.11}$$

for all $x \in \mathscr{D}(L)$ and $z \in L^2[a,b]$. Both L_* and A_* are bounded from the L^2-structure into the graph norm structure of L. We will characterize the adjoint operators L_* and A_* and exhibit their relationships with the adjoint operators $L^\sharp$ and $A^\sharp$. The following two lemmas are proved in a straightforward manner using the 1-1, selfadjoint operators $L^*L + I$ and $LL^* + I$ [**17**, Chapter I].

LEMMA 2.2.

(i) $$L_* z = (L^*L + I)^{-1} L^* z \quad \text{for all } z \in \mathscr{D}(L^*).$$

(ii) $$L_* z = L^*(LL^* + I)^{-1} z \quad \text{for all } z \in L^2[a,b].$$

LEMMA 2.3. $A_* z = (L^*L + I)^{-1} T^* z$ *for all* $z \in \mathscr{D}(T^*)$.

The operator $L^*L + I$, its Green's function $\overline{G}(t,s)$, and [**17**, Chapter IV] give us the following:

$$A_* z(t) = \sum_{i=0}^{m} \int_a^b \left[\left(\frac{\partial}{\partial s}\right)^i \overline{G}(t,s)\right][b_i(s) z(s)]\, ds \tag{2.12}$$

a.e. on $[a,b]$ for all $z \in L^2[a,b]$.

If we combine (2.12) with [**17**, Chapter III], then we obtain the following results for A and its adjoint A_*:

THEOREM 2.4. *Let $\Delta \in P$, $0 \leqslant j < \infty$, and $0 \leqslant k < \infty$, and let T be an mth-order differential operator with $m \leqslant n$.*

(i) *If $z \in C_\Delta^{j,k}[a,b]$, then $A_* z \in C_\Delta^{2n-m+j,k}[a,b]$, and there exists a positive constant α_{jk} independent of Δ such that*

$$\|A_* z\|_{C_\Delta^{2n-m+j,k}} \leqslant \alpha_{jk}\|z\|_{C_\Delta^{j,k}} \quad \text{for all } z \in C_\Delta^{j,k}[a,b].$$

(ii) *If* $x \in \mathscr{D}(L) \cap C_\Delta^{m+j,k}[a,b]$, *then* $Ax \in C_\Delta^{j,k}[a,b]$ *and there exists a positive constant* β_{jk} *independent of* Δ *such that*

$$\|Ax\|_{C_\Delta^{j,k}} \leqslant \beta_{jk}\|x\|_{C_\Delta^{m+j,k}} \quad \textit{for all } x \in \mathscr{D}(L) \cap C_\Delta^{m+j,k}[a,b].$$

Recall from Theorem 1.1 that for a given $y \in L^2[a,b]$ we have $L^\sharp L x_\alpha + \alpha^2 A^\sharp A x_\alpha = L^\sharp y$, and hence, for $u \in \mathscr{D}(L)$,

$$0 = \left(L^\sharp L x_\alpha + \alpha^2 A^\sharp A x_\alpha - L^\sharp y, u\right)_* = \left(L_* L x_\alpha + \alpha^2 A_* A x_\alpha - L_* y, u\right)_L.$$

We conclude that $x_\alpha \in \mathscr{D}(L)$ also satisfies the equation

$$L_* L x_\alpha + \alpha^2 A_* A x_\alpha = L_* y. \tag{2.13}$$

For the operator $L_* L + \alpha^2 A_* A$ we have the following easily-proven result.

LEMMA 2.5. *For each* $\alpha \neq 0$ *the operator* $L_* L + \alpha^2 A_* A$ *is a bounded selfadjoint linear operator on* $\mathscr{D}(L)$ *under the graph norm structure of L and it is invertible.*

(D) *The operator* $K_1 = (L_* L)^+$, *smoothness of the* x_α, *uniform boundedness of the* x_α, *and error estimates.* Our next objective is to show that the smoothness of the x_α depends on the smoothness of the given $y \in L^2[a,b]$, and that the x_α are uniformly bounded in the $C_\Delta^{n,k}$-norm when $y \in C_\Delta^{0,k}[a,b]$, where $\Delta \in P$ and $0 \leqslant k < \infty$. The selfadjoint operator $L_* L: \mathscr{D}(L) \to \mathscr{D}(L)$, which is bounded under the graph norm structure of L, will play a key role. Recall that the range $\mathscr{R}(L)$ is closed in $L^2[a,b]$ under the L^2-structure. Hence, by [3, pp. 481–482] the ranges $\mathscr{R}(L_*)$ and $\mathscr{R}(L_* L)$ are closed in $\mathscr{D}(L)$ under the graph norm structure of L, and $\mathscr{R}(L_* L) = \mathscr{R}(L_*)$ and $\mathscr{N}(L_* L) = \mathscr{N}(L)$. Form the generalized inverse $K_1 = (L_* L)^+ : \mathscr{D}(L) \to \mathscr{D}(L)$, which is given by

$$K_1 = (L_* L)^+ = \left[L_* L | \mathscr{D}(L) \cap \mathscr{N}(L)^\perp\right]^{-1}(I - P),$$

where P is the L^2-orthogonal projection from $L^2[a,b]$ onto $\mathscr{N}(L)$. By the Open Mapping Theorem the linear operator K_1 is bounded under the graph norm structure of L.

Note that $\mathscr{N}(L_*) = \mathscr{R}(L) = \mathscr{N}(L^*)$, and form the generalized inverse of L_*, $(L_*)^+ : \mathscr{D}(L) \to L^2[a,b]$, which is given by

$$(L_*)^+ = \left[L_* | \mathscr{N}(L^*)^\perp\right]^{-1}(I - P).$$

By the Open Mapping Theorem $(L_*)^+$ is bounded from the graph norm structure of L into the L^2-structure.

The following theorem concerns a restriction of the linear operator K_1. Its proof is based on the fact that

$$K_1 = (L_* L)^+ = L^+(L_*)^+ = (I - P) + (L^* L)^+ \tag{2.14}$$

which follows from Lemma 2.2.

THEOREM 2.6. *If* $\Delta \in P$, $0 \leqslant j < \infty$, *and* $0 \leqslant k < \infty$, *then* $K_1: \mathscr{D}(L) \cap C_\Delta^{n+j,k}[a,b] \to \mathscr{D}(L) \cap C_\Delta^{n+j,k}[a,b]$ *and there exists a positive constant* γ_{jk} *independent of* Δ *such that*

$$\|K_1 x\|_{C_\Delta^{n+j,k}} \leqslant \gamma_{jk}\|x\|_{C_\Delta^{n+j,k}} \quad \textit{for all } x \in \mathscr{D}(L) \cap C_\Delta^{n+j,k}[a,b].$$

Applying K_1 to the left-hand side of (2.13) produces $(I - P)x_\alpha + \alpha^2 K_1 A_* A x_\alpha$, while applying it to the right-hand side yields $L^+ y$. Thus, equation (2.13) becomes

$$x_\alpha = \mathscr{P} x_\alpha - \alpha^2 K_1 A_* A x_\alpha + L^+ y. \tag{2.15}$$

It will be used to show how the smoothness of x_α depends on the smoothness of the given $y \in L^2[a, b]$.

THEOREM 2.7. *Assume T is an mth-order differential operator with $0 \leqslant m < n$. If $\Delta \in \mathscr{P}$ and $0 \leqslant k < \infty$, then $y \in C_\Delta^{0,k}[a, b]$ implies that $x_\alpha \in C_\Delta^{n,k}[a, b]$.*

PROOF. Let $\Delta \in \mathscr{P}$ and assume $0 \leqslant m < n$, so that $n - m \geqslant 1$. Then $\mathscr{P} x_\alpha \in C^\infty[a, b] \subseteq C_\Delta^{n,k}[a, b]$ for $0 \leqslant k < \infty$, and $y \in C_\Delta^{0,k}[a, b]$ implies that $L^+ y \in C_\Delta^{n,k}[a, b]$ for $0 \leqslant k < \infty$. Since $x_\alpha \in \mathscr{D}(L) \subseteq H^n[a, b] \subseteq C_\Delta^{n-1,0}[a, b]$, we have from Theorems 2.4 and 2.6 that $A x_\alpha \in C_\Delta^{(n-m)-1,0}[a, b]$, $A_* A x_\alpha \in C_\Delta^{n+2(n-m)-1,0}[a, b]$, and $K_1 A_* A x_\alpha \in C_\Delta^{n+2(n-m)-1,0}[a, b] \subseteq C_\Delta^{n,2(n-m)-1}[a, b]$. If $0 \leqslant k \leqslant 2(n - m) - 1$ and $y \in C_\Delta^{0,k}[a, b]$, then $x_\alpha = P x_\alpha - \alpha^2 K_1 A_* A x_\alpha + L^+ y \in C_\Delta^{n,k}[a, b]$.

If $2(n - m) - 1 < k \leqslant 4(n - m) - 1$ and $y \in C_\Delta^{0,k}[a, b]$, then $x_\alpha = P x_\alpha - \alpha^2 K_1 A_* A x_\alpha + L^+ y \in C_\Delta^{n,2(n-m)-1}[a, b]$, so that $A x_\alpha \in C_\Delta^{n-m,2(n-m)-1}[a, b]$, $A_* A x_\alpha \in C_\Delta^{n+2(n-m),2(n-m)-1}[a, b]$, and $K_1 A_* A x_\alpha \in C_\Delta^{n+2(n-m),2(n-m)-1}[a, b] \subseteq C_\Delta^{n,4(n-m)-1}[a, b]$. Hence, $x_\alpha \in C_\Delta^{n,k}[a, b]$.

Proceed by induction. Q.E.D.

Since we have convergence of the x_α to $x_0 = L^\mu y$ in the $*$-norm by Theorem 1.3, we conjecture from (2.15) that $x_\alpha \to P x_0 + L^+ y$ as $\alpha \to 0$, where convergence is in the $C_\Delta^{n,k}$-norm. We also have

$$x_0 = P x_0 + L^+ y \tag{2.16}$$

and the following consequence of the Schwarz inequality.

LEMMA 2.8. *For $\Delta \in \mathscr{P}$, $0 \leqslant j < \infty$, and $0 \leqslant k < \infty$, there exists a positive constant α_{jk} independent of Δ such that $\|Pz\|_{C_\Delta^{n+j,k}} \leqslant \alpha_{jk} \|z\|$ for all $z \in L^2[a, b]$.*

Next we examine the uniform boundedness and convergence of the x_α. Equation (2.15) will play a key role. Let $\Delta \in \mathscr{P}$ and assume $0 \leqslant k < \infty$ and $0 \leqslant m < n$. Assume $y \in C_\Delta^{0,k}[a, b]$, so that $x_\alpha \in C_\Delta^{n,k}[a, b]$ by Theorem 2.7. Also, since $P x_0 \in C^\infty[a, b] \subseteq C_\Delta^{n,k}[a, b]$ and $L^+ y \in C_\Delta^{n,k}[a, b]$, it follows from (2.16) that $x_0 \in C_\Delta^{n,k}[a, b]$. Thus, from (2.15), (2.16), and Lemma 2.8 we get

$$\begin{aligned} \|x_\alpha\|_{C_\Delta^{n,k}} &\leqslant \|x_\alpha - x_0\|_{C_\Delta^{n,k}} + \|x_0\|_{C_\Delta^{n,k}} \\ &\leqslant \alpha_k \|x_\alpha - x_0\| + \alpha^2 \|K_1 A_* A x_\alpha\|_{C_\Delta^{n,k}} + \|x_0\|_{C_\Delta^{n,k}}. \end{aligned} \tag{2.17}$$

Recall that the H^n-norm, the graph norm of L, and the $*$-norm are all equivalent on $\mathscr{D}(L)$ so that by Theorem 1.3

$$\|x_\alpha - x_0\| \leqslant e_n \|x_0\|_{C_\Delta^{n,k}} \alpha^2 \tag{2.18}$$

for $0 \leqslant |\alpha| \leqslant \min\{\frac{1}{2}, \sqrt{3}/2\sqrt{2\|K\|}\}$.

Finally, from Theorems 2.6 and 2.4 there exists a positive constant b_k independent of Δ such that

$$\|K_1 A_* A x_\alpha\|_{C_\Delta^{n,k}} \leqslant b_k \|x_\alpha\|_{C_\Delta^{n,k}}. \tag{2.19}$$

Combining (2.17), (2.18), and (2.19) and choosing

$$0 < |\alpha| \leqslant \min\left\{\tfrac{1}{2}, \sqrt{3}/2\sqrt{2\|K\|}, 1/\sqrt{2b_k}\right\},$$

we get

$$\|x_\alpha\|_{C_\Delta^{n,k}} \leqslant 2(\alpha_k e_n + 1)\|x_0\|_{C_\Delta^{n,k}} \tag{2.20}$$

for $0 < |\alpha| \leqslant \min\{\frac{1}{2}, \sqrt{3}/2\sqrt{2\|K\|}, 1/\sqrt{2b_k}\}$ and for $0 \leqslant m < n$.

We now show that the x_α converge to x_0 in the $C_\Delta^{n,k}$-norm with rate of convergence of order α^2. Indeed, by (2.15), (2.16), Lemma 2.8, (2.19), (2.18), and (2.20) we get

$$\|x_\alpha - x_0\|_{C_\Delta^{n,k}} \leqslant \left[\alpha_k e_n + 2b_k(\alpha_k e_n + 1)\right]\|x_0\|_{C_\Delta^{n,k}}\alpha^2 \tag{2.21}$$

for $0 < |\alpha| \leqslant \min\{\frac{1}{2}, \sqrt{3}/2\sqrt{2\|K\|}, 1/\sqrt{2b_k}\}$ and for $0 \leqslant m < n$.

We have proved the following

THEOREM 2.9. *Let $\Delta \in \mathscr{P}$, let $0 \leqslant k < \infty$, and assume $y \in C_\Delta^{0,k}[a,b]$. If T is an mth-order differential operator with $0 \leqslant m < n$, then the $x_\alpha \in C_\Delta^{n,k}[a,b]$, the x_α are uniformly bounded in the $C_\Delta^{n,k}$-norm, and the x_α converge to $x_0 = L^\mu y = Px_0 + L^+y$ in the $C_\Delta^{n,k}$-norm with rate of convergence of order α^2 as $\alpha \to 0$, with estimates* (2.20) *and* (2.21).

(E) *T_α and its generalized inverse and associated representor.* In this section we work with the linear operator T_α: $\mathscr{D}(L) \to L^2[a,b] \oplus L^2[a,b]$ given by $T_\alpha x = (Lx, \alpha Tx) = (Lx, \alpha Ax)$, where $\alpha \neq 0$. Recall that T_α is one-to-one, the range $\mathscr{R}(T_\alpha)$ is closed in $L^2[a,b] \oplus L^2[a,b]$ under the L^2-product structure, and T_α is bounded from the graph norm structure of L into the L^2-product structure. Consequently, there exists an adjoint operator T_α^*: $L^2[a,b] \oplus L^2[a,b] \to \mathscr{D}(L)$, which is bounded from the L^2-product structure into the graph norm structure of L, given by

$$\langle T_\alpha x, z\rangle = \left(x, T_\alpha^* z\right)_L \quad \text{for all } x \in \mathscr{D}(L), z \in L^2[a,b] \oplus L^2[a,b].$$

Let Q_α be the orthogonal projection with respect to the L^2-product structure from $L^2[a,b] \oplus L^2[a,b]$ onto $\mathscr{N}(T_\alpha^*) = \mathscr{R}(T_\alpha)^\perp = \mathscr{N}(T_\alpha^\#)$, and consider the generalized inverse T_α^+: $L^2[a,b] \oplus L^2[a,b] \to \mathscr{D}(L)$ given by $T_\alpha^+ = T_\alpha^{-1}(I - Q_\alpha)$. Since the range $\mathscr{R}(T_\alpha)$ is closed in $L^2[a,b] \oplus L^2[a,b]$, we know that T_α^+ is bounded from the L^2-product structure into the graph norm structure of L, or the equivalent $*$-structure or H^n-structure. Choose constants a_n, b_n and c_n such that $\|x\|_L \leqslant a_n\|x\|_*$, $\|x\|_* \leqslant b_n\|x\|_{H^n}$, and $\|x\|_{H^n} \leqslant c_n\|x\|_L$ for all $x \in \mathscr{D}(L)$. It follows that

$$\|T_\alpha^+\|_{H^n} \leqslant c_n\|T_\alpha^+\|_L \leqslant c_n a_n \|T_\alpha^+\|_*, \tag{2.22}$$

where we take the operator norms of T_α^+ from the L^2-product structure into the H^n-structure, the graph norm structure of L, and the $*$-structure, resp.

We are now ready to develop the representor theory for T_α^+. For each integer $j = 0, 1, \dots, n-1$ and for each $t \in [a, b]$, define the linear functional λ_{jt}^α on $L^2[a, b] \oplus L^2[a, b]$ by

$$\lambda_{jt}^\alpha z = (d/dt)^j T_\alpha^+ z(t) \quad \text{for } z \in L^2[a, b] \oplus L^2[a, b]. \tag{2.23}$$

Then $|\lambda_{jt}^\alpha z| \leqslant \|T_\alpha^+\|_{H^n} |z| \leqslant c_n a_n |z|/|\alpha|$ for $0 < |\alpha| \leqslant 1$ by (2.22) and (1.3), where c_n and a_n are independent of t. By the Riesz Representation Theorem there exists a representor $\mathscr{G}_{jt}^\alpha = (l_{jt}^\alpha, a_{jt}^\alpha) \in L^2[a, b] \oplus L^2[a, b]$ with $\lambda_{jt}^\alpha z = \langle \mathscr{G}_{jt}^\alpha, z\rangle$ for all $z \in L^2[a, b] \oplus L^2[a, b]$, and

$$\|\lambda_{jt}^\alpha\| = |\mathscr{G}_{jt}^\alpha| \leqslant c_n a_n/|\alpha| \quad \text{for } 0 < |\alpha| \leqslant 1, \tag{2.24}$$

where c_n and a_n are independent of t. Thus,

$$(d/dt)^j T_\alpha^+ z(t) = \langle \mathscr{G}_{jt}^\alpha, z\rangle = (l_{jt}^\alpha, \zeta) + (a_{jt}^\alpha, \eta) \tag{2.25}$$

for all $z = (\zeta, \eta) \in L^2[a, b] \oplus L^2[a, b]$, for $j = 0, 1, \dots, n-1$, and for $t \in [a, b]$. We will refer to (2.25) as the *generalized Green's function representation of* T_α^+.

Take $x \in \mathscr{D}(L) = \mathscr{D}(T_\alpha)$ and set $T_\alpha x = (Lx, \alpha Ax) = (\zeta, \eta) \in \mathscr{R}(T_\alpha)$. It follows immediately from (2.25) that

$$x^{(j)}(t) = \langle \mathscr{G}_{jt}^\alpha, T_\alpha x\rangle = (l_{jt}^\alpha, \xi) + (a_{jt}^\alpha, \eta) \tag{2.26}$$

for $j = 0, 1, \dots, n-1$ and for $t \in [a, b]$. Applying L^+ to $Lx = \zeta$, we get

$$x = L^+\zeta + Px. \tag{2.27}$$

If we set $u = Px \in N(L)$, then applying (2.26) to u gives

$$(d/dt)^j Px(t) = \alpha(a_{jt}^\alpha, APx). \tag{2.28}$$

To continue this development, we will need to work with the adjoint operator $(L^+)_*\colon \mathscr{D}(L) \to L^2[a, b]$, which is bounded from the graph norm structure of L into the L^2-structure, and is given by

$$(L^+\omega, x)_L = (\omega, (L^+)_* x) \quad \text{for all } \omega \in L^2[a, b], x \in \mathscr{D}(L).$$

We get immediately that

$$(L^+)_* = (L^*)^+ + L \quad \text{on } \mathscr{D}(L), \tag{2.29}$$

and

$$(L^+)_* = (L_*)^+ \quad \text{on } \mathscr{D}(L) \tag{2.30}$$

follows from (2.29) and the fact that $(L_*)^+ = (LL^* + I)(L^*)^+$ on $\mathscr{D}(L)$.

Now let $G(t, s) = G_t(s)$ denote the generalized Green's function of L, and for $j = 0, 1, \dots, n-1$ set $G_j(t, s) = G_{jt}(s) = (\partial/\partial t)^j G(t, s)$. Continuing the above discussion, we have $x^{(j)}(t) - (d/dt)^j Px(t) = (d/dt)^j L^+ Lx(t) = (G_{jt}, Lx)$, while from (2.26), (2.28), and (2.30) we have

$$x^{(j)}(t) - (d/dt)^j Px(t) = \left(l_{jt}^\alpha + \alpha(L_*)^+ A_* a_{jt}^\alpha, Lx\right).$$

Upon combining these two results, we get

$$\left(l_{jt}^{\alpha} + \alpha(L_*)^{+}A_*a_{jt}^{\alpha} - G_{jt}, Lx\right) = 0 \quad \text{for all } x \in \mathscr{D}(L).$$

Since the functions l_{jt}^{α}, $(L_*)^+A_*a_{jt}^{\alpha}$, and G_{jt} all belong to $\mathscr{R}(L) = \mathscr{N}(L^*)^{\perp}$, we must have

$$l_{jt}^{\alpha} = G_{jt} - \alpha(L_*)^{+}A_*a_{jt}^{\alpha} \tag{2.31}$$

for $j = 0, 1, \dots, n-1$ and for $t \in [a, b]$.

Finally, for $j = 0, 1, \dots, n-1$ and for $t \in [a, b]$, we note that $\mathscr{G}_{jt}^{\alpha} \in \mathscr{N}(T_{\alpha}^*)^{\perp} = \mathscr{R}(T_{\alpha})$ by (2.25). Define $g_{jt}^{\alpha} = T_{\alpha}^{-1}\mathscr{G}_{jt}^{\alpha} \in \mathscr{D}(L)$, so that $T_{\alpha}g_{jt}^{\alpha} = (Lg_{jt}^{\alpha}, \alpha Ag_{jt}^{\alpha}) = (l_{jt}^{\alpha}, a_{jt}^{\alpha}) = \mathscr{G}_{jt}^{\alpha}$. Then from (2.27), (2.31), and (2.14) we get

$$g_{jt}^{\alpha} = Pg_{jt}^{\alpha} - \alpha^2 K_1 A_* A g_{jt}^{\alpha} + L^+ G_{jt} \tag{2.32}$$

for $j = 0, 1, \dots, n-1$ and for $t \in [a, b]$.

We will use (2.32) to establish global and local smoothness of g_{jt}^{α} and to bound various of its derivatives. The global smoothness will be established first. Fix an integer j with $0 \leqslant j \leqslant n-1$ and fix $t \in [a, b]$. We know that $Pg_{jt}^{\alpha} \in C^{\infty}[a, b]$, and by [**17**, Chapter IV] $G_{jt} = (\partial/\partial t)^j G(t, \cdot) \in H^{n-j-1}[a, b]$, so that $L^+G_{jt} \in H^{2n-j-1}[a, b]$.

Let $\Delta_0 \in \mathscr{P}$ be the trivial partition of $[a, b]$ given by $a = t_0 < t_1 = b$. Then

$$C_{\Delta_0}^{j,k}[a, b] = H^j[a, b] \cap C^{j+k}[a, b] = C^{j+k}[a, b] \tag{$*$}$$

for all $0 \leqslant j < \infty$ and $0 \leqslant k < \infty$. Assume $0 \leqslant m < n$, so that $n - m \geqslant 1$. Since $g_{jt}^{\alpha} \in \mathscr{D}(L)$, we know that $g_{jt}^{\alpha} \in H^n[a, b] \subseteq C^{n-1}[a, b] = C_{\Delta_0}^{n-1,0}[a, b]$ by $(*)$. Thus, by Theorems 2.4 and 2.6, $Ag_{jt}^{\alpha} \in C_{\Delta_0}^{(n-m)-1,0}[a, b] \subseteq C_{\Delta_0}^{0,0}[a, b]$, $A_*Ag_{jt}^{\alpha} \in C_{\Delta_0}^{2n-m,0}[a, b] \subseteq C_{\Delta_0}^{n+1,0}[a, b]$, and $K_1A_*Ag_{jt}^{\alpha} \in C_{\Delta_0}^{n+1,0}[a, b] = C^{n+1}[a, b]$ by $(*)$.

If $2n - j - 1 \leqslant n + 1$, then $C^{n+1}[a, b] \subseteq H^{n+1}[a, b] \subseteq H^{2n-j-1}[a, b]$, so from (2.32) $g_{jt}^{\alpha} \in H^{2n-j-1}[a, b]$. Otherwise, $n + 1 < 2n - j - 1$, so that $H^{2n-j-1}[a, b] \subseteq H^{n+1}[a, b]$ and $C^{n+1}[a, b] \subseteq H^{n+1}[a, b]$. Thus, from (2.32), $g_{jt}^{\alpha} \in H^{n+1}[a, b] \subseteq C^n[a, b] = C_{\Delta_0}^{n,0}[a, b]$ by $(*)$, and hence, by Theorems 2.4 and 2.6, $Ag_{jt}^{\alpha} \in C_{\Delta_0}^{n-m,0}[a, b] \subseteq C_{\Delta_0}^{1,0}[a, b]$, $A_*Ag_{jt}^{\alpha} \in C_{\Delta_0}^{2n-m+1,0}[a, b] \subseteq C_{\Delta_0}^{n+2,0}[a, b]$ and $K_1A_*Ag_{jt}^{\alpha} \in C_{\Delta_0}^{n+2,0}[a, b] = C^{n+2}[a, b]$ by $(*)$.

If $2n - j - 1 \leqslant n + 2$, then $C^{n+2}[a, b] \subseteq H^{n+2}[a, b] \subseteq H^{2n-j-1}[a, b]$, so from (2.32), $g_{jt}^{\alpha} \in H^{2n-j-1}[a, b]$. Otherwise, $n + 2 < 2n - j - 1$, so that $H^{2n-j-1}[a, b] \subseteq H^{n+2}[a, b]$ and $C^{n+2}[a, b] \subseteq H^{n+2}[a, b]$. Thus, from (2.32), $g_{jt}^{\alpha} \in H^{n+2}[a, b] \subseteq C^{n+1}[a, b] = C_{\Delta_0}^{n+1,0}[a, b]$ by $(*)$.

Continuing this "bootstrap" argument, we conclude that

$$g_{jt}^{\alpha} \in H^{2n-j-1}[a, b] \tag{2.33}$$

for $j = 0, 1, \dots, n-1$, for $t \in [a, b]$, and for $0 \leqslant m < n$.

We establish the local smoothness of g_{jt}^{α} in a similar fashion. Fix an integer j with $0 \leqslant j \leqslant n-1$, fix $t \in [a, b]$, and let $\Delta_t \in \mathscr{P}$ be the simple partition of $[a, b]$ given by $\Delta_t = \{a, t, b\}$. By [**17**, Chapter IV] we have $G_{jt} \in C_{\Delta_t}^{0,\infty}[a, b]$, so that

$$L^+G_{jt} \in C_{\Delta_t}^{n,\infty}[a, b] \subseteq C_{\Delta_t}^{n-1,\infty}[a, b]. \tag{2.34}$$

Assume $0 \leqslant m < n$, so that $n - m \geqslant 1$. Since $g_{jt}^{\alpha} \in \mathscr{D}(L)$, we know that $g_{jt}^{\alpha} \in C_{\Delta_t}^{n-1,0}[a, b]$. By Theorems 2.4 and 2.6, $Ag_{jt}^{\alpha} \in C_{\Delta_t}^{(n-m)-1,0}[a, b] \subseteq C_{\Delta_t}^{0,0}[a, b]$, $A_* Ag_{jt}^{\alpha} \in C_{\Delta_t}^{2n-m,0}[a, b] \subseteq C_{\Delta_t}^{n+1,0}[a, b]$, and $K_1 A_* Ag_{jt}^{\alpha} \in C_{\Delta_t}^{n+1,0}[a, b] \subseteq C_{\Delta_t}^{n-1,2}[a, b]$. Thus, by (2.32), $g_{jt}^{\alpha} \in C_{\Delta_t}^{n-1,2}[a, b]$. Continuing this bootstrap argument, we conclude that $g_{jt}^{\alpha} \in C_{\Delta_t}^{n-1,\infty}[a, b]$ or

$$g_{jt}^{\alpha} \in C^{\infty}[a, t] \cap C^{\infty}[t, b] \cap \mathscr{D}(L) \tag{2.35}$$

for $j = 0, 1, \ldots, n - 1$, for $t \in [a, b]$, and for $0 \leqslant m < n$.

Our next objective is to bound various derivatives of g_{jt}^{α}. Fix an integer j with $0 \leqslant j \leqslant n - 1$, fix $\Delta \in \mathscr{P}$, assume $t \in \Delta$, and assume $0 \leqslant m < n$. From (2.35) it is clear that $g_{jt}^{\alpha} \in C_{\Delta}^{n,k}[a, b]$ for $k = 0, 1, 2, \ldots$ and from (2.34) we have that $L^+ G_{jt} \in C_{\Delta}^{n,k}[a, b]$ for $k = 0, 1, 2, \ldots$. Fix an integer k with $0 \leqslant k < \infty$. Then by Theorems 2.4 and 2.6, $Ag_{jt}^{\alpha} \in C_{\Delta}^{n-m,k}[a, b]$, $A_* Ag_{jt}^{\alpha} \in C_{\Delta}^{3n-2m,k}[a, b]$, $K_1 A_* Ag_{jt}^{\alpha} \in C_{\Delta}^{3n-2m,k}[a, b] \subseteq C_{\Delta}^{n,k}[a, b]$, and

$$\|K_1 A_* Ag_{jt}^{\alpha}\|_{C_{\Delta}^{3n-2m,k}} \leqslant b_k \|g_{jt}^{\alpha}\|_{C_{\Delta}^{n,k}}. \tag{2.36}$$

Next, let

$$\Omega_{jl} = \sup_{\substack{a \leqslant t, s \leqslant b \\ t \neq s}} \left| \frac{\partial^{j+l}}{\partial t^j \partial s^l} G(t, s) \right| < \infty \quad \text{for } l = 0, 1, 2, \ldots,$$

so that

$$\|G_{jt}\|_{C_{\Delta}^{0,k}} \leqslant \sum_{l=0}^{k} \Omega_{jl} = d_k. \tag{2.37}$$

From Lemma 2.8, equations (1.3) and (2.24), and norm equivalence we have

$$\|Pg_{jt}^{\alpha}\|_{C_{\Delta}^{n,k}} \leqslant \alpha_k a_n \|g_{jt}^{\alpha}\|_* = \alpha_k a_n \left\|T_{\alpha}^{-1} \mathscr{G}_{jt}^{\alpha}\right\|_* \leqslant e_k / |\alpha|^2 \tag{2.38}$$

for $0 < |\alpha| \leqslant 1$. Then by (2.32), (2.38), (2.36), and (2.37) and Lemma 2.1(ii), $\|g_{jt}^{\alpha}\|_{C_{\Delta}^{n,k}} \leqslant e_k/|\alpha|^2 + \alpha^2 b_k \|g_{jt}^{\alpha}\|_{C_{\Delta}^{n,k}} + \beta_k d_k$ for $0 < |\alpha| \leqslant 1$, where e_k, b_k, β_k and d_k are independent of Δ. In addition, if α satisfies $0 < |\alpha| \leqslant 1/\sqrt{2b_k}$, then

$$\|g_{jt}^{\alpha}\|_{C_{\Delta}^{n,k}} \leqslant \gamma_k / |\alpha|^2 \tag{2.39}$$

for $0 < |\alpha| \leqslant \min\{1, 1/\sqrt{2b_k}\}$ for $j = 0, 1, \ldots, n - 1$, for $t \in \Delta$, and for $k = 0, 1, 2, \ldots$, where γ_k is independent of Δ.

We have the following theorem.

THEOREM 2.10. *Let T be an mth-order differential operator with $0 \leqslant m < n$, and let $g_{jt}^{\alpha} \in \mathscr{D}(T_{\alpha}) = \mathscr{D}(L)$ with $Tg_{jt}^{\alpha} = \mathscr{G}_{jt}^{\alpha}$ for $j = 0, 1, \ldots, n - 1$ and $t \in [a, b]$. Then $g_{jt}^{\alpha} \in H^{2n-j-1}[a, b] \cap C^{\infty}[a, t] \cap C^{\infty}[t, b]$ for $j = 0, 1, \ldots, n - 1$ and $t \in [a, b]$. Moreover, if $\Delta \in \mathscr{P}$ is any partition of $[a, b]$ and $t \in \Delta$, then $g_{jt}^{\alpha} \in C_{\Delta}^{n,k}[a, b]$ for $k = 0, 1, 2, \ldots$, and $\|g_{jt}^{\alpha}\|_{C_{\Delta}^{n,k}} \leqslant \gamma_k/|\alpha|^2$ for $0 < |\alpha| \leqslant \min\{1, 1/\sqrt{2b_k}\}$, for $j = 0, 1, \ldots, n - 1$, and for $k = 0, 1, 2, \ldots$, where γ_k is independent of α and Δ.*

REMARK 2.11. Fix $t \in [a, b]$ and let $\Delta_t \in \mathscr{P}$ be the simple partition of $[a, b]$ given by $\Delta_t = \{a, t, b\}$. Then from (2.39) it follows that

$$\left\|\left(g_{jt}^{\alpha}\right)^{(2n)}\right\|_{\infty} \leqslant \|g_{jt}^{\alpha}\|_{C_{\Delta_t}^{n,n}} \leqslant \gamma_n/|\alpha|^2 \tag{2.40}$$

for $0 < |\alpha| \leqslant \min\{1, 1/\sqrt{2b_k}\}$, for $j = 0, 1, \ldots, n-1$, and for $t \in [a, b]$, where γ_n is independent of α and $t \in [a, b]$.

(F) *L^∞-error estimates for $x_\alpha - \tilde{x}_\alpha$ and $x_0 - \tilde{x}_\alpha$.* Fix a function $y \in L^2[a, b]$, and let $x_0 = L^\mu y$ be the least squares solution of $Lx = y$ introduced above. Let $\hat{y} = (y, 0) \in L^2[a, b] \oplus L^2[a, b]$, and for each $\alpha \neq 0$ let $x_\alpha = T_\alpha^+ \hat{y}$ be the unique least squares solution of $T_\alpha x = \hat{y}$ (see Lemma 1.2).

Let $\mathscr{P}_0 \subseteq \mathscr{P}$ be a family of uniformly graded partitions of $[a, b]$, i.e., there exists a positive constant σ such that

$$h/\underline{h} \leqslant \sigma \quad \text{for all } \Delta \in \mathscr{P}_0. \tag{2.41}$$

Throughout, we will assume $h < 1$ for all $\Delta \in \mathscr{P}_0$.

For $\alpha \neq 0$ and for $\Delta \in \mathscr{P}_0$, let $\tilde{x}_\alpha \in \mathrm{Sp}(2n-1, \Delta, n-1) \cap \mathscr{D}(L)$ be the least squares approximate to x_α, which is determined by (2.4) or (2.5).

For $x \in C^{n-1}[a, b]$ and Δ: $a = t_0 < t_1 < \cdots < t_N = b$ in $\mathscr{P}_0$, it is well known that there exists a unique spline $\bar{x} \in \mathrm{Sp}(2n-1, \Delta, n-1)$ satisfying

$$\bar{x}^{(j)}(t_i) = x^{(j)}(t_i), \qquad 0 \leqslant i \leqslant N, 0 \leqslant j \leqslant n-1. \tag{2.42}$$

The function $\bar{x}$ is the *piecewise Hermite interpolate* of degree $2n-1$ to x [**25**].

The next theorem establishes superconvergence at the knots for $x_\alpha - \tilde{x}_\alpha$, where α is fixed and $h \to 0$.

THEOREM 2.12. *Assume T is an mth-order differential operator with $0 \leqslant m < n$, assume $0 \leqslant k \leqslant n$, and let $\Delta \in \mathscr{P}_0$. If $y \in C_\Delta^{0,k}[a, b]$, then*

$$\left|x_\alpha^{(j)}(t) - \tilde{x}_\alpha^{(j)}(t)\right| \leqslant \left(\gamma_n/|\alpha|^2\right)\|x_0\|_{C_\Delta^{n,k}} h^{n+k}$$

for $0 < |\alpha| \leqslant \min\{\frac{1}{2}, \sqrt{3}/2\sqrt{2\|K\|}, 1/\sqrt{2b_k}\}$, for $t \in \Delta$, and for $j = 0, 1, \ldots, n-1$, where γ_n is independent of α and Δ.

PROOF. Since x_α and $\tilde{x}_\alpha$ belong to $\mathscr{D}(L)$, we have by (2.26) that

$$x_\alpha^{(j)} - \tilde{x}_\alpha^{(j)}(t) = \langle T_\alpha g_{jt}^\alpha, T_\alpha x_\alpha - T_\alpha \tilde{x}_\alpha \rangle \tag{2.43}$$

for $j = 0, 1, \ldots, n-1$ and $t \in [a, b]$. Let $\bar{x}_\alpha$ and $\overline{g_{jt}^\alpha}$ in S_N be the piecewise Hermite interpolates to x_α and g_{jt}^α, resp. By (2.6) equation (2.43) becomes $x_\alpha^{(j)}(t) - \tilde{x}_\alpha^{(j)}(t) = \langle T_\alpha x_\alpha - T_\alpha \tilde{x}_\alpha, T_\alpha \overline{g_{jt}^\alpha} \rangle$ for $j = 0, 1, \ldots, n-1$ and $t \in [a, b]$, and by the Schwarz inequality, (2.7), and the boundedness of T_α, we have

$$\left|x_\alpha^{(j)}(t) - \tilde{x}_\alpha^{(j)}(t)\right| \leqslant \|T_\alpha\|^2 \|x_\alpha - \bar{x}_\alpha\|_{H^n} \left\|g_{jt}^\alpha - \overline{g_{jt}^\alpha}\right\|_{H^n} \tag{2.44}$$

for $j = 0, 1, \ldots, n-1$ and $t \in [a, b]$, where we work with T_α from $\mathscr{D}(L)$ under the H^n-structure into $L^2[a, b] \oplus L^2[a, b]$ under the L^2-product structure. For $0 < |\alpha| \leqslant 1$ it follows that $|T_\alpha x|^2 = \|Lx\|^2 + \alpha^2\|Ax\|^2 \leqslant \|x\|_*^2 \leqslant b_n^2\|x\|_{H^n}^2$ for all $x \in \mathscr{D}(L)$, and hence

$$\|T_\alpha\| \leqslant b_n \quad \text{for } 0 < |\alpha| \leqslant 1. \tag{2.45}$$

Assume $0 \leqslant k \leqslant n$ and $y \in C_\Delta^{0,k}[a,b]$, so that $x_\alpha \in C_\Delta^{n,k}[a,b]$ by Theorem 2.7. If $t \in \Delta$, then from Theorem 2.10 we have $g_{jt}^\alpha \in C_\Delta^{n,n}[a,b]$. Now by (2.45) and [**18**, Theorem 2.1] equation (2.44) becomes

$$\begin{aligned}
&\left|x_\alpha^{(j)}(t) - \tilde{x}_\alpha^{(j)}(t)\right| \\
&\leqslant b_n^2\left(1+\sqrt{b-a}\right)^2\left\{\sum_{i=0}^n \left\|x_\alpha^{(i)} - \overline{x}_\alpha^{(i)}\right\|_\infty\right\}\left\{\sum_{i=0}^n \left\|\left(g_{jt}^\alpha\right)^{(i)} - \left(\overline{g_{jt}^\alpha}\right)^{(i)}\right\|_\infty\right\} \\
&\leqslant b_n^2\left(1+\sqrt{b-a}\right)^2\gamma^2\left\{\sum_{i=0}^n \left\|x_\alpha^{(n+k)}\right\|_\infty h^k\right\}\left\{\sum_{i=0}^n \left\|\left(g_{jt}^\alpha\right)^{(2n)}\right\|_\infty h^n\right\} \\
&\leqslant \frac{\gamma_n}{|\alpha|^2}\left\|x_\alpha^{(n+k)}\right\|_\infty h^{n+k} \quad \text{by (2.40)}
\end{aligned}$$

for $0 < |\alpha| \leqslant 1$, for $j = 0, 1, \ldots, n-1$, and for $t \in \Delta$, where γ_n is independent of α and Δ.

From (2.20) we have $\|x_\alpha^{(n+k)}\|_\infty \leqslant 2(\alpha_k e_n + 1)\|x_0\|_{C_\Delta^{n,k}}$ for $0 < |\alpha| \leqslant \min\{\frac{1}{2}, \sqrt{3}/2\sqrt{2\|K\|}, 1/\sqrt{2b_k}\}$, so that (2.46) becomes

$$\left|x_\alpha^{(j)}(t) - \tilde{x}_\alpha^{(j)}(t)\right| \leqslant \left(\gamma_n/|\alpha|^2\right)\|x_0\|_{C_\Delta^{n,k}} h^{n+k}$$

$$\text{for } 0 < |\alpha| \leqslant \min\left\{\tfrac{1}{2}, \sqrt{3}/2\sqrt{2\|K\|}, 1/\sqrt{2b_k}\right\},$$

for $j = 0, 1, \ldots, n-1$, and for $t \in \Delta$, where γ_n is independent of α and Δ. Q.E.D.

THEOREM 2.13. *Assume T is an mth-order differential operator with $0 \leqslant m < n$, assume $0 \leqslant k \leqslant n$, and let $\Delta \in \mathscr{P}_0$. If $y \in C_\Delta^{0,k}[a,b]$, then*

$$\left\|x_\alpha^{(j)} - \tilde{x}_\alpha^{(j)}\right\|_\infty \leqslant \left(\gamma_n/|\alpha|^2\right)\|x_0\|_{C_\Delta^{n,k}} h^{n+k-j} \tag{2.46}$$

for $0 < |\alpha| \leqslant \min\{\frac{1}{2}, \sqrt{3}/2\sqrt{2\|K\|}, 1/\sqrt{2b_k}\}$ and for $j = 0, 1, \ldots, n-1$, where γ_n is independent of α and Δ.

PROOF. Let $\Delta \in \mathscr{P}_0$ be given by $a = t_0 < t_1 < \cdots < t_N = b$, and let $\overline{x}_\alpha$ be the piecewise Hermite interpolate to x_α. By Theorem 2.7 we know that $x_\alpha \in C_\Delta^{n,k}[a,b]$. From (2.1) and (2.42) we get

$$\left|\overline{x}_\alpha^{(j)}(t) - \tilde{x}_\alpha^{(j)}(t)\right| \leqslant \sum_{i=0}^{N}\sum_{l=0}^{n-1}\left|x_\alpha^{(l)}(t_i) - \tilde{x}_\alpha^{(l)}(t_i)\right|\left|\varphi_{il}^{(j)}(t)\right|$$

for $j = 0, 1, \ldots, n-1$ and $t \in [a,b]$. Since the support of φ_{il} is the interval $[t_{i-1}, t_{i+1}]$, it follows from [**18**, equation 2.5] and Theorem 2.12 that

$$\begin{aligned}
\left|\overline{x}_\alpha^{(j)}(t) - \tilde{x}_\alpha^{(j)}(t)\right| &\leqslant 2\sum_{l=0}^{n-1}\frac{\gamma_n}{|\alpha|^2}\|x_0\|_{C_\Delta^{n,k}} h^{n+k} h^{l-j} \\
&\leqslant \frac{\alpha_n}{|\alpha|^2}\|x_0\|_{C_\Delta^{n,k}} h^{n+k-j}
\end{aligned}$$

for $0 < |\alpha| \leqslant \min\{\frac{1}{2}, \sqrt{3}/2\sqrt{2\|K\|}, 1/\sqrt{2b_k}\}$, for $j = 0, 1, \ldots, n-1$, and for $t \in [a,b]$, where α_n is independent of α and Δ. Taking the supremum over all $t \in [a,b]$ in the above inequality, we get

$$\left\|\overline{x}_\alpha^{(j)} - \tilde{x}_\alpha^{(j)}\right\|_\infty \leqslant \left(\alpha_n/|\alpha|^2\right)\|x_0\|_{C_\Delta^{n,k}} h^{n+k-j} \tag{2.47}$$

for $0 < |\alpha| \leqslant \min\{\frac{1}{2}, \sqrt{3}/2\sqrt{2\|K\|}, 1/\sqrt{2b_k}\}$ and for $j = 0, 1, \ldots, n-1$ where α_n is independent of α and Δ.

From [**18**, Theorem 2.1] and (2.20)

$$\|x_\alpha^{(j)} - \bar{x}_\alpha^{(j)}\|_\infty \leqslant \beta_n \|x_0\|_{C_\Delta^{n,k}} h^{n+k-j} \tag{2.48}$$

for $0 < |\alpha| \leqslant \min\{\frac{1}{2}, \sqrt{3}/2\sqrt{2\|K\|}, 1/\sqrt{2b_k}\}$ and for $j = 0, 1, \ldots, n-1$, where β_n is independent of α and Δ. Let $\gamma_n = \beta_n + \alpha_n$. Then combining (2.47) and (2.48) and using the triangle inequality, we have the desired result. Q.E.D.

The next theorem, which follows immediately from Theorems 2.9 and 2.13, and the triangle inequality, is the main theorem of this section.

THEOREM 2.14. *Let T be an mth-order differential operator with $0 \leqslant m < n$, and let $x_0 = L^\mu y$. Let $\Delta \in \mathscr{P}_0$, and assume $y \in C_\Delta^{0,k}[a, b]$ so that $x_\alpha \in C_\Delta^{n,k}[a, b]$, where $0 \leqslant k \leqslant n$. Then*

$$\|x_0^{(j)} - \tilde{x}_\alpha^{(j)}\|_\infty \leqslant \gamma_n \|x_0\|_{C_\Delta^{n,k}} \left[|\alpha|^2 + h^{n+k-j}/|\alpha|^2\right]$$

for $0 < |\alpha| \leqslant \min\{\frac{1}{2}, \sqrt{3}/2\sqrt{2\|K\|}, 1/\sqrt{2b_k}\}$ and $j = 0, 1, \ldots, n-1$, where γ_n is independent of α and Δ.

III. Regularization and discrete least squares.

(A) *Introduction of discrete least squares.* In this section we work in the Hilbert space $X = Y = Z = L^2[a, b]$ under the standard inner product and norm. As in §II let L be an nth-order differential operator in $L^2[a, b]$, and let T be an mth-order differential operator in $L^2[a, b]$ with $m \leqslant n$. We assume that $\mathscr{D}(L) \subseteq \mathscr{D}(T)$ and $\mathscr{N}(L) \cap \mathscr{N}(T) = \{0\}$.

For fixed $y \in L^2[a, b]$, for fixed $\alpha \neq 0$, and for $\Delta \in \mathscr{P}_0$, let $\hat{y} = (y, 0) \in L^2[a, b] \otimes L^2[a, b]$ and let $\tilde{x}_\alpha \in S_N = \mathrm{Sp}(2n-1, \Delta, n-1) \cap \mathscr{D}(L)$ be the *continuous least squares approximate* to x_α, which is determined by (2.4), or equivalently, by (2.5). Let $\varphi_1, \ldots, \varphi_M$ be a basis for S_N, where $M = n(N+1) - k_0$. If we write $\tilde{x}_\alpha = \sum_{j=1}^m c_j \varphi_j$, then the coefficients $c_1, \ldots, c_m$ satisfy the linear system (2.8). In the event that the functions in (2.8) are even modestly complicated, exact integration is impossible, and we must resort to numerical quadrature. This transforms the continuous least squares problem to a discrete least squares problem.

To set the problem in a specific context, let $\Delta \in \mathscr{P}_0$ be given by Δ: $a = t_0 < t_1 < \cdots < t_N = b$, and for a positive integer l introduce points $t_{i1}, \ldots, t_{il}$ with $t_{i-1} \leqslant t_{i1} < t_{i2} < \cdots < t_{il} \leqslant t_i$ for $i = 1, \ldots, N$. For $\varphi \in C[t_{i-1}, t_i]$ let

$$\int_{t_{i-1}}^{t_i} \varphi(t)\, dt \approx \sum_{j=1}^{l} \omega_{ij} \varphi(t_{ij})$$

be a quadrature rule with positive weights $\omega_{i1}, \ldots, \omega_{il}$ which integrates polynomials of degree $\leqslant 2n-2$ exactly on $[t_{i-1}, t_i]$. Define pseudo-inner products and pseudo-norms on $[t_{i-1}, t_i]$ and $[a, b]$ by $(\varphi, \psi)_{\Delta_i} = \sum_{j=1}^l \omega_{ij} \varphi(t_{ij}) \psi(t_{ij})$, $\|\varphi\|_{\Delta_i} = (\varphi, \varphi)_{\Delta_i}^{1/2}$, and $(\varphi, \psi)_\Delta = \sum_{i=1}^N (\varphi, \psi)_{\Delta_i}$, $\|\varphi\|_\Delta = (\varphi, \varphi)_\Delta^{1/2}$, resp., for $\varphi, \psi \in C_\Delta^{0,0}[a, b]$. Define a pseudo-inner product and pseudo-norm on $C_\Delta^{0,0}[a, b] \oplus C_\Delta^{0,0}[a, b]$ by $\langle (f, g), (\xi, \eta) \rangle_d = (f, \xi)_\Delta + (g, \eta)_\Delta$ and $|(f, g)|_d = \langle (f, g), (f, g) \rangle_d^{1/2}$, resp., for $(f, g), (\xi, \eta) \in C_\Delta^{0,0}[a, b] \oplus C_\Delta^{0,0}[a, b]$.

Assume $y \in C_\Delta^{0,0}[a, b]$, so that $\hat{y} \in C_\Delta^{0,0}[a, b] \oplus C_\Delta^{0,0}[a, b]$. Define a *discrete least squares approximate* to x_α to be any element $\hat{x}_\alpha \in S_N$ satisfying the equation

$$(3.1) \qquad |T_\alpha \hat{x}_\alpha - \hat{y}|_d = \inf_{\varphi \in S_N} |T_\alpha \varphi - \hat{y}|_d.$$

It will be shown that such an element $\hat{x}_\alpha$ exists, and that conditions exist for which $\hat{x}_\alpha$ is unique.

It is well known that $H^n[a, b]$ is a Banach space under the H^n-norm $\|x\|_{H^n} = \sum_{i=0}^{n-1}\|x^{(i)}\|_\infty + \|x^{(n)}\|$ for $x \in H^n[a, b]$. In this section it will also be convenient to work with $H^n[a, b]$ under the norm

$$\|x\|_{S^n} = \left[\sum_{i=0}^{n} \|x^{(i)}\|^2\right]^{1/2} \quad \text{for } x \in H^n[a, b].$$

It can be shown [17, Chapter II] that these two norms are equivalent on $H^n[a, b]$. For each partition $\Delta \in \mathscr{P}_0$ we will also work with the Banach space

$$H_\Delta^{n,k}[a, b] = \left\{ x \in H^n[a, b] | x \in H^{n+k}[t_{i-1}, t_i], 1 \leqslant i \leqslant N \right\}$$

with norm

$$\|x\|_{H_\Delta^{n,k}} = \left[\sum_{i=1}^{N} \|x\|^2_{S^{n+k}[t_{i-1}, t_i]}\right]^{1/2} \quad \text{for } x \in H_\Delta^{n,k}[a, b].$$

(B) *Existence of* $\hat{x}_\alpha$. We now establish the existence of a discrete least squares approximate $\hat{x}_\alpha \in S_N$ to x_α, which is given by (3.1). It is based on the fact that (3.1) is true iff

$$(3.2) \qquad \langle T_\alpha \hat{x}_\alpha - \hat{y}, T_\alpha \varphi \rangle_d = 0 \quad \text{for all } \varphi \in S_N.$$

If a discrete least squares approximate $\hat{x}_\alpha \in S_N$ to x_α exists, then $\hat{x}_\alpha = \sum_{j=1}^{M} \alpha_j \varphi_j$, and from (3.2) the coefficients $\alpha_1, \ldots, \alpha_M$ satisfy the linear system

$$(3.3) \quad \sum_{j=1}^{M} \left[(L\varphi_i, L\varphi_j)_\Delta + \alpha^2 (A\varphi_i, A\varphi_j)_\Delta \right] \alpha_j = (y, L\varphi_i)_\Delta, \qquad 1 \leqslant i \leqslant M.$$

In order to show that (3.3) has a solution $\underline{\alpha} = [\alpha_1 \cdots \alpha_M]^T$, relabel the lN quadrature points as $s_1 = t_1$, $s_2 = t_{12}, \ldots, s_l = t_{1l}$, $s_{l+1} = t_{21}, \ldots, s_{lN} = t_{Nl}$, and relabel the corresponding weights as $\omega_1 = \omega_{11}$, $\omega_2 = \omega_{12}, \ldots, \omega_l = \omega_{1l}$, $\omega_{l+1} = \omega_{21}, \ldots, \omega_{lN} = \omega_{Nl}$. Let $B = [(L\varphi_i, L\varphi_j)_\Delta]$, an $M \times M$ matrix; $C = [(A\varphi_i, A\varphi_j)_\Delta]$, an $M \times M$ matrix; $\underline{\alpha} = [\alpha_1 \cdots \alpha_M]^T$, an $M \times 1$ matrix; $\underline{y} = [(y, L\varphi_1)_\Delta \cdots (y, L\varphi_M)_\Delta]^T$, an $M \times 1$ matrix, so that (3.3) becomes

$$(3.4) \qquad [B + \alpha^2 C]\underline{\alpha} = \underline{y}.$$

Let $E = [L\varphi_j(s_i)] = [\alpha_{ij}]$, an $lN \times M$ matrix; let $F = [A\varphi_j(s_i)] = [\beta_{ij}]$, an $lN \times M$ matrix; let $D = [\delta_{ij}\omega_j]$, an $lN \times lN$ diagonal matrix (δ_{ij} is the Kronecker delta); and let $\underline{f} = [y(s_1) \cdots y(s_{lN})]^T$, an $lN \times 1$ matrix. Then $B = E^TDE$, $C = F^TDF$, and $\underline{y} = E^TD\underline{f}$, so that (3.4) becomes

$$(3.5) \qquad [E^TDE + \alpha^2 F^TDF]\underline{\alpha} = E^TD\underline{f}.$$

Let $G = D^{1/2}E$, let $H = D^{1/2}F$, and let $\underset{\sim}{g} = D^{1/2}\underset{\sim}{f}$. Then (3.5) becomes

$$[G^TG + \alpha^2 H^T H]\underset{\sim}{\alpha} = G^T\underset{\sim}{g}. \tag{3.6}$$

It can be shown that the range $\mathscr{R}(G^T) \subseteq \mathscr{R}(G^TG + \alpha^2 H^T H)$. It follows immediately that a solution $\underset{\sim}{\alpha} = [\alpha_1 \cdots \alpha_M]^T$ to (3.6) exists, and we have proved the existence of a discrete least squares approximate $\hat{x}_\alpha \in S_N$ to x_α.

(C) *Uniqueness of* $\hat{x}_\alpha$. In this section we state conditions under which a discrete least squares approximate $\hat{x}_\alpha \in S_N$ to x_α is unique. Let $\Delta \in \mathscr{P}_0$ be given by $a = t_0 < t_1 < \cdots < t_N = b$, let $\varphi, \psi \in S_N$, and fix i with $1 \leqslant i \leqslant N$. Clearly $L\varphi$, $L\psi$ and $L\varphi \cdot L\psi$ belong to $H^{2n-1}[t_{i-1}, t_i]$. Let $(f, g)_i = \int_{t_{i-1}}^{t_i} f(t)g(t)\,dt$ for all f, $g \in L^2[a, b]$ and let $\lambda(f) = \int_{t_{i-1}}^{t_i} f(t)\,dt - \sum_{j=1}^{l}\omega_{ij}f(t_{ij})$ for $f \in H^{2n-1}[t_{i-1}, t_i]$.

By the assumptions on our quadrature rule we have $\lambda(P) = 0$ for all polynomials P of degree $\leqslant 2n - 2$, and hence, applying the Peano Kernel Theorem [7, p. 70].

$$\begin{aligned}&|(L\varphi, L\psi)_i - (L\varphi, L\psi)_{\Delta_i}|\\ &\quad\leqslant \frac{1}{(2n-2)!}\int_{t_{i-1}}^{t_i}\left|\left(\frac{d}{d\xi}\right)^{2n-1}[L\varphi(\xi)L\psi(\xi)]\right|\left|\lambda_t\left[(t-\xi)_+^{2n-2}\right]\right|d\xi,\end{aligned}$$

where $(t - \xi)_+^k$ equals $(t - \xi)^k$ for $t \geqslant \xi$ and 0 for $t < \xi$, and $\lambda_t[(t - \xi)_+^k]$ means that λ is applied to $(t - \xi)_+^k$ considered as a function of t with ξ fixed. But

$$\begin{aligned}\left|\lambda_t\left[(t-\xi)_+^{2n-2}\right]\right| &\leqslant \int_{t_{i-1}}^{t_i}(t_i - t_{i-1})^{2n-2}\,dt + \sum_{j=1}^{l}\omega_{ij}(t_i - t_{i-1})^{2n-2}\\ &= 2(t_i - t_{i-1})^{2n-1},\end{aligned}$$

and hence, in terms of $h = \max_{1\leqslant i\leqslant N}(t_i - t_{i-1})$ we have

$$|(L\varphi, L\psi)_i - (L\varphi, L\psi)_{\Delta_i}| \leqslant \frac{2h^{2n-1}}{(2n-2)!}\int_{t_{i-1}}^{t_i}\left|\left(\frac{d}{d\xi}\right)^{2n-1}[L\varphi(\xi)L\psi(\xi)]\right|d\xi \tag{3.7}$$

for all $\varphi, \psi \in S_N$. Similarly, for $A = T|\mathscr{D}(L)$ we get

$$|(A\varphi, A\psi)_i - (A\varphi, A\psi)_{\Delta_i}| \leqslant \frac{2h^{2n-1}}{(2n-2)!}\int_{t_{i-1}}^{t_i}\left|\left(\frac{d}{d\xi}\right)^{2n-1}[A\varphi(\xi)A\psi(\xi)]\right|d\xi \tag{3.8}$$

for all $\varphi, \psi \in S_N$.

We are now in a position to establish conditions which guarantee that $\hat{x}_\alpha$ is unique. Let $\varphi, \psi \in S_N$. Then by (3.7) and (3.8),

$$\begin{aligned}&|\langle T_\alpha\varphi, T_\alpha\psi\rangle - \langle T_\alpha\varphi, T_\alpha\psi\rangle_d|\\ &\quad\leqslant \sum_{i=1}^{N}\frac{2h^{2n-1}}{(2n-2)!}\int_{t_{i-1}}^{t_i}\left|\left(\frac{d}{d\xi}\right)^{2n-1}[L\varphi(\xi)L\psi(\xi)]\right|d\xi\\ &\quad\quad+\alpha^2\sum_{i=1}^{N}\frac{2h^{2n-1}}{(2n-2)!}\int_{t_{i-1}}^{t_i}\left|\left(\frac{d}{d\xi}\right)^{2n-1}[A\varphi(\xi)A\psi(\xi)]\right|d\xi.\end{aligned} \tag{3.9}$$

By Leibnitz's rule and the Schwarz inequality,

$$\begin{aligned}(3.10)\qquad &\int_{t_{i-1}}^{t_i}\left|\left(\frac{d}{d\xi}\right)^{2n-1}[L\varphi(\xi)L\psi(\xi)]\right|d\xi\\ &\leqslant \sum_{q=0}^{2n-1}\binom{2n-1}{q}\left[\int_{t_{i-1}}^{t_i}\left|\left(\frac{d}{d\xi}\right)^{q}L\varphi(\xi)\right|^2 d\xi\right]^{1/2}\\ &\quad\cdot\left[\int_{t_{i-1}}^{t_i}\left|\left(\frac{d}{d\xi}\right)^{2n-1-q}L\psi(\xi)\right|^2 d\xi\right]^{1/2}.\end{aligned}$$

Fix q with $0 \leqslant q \leqslant 2n-1$. Then by Leibnitz's rule and the Schwarz inequality in finite dimensions,

$$(3.11)\qquad \left[\int_{t_{i-1}}^{t_i}\left|\left(\frac{d}{d\xi}\right)^{q}L\varphi(\xi)\right|^2 d\xi\right]^{1/2} \leqslant \gamma_1\|\varphi\|_{S^{3n-1}[t_{i-1},t_i]},$$

where γ_1 is independent of Δ and q. Similarly,

$$(3.12)\qquad \left[\int_{t_{i-1}}^{t_i}\left|\left(\frac{d}{d\xi}\right)^{2n-1-q}L\psi(\xi)\right|^2 d\xi\right]^{1/2} \leqslant \gamma_1\|\psi\|_{S^{3n-1}[t_{i-1},t_i]}.$$

Using (3.11) and (3.12), (3.10) becomes

$$(3.13)\qquad \int_{t_{i-1}}^{t_i}\left|\left(\frac{d}{d\xi}\right)^{2n-1}[L\varphi(\xi)L\psi(\xi)]\right|d\xi \leqslant \gamma_2\|\varphi\|_{S^{3n-1}[t_{i-1},t_i]}\|\psi\|_{S^{3n-1}[t_{i-1},t_i]},$$

where γ_2 is independent of Δ. Since $0 \leqslant m \leqslant n$, it can be shown in the same manner that

$$(3.14)\qquad \int_{t_{i-1}}^{t_i}\left|\left(\frac{d}{d\xi}\right)^{2n-1}[A\varphi(\xi)A\psi(\xi)]\right|d\xi \leqslant \gamma_3\|\varphi\|_{S^{3n-1}[t_{i-1},t_i]}\|\psi\|_{S^{3n-1}[t_{i-1},t_i]},$$

where γ_3 is independent of Δ. Now utilizing (3.13) and (3.14), the Schwarz inequality and the fact that $\varphi, \psi \in \mathrm{Sp}(2n-1, \Delta, n-1)$, estimate (3.9) becomes

$$\begin{aligned}(3.15)\qquad &|\langle T_\alpha\varphi, T_\alpha\psi\rangle - \langle T_\alpha\varphi, T_\alpha\psi\rangle_d|\\ &\leqslant (2/(2n-2)!)\left(\gamma_2+\gamma_3\alpha^2\right)h^{2n-1}\|\varphi\|_{H_\Delta^{0,2n-1}}\|\psi\|_{H_\Delta^{0,2n-1}},\end{aligned}$$

where γ_2 and γ_3 are independent of α and Δ. Finally, invoking [**18**, Lemma 3.3, p. 1156], which follows from Schmidt's inequality [**2**, pp. 734–736] and Markov's inequality [**27**, pp. 507–515], and using the equivalence of the norms $\|\ \|_{S^n}$ and $\|\ \|_{H^n}$, we get

$$(3.16)\qquad |\langle T_\alpha\varphi, T_\alpha\psi\rangle - \langle T_\alpha\varphi, T_\alpha\psi\rangle_d| \leqslant \gamma_4(1+\alpha^2)h\|\varphi\|_{H^n}\|\psi\|_{H^n}$$

for all $\varphi, \psi \in S_n$, where γ_4 is independent of α and Δ.

Recall that T_α^{-1} is bounded from $\mathscr{R}(T_\alpha)$ under the L^2-product structure into $\mathscr{D}(L)$ under the H^n-structure, so that by the norm equivalence and (1.3) we have $\|T_\alpha^{-1}z\|_{H^n} \leqslant \gamma|z|/|\alpha|$ for all $z \in \mathscr{R}(T_\alpha)$ and $0 < |\alpha| \leqslant 1$, or $|T_\alpha x| \geqslant |\alpha|\|x\|_{H^n}/\gamma$ ($\gamma > 0$) for all $x \in \mathscr{D}(T_\alpha) = \mathscr{D}(L)$ and for $0 < |\alpha| \leqslant 1$. Then it follows from (3.16) that $|T_\alpha\varphi|_d^2 \geqslant [|\alpha|^2/\gamma^2 - \gamma_4(1+|\alpha|^2)h]\|\varphi\|_{H^n}^2$ for all $\varphi \in S_N$ and for $0 < |\alpha| \leqslant 1$, where γ_4 is independent of α and Δ (but dependent on the mesh bound σ). For all h

sufficiently small we obtain

$$(3.17) \qquad |T_\alpha\varphi|_d \geqslant \big(|\alpha|/\gamma\big)\|\varphi\|_{H^n}$$

for all $\varphi \in S_N$ and for $0 < |\alpha| \leqslant 1$, where $\gamma > 0$ is independent of α and Δ.

We have the following theorem concerning the existence and uniqueness of $\hat{x}_\alpha$, which follows from (3.2) and (3.17).

THEOREM 3.1. *Let $\Delta \in \mathscr{P}_0$ be given by Δ: $a = t_0 < t_1 < \cdots < t_N = b$, let $y \in C_\Delta^{0,0}[a, b]$, so that $\hat{y} = (y, 0) \in C_\Delta^{0,0}[a, b] \oplus C_\Delta^{0,0}[a, b]$, and assume that the quadrature rule associated with the pseudo-inner product $(\ ,\ )_{\Delta_i}$ integrates polynomials of degree $\leqslant 2n - 2$ exactly on $[t_{i-1}, t_i]$ for $i = 1,\dots,N$. Then a discrete least squares approximate $\hat{x}_\alpha \in S_N$ to x_α, which is given by (3.1), exists and is unique for all h sufficiently small.*

(D) *The quasi-discrete least squares approximate to x_α.* In order to obtain error estimates for $x_\alpha^{(j)} - \hat{x}_\alpha^{(j)}$, we will assume that $y \in C_\Delta^{0,n+k}[a, b]$, so that $x_\alpha \in C_\Delta^{n,n+k}[a, b]$. This additional smoothness is also required in [**1**, **8**, **9**, **18**, and **26**]. We begin by introducing a new spline approximate to x_α. Define a *quasi-discrete least squares approximate* to x_α to be any element $\overset{\triangle}{x}_\alpha \in S_N$ satisfying

$$(3.18) \qquad \left|T_\alpha \overset{\triangle}{x}_\alpha - T_\alpha x_\alpha\right|_d = \inf_{\varphi \in S_N} |T_\alpha\varphi - T_\alpha x_\alpha|_d.$$

The following equation also characterizes a quasi-discrete least squares approximate to x_α. Let $\Delta \in \mathscr{P}_0$, and assume $y \in C_\Delta^{0,0}[a, b]$, so that $x_\alpha \in C_\Delta^{n,0}[a, b]$ and $T_\alpha x_\alpha \in C_\Delta^{0,0}[a, b] \oplus C_\Delta^{0,0}[a, b]$. Then for $\overset{\triangle}{x}_\alpha \in S_N$, equation (3.18) is satisfied iff

$$(3.19) \qquad \langle T_\alpha \overset{\triangle}{x}_\alpha - T_\alpha x_\alpha, T_\alpha\varphi\rangle_d = 0 \quad \text{for all } \varphi \in S_N.$$

Let $\varphi_1,\dots,\varphi_M$ be a basis for S_N, and let $\overset{\triangle}{x}_\alpha = \sum_{j=1}^M \overset{\triangle}{\alpha}_j\varphi_j$ satisfy (3.19). Then

$$(3.20) \qquad \sum_{j=1}^{M} \left[\big(L\varphi_i, L\varphi_j\big)_\Delta + \alpha^2\big(A\varphi_i, A\varphi_j\big)_\Delta\right]\overset{\triangle}{\alpha}_j = \big(Lx_\alpha, L\varphi_i\big)_\Delta + \alpha^2\big(Ax_\alpha, A\varphi_i\big)_\Delta, \qquad 1 \leqslant i \leqslant M.$$

In a fashion analogous to that used in establishing the existence of $\hat{x}_\alpha$, the discrete least squares approximate to x_α, it can be shown that $\underline{\overset{\triangle}{\alpha}} = [\overset{\triangle}{\alpha}_1 \cdots \overset{\triangle}{\alpha}_M]^T$ satisfies the matrix equation

$$(3.21) \qquad [G^TG + \alpha^2H^TH]\underline{\overset{\triangle}{\alpha}} = G^T\underline{h} + \alpha^2H^T\underline{k},$$

where $\underline{h} = D^{1/2}[Lx_\alpha(s_1) \cdots Lx_\alpha(S_{lN})]^T$ and $\underline{k} = D^{1/2}[Ax_\alpha(s_1) \cdots Ax_\alpha(S_{lN})]^T$.

The existence of $\overset{\triangle}{x}_\alpha$ follows from (3.21) and the fact that the ranges $\mathscr{R}(G^T)$ and $\mathscr{R}(H^T)$ are contained in $\mathscr{R}(G^TG + \alpha^2H^TH)$. The following theorem states that it is unique under appropriate conditions. Its proof is based on (3.19) and (3.17).

Theorem 3.2. *Let $\Delta \in \mathscr{P}_0$ be given by Δ: $a = t_0 < t_1 < \cdots < t_N = b$, let $y \in C_\Delta^{0,0}[a, b]$, so that $T_\alpha x_\alpha \in C_\Delta^{0,0}[a, b] \oplus C_\Delta^{0,0}[a, b]$, and assume that the quadrature rule associated with the pseudo-inner product $(\ ,\ )_{\Delta_i}$ integrates polynomials of degree $\leqslant 2n - 2$ exactly on $[t_{i-1}, t_i]$ for $i = 1, \ldots, N$. Then a quasi-discrete least squares approximate $\overset{\triangle}{x}_\alpha \in S_N$ to x_α, which is given by* (3.18), *exists and is unique for all h sufficiently small.*

We now proceed to establish L^∞-estimates for the quantities $x_\alpha^{(j)} - \overset{\triangle}{x}{}_\alpha^{(j)}$ for $j = 0, 1, \ldots, n-1$. This requires superconvergence estimate at the knots of the partition, which will follow from the following three lemmas.

Lemma 3.3. *Let $\Delta \in \mathscr{P}_0$, let $0 \leqslant k \leqslant n$, and let $x \in \mathscr{D}(L) \cap C_\Delta^{n,k}[a, b]$. If $\bar{x} \in S_N$ is the piecewise Hermite interpolate to x, then $|T_\alpha x - T_\alpha \bar{x}|_d \leqslant \gamma \|x^{(n+k)}\|_\infty h^k$ for $0 < |\alpha| \leqslant 1$, where γ is independent of α and Δ.*

Proof. We have from the definition of our pseudo-inner product and [**18**, Theorem 2.1, p. 1154] that

$$|T_\alpha x - T_\alpha \bar{x}|_d^2 \leqslant \left[\|Lx - L\bar{x}\|_\infty^2 + \alpha^2 \|Ax - A\bar{x}\|_\infty^2\right] \sum_{i=1}^{N} \sum_{j=1}^{l} \omega_{ij}$$
$$= (b-a)\left[\left\| \sum_{i=0}^{n} a_i [x^{(i)} - \bar{x}^{(i)}] \right\|_\infty^2 + \alpha^2 \left\| \sum_{i=0}^{m} b_i [x^{(i)} - \bar{x}^{(i)}] \right\|_\infty^2 \right]$$
$$\leqslant \gamma^2 \|x^{(n+k)}\|_\infty^2 h^{2k}$$

for $0 < |\alpha| \leqslant 1$, where γ is independent of α and Δ. Q.E.D.

Lemma 3.4. *Let $\Delta \in \mathscr{P}_0$, let $0 \leqslant k \leqslant n$, and let $y \in C_\Delta^{0,n+k}[a, b]$, so that $x_\alpha \in C_\Delta^{n,n+k}[a, b]$. Then $\|Lx_\alpha - L\overset{\triangle}{x}_\alpha\|_{C_\Delta^{0,n+k}} \leqslant (\gamma/|\alpha|)\|x_\alpha\|_{C_\Delta^{n,n+k}}$ for all h sufficiently small and $0 < |\alpha| \leqslant 1$, where γ is independent of α and Δ.*

Proof. Let $\bar{x}_\alpha \in S_N$ be the piecewise Hermite interpolate to x_α. Then by Leibnitz's rule,

$$\left\|Lx_\alpha - L\overset{\triangle}{x}_\alpha\right\|_{C_\Delta^{0,n+k}} \leqslant \gamma_1 \left[\|x_\alpha - \bar{x}_\alpha\|_{C_\Delta^{n,n+k}} + \left\|\bar{x}_\alpha - \overset{\triangle}{x}_\alpha\right\|_{C_\Delta^{n,n+k}}\right], \tag{3.22}$$

where γ_1 is independent of α and Δ. Also, by [**18**, Theorem 2.1, p. 1154]

$$\|x_\alpha - \bar{x}_\alpha\|_{C_\Delta^{n,n+k}} \leqslant \sum_{i=0}^{2n-1} \gamma_2 \|x_\alpha^{(2n-1)}\|_\infty h^{2n-1-i} + \sum_{i=0}^{k} \|x_\alpha^{(2n+i)}\|_\infty \tag{3.23}$$
$$\leqslant \gamma_3 \|x_\alpha\|_{C_\Delta^{n,n+k}},$$

where γ_3 is independent of α and Δ, and

$$\left\|\bar{x}_\alpha - \overset{\triangle}{x}_\alpha\right\|_{C_\Delta^{n,n+k}} = \left\|\bar{x}_\alpha - \overset{\triangle}{x}_\alpha\right\|_{C_\Delta^{0,2n-1}} \leqslant \gamma_4 h^{-n} \left\|\bar{x}_\alpha - \overset{\triangle}{x}_\alpha\right\|_{C^{n-1}} \tag{3.24}$$

by [**18**, Lemma 3.3, p. 1156], where γ_4 is independent of α and Δ.

Finally, by (3.17), (3.18), and Lemma 3.3,

$$(3.25)\qquad \left\|\bar{x}_\alpha - \overset{\triangle}{x}_\alpha\right\|_{C^{n-1}} \leqslant \left\|\bar{x}_\alpha - \overset{\triangle}{x}_\alpha\right\|_{H^n} \leqslant \left(\gamma_5/|\alpha|\right)\left[|T_\alpha x_\alpha - T_\alpha \bar{x}_\alpha|_d + \left|T_\alpha \overset{\triangle}{x}_\alpha - T_\alpha x_\alpha\right|_d\right] \leqslant \left(\gamma_6/|\alpha|\right)\|x_\alpha\|_{C_\Delta^{n,n+k}} h^n$$

for $0 < |\alpha| \leqslant 1$ and h sufficiently small, where γ_6 is independent of α and Δ.

Combining (3.22)–(3.25), we get the desired result. Q.E.D.

The following lemma is proved in a similar fashion.

LEMMA 3.5. *Let $\Delta \in \mathscr{P}_0$, let $0 \leqslant k \leqslant n$, and let $y \in C_\Delta^{0,n+k}[a, b]$, so that $x_\alpha \in C_\Delta^{n,n+k}[a, b]$. Then $\|Ax_\alpha - A\overset{\triangle}{x}_\alpha\|_{C_\Delta^{0,n+k}} \leqslant (\gamma/|\alpha|)\|x_\alpha\|_{C_\Delta^{n,n+k}}$ for all h sufficiently small and $0 < |\alpha| \leqslant 1$, where γ is independent of α and Δ.*

We can now state the following superconvergence result. It requires additional precision of the quadrature rule.

THEOREM 3.6. *Let $\Delta \in \mathscr{P}_0$ be given by Δ: $a = t_0 < t_1 < \cdots < t_N = b$, let $0 \leqslant k \leqslant n$, and let $y \in C_\Delta^{0,n+k}[a, b]$, so that $x_\alpha \in C_\Delta^{n,n+k}[a, b]$. Assume that the quadrature rule associated with the pseudo-inner product $(\ ,\ )_{\Delta_i}$ integrates polynomials of degree $\leqslant 2n - 1$ exactly on $[t_{i-1}, t_i]$ for $i = 1, \ldots, N$. Then*

$$\left|x_\alpha^{(j)}(t_i) - \overset{\triangle}{x}{}_\alpha^{(j)}(t_i)\right| \leqslant \left(\gamma/|\alpha|^3\right)\|x_\alpha\|_{C_\Delta^{n,n+k}} h^{n+k}$$

for h sufficiently small, for $0 < |\alpha| \leqslant 1$, for $0 \leqslant i \leqslant N$, and for $0 \leqslant j \leqslant n - 1$, where γ is independent of α and Δ.

PROOF. Fix integers i and j with $0 \leqslant i \leqslant N$ and $0 \leqslant j \leqslant n - 1$. By (2.26), $x_\alpha^{(j)}(t_i) - \overset{\triangle}{x}{}_\alpha^{(j)}(t_i) = \langle \mathscr{G}_{jt_i}^\alpha, T_\alpha x_\alpha - T_\alpha \overset{\triangle}{x}_\alpha\rangle = \langle \mathscr{G}_{jt_i}^\alpha, T_\alpha x_\alpha - T_\alpha \overset{\triangle}{x}_\alpha\rangle + E(t_i)$, where $E(t_i) = \langle \mathscr{G}_{jt_i}^\alpha, T_\alpha x_\alpha - T_\alpha \overset{\triangle}{x}_\alpha\rangle - \langle \mathscr{G}_{jt_i}^\alpha, T_\alpha x_\alpha - T_\alpha \overset{\triangle}{x}_\alpha\rangle_d$. Recall that $g_{jt_i}^\alpha = T^{-1}\mathscr{G}_{jt_i}^d \in C_\Delta^{n,n}[a, b]$ by Theorem 2.10, and let $\overline{g_{jt_i}^\alpha} \in S_N$ be the piecewise Hermite interpolate to $g_{jt_i}^\alpha$. Then by (3.19), the Schwarz inequality (in a pseudo-inner product space), Lemma 3.3, and equations (3.18) and (2.40) we have

$$(3.26)\qquad \left|\langle \mathscr{G}_{jt_i}, T_\alpha x_\alpha - T_\alpha \overset{\triangle}{x}_\alpha\rangle_d\right| \leqslant \gamma_1\left\|\left(g_{jt_i}^\alpha\right)^{(2n)}\right\|_\infty h^n \gamma_1 \|x_\alpha^{(2n)}\|_\infty h^n \leqslant \left(\gamma_2/|\alpha|^2\right)\|x_\alpha\|_{C_\Delta^{n,n+k}} h^{n+k},$$

where γ_2 is independent of α and Δ.

We next investigate the quadrature error $E(t_i)$. Since $x_\alpha \in C_\Delta^{n,n+k}[a, b]$, it follows that $Lx_\alpha \in C_\Delta^{0,n+k}[a, b]$, so that $Lx_\alpha \in C^{n+k}[t_{i-1}, t_i] \subseteq H^{n+k}[t_{i-1}, t_i]$, and $Ax_\alpha \in C_\Delta^{n-m,n+k}[a, b] \subseteq C_\Delta^{0,n+k}[a, b]$, so that $Ax_\alpha \in C^{n+k}[t_{i-1}, t_i] \subseteq H^{n+k}[t_{i-1}, t_i]$. If $\varphi \in S_N$, then $L\varphi$, $A\varphi \in C^\infty[t_{i-1}, t_i]$ so that $L\varphi$, $A\varphi \in H^{n+k}[t_{i-1}, t_i]$. By assumption, the quadrature rule is exact for polynomials of degree $\leqslant n + k - 1$. Thus, by the Peano Kernel Theorem [7, p. 70], Leibnitz's rule, and

Theorem 2.10 (see the derivation of (3.7))

$$|E(t_i)| \leqslant \sum_{r=1}^{N} \left|\lambda\left(Lg_{jt_i}^{\alpha}\left[Lx_\alpha - L\overset{\Delta}{x}_\alpha\right]\right)\right| + \alpha^2 \sum_{r=1}^{N} \left|\lambda\left(Ag_{jt_i}^{\alpha}\left[Ax_\alpha - A\overset{\Delta}{x}_\alpha\right]\right)\right|$$

$$\leqslant \frac{(b-a)\gamma_3}{(n+k-1)!}\frac{h^{n+k}}{|\alpha|^2}\left\|Lx_\alpha - L\overset{\Delta}{x}_\alpha\right\|_{C_\Delta^{0,n+k}}$$

$$+ \frac{(b-a)\gamma_4}{(n+k-1)!}h^{n+k}\left\|Ax_\alpha - A\overset{\Delta}{x}_\alpha\right\|_{C_\Delta^{0,n+k}}.$$

Finally, by Lemmas 3.4 and 3.5, we get

$$(3.27) \qquad |E(t_i)| \leqslant \left(\gamma_5/|\alpha|^3\right)\|x_\alpha\|_{C_\Delta^{n,n+k}} h^{n+k},$$

where γ_5 is independent of α and Δ.

Combining (3.26) and (3.27), we get the desired result. Q.E.D.

The principal error estimates for $x_\alpha^{(j)} - \overset{\Delta}{x}{}_\alpha^{(j)}$, $0 \leqslant j \leqslant n-1$, are established in the next theorem.

THEOREM 3.7. *Let $\Delta \in \mathscr{P}_0$ be given by Δ: $a = t_0 < t_1 < \cdots < t_N = b$, let $0 \leqslant k \leqslant n$, and let $y \in C_\Delta^{0,n+k}[a,b]$, so that $x_\alpha \in C_\Delta^{n,n+k}[a,b]$. Assume that the quadrature rule associated with the pseudo-inner product $(\ ,\ )_{\Delta_i}$ integrates polynomials of degree $\leqslant 2n-1$ exactly on $[t_{i-1}, t_i]$ for $i = 1,\ldots,N$. Then*

$$\left\|x_\alpha^{(j)} - \overset{\Delta}{x}{}_\alpha^{(j)}\right\|_\infty \leqslant \left(\gamma/|\alpha|^3\right)\|x_\alpha\|_{C_\Delta^{n,n+k}} h^{n+k-j}$$

for all h sufficiently small, for $0 < |\alpha| \leqslant 1$, and for $0 \leqslant j \leqslant n-1$, where γ is independent of α and Δ.

PROOF. Fix an integer j with $0 \leqslant j \leqslant n-1$, let $\bar{x}_\alpha \in S_N$ be the piecewise Hermite interpolate to x_α, and let $\{\varphi_{rs} | 0 \leqslant r \leqslant N, 0 \leqslant s \leqslant n-1\}$ be the cardinal basis for $\mathrm{Sp}(2n-1, \Delta, n-1)$. We know that the support of φ_{rs} is $[t_{r-1}, t_{r+1}]$, and (2.1) gives us

$$\bar{x}_\alpha(t) = \sum_{r=0}^{N}\sum_{s=0}^{n-1} x_\alpha^{(s)}(t_r)\varphi_{rs}(t)$$

and

$$\overset{\Delta}{x}_\alpha(t) = \sum_{r=0}^{N}\sum_{s=0}^{n-1} \overset{\Delta}{x}{}_\alpha^{(s)}(t_r)\varphi_{rs}(t)$$

for all $t \in [a,b]$. Then by Theorem 3.6 and [**18**, equations 2.5] it follows that

$$\left|\bar{x}_\alpha^{(j)}(t) - \overset{\Delta}{x}{}_\alpha^{(j)}(t)\right| \leqslant \left(\gamma_1/|\alpha|^3\right)\|x_\alpha\|_{C_\Delta^{n,n+k}} h^{n+k-j}$$

for all $t \in [a,b]$, so that

$$(3.28) \qquad \left\|\bar{x}_\alpha^{(j)} - \overset{\Delta}{x}{}^{(j)}\right\|_{\alpha\infty} \leqslant \left(\gamma_1/|\alpha|^3\right)\|x_\alpha\|_{C_\Delta^{n,n+k}} h^{n+k-j},$$

where γ_1 is independent of α and Δ. The conclusion follows from [**18**, Theorem 2.1, p. 1154], equation (3.28), and the triangle inequality. Q.E.D.

The following estimates for the higher order derivatives $x_\alpha^{(j)} - \overset{\triangle}{x}{}_\alpha^{(j)}$, $n \leqslant j \leqslant n + k$, are indirect consequences of Markov's inequality [27, pp. 507–515].

THEOREM 3.8. *Under the hypotheses of Theorem* 3.7,

$$\left\| x_\alpha^{(j)} - \overset{\triangle}{x}{}_\alpha^{(j)} \right\|_\infty \leqslant \left(\gamma / |\alpha|^3 \right) \|x_\alpha\|_{C_\Delta^{n,n+k}} h^{n+k-j}$$

for all h sufficiently small, for $0 < |\alpha| \leqslant 1$, *and for* $n \leqslant j \leqslant n + k$, *where* γ *is independent of* α *and* Δ.

PROOF. The result follows immediately from [18, Theorem 2.1, p. 1154, and Lemma 3.3, p. 1156], equation (3.28), and the triangle inequality. Q.E.D.

(E) L^∞*-error estimates for* $\overset{\triangle}{x}{}_\alpha - \hat{x}_\alpha$ *and* $\hat{x}_\alpha - x_0$. In this section we first establish L^∞-error estimates for $\overset{\triangle}{x}{}_\alpha^{(j)} - \hat{x}_\alpha^{(j)}$, where $\hat{x}_\alpha$ is the discrete least squares approximate to x_α given by (3.1), and $\overset{\triangle}{x}{}_\alpha$ is the quasi-discrete least squares approximate to x_α given by (3.18). These estimates, combined with those in Theorems 2.9 and 3.7, will give the desired estimates for $\hat{x}_\alpha^{(j)} - x_0^{(j)}$. As in (D) we require that $y \in C_\Delta^{0,n+k}[a, b]$. We need the following preliminary result.

THEOREM 3.9. *Under the hypotheses of Theorem* 3.7,

$$\left| T_\alpha \overset{\triangle}{x}{}_\alpha - T_\alpha \hat{x}_\alpha \right|_d \leqslant \left(\gamma / |\alpha| \right) \left[\|x_\alpha\|_{C_\Delta^{n,n+k}} + \|y\|_{C_\Delta^{0,n+k}} \right] h^k$$

for all h sufficiently small and $0 < |\alpha| \leqslant 1$, *where* γ *is independent of* α *and* Δ.

PROOF. We have from equations (3.2), (3.19), and (2.3),

$$\begin{aligned} &|T_\alpha \overset{\triangle}{x}{}_\alpha - T_\alpha \hat{x}_\alpha|_d^2 \\ &\quad = \langle T_\alpha x_\alpha - \hat{y}, T_\alpha \overset{\triangle}{x}{}_\alpha - T_\alpha \hat{x}_\alpha \rangle_d - \langle T_\alpha x_\alpha - \hat{y}, T_\alpha \overset{\triangle}{x}{}_\alpha - T_\alpha \hat{x}_\alpha \rangle. \end{aligned} \tag{3.29}$$

From (3.29), the definition of the pseudo-inner product, and the Peano Kernel Theorem [7, p. 70], it follows that

$$\begin{aligned} &\left| T_\alpha \overset{\triangle}{x}{}_\alpha - T_\alpha \hat{x}_\alpha \right|_d^2 \\ &\quad \leqslant \sum_{r=1}^N \left| \lambda\left([L x_\alpha - y] \left[L \overset{\triangle}{x}{}_\alpha - L \hat{x}_\alpha \right] \right) \right| + \alpha^2 \sum_{r=1}^N \left| \lambda\left(A x_\alpha \left[A \overset{\triangle}{x}{}_\alpha - A \hat{x}_\alpha \right] \right) \right| \\ &\quad \leqslant \gamma_1 h^{n+k} \left[\|x_\alpha\|_{C_\Delta^{n,n+k}} + \|y\|_{C_\Delta^{0,n+k}} \right] \left\| L \overset{\triangle}{x}{}_\alpha - L \hat{x}_\alpha \right\|_{C_\Delta^{0,n+k}} \\ &\quad\quad + \gamma_2 \alpha^2 h^{n+k} \|x_\alpha\|_{C_\Delta^{n,n+k}} \left\| A \overset{\triangle}{x}{}_\alpha - A \hat{x}_\alpha \right\|_{C_\Delta^{0,n+k}}, \end{aligned} \tag{3.30}$$

where γ_1 and γ_2 are independent of α and Δ.

From Lemma 2.1(i), the proof of Lemma 3.4 (see (3.24)), and (3.17) we have

$$\left\|L\overset{\Delta}{x}_\alpha - L\hat{x}_\alpha\right\|_{C_\Delta^{0,n+k}} \leqslant (\gamma_3/|\alpha|)h^{-n}\left|T_\alpha\overset{\Delta}{x}_\alpha - T_\alpha\hat{x}_\alpha\right|_d \tag{3.31}$$

and

$$\left\|A\overset{\Delta}{x}_\alpha - A\hat{x}_\alpha\right\|_{C_\Delta^{0,n+k}} \leqslant (\gamma_4/|\alpha|)h^{-n}\left|T_\alpha\overset{\Delta}{x}_\alpha - T_\alpha\hat{x}_\alpha\right|_d \tag{3.32}$$

for $0 < |\alpha| \leqslant 1$ and h sufficiently small, where γ_3 and γ_4 are independent of α and Δ. Combine (3.30), (3.31), and (3.32) for the desired result. Q.E.D.

THEOREM 3.10. *Under the hypotheses of Theorem* 3.7,

$$\left\|\overset{\Delta}{x}{}_\alpha^{(j)} - \hat{x}_\alpha^{(j)}\right\|_\infty \leqslant (\gamma/|\alpha|^2)\left[\|x_\alpha\|_{C_\Delta^{n,n+k}} + \|y\|_{C_\Delta^{0,n+k}}\right]h^k$$

for all h sufficiently small, for $0 < |\alpha| \leqslant 1$, and for $0 \leqslant j \leqslant n-1$, where γ is independent of α and Δ.

PROOF. The result follows from (3.17) and Theorem 3.9. Q.E.D.

We now present the main result of §III.

THEOREM 3.11. *Let T be an mth-order differential operator with $0 \leqslant m < n$ and let $x_0 = L^\mu y$. Let $\Delta \in \mathscr{P}_0$ be given by Δ: $a = t_0 < t_1 < \cdots < t_N = b$, let $0 \leqslant k \leqslant n$, and let $y \in C_\Delta^{0,n+k}[a,b]$, so that $x_\alpha \in C_\Delta^{n,n+k}[a,b]$. Assume that the quadrature rule associated with the pseudo-inner product $(\ ,\ )_{\Delta_i}$ integrates polynomials of degree $\leqslant 2n-1$ exactly on $[t_{i-1}, t_i]$ for $i = 1,\ldots,N$. Let $\hat{x}_\alpha \in S_N$ be the discrete least squares approximate to x_α given by* (3.1). *Then*

$$\|\hat{x}_\alpha^{(j)} - x_0^{(j)}\|_\infty \leqslant \gamma\left[\|x_0\|_{C_\Delta^{n,n+k}} + \|y\|_{C_\Delta^{0,n+k}}\right]\left[\frac{h^k}{|\alpha|^2} + \frac{h^{n+k-j}}{|\alpha|^3} + |\alpha|^2\right]$$

for all h sufficiently small, for $0 \leqslant |\alpha| \leqslant \{\frac{1}{2}, \sqrt{3}/2\sqrt{2\|K\|}, 1/\sqrt{2b_k}\}$, and for $0 \leqslant j \leqslant n-1$, where γ is independent of α and Δ.

PROOF. This follows from Theorems 3.10, 3.7, 2.9 and the triangle inequality. Q.E.D.

BIBLIOGRAPHY

1. U. Ascher, *Discrete least squares approximations for ordinary differential equations*, SIAM J. Numer. Anal. **15** (1978), 478–496.

2. R. Bellman, *A note on an inequality of E. Schmidt*, Bull. Amer. Math. Soc. **50** (1944), 734–736.

3. F. J. Beutler and W. L. Root, *The operator pseudoinverse in control and systems identification*, Generalized Inverses and Applications (M. Z. Nashed, ed.), Academic Press, New York, 1976, pp. 397–494.

4. L. Cesari, *Nonlinear analysis*, Lecture Notes, C.I.M.E., Bressanone, 1972.

5. ______, *Functional analysis, nonlinear differential equations, and the alternative method*, Nonlinear Functional Analysis and Differential Equations (L. Cesari, R. Kannan and J. D. Schur, eds.), Marcel Dekker, New York, 1976, pp. 1–197.

6. S. D. Conte and C. deBoor, *Elementary numerical analysis*, 2nd ed., McGraw-Hill, New York, 1972.

7. P. J. Davis, *Interpolation and approximation*, Dover, New York, 1975.

8. C. deBoor and B. Swartz, *Collocation at Gaussian points*, SIAM J. Numer. Anal. **10** (1973), 582–606.

9. J. Douglas, Jr. and T. Dupont, *Galerkin approximations for the two point boundary value problem using continuous, piecewise polynomial spaces*, Numer. Math. **22** (1974), 99–109.

10. J. K. Hale, *Applications of alternative problems*, Lecture Notes, Brown Univ., Providence, R.I., 1971.

11. E. Isaacson and H. B. Keller, *Analysis of numerical methods*, Wiley, New York, 1966.

12. V. V. Ivanov, *The theory of approximate methods and their application to the numerical solution of singular integral equations*, Noordhoff, Leyden, 1976.

13. R. Kannan and J. Locker, *Continuous dependence of least squares solutions of linear boundary value problems*, Proc. Amer. Math. Soc. **59** (1976), 107–110.

14. T. Kato, *Perturbation theory for linear operators*, 2nd ed., Springer-Verlag, Berlin, Heidelberg and New York, 1976.

15. D. Kouba, *Regularization for nth-order linear boundary value problems using mth-order differential operators*, Doctoral Dissertation, Colorado State Univ., 1982.

16. J. Locker, *The generalized Green's function for an nth-order linear differential operator*, Trans. Amer. Math. Soc. **288** (1977), 243–268.

17. ______, *Functional analysis and two-point differential operators*, Lecture Notes, Colorado State Univ., Fort Collins, 1982.

18. J. Locker and P. M. Prenter, *Optimal L^2 and L^∞ error estimates for continuous and discrete least squares methods for boundary value problems*, SIAM J. Numer. Anal. **15** (1978), 1151–1160.

19. ______, *Regularization with differential operators*. I. *General theory*, J. Math. Anal. Appl. **74** (1980), 504–529.

20. ______, *Regularization with differential operators*. II. *Weak least squares finite element solutions to first kind integral equations*, SIAM J. Numer. Anal. **17** (1980), 247–267.

21. ______, *Regularization and linear boundary value problems* (to appear).

22. ______, *Representors and superconvergence of least squares finite element approximates*, Applicable Anal. (to appear).

23. M. Z. Nashed, *Approximate regularized solutions to improperly posed linear integral and operator equations*, Constructive and Computational Methods for Differential and Integral Equations (D. Colton and R. P. Gilbert, eds.), Lecture Notes in Math., vol. 430, Springer-Verlag, Berlin, Heidelberg and New York, 1974, pp. 289–332.

24. D. L. Phillips, *A technique for the numerical solution of certain integral equations of the first kind*, J. Assoc. Comput. Mach. **9** (1962), 84–97.

25. P. M. Prenter, *Splines and variational methods*, Wiley, New York, 1975.

26. P. Sammon, *The discrete least squares method*, Math. Comp. **31** (1977), 60–65.

27. M. H. Schultz, *Error bounds for polynomial spline interpolation*, Math. Comp. **24** (1970), 507–515.

28. A. N. Tikhonov, *Solution of incorrectly formulated problems and the regularization method*, Soviet Math. Dokl. **4** (1963), 1035–1038.

29. ______, *Regularization of incorrectly posed problems*, Soviet Math. Dokl. **4** (1963), 1624–1627.

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