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DERIVATION, L^Ψ-BOUNDED MARTINGALES AND COVERING CONDITIONS

BY

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ABSTRACT. Let (Ω, Σ, P) be a complete probability space. Let $(\Sigma_t)_{t\in J}$ be a directed family of sub-σ-algebras of Σ. Let (Φ, Ψ) be a pair of conjugate Young functions. We investigate the covering conditions that are equivalent to the essential convergence of L^Ψ-bounded martingales. We do not assume that either Φ or Ψ satisfy the Δ_2 condition. We show that when Φ satisfies condition Exp, that is when there exists an $a > 0$ such that $\Phi(u) \leqslant \exp au$ for each $u \geqslant 0$, the essential convergence of L^Ψ-bounded martingales is equivalent to the classical covering condition V_Φ. This covers in particular the classical case $\Psi(t) = t(\log t)^+$. The growth condition Exp on Φ cannot be relaxed. When J contains a countable cofinite set, we show that the essential convergence of L^Ψ-bounded martingales is equivalent to a covering condition D_Φ (that is weaker than V_Φ). When Φ fails condition Exp, condition D_Φ is optimal. Roughly speaking, in the case of L^1-bounded martingales, condition D_Φ means that, locally, the Vitali condition with finite overlap holds. We also investigate the case where J does not contain a countable cofinal set and Φ fails condition Exp. In this case, it seems impossible to characterize the essential convergence of L^Ψ-bounded martingales by a covering condition. Using the Continuum Hypothesis, we also produce an example where all equi-integrable L^1-bounded martingales, but not all L^1-bounded martingales, converge essentially. Similar results are also established in the derivation setting.

1. Results. Let J be a directed set filtering to the right, i.e. a set of indices partially ordered by $\leqslant$, such that for each pair t_1, t_2 of elements of J there exists an element t_3 such that $t_1, t_2 \leqslant t_3$. A subset J_0 of J is called cofinal if each element of J is dominated by an element of J_0. Given a finite subset I of J, and t in J, we write $I \leqslant t$ if t is greater than each element of I, and $t \leqslant I$ if each element of I is greater than t.

Let (Ω, Σ, P) be a probability space. Sets and random variables are considered equal if they are equal almost surely. All sets and functions considered are measurable unless otherwise specified. Let $X = (X_t)_{t\in J}$ be a family of random variables taking values in $\overline{\mathbf{R}}$. The essential upper limit of X, e-limsup X_t, is defined by

$$X^* = \text{e-limsup}\, X_t = \operatorname*{ess\,inf}_s \Big(\operatorname*{ess\,sup}_{t \geqslant s} X_t \Big)$$

while the essential lower limit of X is defined by

$$X_* = \text{e-liminf}\, X_t = -\text{e-limsup}(-X_t).$$

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The directed family X is said to converge essentially if $X^* = X_*$ a.s. Since no other notion of convergence is used, we shall at times omit the word "essentially". For a family $A = (A_t)$ of sets indexed by J we define A^* by $1_{A^*} = \text{e-limsup}\, I_{A_t}$.

A stochastic basis (Σ_t) is an increasing family of sub-σ-algebras of Σ indexed by J, i.e. for $s \leqslant t$ we have $\Sigma_s \subset \Sigma_t$. We denote by E^t the conditional expectation with respect to Σ_t. A process X is a family (X_t) of random variables such that X_t is Σ_t-measurable for every t. We say that (X_t) is L^1-bounded if $\sup_t E|X_t| < \infty$; and that it is a martingale (resp. submartingale) if for $s \leqslant t$, $E^s(X_t) = X_s$ (resp. $\geqslant X_s$). A family (A_t) of sets is adapted if $A_t \in \Sigma_t$ for every $t \in J$. A finite adapted family is an adapted family A_t such that only finitely many of the sets A_t are not empty.

An incomplete multivalued stopping time τ is a collection of sets $\{\tau = t\}$ for each $t \in J$, such that $\{\tau = t\} \in \Sigma_t$ and that the set

$$R(\tau) = \{t \in J: \{\tau = t\} \neq \varnothing\}$$

is finite and nonempty [8]. The set of incomplete multivalued stopping times is denoted by IM. For $\tau \in IM$, $t \in J$ we write $t \leqslant \tau$ if $t \leqslant R(\tau)$, $\tau \leqslant t$ if $R(\tau) \leqslant t$. The sum function S_τ of $\tau \in IM$ is given by $S_\tau = \Sigma_t 1_{\{\tau \in t\}}$, and the excess function e_τ by $e_\tau = S_\tau - 1_{\{S_\tau \geqslant 1\}}$. If $\tau \in IM$ and A is an adapted family of sets, we set

$$A(\tau) = \bigcup(\{\tau = t\} \cap A_t).$$

We now recall some classical facts about Orlicz spaces. References can be found in [12]. Let $\psi: \mathbf{R}^+ \to \overline{\mathbf{R}}^+$ be left continuous, nondecreasing zero at the origin. We set $\Psi(t) = \int_0^t \psi(u)\, du$. Let ϕ be given by $\phi(u) = \sup\{s: \psi(s) < u\}$ for $u > 0$. Let $\Phi(u) = \int_0^u \phi(t)\, dt$. Then (Φ, Ψ) are called conjugate Young's functions. For a measurable function f, let

$$\|f\|_\Psi = \sup\{E(|fg|): E(\Phi(|g|)) \leqslant 1\}.$$

We denote by L^Ψ the space of those f for which $\|f\|_\Psi < +\infty$. Provided with the norm $\|\cdot\|_\Psi$, it is a Banach lattice. We define L^Φ in a similar way. We have $L^\Psi = L^\infty$ if and only if Ψ is always finite.

An essential fact is that conditional expectation with respect to a sub-σ-algebra is a norm-one operator on L^Ψ.

We shall use the following facts:

(1) If $\|f\|_\Phi \leqslant 1$, $\|g\|_\Psi \leqslant 1$, then $|E(fg)| \leqslant 1$.

(2) If $0 < \|f\|_\Phi < \infty$, then $\int \Phi(|f|/\|f\|_\Phi)\, dP \leqslant 1$.

(3) $\|f\|_\Phi \leqslant 1 + \int \Phi(|f|)\, dP$.

For the simplicity of the discussion, we shall assume that

(4) $\lim_{t \to \infty} \phi(t) = \infty$.

The only case that this assumption eliminates is the case $L^\Psi = L^\infty$, but this case is covered by the classical work of Krickeberg [5].

We say that a process (X_t) is L^Ψ-bounded if $\sup \|X_t\|_\Psi < \infty$. Since we are interested in L^Ψ-bounded martingales, we can replace $\|\cdot\|_\Psi$ by an equivalent norm. Hence it is no loss of generality to assume that the following holds:

(5) $\Psi(t) > 0$ and $\Phi(t) > 0$ for $t > 0$.

Without this unrestrictive hypothesis, the formulation of condition D_Φ (see Definition 3) would have to be more complicated.

It is said that Ψ satisfies the Δ_2 condition if the following holds:

$$(\Delta_2) \qquad \exists M > 0, \forall u > 1, \quad \Psi(2u) \leqslant M\Psi(u).$$

Condition Δ_2 holds if and only if L^{∞} is dense in L^{Ψ} for the norm $\|\cdot\|_{\Psi}$ [6].

We shall need a growth condition that is weaker than condition Δ_2. We say that Φ satisfies condition Exp if the following holds:

$$(\mathrm{Exp}) \qquad \exists a > 0, \forall u \geqslant 0, \qquad \Phi(u) \leqslant \exp au.$$

Before we describe the covering-type conditions that we shall use, a comment is in order. The first thought of the reader glancing at these conditions is likely to be that no natural example of filtration (e.g. in multiparameter processes) seems to satisfy them. It is actually not a deep fact that a covering condition implies the convergence of certain types of martingales. Since multiparameter martingale convergences are deep facts, there is little hope to find easy proofs that the corresponding filtrations satisfy covering conditions. However, since this paper will show that covering conditions are actually necessary for martingale convergence, all the natural examples of filtrations actually satisfy some of the conditions we now state.

DEFINITION 1. We say that the stochastic basis (Σ_t) satisfies condition V_{Φ} if for each adapted family (A_t) and each $\varepsilon > 0$ there exists $\tau \in IM$ with $P(A^* \setminus A(\tau)) \leqslant \varepsilon$ and $\|e_{\tau}\|_{\Phi} \leqslant \varepsilon$.

It is also possible to formulate condition V_{Φ} without using multivalued stopping times. In that language, V_{Φ} states that, for each adapted family (A_t) and each $\varepsilon > 0$, there exists an adapted family (B_t), with $B_t \subset A_t$, all of the sets B_t empty except finitely many ones, and such that $P(A^* \setminus \bigcup B_t) \leqslant \varepsilon$ and $\|\Sigma 1_{B_t} - 1_{\bigcup B_t}\|_{\Phi} \leqslant \varepsilon$. The other covering conditions can be reformulated in a similar way.

DEFINITION 2. We say that the stochastic basis (Σ_t) satisfies condition FV_{Φ} if for each adapted family $A = (A_t)$ and each $\varepsilon > 0$ there exist functions $(\xi_t)_{t \in J}$, $\xi_t \geqslant 0$, $\xi_t = 0$ outside A_t, ξ_t bounded, ξ_t is Σ_t-measurable, such that only finitely many functions ξ_t are not zero, and such that $\int (\Sigma \xi_t)\, dP \geqslant P(A^*) - \varepsilon$, and $\|\Sigma \xi_t - 1 \wedge \xi_t\|_{\Phi} \leqslant \varepsilon$.

DEFINITION 3. We say that the stochastic basis (Σ_t) satisfies condition D_{Φ} if one can write Ω as an increasing union $\Omega = \bigcup_n \Omega_n$, such that, for each adapted family (A_t), each n and each $\gamma > 0$, there is $\tau \in IM$ with

$$P((A^* \cap \Omega_n) \setminus A(\tau)) \leqslant \gamma$$

and $\int \Phi(e_{\tau}/n)\, dP \leqslant \gamma$.

DEFINITION 4. We say that the stochastic basis (Σ_t) satisfies condition D'_{Φ} if for each $\varepsilon > 0$ there is $\eta(\varepsilon) > 0$ such that, for each $\gamma > 0$ and each adapted family of sets (A_t), there is $\tau \in IM$ with $P(A^* \setminus A(\tau)) \leqslant \varepsilon$ and $\int \Phi(\eta(\varepsilon) e_{\tau})\, dP \leqslant \gamma$.

DEFINITION 5. We say that the stochastic basis (Σ_t) satisfies condition C if for each $\varepsilon > 0$ there is $M_{\varepsilon} > 0$ such that, for each adapted family $A = (A_t)$ with $P(A^*) > \varepsilon$, there is $\tau \in IM$ with $P(A^* \setminus A(\tau)) \leqslant \varepsilon$ and $\|S_{\tau}\|_{\Phi} \leqslant M_{\varepsilon}$.

Condition V_{Φ} is classical. C. A. Hayes proved (in the setting of derivation [4]) that when Φ satisfies the Δ_2 condition, condition V_{Φ} is necessary and sufficient for the convergence of L^{Ψ}-bounded martingales. Condition FV_{Φ} is a "function" version of condition V_{Φ}, where the functions $1_{\{\tau = t\}}$ are replaced by more general functions. It is weaker than V_{Φ}. Other trivial implications are $V_{\Phi} \Rightarrow D'_{\Phi} \Rightarrow D_{\Phi} \Rightarrow C_{\Phi}$.

In [**10**] A. Millet and L. Sucheston introduced a condition C, and they proved that this condition is sufficient for the convergence of L^1-bounded martingales. Our condition C_Φ is an obvious adaptation of their idea. They asked whether condition C is necessary for the convergence of L^1-bounded martingales. This question, (communicated by L. Sucheston) to which we give a positive answer is at the origin of this work.

Our first result is of a very general nature.

THEOREM 6. *Condition FV_Φ is sufficient for the convergence of L^Ψ-bounded martingales. If $L^\Psi \neq L^1$, condition FV_Φ is also necessary.*

This result is not satisfactory, since condition FV_Φ involves functions instead of sets, so is not a Vitali-type condition. However, we shall see later that it is not readily possible to improve upon Theorem 6 without further hypothesis. But the situation is much better when Φ satisfies condition Exp.

THEOREM 7. *When Φ satisfies condition* Exp, *condition V_Φ is necessary and sufficient for the convergence of L^Ψ-bounded martingales.*

We can also improve Theorem 6 when the structure of J is simple.

THEOREM 8. *Condition D_Φ (resp. D'_Φ) is sufficient for the convergence of L^Ψ-bounded martingales. When J contains a countable cofinal set, it is also necessary.*

This theorem is proved in paragraph 4. Since the proof is long, we give a simpler argument for the important case $L^\Psi = L^1$ in paragraph 5. When Φ fails condition Exp, D_Φ is the sharpest necessary condition that we know. This condition seems to be actually very sharp. In the case $L^\Psi = L^1$, an example of A. Millet and L. Sucheston shows that in general one cannot take the sets Ω_n in $\bigcup \Sigma_t$ [**10**]. (Such an example could actually be produced whenever Φ fails condition Exp, along the lines of Theorem 10).

It can be useful to have whenever possible a sufficient condition that is simpler than condition D'_Φ.

THEOREM 9. *Assume that Ψ satisfies the Δ_2 condition. Then condition C_Φ is sufficient for the convergence of L^Ψ-bounded martingales. (When J contains a countable cofinal set, it is also necessary.)*

The proof will show that actually the "function version" of C_Φ is also sufficient.

We now give results that show that the preceding cannot be readily improved upon. First, condition Exp in Theorem 6 cannot be relaxed.

THEOREM 10. *Assume that Φ fails condition* Exp. *Then there exist a countable index set J and finite σ-algebras $(\Sigma_t)_{t \in J}$ of* $[0,1]$ *such that condition D_Φ holds (and hence L^Ψ-bounded martingales converge) but that condition V_Φ fails.*

The filtration used in the proof of Theorem 10 is somehow artificial. It would be interesting to know what happens for "natural" filtrations. The same comment applies to Theorems 11 and 12.

It is not possible either to relax condition Δ_2 in Theorem 9.

THEOREM 11. *Assume that Ψ fails the Δ_2 condition. Then there exists a countable index set J and finite σ-algebras $(\Sigma_t)_{t\in J}$ on $[0,1]$ such that condition C_Φ holds but not all L^{Ψ}-bounded martingales converge essentially (and hence condition D_Φ fails).*

One of the consequences of Theorem 7 is that when Φ satisfies condition Exp, the convergence of L^{Ψ}-bounded martingales is equivalent to V_Φ, a covering type condition. The same conclusion holds from Theorem 8, but this time under the hypothesis that J contains a countable cofinal set. Does this conclusion always hold? The weakest useful covering type condition we know of is condition C_Φ, so the following seems to show that the answer is no.

THEOREM 12. *Assume that Φ does not satisfy condition* Exp. *Then there exist an index set J and a stochastic basis of finite algebras $(\Sigma_t)_{t\in J}$ on $[0,1]$ that satisfy condition FV_Φ but fail condition C_Φ.*

It has been shown by K. Astbury [1] that when J contains a countable cofinal subset, the convergence of L^1-bounded martingales is equivalent to the convergence of equi-integrable L^1-bounded martingales. Using the Continuum Hypothesis, we show that this is not the case in general.

THEOREM 13. *Under the Continuum Hypothesis, there exists a stochastic basis of finite algebras on $[0,1]$ such that all equi-integrable L^1-bounded martingales converge, but some L^1-bounded martingales fail to converge.*

It is also possible, using the Continuum Hypothesis, to show that condition FV_∞ (that is condition FV_Φ for $L^{\Psi} = L^1$) is not necessary for the convergence of L^1-bounded martingales, but we shall not do it here.

We now turn to derivation. It is essentially a routine to transform a proof of a theorem concerning martingales on an index set to a proof of the corresponding theorem concerning derivation, except that the exhaustion arguments are less straightforward (but nevertheless a standard technique). In order to keep this paper at a reasonable length, we have chosen to give only the proofs in the martingale case, except for Theorem 21, that is proved in §10.

Let (Ω, Σ, μ) be a probability space. (We do not consider the case of infinite measure. Our results can be adapted to this case, but the statements are not so simple since we no longer have $L^\infty \subset L^{\Psi}$.) Suppose that for each x in Ω, we are given a family $\mathscr{B}(x)$ of Moore–Smith sequences (i.e. families filtering to the right) of measurable sets of positive measure, such that if a sequence belongs to $\mathscr{B}(x)$, the same holds for each of its cofinal subsequences. The collection $\mathscr{B}$ of the families $\mathscr{B}(x)$ is called a derivation basis. For $f \in L^1(\mu)$ we set

$$D^*f(x) = \sup\left[\limsup \frac{\int_{A_t} f\,d\mu}{\mu(A_t)}\right]$$

where the limsup refers to a family $(A_t)_{t\in J}$ in $\mathscr{B}(x)$ and the sup to the choice of (A_t) in $\mathscr{B}(x)$. We define $D_* f(x)$ in a similar way, using inf instead of sup. We say that $\mathscr{B}$ differentiates f if $D^*f(x) = D_*f(x) = f(x)$ μ a.e. If H is a subspace of L^1, we say that $\mathscr{B}$ differentiates H if it differentiates each f in H.

If for each point x of a (not necessarily measurable) subset X of Ω we are given a sequence $(A_t(x))$ of $\mathscr{B}(x)$, we say that the collection $\mathscr{V}$ of these sequences is a Vitali cover of X, and we call a $\mathscr{V}$-set any set of the type $A_t(x)$.

Given a finite family $\mathscr{F}$ of measurable sets, we denote

$$e_{\mathscr{F}} = \sum_{\mathscr{F}} 1_A - 1 \wedge \sum_{\mathscr{F}} 1_A; \qquad d_{\mathscr{F}} = \bigcup_{\mathscr{F}} A.$$

DEFINITION 14. We say that $\mathscr{B}$ satisfies condition V_Φ if, for each $X \subset \Omega$, each Vitali cover $\mathscr{V}$ of X, and each $\varepsilon > 0$, there exists a finite family $\mathscr{F}$ of $\mathscr{V}$-sets with $\|e_{\mathscr{F}}\|_\Phi \leqslant \varepsilon$ and $\mu^*(X \setminus d_{\mathscr{F}}) \leqslant \varepsilon$.

DEFINITION 15. We say that $\mathscr{B}$ satisfies condition FV_Φ if, for each $X \subset \Omega$, each Vitali cover $\mathscr{V}$ of X, and each $\varepsilon > 0$, there exists a finite family $\mathscr{F}$ of $\mathscr{V}$-sets and for $A \in \mathscr{F}$ numbers $\alpha_A \geqslant 0$, such that if $\xi = \sum_{\mathscr{F}} \alpha_A 1_A$, we have $\|\xi\|_1 \geqslant \mu^*(X) - \varepsilon$ and $\|\xi - \xi \wedge 1\|_\Phi \leqslant \varepsilon$.

DEFINITION 16. We say that $\mathscr{B}$ satisfies condition D'_Φ (resp. C_Φ) if for each $\varepsilon > 0$ there is $\eta > 0$ (resp. $M_\varepsilon < \infty$) such that, for each $X \subset \Omega$, each Vitali cover V of X, and each $\gamma > 0$, there is a finite family $\mathscr{F}$ of $\mathscr{V}$-sets with $\mu^*(X \setminus d_{\mathscr{F}}) \leqslant \varepsilon$ and $\int \Phi(\eta e_{\mathscr{F}})\, dP \leqslant \gamma$ (resp. $\|e_{\mathscr{F}}\|_\Phi \leqslant M_\varepsilon$).

DEFINITION 17. We say that $\mathscr{B}$ satisfies condition D_Φ if one can write Ω as an increasing union of measurable sets Ω_n, such that for each $X \subset \Omega$, each Vitali cover V of X, and each $\gamma > 0$, there is a finite family $\mathscr{F}$ of $\mathscr{V}$-sets with $\mu^*((\Omega_n \cap X)) \setminus d_{\mathscr{F}}) \leqslant \gamma$ and $\int \Phi(e_{\mathscr{F}}/n)\, dP \leqslant \gamma$.

We then have

THEOREM 18. *Condition FV_Φ is sufficient for $\mathscr{B}$ to differentiate L^Ψ. If $L^\Psi \neq L^1$, it is also necessary.*

THEOREM 19. *If Φ satisfies condition* Exp, *condition V_Φ is necessary and sufficient for $\mathscr{B}$ to differentiate L^Ψ.*

THEOREM 20. *Condition D'_Φ is sufficient for $\mathscr{B}$ to differentiate L^Ψ. When Ψ satisfies the Δ_2 condition, condition C_Φ is also sufficient.*

One could think at first that the usual hypothesis that the Moore–Smith sequences of sets in each $\mathscr{B}(x)$ are actual sequences would correspond to the hypothesis that the index set has countable cofinality in the martingale setting. This is not the case, as the following shows.

THEOREM 21. *Suppose that Φ fails condition* Exp. *There exists a compact metric space L and a derivation basis $\mathscr{B}$ on L, such that each $\mathscr{B}(x)$ consists of the subsequences of a sequence $W_n(x)$ of open sets of diameter going to zero, and such that $\mathscr{B}$ differentiates L^Ψ but that condition C_Φ fails.*

Let $\mathscr{U}$ be a family of measurable sets of Ω and δ a function from $\mathscr{U}$ to $\mathbf{R}^+$. If for each $x \in \Omega$, $\mathscr{B}(x)$ consists of the sequences (U_n) such that $x \in U_n \in \mathscr{U}$ and $\delta(U_n) \to 0$, we say that $\mathscr{B}$ is a D-basis [3]. It turns out that D-bases behave like martingales whose index set have a countable cofinality.

THEOREM 22. *Suppose that $\mathscr{B}$ is a D-basis. Then condition D_{Φ} is necessary and sufficient for $\mathscr{B}$ to differentiate L^{Ψ}.*

ACKNOWLEDGMENT. The author was introduced to these problems by Professor Sucheston, with whom he had several stimulating conversations.

2. Proof of Theorem 6. The proof that condition FV_{Φ} is sufficient for the convergence of L^{Ψ}-bounded martingales is routine, as will be the proof of the sufficiency part of Theorems 8 and 9. We reproduce it for completeness.

PROPOSITION 23. *Assume that condition FV_{Φ} holds. Let (X_t) be a positive submartingale that is L^{Ψ}-bounded. Let $s \in J$ and $B \in \Sigma_s$. Then for each $\lambda > 0$ we have*

$$P(B \cap \{X^* \geqslant \lambda\}) \leqslant \frac{1}{\lambda} \sup_t E(X_t 1_B).$$

PROOF. Let k be such that $\|X_t\|_{\Psi} \leqslant k$ for each t. Let $0 < \beta < \lambda$. For $t \geqslant s$, let $A_t = \{X_t > \beta\} \cap B$ and let $A_t = \varnothing$ otherwise. The family (A_t) is adapted. We have $\{X^* \geqslant \lambda\} \cap B \subset A^*$.

Let $\varepsilon > 0$. According to condition FV_{Φ}, there is a finite set $I \subset J$, and for $i \in J$, there is a function $\xi_i > 0$, ξ_i bounded, ξ_i zero outside A_i, where ξ_i is Σ_i-measurable, such that $\int \Sigma_I \xi_i \, dP \geqslant P(A^*) - \varepsilon$ and $\|\xi'\|_{\Phi} \leqslant \varepsilon$, where $\xi'' = 1 \wedge \Sigma_I \xi_i$, $\xi' = \Sigma_I \xi_i - \xi''$.

Let $t \in I$ with $I \leqslant t$. We can write

$$\begin{aligned} P(A^*) - \varepsilon &\leqslant E\Big(\sum_I \xi_i\Big) = E\Big(\sum_I \xi_i 1_{A_i}\Big) \leqslant \frac{1}{\beta} E\Big(\sum_I \xi_i X_i\Big) \leqslant \frac{1}{\beta} \sum_I E(\xi_i X_i) \\ &\leqslant \frac{1}{\beta} \sum_I E(\xi_i X_t) = \frac{1}{\beta} E\Big(\sum_I \xi_i X_t\Big) = \frac{1}{\beta} E(\xi' X_t) + \frac{1}{\beta} E(\xi'' X_t). \end{aligned}$$

Since $\|\xi'\|_{\infty} \leqslant 1$, and $\xi' = 0$ outside B, we get $E(\xi' X_t) \leqslant E(X_t 1_B)$. Since $\|\xi'\|_{\Phi} \leqslant \varepsilon$ and $\|X_t\|_{\Psi} \leqslant k$, we get $E(\xi' X_t) \leqslant \varepsilon k$, so

$$P(\{X^* \geqslant \lambda\} \cap B) \leqslant (1/\beta) \sup E(X_t 1_B) + \varepsilon k/\beta.$$

Letting $\varepsilon \to 0$, and then $\beta \to \lambda$, we get the result.

PROPOSITION 24. *Assume that for every L^{Ψ}-bounded positive submartingale X_t, every $s \in J$, every $B \in \Sigma_s$ and every $\lambda > 0$, we have*

$$P(B \cap \{X^* \geqslant \lambda\}) \leqslant (1/\lambda) \sup E(X_t 1_B).$$

Then every L^{Ψ}-bounded martingale converges.

PROOF. The case of $L^{\Psi} = L^1$ will be covered by the more general Proposition 39, so we assume $L^{\Psi} \neq L^1$. In this case, an L^{Ψ}-bounded martingale is equi-integrable, so is of the type $X_t = E^t(f)$ for some $f \in L^{\Psi}$. Let $\varepsilon > 0$. There is $h \in L^{\infty}$ with

$\|f - h\|_1 < \varepsilon^2$, so there is $s \in J$, and a bounded g that is Σ_s-measurable, such that $\|f - g\| \leqslant \varepsilon^2$. Let $Y_t = E^t(|f - g|)$. This is a positive martingale, bounded in L^Ψ, and such that $E(Y_t) \leqslant \varepsilon^2$ for each t, so we have $P(Y^* \geqslant \varepsilon) \leqslant \varepsilon$. Since $|X_t - E^t(g)| \leqslant Y_t$, and since $E^t(g) = g$ for $t \geqslant s$, the result follows easily. One should note that if Ψ fails the Δ_2 condition, one cannot in general take g such that $\|f - g\|_\Psi < \|f\|_\Psi$.

We have proved that condition FV_Φ implies convergence of L^Ψ-bounded martingales. We now suppose that $L^\Psi \neq L^1$, and start to prove the converse. Suppose that condition FV_Φ fails. Then there is an adapted family (A_t), an $\varepsilon > 0$, such that for each family (ξ_t) of functions that satisfy the condition

(6) $\xi_t \geqslant 0$, $\xi_t = 0$ outside A_t, ξ_t is Σ_t-measurable, ξ_t is bounded, only finitely many functions ξ_t are nonzero, and $\int \Sigma\xi_t \, dP \geqslant P(A) - \varepsilon$;

then we have $\|\Sigma\xi_t - 1 \wedge \Sigma\xi_t\|_\Phi \geqslant \varepsilon$. We consider the following three subsets of L^Φ.

$$C_1 = \{\xi = \Sigma\xi_t;\ (\xi_t) \text{ satisfies } (6)\},$$

$$C_2 = \{\xi;\ \xi \leqslant 1\}, \qquad C_3 = \{\xi;\ \|\xi\|_\Phi < \varepsilon\}.$$

These sets are convex. Since C_3 is open, $C_2 + C_3$ is open. Also, $C_1 \cap (C_2 + C_3)$ is empty. The theorem of Hahn–Banach implies that there is $h \in (L^\Phi)^*$ such that $h \geqslant 1$ on C_1 and $h < 1$ on $C_2 + C_3$. Since $h < 1$ on C_2, we have $h \geqslant 0$.

We note that, for a set A, if $P(A) \leqslant 1/\Phi(a)$, we have $\int \Phi(a1_A)\, dP \leqslant 1$, so $\|1_A\|_\Phi \leqslant 1/a$. At this point we use that Φ is always finite, that is $L^\Phi \neq L^\infty$.) It follows that $h(1_A) \to 0$ when $P(A) \to 0$, so the finitely additive measure m on Σ given by $m(A) = h(1_A)$ is in fact absolutely continuous with respect to P. It follows that there is $f \in L^1, f \geqslant 0$, such that, for $g \in L^\infty$, we have $h(g) = E(fg)$.

Let $g \in L^\Phi$, $g > 0$, with $\|g\|_\Phi \leqslant \varepsilon$. For $a > 0$, let g_a be the truncation of g at a. Since $g_a \in L^\infty$ and $g_a \in C_3$, we get

$$E(fg_a) = h(g_a) \leqslant 1.$$

Letting $a \to \infty$, Fatou's theorem shows that $E(fg) \leqslant 1$. This implies $\|f\|_\Psi \leqslant 1/\varepsilon$, so $f \in L^\Psi$. Moreover, $E(f) = h(1) \leqslant 1$, since $h < 1$ on C_2.

For $t \in \Sigma$, and a function ξ_t that satisfies $\xi_t \geqslant 0$, $\xi_t = 0$ outside A_t, $E(\xi_t) = P(A^*) - \varepsilon$, ξ_t is Σ_t-measurable, bounded, we have $\xi_t \in C_1$, so $E(\xi_t E^t(f)) = E(f\xi_t) = h(\xi_t) \geqslant 1$. It follows that $E^t(f) \geqslant 1/(P(A^*) - \varepsilon)$ on A_t. If $X_t = E^t(f)$, we then have

$$P(X^* \geqslant 1/(P(A^*) - \varepsilon)) \geqslant P(A^*).$$

Since $\|f\|_1 \leqslant 1$, we also have

$$P(X_* \geqslant 1/(P(A^*) - \varepsilon)) \leqslant P(A^*) - \varepsilon.$$

This shows that $X^* \neq X_*$, and hence (X_t) does not converge essentially. The proof of Theorem 6 is complete.

3. Proof of Theorem 7. The core of the proof will be a random choice argument. It relies on the following standard inequality:

LEMMA 25. *Let Y be a Poisson random variable with $E(Y) = \mu$. Then for $u \geqslant 0$,*

$$P(Y \geqslant \mu(1 + u)) \leqslant \exp(-u\theta(u)\mu)$$

where $\theta(u) = (1 + 1/u)\log(1 + u) - 1$.

PROOF. Use $P(Y \geqslant t) \leqslant E(\exp(hY - ht))$ for $h \geqslant 0$, and minimize over h.

Before we embark on the proof of Theorem 7, we settle a purely technical point.

LEMMA 26. *Assume that condition FV_Φ holds. Then for each adapted family (A_t) of sets, each $\gamma > 0$, each $\varepsilon > 0$, there exists a family (ξ_t) of functions, $\xi_t \geqslant 0$, $\xi_t = 0$ outside A_t, ξ_t is bounded, Σ_t-measurable, only finitely many functions ξ_t are nonzero, such that if we set $\xi = \Sigma\,\xi_t$, $\xi'' = \xi \wedge 1$, $\xi' = \xi - \xi''$, we have*

$$E(\xi'') \geqslant P(A^*) - \varepsilon, \qquad \int \Phi(\xi'/\gamma)\, dP \leqslant \varepsilon.$$

PROOF. Let $u_0 \geqslant 0$ and $\alpha > 0$ be such that $\Phi(u) \geqslant \alpha u$ for $u \geqslant u_0$. For any function $f \geqslant 0$ we have

$$\int_{\{f \geqslant \gamma u_0\}} f\, dP \leqslant \frac{\gamma}{\alpha} \int \Phi(f/\gamma)\, dP.$$

so

$$\int f\, dP \leqslant \gamma u_0 + \frac{\gamma}{\alpha} \int \Phi(f/\gamma)\, dP.$$

Let $0 < \varepsilon < 1$. There is $\gamma_0 > 0$ such that, for $\gamma \leqslant \gamma_0$, we have $E(f) \leqslant \varepsilon/2$ whenever $\int \Phi(f/\gamma)\, dP \leqslant 1$. Let $\gamma \leqslant \gamma_0$. Condition FV_Φ shows that there exists a family (ξ_t) of functions, $\xi_t \geqslant 0$, $\xi_t = 0$ outside A_t, ξ_t is bounded, Σ_t-measurable, only finitely many functions ξ_t are nonzero, and $E(\Sigma\xi_t) \geqslant P(A^*) - \varepsilon/2$, $\|\xi'\|_\Phi \leqslant \varepsilon\gamma$, where $\xi = \Sigma\xi_t$, $\xi'' = \xi \wedge 1$, $\xi' = \xi - \xi''$. Since $\int \Phi(\xi'/\varepsilon\gamma)\, dP \leqslant 1$, we have $\int \Phi(\xi'/\gamma)\, dP \leqslant \varepsilon \leqslant 1$. This implies $E(\xi') \leqslant \varepsilon/2$, so $E(\xi'') \geqslant P(A^*) - \varepsilon$. The proof is complete.

We now start the proof of Theorem 7.

First step. Since Φ satisfies condition Exp, there is $a > 0$ such that $\Phi(u) \leqslant \exp au$ for $u \geqslant 0$. We fix $\gamma > 0$, $b > 0$, and we fix an adapted family (A_t) with $P(A^*) > \gamma$. Let u_0 be large enough that

$$e^{2ab+4} \leqslant 2^{-18}\gamma^2(\theta(u_0) - ab); \qquad e^{ab} \leqslant 1 + u_0. \tag{7}$$

We let $\eta = 1/(u_0 + 1)$. We note that η depends only on b and γ.

Second step. Let $t_0 \in J$ with $P(D) \leqslant \frac{8}{7}P(A^*)$, where D is the essential union of the sets A_t for $t \geqslant t_0$. Let $A'_t = A_t$ for $t \geqslant t_0$, and $A'_t = \varnothing$ otherwise. Then $A'^* = A^*$. By Lemma 26, there is a finite set $I \subset J$, with $t_0 \leqslant I$, and for $i \in I$ a bounded Σ_i-measurable function ξ_i, with $\xi_i \geqslant 0$, $\xi_i = 0$ outside A_i, and

$$E(\xi'') \geqslant \tfrac{6}{7}P(A^*); \qquad \int \Phi(2b\xi')\, dP \leqslant 2^{-17}\eta^2\gamma^2 \tag{8}$$

where $\xi = \Sigma\xi_i$, $\xi'' = \xi \wedge 1$, $\xi' = \xi - \xi''$. Since $\xi'' = 0$ outside D, and $E(\xi'') \geqslant \frac{3}{4}P(D)$, we have $P(F_0) \geqslant \frac{1}{2}P(D)$, where $F_0 = \{\xi'' \geqslant \frac{1}{2}\}$. In particular, $P(F_1) \geqslant \frac{3}{8}P(D) \geqslant \frac{3}{8}P(A^*)$, where $F_1 = A^* \cap F_0$. Note also that we can assume $0 \leqslant \xi_i \leqslant 1$ for each i.

For $i \in I$, $k \geqslant 1$, let $B_{i,k} = \{2^{-k} < \xi_i \leqslant 2^{-k+1}\}$. We note that

$$\sum_{i,k} 2^{-k} 1_{B_{i,k}} < \xi \leqslant \sum_{i,k} 2^{-k+1} 1_{B_{i,k}}.$$

We fix k_0 such that if

$$F = F_1 \cap \left\{ \sum_{i \in I,\, k \leqslant k_0} 2^{-k} 1_{B_{i,k}} \geqslant 2^{-3} \right\},$$

then $P(F) \geqslant \frac{1}{4} P(A^*) \geqslant \gamma/4$.

Third step. We now consider a standard probability space (Ω', Ξ, Q). On Ω', we consider independent Poisson random variables $\varepsilon_{i,k}$, for $i \in I$, $k \leqslant k_0$, such that $E(\varepsilon_{i,k}) = \eta 2^{-k}$.

On the product $\Omega \times \Omega'$, we consider the random variable $Z(\omega, \omega')$ given by

$$Z(\omega, \omega') = \sum_{k \leqslant k_0;\, i \in I} \varepsilon_{i,k}(\omega') 1_{B_{i,k}}(\omega).$$

We shall bound the integral

$$\mathscr{I} = \int_{\{Z > 1\}} \Phi(bZ)\, dP\, dQ.$$

We fix $\omega \in \Omega$. Let $\mu(\omega) = \int Z(\omega, \omega')\, dQ(\omega')$. If $L = \{i, k;\, \omega \in B_{i,k}\}$, we have $\mu(\omega) = \eta \sum_{(i,k) \in L} 2^{-k}$, so in particular $\mu(\omega) \leqslant \eta \xi(\omega)$; that is, $(1 + u_0)\mu(\omega) \leqslant \xi(\omega)$.

Let us first suppose that $2 \leqslant (1 + u_0)\mu(\omega)$. Since $\xi(\omega) \geqslant 2$, we have

$$\begin{aligned} \int_{\{Z \leqslant \mu(\omega)(1 + u_0)\}} \Phi(bZ(\omega', \omega))\, dQ(\omega') &\leqslant \Phi(b(1 + u_0)\mu(\omega)) \\ &\leqslant \Phi(b\xi(\omega)) \leqslant \Phi(2b(\xi(\omega) - 1)) \leqslant \Phi(2b\xi'(\omega)). \end{aligned}$$

Now $\omega' \to Z(\omega, \omega')$ is Poisson of parameter $\mu(\omega)$, so, since θ is increasing, we have from Lemma 26, for $u \geqslant u_0$,

$$Q(\{Z > \mu(\omega)(1 + u)\}) \leqslant \exp(-\theta(u_0) u \mu(\omega));$$

so, with some elementary computations, we get, using (7).

$$\begin{aligned} \int_{\{Z \geqslant \mu(\omega)(1 + u_0)\}} \Phi(bZ(\omega', \omega))\, dQ(\omega') &= \int_{\mu(\omega)(1 + u_0)}^{\infty} \Phi(bu) Q(Z \geqslant u)\, du \\ &\leqslant \int_{\mu(\omega)(1 + u_0)}^{\infty} \exp(abu - \theta(u_0)(u - \mu(\omega)))\, du \\ &\leqslant (\theta(u_0) - ab)^{-1} (\eta e^{ab+1})^{\mu(\omega)(1 + u_0)} \\ &\leqslant \eta^2 (\theta(u_0) - ab)^{-1} e^{2ab+2} \leqslant 2^{-18} \eta^2 \gamma^2. \end{aligned}$$

Suppose now that $(1 + u_0)\mu(\omega) \leqslant 2$. Then $Z > 1$ forces $Z \geqslant 2$, so $\{Z > 1\} \subset \{Z \geqslant (1 + u_0)\mu(\omega)\}$. The second part of the computation above shows that

$$\begin{aligned} \int_{\{Z > 1\}} \Phi(bZ(\omega', \omega))\, dQ(\omega') &\leqslant \int_2^{\infty} \exp(abu - \theta(u_0)(u - \mu(\omega)))\, du \\ &\leqslant (\theta(u_0) - ab)^{-1} \exp(2ab + \theta(u_0)\mu(\omega) - 2\theta(u_0)) \\ &\leqslant \eta^2 (\theta(u_0) - ab) e^{2ab+4} \leqslant 2^{-18} \eta^2 \gamma^2 \end{aligned}$$

since $\mu(\omega)\theta(u_0) \leqslant 2\eta\theta(u_0) \leqslant 2$ and $2\theta(u_0) \geqslant 2\log(1 + u_0) - 2$.

Integrating over Ω and using (8) we get $\mathscr{I} \leqslant 2^{-16} \eta^2 \gamma^2$.

Let

$$U = \left\{ \omega';\ \int_{\{Z>1\}} \Phi(bZ(\omega', \omega))\, dP(\omega) \leqslant 2^{-7}\eta\gamma \right\}.$$

From Fubini's theorem and the majoration $\mathscr{I} \leqslant 2^{-16}\eta^2\gamma^2$ we get $Q(U) \geqslant 1 - 2^{-9}\eta\gamma$.

Fourth step. Again we fix θ. We have

$$Q(\{\omega';\ Z(\omega, \omega') \geqslant 1\}) = 1 - \exp(-\mu(\omega)).$$

Note that $1 - e^{-x} \geqslant x/2$ for $0 \leqslant x \leqslant 1$. For $\omega \in F_1$ we have $\mu(\omega) \geqslant \eta 2^{-3}$ so $1 - \exp(-\mu(\omega)) \geqslant \eta 2^{-4}$, so we get

$$P \otimes Q(A^* \times \Omega' \cap \{Z \geqslant 1\}) \geqslant 2^{-7}\eta\gamma.$$

Let

$$V = \{\omega';\ P(A^* \cap \{\omega;\ Z(\omega, \omega') \geqslant 1\}) \geqslant 2^{-8}\eta\gamma\}.$$

We then have $Q(V) \geqslant 2^{-8}\eta\gamma$. This shows that $U \cap V \neq \varnothing$. We fix $\omega' \in U \cap V$.

Define $\tau \in IM$ by $\{\tau = t\} = \varnothing$ for $t \notin I$, and for $i \in I$ by

$$\{\tau = i\} = \bigcup B_{i,k}$$

where the union is taken over all the $k \leqslant k_0$ for which $\varepsilon_{i,k}(\omega') = 1$. Since $B_{i,k} \subset A_i$, we have $A(\tau) \supset B_{i,k}$ whenever $\varepsilon_{i,k}(\omega') = 1$, so $\{\omega\colon Z(\omega, \omega') \geqslant 1\} \subset A(\tau)$. Since $\omega' \in V$, we have $P(A(\tau) \cap A^*) \geqslant 2^{-4}\eta\gamma$. We have $S_\tau(\omega) \leqslant Z(\omega, \omega')$ and since $\omega' \in U$, we get

$$\int \Phi(be_\tau)\, dP \leqslant \int_{\{S_\tau > 1\}} \Phi(bS_\tau)\, dP \leqslant 2^{-4}\eta\gamma.$$

Fifth step. Now let $\varepsilon > 0$. In the above construction, let $\gamma = \varepsilon/2$, $b = 2/\varepsilon$. We have shown that there is $\delta = 2^{-8}\eta\varepsilon$, depending only on ε, such that whenever (A_t) is an adapted family with $P(A^*) > \varepsilon$, there is $\tau \in IM$ with $P(A(\tau) \cap A^*) \geqslant \delta$ and $\int \Phi(2e_\tau/\varepsilon)\, dP \leqslant \delta$. The conclusion of the proof follows the standard exhausting procedure. Let us fix an adapted family (A_t) with $P(A^*) > \varepsilon$. By induction over the integer k, we construct $\tau_k \in IM$ such that $P(A(\tau_k) \cap A^*) \geqslant \delta k$ and $\int \Phi(2e_{\tau_k}/\varepsilon)\, dP \leqslant k\delta$. The induction continues as long as $P(A^* \setminus A(\tau_k)) > \varepsilon$, so it finishes in at most $1/\delta$ steps. We have just proved the possibility of the first step. Assume now that τ_k has been constructed, and that $P(A^* \setminus A(\tau_k)) > \varepsilon$. We can assume $\{\tau_k = t\} \subset A_t$ for each t. Let $t_0 \in I$ be such that $\tau_k \leqslant t_0$. For $t \geqslant t_0$, let $A'_t = A_t \setminus A(\tau_k)$, and let $A'_t = \varnothing$ otherwise. We have $P(A'^*) > \varepsilon$, so there is $\tau \in IM$ with $P(A'^* \cap A(\tau)) \geqslant \delta$ and $\int \Phi(2e_\tau/\varepsilon)\, dP \leqslant \delta$. We define $\{\tau_{k+1} = t\} = \{\tau_k = t\}$ for $t \leqslant t_0$, $\{\tau_{k+1} = t\} = \{\tau = t\} \setminus A(\tau_k)$ for $t \geqslant t_0$ and $\{\tau_k = t\} = \varnothing$ otherwise, and it is straightforward to check that τ_{k+1} satisfies the requirements. When the induction stops, we have constructed $\tau' \in IM$ with $P(A^* \setminus A(\tau')) < \varepsilon$ and

$$\int \Phi(2e_{\tau'}/\varepsilon)\, dP \leqslant P(A^* \cap A(\tau')) \leqslant 1,$$

so $\|2e_{\tau'}/\varepsilon\|_\Phi \leqslant 2$ and $\|e_{\tau'}\|_\Phi \leqslant \varepsilon$. The proof is complete.

4. Proof of Theorem 8. We start a series of lemmas that will culminate in the proof of the necessity of condition D'_Φ when J contains a countable cofinal set. For clarity we suppose in this paragraph that $L^\Phi \neq L^\infty$. The case of L^1-bounded martingales will be investigated in the next paragraph.

LEMMA 27. *Let N_1, N_2 be two lattice norms on L^Φ, that are equivalent to the norm $\|\cdot\|_\Phi$. Let $I \subset J$ be a finite set, and for $i \in I$ let A_i be a Σ_i-measurable set. Assume that for each family $(\xi_i)_{i\in I}$ of bounded functions, with $\xi_i \geqslant 0$, each ξ_i is Σ_i-measurable, $\xi_i = 0$ outside A_i, and $\|\Sigma_{i\in I}\xi_i\|_1 = 1$, we have either $N_1(\Sigma\xi_i) \geqslant 1$ or $N_2(\Sigma\xi_i) \geqslant 1$. Then there exist two functions $f_1, f_2 \in L^\Psi$ such that for $j = 1, 2$ we have*

$$\forall g \in L^\Phi, \quad N_j(g) \leqslant 1 \Rightarrow E\big(|f_j g|\big) \leqslant 1$$

and that for i in I we have

$$A_i \subset \left\{E^i(f_1 + f_2) \geqslant 1\right\}.$$

PROOF. Consider the set C_1 of functions $\xi = \Sigma_{i\in I}\xi_i$, where the functions ξ_i satisfy the conditions of the lemma. It is a convex set. Let $C_2 = \{g \in L^\Phi, N_1(g) < 1, N_2(g) < 1\}$. It is a convex open set of L^Φ, and $C_1 \cap C_2 = \varnothing$ by hypothesis. The theorem of Hahn–Banach gives an h in the dual G of L^Φ with $h < 1$ on C_2 and $h \geqslant 1$ on C_1. For $j = 1, 2$, let

$$M_j = \left\{l \in G \colon |l(g)| \leqslant 1 \text{ for } g \in L^\Phi, N_j(g) \leqslant 1\right\}.$$

Let

$$M = \left\{l \in G \colon |l(g)| \leqslant 1 \text{ for } g \in L^\Phi, N_1(g) \leqslant 1, N_2(g) \leqslant 1\right\}.$$

A straightforward use of the theorem of Hahn–Banach shows that M is the convex hull of M_1 and M_2; that is, we can write $h = \lambda_1 h_1 + \lambda_2 h_2$ where $\lambda_1, \lambda_2 \geqslant 0$, $\lambda_1 + \lambda_2 = 1$, $h_1 \in M_1$, $h_2 \in M_2$.

For $i \in I$, consider the restriction of h_1 (resp. h_2) to $L^\infty(\Omega, \Sigma_i, P)$. Since $L^\Phi \neq L^\infty$, we have shown in the proof of Theorem 6 that for $j = 1, 2$ there is $f_j' \in L^1$ such that $h_j(g) = E(f_j' g)$ for $g \in L^\infty$. Let $f_j = |f_j'|$. If $g \in L^\Phi$ with $N_j(g) \leqslant 1$, $g \geqslant 0$, for $a > 0$ its truncation g_a at a satisfies $N_j(g_a) \leqslant 1$, so if $g_a' = g_a \operatorname{sign} f_j'$ we have

$$E\big(f_j g_a\big) = E\big(f_j' g_a'\big) = h\big(g_a'\big) \leqslant 1.$$

Letting $a \to \infty$, Fatou's lemma gives $E(f_j g) \leqslant 1$. For $g \in C_1$, we have

$$1 \leqslant h(g) = \lambda_1 E\big(f_1' g\big) + \lambda_2 E\big(f_2' g\big) \leqslant E\big((f_1 + f_2) g\big)$$

and this shows that $A_i \subset \{E^i(f_1 + f_2) \geqslant 1\}$, and concludes the proof.

LEMMA 28. *Assume that J contains a countable cofinal set. Let $\varepsilon, \delta > 0$ be fixed. Assume that for each $s \in J$ and each $\eta \geqslant 0$, there is a finite set $I \subset J$, $s \leqslant I$ and $g \in L^\Psi$ with $\int \Psi(g\delta)\, dP < \eta$, and $P(\bigcup_{i\in I} B_i) \geqslant \varepsilon/2$ where $B_i = \{E^i(g) \geqslant \frac{1}{2}\}$. Then there is f in L^Ψ such that the martingale $(E^t(f))$ fails to converge essentially.*

PROOF. We first note that, for each $\gamma > 0$, there is $\eta > 0$ such that $\int \Psi(g\delta)\, dP \leqslant \eta$ implies $\|g\|_1 \leqslant \gamma$. Indeed, there is α, $u_0 > 0$ such that $\Psi(u) - \Psi(u_0) \geqslant \alpha(u - u_0)$ for $u \geqslant u_0$, and hence

$$\int_{\{g\delta > u_0\}} g\, dP \leqslant \delta^{-1}\left(\alpha^{-1}\int \Psi(\delta g)\, dP + P(\{g\delta > u_0\})\right).$$

Moreover, since we assume (5), we have $\Psi(u) > 0$ for $u > 0$, so g goes to zero in measure when $\int \Psi(\delta g)\, dP$ goes to zero; so the claim follows.

Let (s_k) be a countable cofinal set of J. By induction, we construct finite sets I_k, with $s_k \leqslant I_k$, sets $(A_k)_{i \in I_k}$, a decreasing sequence of numbers $a_k \geqslant 0$, such that if we set $p_k = \operatorname{card} I_k$, the following conditions hold:

(9) $a_k < a_{k-1}/2$, and if $B \in \Sigma$, $P(B) < a_k$, then for $j \leqslant k$ we have $\int_B f_j \, dP \leqslant 2^{-j-3} p_j^{-1}$.

$$\int \Psi(\delta f_k)\, dP \leqslant 2^{-k}; \qquad \|f_k\|_1 \leqslant \operatorname{Inf}\left(2^{-k-5}\varepsilon, a_{k-1}/8\right). \tag{10}$$

$$P\left(\bigcup_{i \in I_k} A_i\right) \geqslant \varepsilon/2 \quad \text{where for } i \in I_k, A_i = \left\{E^i(f_k) \geqslant \tfrac{1}{2}\right\}. \tag{11}$$

Let $B_k = \{f_k \geqslant \frac{1}{4}\}$. Since $\|f_k\|_1 \leqslant a_{k-1}/8$, we have $P(B_k) \leqslant a_{k-1}/2$. Let $h_k = f_k I_{B_k}$. Since $E^i(f_k - h_k) \leqslant \frac{1}{4}$ for each i we have $A_i \subset \{E^i(h_k) \geqslant \frac{1}{4}\}$. Let $C_k = \bigcup_{j > k} B_j$, and $g_k = h_k 1_{X/C_k}$. We note that the functions g_k live on disjoint sets. Also, since $P(C_k) \leqslant a_k$, we have $\|h_k - g_k\|_1 \leqslant 2^{-k-3}/p_k$. So if $A_i' = \{E^i(g_k) \geqslant \frac{1}{8}\}$ then $P(A_i \setminus A_i') \leqslant 2^{-k}/p_k$. In particular, we get

$$P\left(\bigcup_{i \in I_k} \left\{E^i(g_k) \geqslant \tfrac{1}{8}\right\}\right) \geqslant \varepsilon/2 - 2^{-k}. \tag{12}$$

Let $f = \sum_{k \geqslant 1} g_k$. Since $g_k \leqslant f_k$, we have $\int \Psi(\delta^{-1} g_k)\, dP \leqslant 2^{-k}$. Since the functions g_k live on disjoint sets, we have $\int \Psi(\delta^{-1} f)\, dP \leqslant 1$. In particular, $f \in L^\Psi$. Since $g_k \leqslant f_k$, we have $\|g_k\|_1 \leqslant 2^{-k-5}\varepsilon$, so $\|f\|_1 \leqslant 2^{-5}\varepsilon$. Let $X_t = E^t(f)$. It follows from (12) that $P(X^* \geqslant \frac{1}{8}) \geqslant \varepsilon/2$. On the other hand, we have $\int X_* \, dP \leqslant \int f \, dP \leqslant 2^{-5}\varepsilon$, so $P(X_* \geqslant \frac{1}{8}) \leqslant \varepsilon/4$. This shows that (X_t) does not converge essentially, and concludes the proof.

LEMMA 29. *Suppose that J contains a countable cofinite set, and that for each $f \in L^\Psi$, the martingale $E^t(f)$ converges essentially. Let $\varepsilon > 0$. Then there is η_0, depending only on ε, such that, for each $\delta > 0$, there are $\eta > 0$ and s in J, depending only on ε and δ, such that, for each finite set I with $s \leqslant I$, and each family $(A_i)_{i \in I}$, where A_i is a Σ_i-measurable set, such that $P(\bigcup A_i) \geqslant \varepsilon$, there is a family (ξ_i) of functions, where ξ_i is Σ_i-measurable, $\xi_i \geqslant 0$, $\xi_i = 0$ outside A_i, $\|\sum \xi_i\|_1 = 1$ and*

$$\int \Phi\left(\eta_0 \sum \xi_i\right) dP \leqslant \eta_0, \qquad \int \Phi\left(\eta \sum \xi_i\right) dP \leqslant \eta\delta. \tag{13}$$

PROOF. *First step.* We show that there are η_0 and s_0 such that, for each finite set I with $s_0 \leqslant I$, and each family $(A_i)_{i \in I}$ of Σ_i-measurable sets, with $P(\bigcup A_i) \geqslant \varepsilon/2$ there is a family (ξ_i) of bounded functions, where ξ_i is Σ_i-measurable, $\xi_i \geqslant 0$, $\xi_i = 0$ outside A_i, $\|\sum \xi_i\|_1 = 1$ and $\int \Phi(4\eta_0 \sum \xi_i)\, dP \leqslant \eta_0$.

Suppose this fails. So, for each $\eta_0 > 0$ and $s_0 \in J$, there is a finite set I with $s_0 \leqslant I$, and a family $(A_i)_{i \in I}$ of Σ_i-measurable sets with $P(\bigcup A_i) \geqslant \varepsilon/2$, such that we cannot find a family (ξ_i) of functions, where ξ_i is Σ_i-measurable, $\xi_i \geqslant 0$, $\xi_i = 0$ outside A_i, $\|\sum \xi_I\|_1 = 1$ and $\int \Phi(4\eta_0 \sum \xi_i)\, dP \leqslant \eta_0$.

We now use Lemma 27 with $N_1(\cdot) = N_2(\cdot) = \|\cdot\|_{\Phi'}$, where $\Phi'(u) = \eta_0^{-1}\Phi(4\eta_0 u)$. (The conjugate Young function Ψ' is given by $\Psi'(u) = \eta_0^{-1}\Psi(u/4)$.) So there is $f \in L^{\Psi'}$ such that $E(|fg|) \leqslant 2$ for $\|g\|_{\Phi'} \leqslant 1$, and such that $A_i \subset \{E(f^i) \geqslant 1\}$. We see that $\|f\|_{\Psi'} \leqslant 2$, so $\int \Psi'(f/2)\,dP \leqslant 1$, so $\int \Psi(f/8)\,dP \leqslant \eta_0$. The conclusion follows from Lemma 28 (with $\delta = \frac{1}{8}$).

Second step. We finish the proof. Suppose that the conclusion fails. Then there exists $\delta > 0$ such that, for each $\eta > 0$ and $s \geqslant s_0$, there exist a finite set I with $s \leqslant I$ and a family $(A_i)_{i\in I}$ of Σ_i-measurable sets with $P(\bigcup A_i) \geqslant \varepsilon/2$, such that for each family (ξ_i) of functions, where ξ_i is Σ_i-measurable, $\xi_i \geqslant 0$, $\xi_i = 0$ outside A_i, $\|\Sigma\xi_i\|_1 = 1$, we get either $\int \Phi(\eta_0\Sigma\xi_i)\,dP \geqslant \eta_0$ or $\int \Phi(\eta\Sigma\xi_i)\,dP \geqslant \eta\delta$.

We now use Lemma 27 with $N_1(\cdot) = \|\cdot\|_{\Phi''}$, $N_2(\cdot) = \|\cdot\|_{\Phi^{\sim}}$, where $\Phi''(u) = \eta_0^{-1}\Phi(\eta_0 u)$, $\Phi^{\sim}(u) = (\delta\eta)^{-1}\Phi(\eta u)$. (The conjugate Young functions are given, respectively, by $\Psi''(u) = \eta_0^{-1}\Psi(u)$ and $\Psi^{\sim}(u) = (\delta\eta)^{-1}\Psi(\delta u)$.) It follows that there are g_1, g_2 with $\|g_1\|_{\Psi''} \leqslant 1$, $\|g_2\|_{\Psi^{\sim}} \leqslant 1$ such that $A_i \subset \{E^i(g_1 + g_2) \geqslant 1\}$.

Let $C_i = \{E^i(g_1) > \frac{1}{2}\}$. Suppose, if possible, that $P(\bigcup C_i) \geqslant \varepsilon/2$. Then the first step gives functions ξ_i, $\xi_i \geqslant 0$, $\xi_i = 0$ outside C_i, ξ_i is Σ_i-measurable, $\|\Sigma\xi_i\|_1 = 1$, and $\int \Phi(4\eta\Sigma\xi_i)\,dP \leqslant \eta_0$, so $\int\Phi'(4\Sigma\xi_i)\,dP \leqslant 1$, so $\|\Sigma\xi_i\|_{\Phi'} \leqslant \frac{1}{2}$. It follows that $E(g_1\Sigma\xi_i) \leqslant \frac{1}{2}$. However, since ξ_i is Σ_i-measurable,

$$E\left(g_1\sum\xi_i\right) = \sum E(g_1\xi_i) = \sum\left(\xi_i E^i(g_1)\right)$$
$$> \frac{1}{2}\sum E(\xi_i) \geqslant \frac{1}{2}$$

since $E^i(g_1) > \frac{1}{2}$ when $\xi_i > 0$. This contradiction proves that $P(\bigcup C_i) \leqslant \varepsilon/2$. Let $A_i' = \{E^i(g_2) \geqslant \frac{1}{2}\}$. Since $P(\bigcup A_i) \geqslant \varepsilon$ we have $P(\bigcup A_i') \geqslant \varepsilon/2$. Since $\|g_2\|_{\Psi^{\sim}} \leqslant 1$, we have $\int \Psi(\delta g_2)\,dP \leqslant \eta\delta$. But Lemma 28 shows this is impossible. This concludes the proof.

Lemma 30. *Suppose that J contains a countable cofinite set, and that L^Ψ-bounded martingales converge essentially. Let $\varepsilon > 0$. Then there is η_1, depending only on ε, such that for each $\delta > 0$ there are $\eta > 0$, $s \in J$, $\gamma > 0$, $b > 0$, depending only on ε and δ, such that, for every finite adapted famly of sets (A_t) with*

$$\varepsilon < P\left(\bigcup_{t\geqslant s} A_t\right); \qquad \sum_{t\geqslant s} P(A_t) \leqslant (1+\gamma)P\left(\bigcup_{t\geqslant s} A_t\right), \tag{14}$$

there is $\tau \in IM$ with $\tau \geqslant s$ such that $P(A(\tau)) \geqslant b$ and

$$\int \Phi(\eta_1 S_\tau)\,dP \leqslant 1; \qquad \int \Phi(\eta S_\tau)\,dP \leqslant \delta\eta P(A(t)). \tag{15}$$

Proof. *First step.* Let $\delta > 0$, and let $\eta_0 = \eta_0(\varepsilon)$, $\eta_1 = \eta_0/4$, and $\eta = \eta(\varepsilon, \delta 2^{-4})$ be as in Lemma 29. Since we assumed that $\lim\phi(t) = \infty$, there is $l \in \mathbf{N}$ such that for $u \geqslant 2^l$ we have $\Phi(\eta_0 u) \geqslant 4\eta_0 u$. Let $b = 2^{-l-5}/(l+2)$, and $\gamma = 2^{-l-2}$. Let

$$C = \left\{\omega; \exists i \neq j, i, j \geqslant s, x \in A_i \cap A_j\right\}.$$

Then (14) implies that $P(C) \leqslant \gamma$. Let $D = \bigcup_{t\geqslant s} A_t$.

Second step. Let $(\xi_i)_{i\in I}$ be as in Lemma 29 but with $\delta 2^{-4}$ instead of δ, and let $\xi = \Sigma\xi_i$. We have $\int_{C\cap\{\xi\leqslant 2^l\}} \xi\,dP \leqslant \frac{1}{4}$, and also

$$\int_{\{\xi\geqslant 2^l\}} \xi\,dP \leqslant \frac{1}{4}\int \eta_0^{-1}\Phi(\eta_0\xi)\,dP \leqslant \frac{1}{4}.$$

It follows that $\int_{D\setminus C} \xi \, dP \geqslant \frac{1}{2}$. For each $-2 < k \leqslant l-1$ and $i \in I$, let

$$A_i^k = \left\{2^k < \xi_i \leqslant 2^{k+1}\right\}$$

so that $A_i^k \in \Sigma_i$. Let $\tau_k \in IM$ be given by $\{\tau_k = i\} = A_i^k$ for $i \in I$, $\{\tau_k = i\} = \varnothing$ otherwise. The rest of the proof consists of showing that for τ one can take one of the τ_k. We first note that, for each i, we have $\sum_k 2^k 1_{\{\tau_k = i\}} < \xi_i$ so we have

$$\sum_k 2^k S_{\tau_k} \leqslant \xi. \tag{16}$$

Let $H = (D \setminus C) \cap \{\xi > \frac{1}{4}\}$. Since $\int_{D\setminus C} \xi \, dP \geqslant \frac{1}{2}$, we have $\int_H \xi \, dP \geqslant \frac{1}{4}$. For $\omega \in H$, there exists a unique i such that $\xi_i(\omega) > 0$, so we have $\xi_i(\omega) > \frac{1}{4}$, so

$$\xi_i(\omega) \leqslant \sum_{k=-2}^{l-1} 2^{k+1} 1_{\{\tau_k = i\}}(\omega).$$

Summation over i gives $\xi(\omega) \leqslant \sum_{k=-2}^{l-1} 2^{k+1} S_{\tau_k}(\omega)$. For $k \in [-2, l-1]$, let $a_k = \int_H 2^{k+1} S_{\tau_k} \, dP$. Since $\int_H \xi \, DP \geqslant \frac{1}{4}$, w have $\sum_k a_k \geqslant \frac{1}{4}$. Now from (13), (16) we deduce that

$$\sum_k \int \Phi\left(\eta 2^k S_{\tau_k}\right) dP \leqslant 2^{-4} \delta \eta,$$

so we have $\sum a_k b_k \leqslant 2^{-4} \delta \eta$, where $b_k = a_k^{-1} \int \Phi(\eta 2^k S_{\tau_k}) \, dP$ for $a_k \neq 0$, and zero otherwise. Let

$$L = \{k;\ b_k \geqslant \delta n/2\}.$$

Then $\sum_{k \in L} a_k \leqslant 2^{-3}$, so $\sum_{k \in L} a_k \geqslant \frac{1}{8}$. Let

$$L' = \{k \in [-2, l-1];\ a_k \geqslant 1/16(l+2);\ k \notin L\}.$$

We have $\sum_{k \in L'} a_k \geqslant 2^{-4}$. It follows that there is k with $b_k \leqslant \delta\eta/2$ and $a_k \geqslant 1/16(l+2)$.

Since the sets A_i^k are disjoint on H, and since $\{\tau = i\} \subset A_i$ for each i, we get

$$P(H \cap A(\tau_k)) = \int_H S_{\tau_k} \, dP = 2^{-k-1} a_k \geqslant 2^{-l-5}/(l+2) = b.$$

Since $b_k \leqslant \delta\eta/2$, we get $2^k \int \Phi(\eta_0 S_{\tau_k}) \, dP \leqslant \delta \eta a_k/2$, so, since $2^{-k} a_k \leqslant 2P(A(\tau_k))$, we get

$$\int \Phi\left(\eta S_{\tau_k}\right) dP \leqslant \delta \eta P(A(\tau_k)).$$

Moreover,

$$\int \Phi\left(\eta_1 S_{\tau_k}\right) dP \leqslant \int \Phi(4\eta_1 \xi) \, dP \leqslant \int \Phi(\eta_0 \xi) \, dP \leqslant 1.$$

The proof is complete.

Lemma 31. *Suppose that J contains a countable cofinal set, and that L^{Ψ}-bounded martingales converge essentially. Let $\varepsilon > 0$. Then there is η_1, depending only on ε, such that, for each $\delta > 0$, there are $s_1 \in J$, $\gamma_1 > 0$, $b_1 > 0$, depending only on ε and δ, such that for every finite adapted family of sets (A_t) which satisfies*

$$\varepsilon < P\left(\bigcup_{t \geqslant s_1} A_t\right); \qquad \sum_{t \geqslant s_1} P(A_t) \leqslant (1+\gamma_1) P\left(\bigcup_{t \geqslant s_1} A_t\right), \tag{17}$$

there is $\tau \in IM$ *with* $\tau \geqslant s_1$ *such that*

$$P(A(\tau)) \geqslant b_1 \quad \text{and} \quad \int \Phi(\eta_1 e_\tau)\, dP \leqslant \delta P(A(\tau)). \tag{18}$$

PROOF. Fix $\varepsilon > 0$. Let η_1 be as in Lemma 30. We can assume $24\,\eta_1 \leqslant 1$. Let $\delta > 0$. Let s, γ, b, η be as in Lemma 30. Let $d = \eta/2\eta_1$. Let k be large enough that $k \geqslant 4/\eta\delta bd$. Let $\gamma_1 \leqslant \gamma$ be small enough that $\gamma_1\Phi(\eta_1 k) \leqslant n\delta d$. Let $b_1 = db/4$. Let $s_1 = s$. Let (A_t) be as in the statement of the present lemma. From Lemma 30 there exists $\tau' \in IM$ such that $P(A(\tau')) \geqslant b$ and

$$\int \Phi(\eta_1 S_{\tau'})\, dP \leqslant 1, \qquad \int \Phi(\eta S_{\tau'})\, dP \leqslant \delta\eta P(A(\tau')). \tag{19}$$

We can suppose that $\{\tau' = t\} \subset A_t$ for each t. Let

$$I = \{i \in J; \{\tau' = 1\} \neq \varnothing\}.$$

Let

$$C = \{\omega; \exists i, j \in I, i \neq j, \omega \in \{\tau' = i\} \cap \{\tau' = j\}\}.$$

From (17), we have $P(C) \leqslant \gamma_1$. For $i \in I$, let $a_i = P(\{\tau' = i\} \cap C)$. Since we can assume $\gamma_1 < b/2$, we have $\Sigma a_i \geqslant P(A(\tau'))/2$.

We shall obtain τ by a random choice. More precisely, let $Y = \{0,1\}^I$, provided with the probability Q that makes the coordinate functions ε_i independent of expectation d. For $y \in Y$, let τ_y be given by $\{\tau_y = i\} = \{\tau' = i\}$ if $y(i) = 1$, and $\{\tau_y = i\} = \varnothing$ otherwise. We shall show that, with positive probability, τ_y satisfies (18). For y in Y, we have

$$P(A(\tau_y)) \geqslant \sum \varepsilon_i(y) a_i \overset{\text{def}}{=} R(y).$$

We have $E(R) = d\Sigma a_i$. Let $U = \{y; R(y) > E(R)/2\}$. Then

$$E(R)/2 \leqslant \int_U R\, dQ \leqslant Q(U)^{1/2}(E(R^2))^{1/2}.$$

Since $E(R^2) \leqslant d(\Sigma a_i)^2$, we get $Q(U) \geqslant d/4$. And for $y \in U$, we have

$$P(A(\tau_y)) > d\sum a_i/2 \geqslant dP(A(\tau'))/4 \geqslant bd/4 = b_1.$$

Let $D = \{S_{\tau'} \leqslant k\}$. Since $e_{\tau'} = 0$ outside C, we get, by the choice of γ_1,

$$\int_D \Phi(\eta_1 e_{\tau'})\, dP \leqslant \gamma_1\Phi(\eta_1 k) \leqslant \eta\delta b.$$

Let $\omega \notin D$. Let $V_\omega = \{y \in Q; \Sigma\varepsilon_i(y) \geqslant 2dS_{\tau'}(\omega)\}$ where the summation is over the indexes i for which $\omega \in \{\tau' = i\}$. For $y \notin V_\omega$, we have $\Phi(\eta_1 e_{\tau_y}) \leqslant \Phi(\eta S_{\tau'}(\omega))$. For $y \in V_\omega$, we have $\Phi(\eta_1 e_{\tau_y}) \leqslant \Phi(\eta_1 S_{\tau'}(\omega))$. Let

$$\theta(\omega, y) = \sup\Big(\Phi\big(\eta_1 e_{\tau_y}(\omega)\big), \Phi\big(\eta S_{\tau'}(\omega)\big)\Big) - \Phi(\eta S_{\tau'}(\omega)).$$

We have shown that

$$\int \theta(\omega, y)\, dQ(y) \leqslant Q(V_\omega)\Phi(\eta_1 S_{\tau'}(\omega)).$$

The choice of k implies that $Q(V_\omega) < \eta\delta db/4$. Integrating over $C \setminus D$ and using (19) gives

$$\int_{C \setminus D} \theta(\omega, y)\, dQ(y)\, dP(\omega) < \eta\delta db/4.$$

If $V = \{y \in Q; \int_D \theta(\omega, y)\, dP(\omega) \leqslant \eta\delta b\}$, then $Q(V) > 1 - d/4$. For $y \in V$, we have

$$\begin{aligned}\int \theta(\eta_1 e_{\tau_y})\, dP &= \int_C \Phi(\eta_1 e_{\tau_y})\, dP \\ &\leqslant \int_D \Phi(\eta_1 e_{\tau'})\, dP + \int_{C\setminus D} \Phi(\eta S_{\tau'})\, dP + \int_{C \setminus D} \theta(\omega, y)\, dP(\omega) \\ &\leqslant 2\eta\delta b + \eta\delta P(A(\tau')) \leqslant 3\eta S P(A(\tau')).\end{aligned}$$

We have $Q(U \cap V) > 0$. And for $y \in U \cap V$, we have

$$b_1 \leqslant P(A(\tau')) \leqslant (4/d) P\big(A(\tau_y)\big) = (8\eta_1/\eta) P(A(\tau)),$$

so

$$\int \Phi(\eta_1 e_{\tau_y})\, dP \leqslant 24\eta_1 \delta P\big(A(\tau_y)\big) \leqslant \delta P\big(A(\tau_y)\big).$$

The proof is complete.

LEMMA 32. *Suppose that J contains a countable cofinite set, and that L^Ψ-bounded martingales converge essentially. Then given $\varepsilon > 0$, there exists $\eta_1 = \eta_1(\varepsilon)$, such that for $\delta > 0$, there is $s_2 \in J$, $\gamma_2 > 0$, $b_2 > 0$ (depending only on ε and δ) and a set $T \subset \Sigma$ with $P(T) \geqslant 1 - \varepsilon - \delta$ such that, for every finite adapted family of sets (A_t) which satisfy*

$$P\Big(\bigcup_{t \geqslant s_2} A_t \cap T\Big) \geqslant \delta, \tag{20}$$

$$\sum_{t \geqslant s_2} P(A_t) \leqslant (1 + \gamma_2) P\Big(\bigcup_{t \geqslant s_2} A_t\Big), \tag{21}$$

there is $\tau \in IM$ with $\tau \geqslant s_2$ such that

$$P(A(\tau)) \geqslant b_2 \quad \text{and} \quad \int \Phi(\eta_1 e_\tau)\, dP \leqslant \delta b_2.$$

PROOF. Let η_1 be as in Lemma 31 and let $\delta' \leqslant \delta$. For an adapted family $A = (A_t)$, write

$$\begin{aligned} f(A, \delta') &= \sup\Big\{b > 0; \exists \tau \in IM; P(A(\tau)) \geqslant b, \int \Phi(\eta_1 e_\tau)\, dP \leqslant \delta' b\Big\} \\ &= \sup\Big\{P(A(\tau)); \tau \in IM; \int \Phi(\eta_1 e_\tau)\, dP \leqslant \delta' P(A(\tau))\Big\}.\end{aligned}$$

This is an increasing function of δ'.

Given $\varepsilon', \gamma' > 0$, $s' \in J$, let $F(\varepsilon', \gamma', s')$ denote the collection of the finite adapted families $A = (A_t)$ such that the following holds:

$$A_t \neq \varnothing \Rightarrow t \geqslant s', \quad P\big(\bigcup A_t\big) > \varepsilon', \quad \sum P(A_t) \leqslant (1+\gamma')P\big(\bigcup A_t\big).$$

Write

$$g(\varepsilon', \gamma', s', \delta') = \operatorname{Inf}\{f(A, \delta'); A \in F(\varepsilon', \gamma', s')\}.$$

This function is increasing in ε', δ' and s', decreasing in γ'. Let

$$h(\varepsilon', \delta') = \sup_{s' \in i,\, \gamma' > 0} g(\varepsilon', \gamma', s', \delta').$$

If $h(\delta, \delta/2) > 0$, the result is true. Actually, in this case it is enough to take $b_2 = h(\delta, \delta/2)/2$, and s_2, γ_2, with $g(\delta, \gamma_2, s_2, \delta/2) > b_2$.

We now suppose $h(\delta, \delta/2) = 0$. Let

$$\varepsilon_0 = \inf\{\varepsilon' \geqslant 0; \forall \delta' > 0, h(\varepsilon', \delta') > 0\}.$$

Lemma 31 means that, for each $\delta' > 0$, we have $h(\varepsilon, \delta') > 0$, so we have $\delta \leqslant \varepsilon_0 < \varepsilon$. Since $\varepsilon_0 - \delta/2 < \varepsilon_0$, there is $\delta' > 0$ with $h(\delta_0 - \delta/2, \delta') = 0$. Let $b_2 = h(\varepsilon_0 + \delta/2, \delta'/4)/4 > 0$. Let $s_3 \in j$, $\gamma_2 > 0$ such that

$$2b_2 < g(\varepsilon_0 + \delta/2, 2\gamma_2, s_3, \delta'/4) \leqslant 4b_2.$$

Since $h(\varepsilon_0 - \delta/2, \delta') = 0$, we have $g(\varepsilon_0 - \delta/2, \gamma_2, s_3, \delta') = 0$. So there is $B = (B_t) \in F(\varepsilon_0 - \delta/2, \gamma_2, s_3)$ such that $f(B, \delta') < b_2$. Let $I = \{t; B_t \neq \varnothing\}$. Let $U = \bigcup B_t$. We have $P(U) \geqslant \varepsilon_0 - \delta/2$. We have $P(U) \leqslant \varepsilon_0 + \delta/2$, for otherwise $B \in F(\varepsilon_0 + \delta/2, \gamma_2, s_3)$, which is impossible because

$$f(B, \delta') < b_2 < g(\varepsilon_0 + \delta/2, 2\gamma_2, s_3, \delta'/4) < g(\varepsilon_0 + \delta/2, \gamma_2, s_3, \delta').$$

Let $T = \Omega \setminus U$, so $P(T) \geqslant 1 - \varepsilon - \delta$. Let $s_2 \in J$ with $s_2 \geqslant I$.

Let $A = (A_t)$ be an adapted family such that (20) and (21) hold. Let $A'_t = B_t$ for $t \in I$, $A'_t = A_t \setminus U = A_t \cap T$ for $t \geqslant s_2$, and $A'_t = \varnothing$ otherwise. It follows from (20) that $P(\bigcup_{t \geqslant s_3} A'_t) \geqslant \varepsilon_0 + \delta/2$. Also,

$$\begin{aligned}
\sum_{t \geqslant s_3} P(A'_t) &= \sum_{t \in I} P(B_t) + \sum_{t \geqslant s_2} P(A'_t) \\
&\leqslant (1+\gamma_2)\left[P(U) + P\Big(\bigcup_{t \geqslant s_2} A_t\Big)\right] - \sum_{t \geqslant s_2} P(A_t \cap U) \\
&\leqslant (1+\gamma_2)\left[P(U) + P\Big(\bigcup_{t \geqslant s_2} A_t \setminus U\Big)\right] \\
&\quad + (1+\gamma_2)P\Big(\bigcup_{t \geqslant s_2} A_t \cap U\Big) - \sum_{t \geqslant s_2} P(A_t \cap U) \\
&\leqslant (1+2\gamma_2)P\big(\bigcup A'_t\big).
\end{aligned}$$

It follows that

$$A' = (A'_t) \in F(\varepsilon_0 + \delta/2, 2\gamma_2, s_3).$$

The definition of $g(\varepsilon_0 + \delta/2, 2\gamma_2, s_3, \delta'/4)$ shows that there is a $c > 0$ with

$$2b_2 \leqslant c \leqslant g(\varepsilon_0 + \delta/2, 2\gamma_2, s_3, \delta'/4) \leqslant 4b_2$$

and $\tau' \in IM$ with

$$P(A'(\tau')) = c \geqslant 2b_2, \qquad \int \Phi(\eta_1 e_{\tau'})\, dP \leqslant c\delta'/4 \leqslant \delta b_2.$$

Let $\tau_1 \in IM$ be given by $\{\tau_1 = t\} = \{\tau' = t\}$ for $t \in I$ and $\{\tau_1 = t\} = \varnothing$ otherwise. We have $\int \Phi(\eta_1 e_{\tau_1})\, dP \leqslant \delta b_2$. Since $f(B, \delta') < b_2$, this means that $P(B(\tau_1)) \leqslant b_2$. Let $\tau \in IM$ be given by $\{\tau = t\} = \{\tau' = t\}$ for $t \geqslant s_2$ and $\{\tau = t\} = \varnothing$ otherwise. Then $A'(\tau') \subset B(\tau_1) \cup A(\tau)$, so we have $P(A(\tau)) \geqslant b_2$. Since $\int \Phi(\eta_1 e_\tau)\, dP \leqslant \delta b_2$, the proof is complete.

We can start the proof of Theorem 8.

PROPOSITION 33. *Assume that J contains a countable cofinal set and that L^Ψ-bounded martingales converge essentially. Then for each $\varepsilon > 0$ there are $\eta = \eta(\varepsilon)$ and a set Ω_ε with $P(\Omega_\varepsilon) \geqslant 1 - \varepsilon$ such that, for each adapted family $A = (A_t)$ and each $\delta > 0$, there is $\tau \in IM$ with $P((A^* \cap \Omega_\varepsilon)\setminus A(\tau)) \leqslant \delta$ and $\int \Phi(\eta e_\tau)\, dP \leqslant \delta$.*

PROOF. *First step*. Let $\eta = \eta_1(\varepsilon/2)$. According to Lemma 32, for each η there are $s(n) \in J$, $\gamma(n)$, $b(n) > 0$ and a set $T_n \in \Sigma$ with $P(T_n) \geqslant 1 - \varepsilon/2 - 2^{-n}$ such that for every adapted family of sets (A_t) satisfying

$$P\Big(\bigcup_{t \geqslant s(n)} A_t \cap T_n\Big) \geqslant 2^{-n}, \tag{22}$$

$$\sum_{t \geqslant s(n)} P(A_t) \leqslant (1 + \gamma(n)) P\Big(\bigcup_{t \geqslant s(n)} A_t\Big), \tag{23}$$

there is $\tau \in IM$ with $P(A(\tau)) \geqslant b(n)$ and $\int \Phi(\eta e_\tau)\, dP \leqslant 2^{-n} b(n)$.

Let f be a cluster point of the sequence 1_{T_n} for the weak topology of $L^2(P)$. We have $f \leqslant 1$ and $\int f\, dP \geqslant 1 - \varepsilon/2$. Let $\Omega_\varepsilon = \{f > \frac{1}{2}\}$, so $P(\Omega_\varepsilon) \geqslant 1 - \varepsilon$.

Second step. Since $L^\infty \subset L^\Psi$, all L^∞ bounded martingales converge essentially. So condition V_1 of Krickeberg holds [**5**], that is for each adapted family A, and for each $\gamma > 0$, there is $\tau \in IM$ with $P(A(\tau)) \geqslant P(A) - \gamma$ and $\|e_\tau\|_1 \leqslant \gamma$.

Third step. Let A be an adapted family with $P(A^* \cap \Omega_\varepsilon) > 0$, and let $\delta > 0$. Since $\int_{A^* \cap \Omega_\varepsilon} f\, dP > P(A^* \cap \Omega_\varepsilon)/2$, there is n with $2^{-n} < \delta$, $2^{-n+2} < P(A^* \cap \Omega_\varepsilon)$ such that $P(A^* \cap \Omega_\varepsilon \cap T_n) \geqslant P(A^* \cap \Omega_\varepsilon)/2$.

By induction over k, we construct $\tau_k \in IM$ such that

$$P(A(\tau_k)) \geqslant k b(n), \qquad \int \Phi(\eta e_{\tau_k})\, dP \leqslant k\delta b(n). \tag{24}$$

The induction continues as long as $P((A^* \cap T_n)\setminus A(\tau_k)) \geqslant 2^{-n}$, so it stops in a finite number of steps, and produces $\tau \in IM$ with $P(A(\tau)) \geqslant P(A^* \cap \Omega_\varepsilon)/4$ and $\int \Phi(\eta e_\tau)\, dP \leqslant \delta P(A(\tau)) \leqslant \delta$.

The second step shows that there is $\tau' \in IM$ with $P(A(\tau') \cap T_n) > b(n)$ and $\|e_{\tau'}\|_1 \leqslant \gamma(n)$. In other words, the family (A'_t) given by $A'_t = A_t \cap \{\tau' = t\}$ satisfies (22) and (23). The existence of τ_1 follows from the first step.

Assume now that τ_k has been constructed, and that $P((A^* \cap T_n)\setminus A(\tau_k)) > 2^{-n}$. Let $s \geqslant \tau_k$. Define $B_t = A_t \setminus A(\tau_k)$ for $t \geqslant s$, $B_t = \varnothing$ otherwise. Then $P(B^* \cap T_n) > 2^{-n}$, so the second step shows that there is $\tau' \in IM$ with $P(B(\tau') \cap T_n) > 2^{-n}$ and

$\|e_{\tau'}\|_1 \leqslant \gamma(n)$. In particular, the family (A'_t) given by $A'_t = A_t \cap \{\tau' = t\}$ satisfies (22) and (23), so there is τ'' with $P(A'(\tau'')) > b(n)$ and $\int \Phi(\eta e_{\tau''})\, dP \leqslant \delta b(n)$. We now define $\{\tau_{k+1} = t\} = \{\tau_k = t\}$ for $t \leqslant s$, $\{\tau_{k+1} = t\} = \{\tau'' = t\}$ for $t \geqslant s$, $\{\tau_{k+1} = t\} = \varnothing$ otherwise, and the induction continues.

Fourth step. Using one more standard exhaustion, that we leave to the reader, we find that there is $\tau \in IM$ with $P(A(\tau)) \geqslant P(A^* \cap \Omega_\varepsilon) - \delta$ and $\int \Phi(\eta e_\tau)\, dP \leqslant \delta$. This is formally weaker than the property we look for. However, there is $s \in J$ and $C \subset \Sigma_s$ with $P(C \cap A^* \cap \Omega_\varepsilon) \geqslant P(A^* \cap \Omega_\varepsilon) - \delta$ and $P(C \setminus \Omega_\varepsilon) \leqslant \varepsilon$. Define $B_t = A_t \cap C$ for $t \geqslant s$, $B_t = \varnothing$ otherwise. The above result applied to B yields a $\tau \in IM$ with $\int \Phi(\eta e_\tau)\, dP \leqslant \delta$ and $P((A^* \cap \Omega_\varepsilon) \setminus A(\tau)) \leqslant 3\delta$. The proof is complete.

We now prove Theorem 8. The preceding proposition asserts that condition D_Φ is necessary when J contains a countable cofinal set. Since we assume $L^\Psi \neq L^1$, we consider only martingales of the type $(E^t(f))$ for $f \in L^\Psi$; the fact that condition D_Φ is sufficient will follow from the proof of Proposition 24 and the following

PROPOSITION 34. *Assume that condition D_Φ holds. Let $f \in L^\Psi, f \geqslant 0$. Let $X_t = E^t(f)$. Then for $\lambda > 0$ we have $P(X^* > \lambda) \leqslant E(f)/\lambda$.*

PROOF. Let $\alpha > 0$. There is $\gamma > 0$ such that $\int \Psi(f\alpha^{-1}\gamma)\, dP < \infty$. Since we assume (5), we have $\Phi(x) > 0$ for $x > 0$, so for each u, $\gamma \to \gamma^{-1}\Psi(u\gamma)$ is increasing from zero. Lebesgue's theorem shows that there is $\gamma > 0$ such that $\int \gamma^{-1}\Psi(f\alpha^{-1}\gamma)\, dP \leqslant 1$. Let $\lambda > 0$, and let $0 < \beta < \lambda$. Let $A_t = \{X_t > \beta\}$. Then $P(X^* > \lambda) \leqslant P(A^*)$. Let $0 < \varepsilon < P(A^*)$. Then condition D_Φ gives $\eta(\varepsilon)$ and $\tau \in IM$ such that $\int \gamma^{-1}\Phi(\eta(\varepsilon)e_\tau)\, dP \leqslant 1$ and $P(A^* \setminus A(\tau)) < \varepsilon$. We can assume $\{\tau = t\} \subset A_t$ for each t. We get

$$E(S_\tau f) = \sum E(f 1_{\{\tau = i\}}) = \sum E(X_i 1_{\{\tau = i\}}) \geqslant \beta P(A(\tau)).$$

On the other hand, $\gamma^{-1}\Phi(x)$ and $\gamma^{-1}\Psi(x\gamma)$ are conjugate Young functions, so Young's inequality (1) gives

$$E(e_\tau \eta(\varepsilon) f\alpha^{-1}) \leqslant 1, \quad \text{so } E(e_\tau f) \leqslant \alpha/\eta(\varepsilon);$$

so we get

$$E(S_\tau f) \leqslant E(f) + E(e_\tau f) \leqslant E(f) + \alpha/\eta(\varepsilon)$$

and

$$P(\{X^* > \lambda\}) \leqslant P(A^*) \leqslant \varepsilon + \beta^{-1}(E(f) + \alpha/\eta(\varepsilon)).$$

Letting β, ε be fixed, we let α go to zero. We then let ε go to zero and β go to λ to get the required inequality. The proof is complete.

5. Proof of Theorem 9. Since $\int \Phi(e_\tau/n)\, dP \leqslant 1$ implies $\|e_\tau\|_\Phi \leqslant 2n$, condition D_Φ implies condition C_Φ. So we have already proved that when $L^\Psi \neq L^1$, condition C_Φ is necessary when J contains a countable cofinal set. Actually using our techniques, the direct proof that C_Φ is necessary is much easier than the proof that D_Φ is necessary. (But, as Theorem 11 shows, C_Φ is not sufficient in general). We suppose now $L^\Psi = L^1$, and we prove in that case that D_Φ is necessary when J contains a countable cofinal set.

LEMMA 35. *Let $I \subset J$ be a finite set, and for each $i \in I$, let A_i be a Σ_i-measurable set. Let $a > 0$. Assume that for each family $(\xi_i)_{i\in I}$ of functions, $\xi_i \geqslant 0$, $\xi_i = 0$ outside A_i, ξ_i is bounded Σ_i-measurable, and $\|\Sigma_{i\in I}\xi_i\|_1 = 1$, we have $\|\Sigma_{i\in I}\xi_i\|_\infty \geqslant a$. Then for each $\gamma > 0$ there exists $f \in L^1$ with $\| f \|_2 \leqslant 1/a$, $f \geqslant 0$, such that, for $i \in I$, we have $P(A_i \setminus A_i') \leqslant \gamma$, where $A_i' = \{E^i(f) \geqslant \frac{1}{2}\}$.*

PROOF. We denote by C_1 the set of functions $\Sigma_{i\in I}\xi_i$ where ξ_i is bounded, Σ_i-measurable, $\xi_i \geqslant 0$, $\xi_i = 0$ outside A_i, $\|\Sigma\xi_i\|_1 = 1$, $\|\Sigma\xi_i\|_\infty \leqslant 1/\gamma$.

We denote by C_2 the set of ξ with $\sup \xi \leqslant a/2$. Then C_1, C_2 are convex. Moreover, C_1 is $\sigma(L^\infty, L^1)$-compact and C_2 is $\sigma(L^\infty, L^1)$-closed. The theorem of Hahn–Banach gives $f \in L^1$ with $f < \frac{1}{2}$ on C_2 and $f > \frac{1}{2}$ on C_1. Since $f < \frac{1}{2}$ on C_2, we have $f \geqslant 0$ and $\| f \|_1 \leqslant 1/a$. Suppose, if possible, that for some i we have $P(B_i) > \gamma$, where $B_i = A_i \setminus A_i'$. Then the function $h = P(B_i)^{-1}1_{B_i}$ belongs to C_1, so $E(fh) > \frac{1}{2}$. However, $E(fh) = P(B_i)^{-1}\int_{B_i} E^i(f) \leqslant \frac{1}{2}$. This contradiction proves the lemma.

LEMMA 36. *Assume that J contains a countable cofinite set and that for each $f \in L^1$ the martingale $(E^t(f))$ converges essentially. Then for each $\varepsilon > 0$ there is $s \in J$ and $N_\varepsilon > 0$ such that, for each finite set $I \subset J$ with $s \leqslant I$, and each family $(A_i)_{i\in I}$, A_i is Σ_i-measurable, $P(\bigcup A_i) \geqslant \varepsilon$, there exist bounded functions ξ_i, ξ_i is Σ_i-measurable, $\xi_i \geqslant 0$, $\xi_i = 0$ outside A_i, $\|\Sigma\xi_i\|_1 = 1$, $\|\Sigma\xi_i\|_\infty \leqslant N_\varepsilon$.*

PROOF. Otherwise there is $\varepsilon > 0$ such that, for each $s \in J$ and $N > 0$, there is a finite set $I \subset J$, with $s \leqslant I$ and a family $(A_i)_{i\in I}$ of Σ_i-measurable sets, with $P(\bigcup A_i) \geqslant \varepsilon$, such that for each family ξ_i of bounded functions ξ_i, $\xi_i \geqslant 0$, $\xi_i = 0$ outside A_i, $\|\Sigma\xi_i\|_1 = 1$, then $\|\Sigma\xi_i\|_\infty \geqslant N$.

Let (s_k) be a cofinal sequence in I. Using the preceding statement and Lemma 35, we construct for each k a finite set I_k with $s_k \leqslant I_k$ and $f_k \in L^1, f \geqslant 0, \| f_k\|_1 \leqslant \varepsilon 2^{-k-2}$ such that

$$(25) \qquad P\left(\bigcup_{i\in I_k}\left\{E^i(f_k) > \frac{1}{2}\right\}\right) \geqslant \varepsilon - \frac{1}{k}.$$

Let $f = \Sigma f_k$. Then $\| f \|_1 \leqslant \varepsilon/4$. Let $X_t = E^t(f)$. From (25) we see that $P(X^* \geqslant \frac{1}{2}) \geqslant \varepsilon$. On the other hand, $P(X_* \geqslant \frac{1}{2}) \leqslant \varepsilon/2$, so $X^* \neq X_*$. This contradiction concludes the proof.

LEMMA 37. *Assume that J contains a countable cofinite set and that for each $f \in L^1$ the martingale $(E^t(f))$ converges essentially. Then for each $\varepsilon > 0$ there is $M_\varepsilon > 0$, $b > 0$, $s \in J$ and $\gamma > 0$ such that, for each finite adapted family (A_t),*

$$(26) \qquad P\left(\bigcup_{t>s} A_t\right) > \varepsilon, \qquad \sum_{t>s} P(A_t) \leqslant (1+\gamma)P\left(\bigcup_{t>s} A_t\right),$$

there exists $\tau \in IM$ with $P(A(\tau)) > b$ and $\|S_\tau\|_\infty \leqslant M_\varepsilon$.

PROOF. Let N_ε and s be as in Lemma 36. Let $b = 1/4N_\varepsilon$, $\gamma = 1/2N_\varepsilon$. Let (A_t) be an adapted family satisfying (26). We can assume there is $I \geqslant s$ such that $A_t = \varnothing$ for $t \notin I$. Let

$$C = \{\omega;\ \exists i, j \in I, i \neq j, \omega \in A_i \cap A_j\}.$$

Then $P(C) \leqslant 1/2N_\varepsilon$. Let $(\xi_i)_{i \in J}$ be the functions given by Lemma 36, and let $\xi = \Sigma \xi_i$. Since $\|\xi\|_\infty \leqslant N_\varepsilon$, we have $\int_C \xi \, dP \leqslant \frac{1}{2}$, so $\int_{\Omega \setminus C} \xi \, dP \geqslant \frac{1}{2}$. It follows that if $H = \{\xi > \frac{1}{4}\} \setminus C$ we get $P(H) \geqslant 1/4N_\varepsilon$. For $\omega \in H$, there is a unique $i \in I$ with $\xi_i(\omega) > 0$, so $\xi_i(\omega) \geqslant \frac{1}{4}$. Define $\tau \in IM$ by $\{\tau = i\} = \{\xi_i \geqslant \frac{1}{4}\}$ for $i \in I$, $\{\tau = i\} = \varnothing$ otherwise. We have shown that $P(A(\tau)) \geqslant P(H) \geqslant b$. On the other hand, $S_\tau \leqslant 4\xi$, so $\|S_\tau\|_\infty \leqslant 4N_\varepsilon$. The proof is complete.

At this point, the fact that condition V_1 holds and the standard exhaustion procedure make it clear that condition C_Φ is necessary for the convergence of equi-integrable L^1-bounded martingales. The proof that condition D_Φ is also necessary follows the line of Lemmas 32 and 33 with some simplifications.

It remains to show that condition C_Φ implies the convergence of L^Ψ-bounded martingales if Ψ satisfies the Δ_2 condition.

PROPOSITION 38. *Assume that condition C_Φ holds. Let $s \in J$, $B \in \Sigma_s$. Then for each positive submartingale (X_t) and $\lambda > 0$ we have*

$$P(\{X^* \geqslant \lambda\} \cap B) \leqslant \varepsilon + (2M_\varepsilon/\lambda) \sup_t \|X_t 1_B\|_\Psi.$$

PROOF. We can assume $P(\{X^* \geqslant \lambda\} \cap B) > \varepsilon$. Let $A_t = \{X_t \geqslant \lambda/2\} \cap B$ for $t \geqslant s$. Then $\varepsilon < P(\{X^* \geqslant \lambda\} \cap B) \leqslant P(A^*)$. From condition C_Φ, there is a $\tau \in IM$ such that $P(A^* \setminus A(\tau)) \leqslant \varepsilon$ and $\|S_\tau\|_\Phi \leqslant M_\varepsilon$. Let $t \geqslant \tau$. We have

$$E(X_t S_\tau 1_B) \geqslant \sum E(X_t 1_{\{\tau = i\}} 1_B) \geqslant \sum E(X_t 1_{\{\tau = i\} \cap B}) \geqslant \lambda P(A(\tau)).$$

On the other hand, $E(X_t S_\tau 1_B) \leqslant \|X_t 1_B\|_\Psi M_\varepsilon$, so we get

$$P(A^*) \leqslant \varepsilon + (2M_\varepsilon/\lambda) \sup_t \|X_t 1_B\|_\Psi.$$

PROPOSITION 39. *Assume that Ψ satisfies the Δ_2 condition. Assume that for every $s \in J$, every $B \in \Sigma_s$, every positive submartingale (X_t), we have for $\lambda > 0$*

$$P(\{X^* > \lambda\} \cap B) \leqslant \varepsilon + (2M_\varepsilon/\lambda) \sup_t \|X_t 1_B\|_\Psi. \tag{27}$$

Then every L^Ψ-bounded martingale converges.

PROOF. Let $f \in L^\Psi$ and $\varepsilon > 0$. Denote by f_a the truncation of f at $-a$ and a. Since Ψ satisfies the Δ_2 condition, $\|f - f_a\|_\Psi \to 0$ when $a \to \infty$. It follows that there are $s \in J$ and $g \in L^\Psi$, g is Σ_s-measurable, with $\|f - g\|_\Psi < \varepsilon/2M_\varepsilon$. Let $Y_t = E^t(|f - g|)$. From (27) we get $P(Y^* > \varepsilon) \leqslant 2\varepsilon$. Since $|E^t(f) - E^t(g)| \leqslant Y_t$ and $E^t(g) = g$ for $t \geqslant s$, we see that the martingale $(E^t(f))$ converges essentially. If $L^\Psi \neq L^1$, all the L^Ψ-bounded martingales are equi-integrable, so are of this type. We now investigate the case $L^\Psi = L^1$. Let Ξ denote the union of all the algebras Σ_t. For $A \in \Xi$, let $m(A) = \lim_t \int_A X_t \, dP$. This is a finitely additive measure of bounded variation. As in **[11]**, we write $m = m_1 + m_2$ where m_1 is absolutely continuous and m_2 is singular. Let $m_1(A) = \int_A f \, dP$. We have shown the convergence of $(E^t(f))$. It remains to show that $Y_t = X_t - E^t(f)$ converges essentially to zero. Let $Z_t = |Y_t|$. This is a submartingale. Let $\varepsilon > 0$. Let $s \in J$ and $B \in J$ with $P(B) \geqslant 1 - \varepsilon$ and $|m_2|(B) < \varepsilon/2M_\varepsilon$. Since $E(Z_t 1_B) \leqslant |m_2|(B)$, from (27) we get $P(\{Z^* \geqslant \varepsilon\} \cap B) \leqslant 2\varepsilon$, so $P(Z^* \geqslant \varepsilon) \leqslant 3\varepsilon$, and this concludes the proof.

6. Proof of Theorem 10.

LEMMA 40. *If Φ does not satisfy condition* Exp, *for each $a > 0$ there is an integer l with $\Phi(2l) \geqslant a^l \Phi(l)$.*

PROOF. If for each n we have $\Phi(2^{n+1}) \leqslant a^{2^n}\Phi(2^n)$, we get $\Phi(2^n) \leqslant a^{2^n}\Phi(1)$ for each n, so for each $u \geqslant 1$ we have $\Phi(u) \leqslant a^{2u}\Phi(1)$, which implies condition Exp.

We now start the construction. Since the details differ whether Φ is always finite or not, we assume Φ always finite, leaving the other (simpler) case to the reader. By the lemma, there is a sequence (l_n) of integers with

$$\Phi(2l_n) \geqslant 2^{n+3} 2^{2(n+4)l_n} \Phi(l_n). \tag{28}$$

We can assume $l_n \to \infty$ and $\Phi(l_n) \geqslant 2^3$. Let $k_n = 2^{n+4} l_n$. Let $G_n = \{1, \ldots, k_n\}$, $H_n = \{1, \ldots, l_n\}$. Let U_n be the uniform probability on G_n. Let $M_n = G_n^{H_n}$, and Q_n be the power of U_n on M_n. Let $L_n = G_n \cup M_n$. Let $b_n = 2^{-n}/\Phi(l_n)$, so $b_n \leqslant 2^{-n-5}$. On L_n, consider the probability P_n given by

$$P_n(B) = (1 - b_n)U_n(B \cap G_n) + b_n Q_n(B \cap M_n).$$

For $i \leqslant k_n$, let

$$C_{n,i} = \{i\} \cup \{y \in M_n, \exists p \in H_n;\ y(p) = i\}.$$

Let $L = \prod L_n$, and P be the product probability. Then $(L, \mathscr{B}, P)$ is isomorphic to $([0,1], \mathscr{B}, \lambda)$ where $\mathscr{B}$ is the Borel σ-algebra. We denote by Σ_n the σ-algebra of sets that depend only on the first $n-1$ coordinates. Let $B_{n,i} = \{z = (z_n) \in L;\ z_n \in C_{n,i}\}$. We denote by $\Sigma_{n,i}$ the σ-algebra generated by Σ_n and $B_{n,i}$. Let $J = \{(n,i);\ i \in H_n\}$. For $(n', i') \in J$, we say that $(n, i) < (n', i')$ if either $n < n'$ or $(n, i) = (n', i')$. The map $t \to \Sigma_t$ is increasing.

We first show that condition D_Φ holds. Let (A_t) be an adapted family of sets. Let $\varepsilon > 0$. Let n_0 with $2^{-n_0} \leqslant \varepsilon$. Let $A_t'' = \varnothing$ if $t = (m, i)$ for $m \leqslant n_0$ and $A_t'' = A_t$ otherwise. Let $A_t' = A_t'' \setminus \bigcup_{s<t} A_s'$. For each, let $C_n = \bigcup_{i \in H_n} A'_{n,i}$. Then the sets C_n are disjoint, and $\bigcup C_n \supset A^*$. Let q be such that $P(A^* \setminus \bigcup_{n \leqslant q} C_n) \leqslant \varepsilon$. Define $\tau \in IM$ by $\{\tau = (n,i)\} = A'_{n,i}$ for $n \leqslant q$, and $\{\tau = (n,i)\} = \varnothing$ otherwise. Then $A(\tau) = \bigcup_{n \leqslant q} C_n$, so $P(A^* \setminus A(\tau)) \leqslant \varepsilon$. Let $f = \sum_{i \in H_n} 1_{B_{n,i}}$, $g_n = f - f \wedge 1$. Then for $x = (x_n) \in L$, $g_n(x) = 0$ except when $x_n \in M_n$, in which case $g_n \leqslant l_n$, so $\int \Phi(g_n)\, dP \leqslant b_n \Phi(l_n) \leqslant 2^{-n}$. But one sees that $e_\tau \leqslant g_n$ on C_n. Since $C_n = \varnothing$ for $n \leqslant n_0$, we get

$$\int \Phi(e_\tau)\, dP \leqslant \sum_{n > n_0} \int_{C_n} \Phi(g_n)\, dP \leqslant 2^{-n_0} \leqslant \varepsilon$$

so condition D_Φ holds.

We now show that condition V_Φ fails. Let $A_{n,i} = B_{n,i}$ for each n, i. Then $A^* = L$. Let $\tau \in IM$ be such that $P(A(\tau)) > \frac{1}{2}$. We are going to show that $\|e_\tau\|_\Phi \geqslant \frac{1}{2}$.

For each n, let

$$B_n = \bigcup_{i \in H_n} B_{n,i} \cap \{\tau = (n,i)\}.$$

We have $A(\tau) = \bigcup_n B_n$, so there exists n with $P(B_n) \geqslant 2^{-n-1}$. Let $\mathscr{Y}$ be the family of atoms Y of Σ_n such that $P(Y \cap B_n) \geqslant 2^{-n-2}P(Y)$. Then the union of $\mathscr{Y}$ has probability $\geqslant 2^{-n-2}$. Let us fix $Y \in \mathscr{Y}$. Let $l = \operatorname{card} K$, where

$$K = \{i \in H_n;\, Y \cap B_{n,i} \subset \{\tau = (n, i)\}\}.$$

We have

$$P(Y \cap B_n) \leqslant P(Y)(b_n + l(1 - b_n)/k_n).$$

Since $b_n \leqslant 2^{-n-3}$, it follows that $l \geqslant 2^{-n-3}k_n \geqslant 2l_n$. Let

$$N = \{y \in M_n;\, \text{the numbers } y(p) \text{ for } p \in H_n \text{ are distinct and in } K\}.$$

Since $l \geqslant 2l_n$, we have

$$P_n(N) \geqslant b_n\left(\frac{l}{k_n}\frac{l-1}{k_n}\cdots\frac{l-l_n+1}{k_n}\right) \geqslant b_n(l/2k_n)^{l_n} \geqslant b_n(2^{-n-4})^{l_n}.$$

Let $N' = \{z \in Y;\, z_n \in N\}$. On N' we have $e_\tau \geqslant l_n$. It follows from (28) that

$$\int_{B_n \cap Y} \Phi(2e_\tau)\, dP \geqslant P(N')\Phi(2l_n) \geqslant P(Y)P_n(N)\Phi(2l_n)$$

$$\geqslant P(Y)(2^{-n-4})^{l_n} b_n \Phi(2l_n) \geqslant 2^{n+3}P(Y).$$

Summation over Y in $\mathscr{Y}$ gives $\int \Phi(2e_\tau)\, dP \geqslant 2$, so $\|2e_\tau\|_\Phi \geqslant 1$ and $\|e_\tau\|_\Phi \geqslant \frac{1}{2}$, and this concludes the proof.

7. Proof of Theorem 11.

LEMMA 41. *If Ψ fails condition Δ_2, one can find numbers $a_n, b_n > 0$, integers $k_n > 0$ such that the following hold*:

(29) $$b_n\Psi(a_n) \leqslant 2^{-n},$$

(30) $$k_n a_n b_n \geqslant \tfrac{1}{4},$$

(31) $$b_n\Phi(k_n) \leqslant 1.$$

PROOF. We can find a_n such that $\Psi(a_n) \leqslant 2^{-n}\Psi(2a_n)$. Let $b_n = 1/2a_n\psi(2a_n)$. Then the equality case in Young's inequality shows that

$$\Psi(2a_n) + \Phi(\psi(2a_n)) = 2a_n\Psi(2a_n) = 1/b_n.$$

It follows that $b_n\Psi(a_n) \leqslant 2^{-n}b_n\Psi(2a_n) \leqslant 2^{-n}$. Also $b_n\Phi(\frac{1}{2}a_nb_n) \leqslant 1$. Since $a_nb_n \to 0$, one can suppose $2a_nb_n \leqslant 1$ so it is enough to take for k_n the integer part of $1/2a_nb_n$.

We now proceed with the construction. Since $b_n \to 0$, we suppose $b_n \leqslant \frac{1}{2}$ for each n. Let $L_n = \{0, \ldots, k_n\}$. Let $M_n = L_n \setminus \{0\}$. On L_n, let P_n be the probability given by $P_n(\{0\}) = b_n$, $P_n(\{i\}) = (1 - b_n)/k_n$ for $i \leqslant i \leqslant k_n$. Let $L = \prod L_n$, and P be the product probability. Then L is a compact metric space, and $(L, \mathscr{B}, P)$ is isomorphic to $([0,1], \mathscr{B}, \lambda)$, where λ is Lebesgue's measure and $\mathscr{B}$ is the Borel σ-algebra.

For each n, let Σ_n be the σ-algebra of subsets of L that depend only on the first $(n-1)$ coordinates. For $i \in M_n$, let

$$B_{n,i} = \{x \in L;\, x = (x_n),\, x_n \in \{0, i\}\}.$$

Let $\Sigma_{n,i}$ be generated by Σ_n and $B_{n,i}$. We order the algebras $\Sigma_{n,i}$ by inclusion, and we denote as usual the index set by J, so $J = \{(n, i); n \in \mathbf{N}, i \in M_n\}$.

For each n, define f_n by $f_n(x) = a_n$ if $x_n = 0$, $f_n(x) = 0$ otherwise. From (29) we see that $\int \Psi(f_n)\, dP \leqslant 2^{-n}$. On the other hand, we get that $E^{n,i}(f_n) \geqslant k_n a_n b_n \geqslant \frac{1}{4}$ on $B_{n,i}$. Since $P(\bigcup_{i \in M_n} B_{n,i}) = 1 - b_n$, we get from Lemma 28 that there is $f \in L^{\Psi}$ such that $(E^t(f))$does not converge essentially.

We show now that condition C_{Φ} holds. Let (A_t) be an adapted family. For $t \in J$, let $A'_t = A_t \setminus \bigcup_{s<t} A_s$. This is an adapted family, and $\bigcup A'_t \supset A^*$. For each n, let $C_n = \bigcup_{i \in M_n} A'_{n,i}$, so $\bigcup A'_t = \bigcup C_n$. Let $\varepsilon > 0$ and let p be such that $P(A^* \setminus \bigcup_{n \leqslant p} C_n) \leqslant \varepsilon$. For $n \leqslant p$, define $\{\tau = (n, i)\} = A'_{n,i}$ and define $\{\tau = (n, i)\} = \varnothing$ otherwise. We have $A(\tau) = \bigcup_{n \leqslant p} C_n$, so $P(A^* \setminus A(\tau)) \leqslant \varepsilon$. Since the sets C_n are disjoint,

$$\int \Phi(e_\tau)\, dP \leqslant \sum_{n \leqslant p} \int_{C_n} \Phi(e_\tau)\, dP.$$

Let Y be an atom of Σ_n, and l be the number of i in M_n such that $A'_{n,i} \cap Y \neq \varnothing$. For $x \in Y \cap C_n$, we have $e_\tau(x) = l - 1$ if $x_n = 0$, and zero otherwise. It follows that

$$\int_{Y \cap C_n} \Phi(e_\tau)\, dP = P(Y) b_n \Phi(l-1) \leqslant l P(Y) b_n \Phi(k_n)/k_n \leqslant l P(Y)/k_n.$$

On the other hand, $P(Y \cap C_n) \geqslant P(Y)(1 - b_n) l/k_n \geqslant P(Y) l/2k_n$ so finally

$$\int_{Y \cap C_n} \Phi(e_\tau)\, dP \leqslant 2P(Y \cap C_n).$$

By summation over Y, we get $\int_{C_n} \Phi(e_\tau)\, dP \leqslant 2P(C_n)$ and summation over n gives $\int \Phi(e_\tau)\, dP \leqslant 2$, so $\|e_\tau\|_\Phi \leqslant 3$ and this concludes the proof.

8. Proof of Theorem 12. The details of the proof differ slightly depending upon $L^{\Psi} = L^1$ or $L^{\Psi} \neq L^1$, although the ideas are the same. We shall consider only the case $L^{\Psi} \neq L^1$; that is, $\Phi(u) < \infty$ for each u (the case $L^{\Psi} = L^1$ is somewhat simpler).

Since Φ fails condition Exp, the method of Lemma 40 shows that there is a sequence (a_n) of integers with

$$\Phi(2na_n) 2^{-2n-2} (\alpha_n/2)^{n^2 a_n} \geqslant \Phi(na_n)$$

where $\alpha_n = 1 - ((1 - 2^{-n-2})/(1 - 2^{-n-3}))^{1/n}$. Let k_n be large enough that $k_n \alpha_n \geqslant 2n^2 a_n$ and that $(1 - 1/k_n)^n \leqslant 1 - 2n/3k_n$.

Let $H_n = \{1, \ldots, k_n\}$ and let U_n be the uniform probability on H_n. Let $M_n = H_n^n$, and let Q_n be the power of U_n on M_n. Let $N_n = H_n^{n^2 a_n}$, and let R_n be the power of U_n on N_n. Let $L_n = M_n \cup N_n$. Let $b_n = 2^n/\Phi(na_n)$. We can assume $\Phi(na_n) \geqslant 8$, so $b_n \leqslant 2^{-n-3}$. For $A \subset L_n$, let

$$P_n(A) = (1 - b_n) Q_n(A \cap M_n) + b_n R_n(A \cap N_n).$$

Let $L = \prod L_n$ and on L let P be the product of the P_n. For $i \in H_n$, let $D_i = B_i \cup C_i$, where

$$B_i = \{x \in M_n; \exists r \leqslant n; x(r) = i\},$$
$$C_i = \{x \in N_n; \exists r \leqslant n^2 a_n; x(r) = i\}.$$

We note that

$$(32)\qquad Q_n(B_i) = 1 - (1 - 1/k_n)^n, \qquad R_n(C_i) = 1 - (1 - 1/k_n)^{n^2 a_n}.$$

Let $L'_n = \prod_{i<n} L_i$, $L''_n = \prod_{i>n} L_i$ and let P'_n be the product probability on L'_n. Let $G_n = L'_n \times H_n$, and $S_n = P'_n \otimes U_n$. Let

$$\mathscr{T} = \{T = (T_n); \forall n, T_n \subset G_n, S_n(T_n) \to 0\}.$$

We order $\mathscr{T}$ by inclusion; that is, $T \leqslant T'$ if $T_n \subset T'_n$ for each n. This order makes $\mathscr{T}$ a directed set.

We can now define the index set J:

$$J = \{(n, f, i, T); n \in \mathbf{N}, f \in L'_n, i \in H_n, T \in \mathscr{T}, (f, i) \in T_n\}.$$

The order on J is defined by $(n, f, i, T) \leqslant (n', f', i', T')$ if either $n < n'$ and $T \leqslant T'$ or $n = n'$, $T \leqslant T'$, $f = f'$, $i = i'$.

We denote by Σ_n the algebra of subsets of L of the type $\pi_n^{-1}(A)$, where $A \subset L'_n$ and π_n is the natural projection $L \to L'_n$. For $t = (n, f, i, T) \in J$, we denote by Σ_t the algebra generated by Σ_n and $\{f\} \times D_i \times L''_n$. It is clear that the map $t \to \Sigma_t$ is increasing.

We now show that condition C_Φ fails. For $f \in L'_n$, $i \in H_n$, let $F_{f,i} = \{f\} \times D_i \times L''_n$. Let us fix m. For $t = (n, f, i, T)$, let $A_t = F_{f,i}$ if $n \geqslant m$, and $A_t = \varnothing$ otherwise. We first show that $P(A^*) = 1$. Let $T \in \mathscr{T}$ be fixed. It is enough to show that

$$\lim_n P\Big(\bigcup_{(f,i) \in T_n} F_{f,i}\Big) = 1.$$

Let $f \in L'_n$ be fixed, let $V = \{i \in H_n; (f, i) \in T_n\}$, and let $l = \operatorname{card} V$. We get

$$P\Big(\bigcup_{i \in V} F_{f,i}\Big) = P'_n(\{f\}) d_n,$$

where

$$d_n = (1 - b_n)\big(1 - (1 - l/k_n)^n\big) + b_n\Big(1 - (1 - l/k_n)^{n^2 a_n}\Big) \geqslant 1 - l/k_n.$$

Summation over f gives

$$P\Big(\bigcup_{(f,i) \in T_n} F_{f,i}\Big) \geqslant 1 - S_n(T_n),$$

and this completes the proof that $P(A^*) = 1$.

Now let $\tau \in IM$ with $P(A(\tau)) \geqslant \frac{1}{2}$. We show that $\|S_\tau\|_\Phi \geqslant m/2$. For $t \in J$, let $B_t = \{\tau = t\} \cap A_t$. Since A_t is an atom of Σ_t, we have either $B_t = A_t$ or $B_t = \varnothing$. For each n, let

$$\begin{aligned}\Gamma_n &= \{(f, i) \in L'_n \times H_n; \exists T \in \mathscr{T}; B_t = A_t \text{ where } t = (n, f, i, T)\}\\ &= \{(f, i) \in L'_n \times H_n; \exists t \in J, B_t = F_{f,i}\}.\end{aligned}$$

Let $\Delta_n = \bigcup_{(f,i) \in \Gamma_n} F_{f,i}$. We have $\bigcup B_t = \bigcup_{n \geqslant m} \Delta_n$, so there is $n \geqslant m$ such that $P(\Delta_n) \geqslant 2^{-n-2}$. Let

$$W = \Big\{f \in L'_n; P_n\Big(\bigcup_{(f,i) \in \Gamma_n} D_i\Big) \geqslant 2^{-n-2} P'_n(\{f\})\Big\}.$$

From Fubini's theorem one gets $P_n'(W) \geqslant 2^{-n-2}$. We fix $f \in W$. Let $V = \{i \in H_n; (f, i) \in \Gamma_n\}$, and let $l = \operatorname{card} V$. We have

$$2^{-n-2} \leqslant P_n\Big(\bigcup_{i \in V} D_i\Big) = (1 - b_n)\big(1 - (1 - l/k_n)^n\big) + b_n\big(1 - (1 - l/k_n)^{n^2 a_n}\big)$$
$$\leqslant 1 - (1 - b_n)(1 - l/k_n)^n \leqslant 1 - (1 - 2^{-n-3})(1 - l/k_n)^n.$$

It follows that $l/k_n \geqslant \alpha_n$. The choice of k_n implies that $k_n \alpha_n \geqslant 2n^2 a_n$. We now estimate

$$h_n = P_n\Big(\Big\{x \in L_n;\ \sum_{i \in V} 1_{D_i}(x) \geqslant n^2 a_n\Big\}\Big).$$

We have

$$h_n \geqslant b_n R_n\Big(\Big\{y \in N_n;\ \sum_{i \in V} 1_{C_i}(y) \geqslant n^2 a_n\Big\}\Big)$$
$$= b_n\left(\frac{l}{k_n}\frac{l-1}{k_n}\cdots\frac{l - n^2 a_n + 1}{k_n}\right) \geqslant b_n\left(\frac{l}{2k_n}\right)^{n^2 a_n} \geqslant b_n(\alpha_n/2)^{n^2 a_n}.$$

It follows that if $G = \{f\} \times L_{n-1}''$, we have

$$\int_G \Phi\left(\frac{2}{n}\sum_{i \in V} 1_{D_i}\right) dP \geqslant b_n(\alpha_n/2)^{n^2 a_n}\Phi(2na_n)P_n'(\{f\}).$$

Summation for $f \in W$ gives

$$\int \Phi\left(\frac{2}{n}S_\tau\right) dP \geqslant \int \Phi\left(\frac{2}{n}\sum_{(f,i) \in \Gamma_n} 1_{F_{f,i}}\right) dP$$
$$\geqslant 2^{-n-2} b_n(\alpha_n/2)^{n^2 a_n}\Phi(2na_n) \geqslant 1.$$

So $\|S_\tau\|_\Phi \geqslant n/2 \geqslant m/2$.

We now start proving that condition FV_Φ holds.

LEMMA 42. *Let $(A_t)_{t \in J}$ be an adapted family such that, for each $t = (n, f, i, T)$, A_t is Σ_n-measurable. Then there is a disjoint adapted family $(B_t)_{t \in J}$, with $B_t \subset A_t$ and $P(\bigcup B_t) \geqslant P(\bigcup A_t)$.*

PROOF. This is actually almost obvious. Let

$$J_n = \{(n, f, i, T) \in J;\ f \in L_n',\ i \in I_n,\ T \in \mathscr{T}\}.$$

Let $\Delta_n' = \bigcup_{t \in J_n} A_t$, $\Delta_n = \Delta_n' \setminus \bigcup_{m<n} \Delta_m'$. We have $\bigcup \Delta_n = \bigcup A_t$ and the sets Δ_n are disjoint. For $t \in J_n$, let $A_t' = A_t \cap \Delta_n \in \Sigma_n$, so $\bigcup A_t' = \bigcup A_t$. We can find a disjoint family $(B_t)_{t \in J_n}$, $B_t \in \Sigma_n$, $B_t \in A_t'$ with $\bigcup_{J_n} B_t = \bigcup_{J_n} A_t$, so the family $(B_t)_{t \in J}$ is disjoint and has the same union as (A_t).

LEMMA 43. *Let (A_t) be an adapted family such that for $t = (n, f, i, T) \in J$, we have $A_t \subset \{f\} \times D_i^c \times L_n''$, where $D_i^c = L_n \setminus D_i$. Then for each $\varepsilon > 0$ there is a disjoint adapted family (B_t), $B_t \subset A_t$, with $P(\bigcup B_t) \geqslant P(A^*) - \varepsilon$.*

PROOF. Let m be such that for $n \geqslant m$ and $i \in H_n$ we have $P_n(D_i) \leqslant \varepsilon$. For $t = (n, f, i, T) \in J$, $n \geqslant m$, let $A'_t = \{f\} \times L''_{n-1}$ if $A_t = \{f\} \times D_i^c \times L''_n$, and $A'_t = \varnothing$ if $A_t = \varnothing$ (note that $\{f\} \times D_i^c \times L''_n$ is an atom of Σ_t, so no other case is possible). Now $A'_t \in \Sigma_n$, and $P(\bigcup A'_t) \geqslant P(A^*)$. Lemma 42 gives a disjoint adapted family $B'_t \subset A'_t$ with $P(\bigcup B'_t) = P(\bigcup A'_t)$, and the proof of Lemma 42 shows that, for $t = (n, f, i, T)$, B'_t is actually Σ_n-measurable, so either $B'_t = \varnothing$ or $B'_t = \{f\} \times L''_{n-1}$. Define $B_t = A_t$ whenever $B'_t \neq \varnothing$. Since $B_t \subset B'_t$, the family (B_t) is disjoint. Also, since $P(B_t) \geqslant (1-\varepsilon)P(B'_t)$, we have $P(\bigcup B_t) \geqslant (1-\varepsilon)P(\bigcup A'_t) \geqslant (1-\varepsilon)P(A^*)$, and the proof is complete.

We now come to the main argument.

PROPOSITION 44. *Assume that for each $t = (n, f, i, T) \in J$, either $A_t = \varnothing$ or $A_t = F_{f,i}$. Then for each $m > 0$ and each $\varepsilon > 0$ there are numbers γ_t such that if $\eta = \sum \gamma_t 1_{A_t}$, we have $\int_{\{\eta > 1\}} \Phi(m\eta)\, dP \leqslant 1$ and $P(A^* \cap \{\eta \geqslant 1\}) \geqslant P(A^*) - \varepsilon$.*

PROOF. *First step*. For each n let

$$\Gamma_n = \left\{(f, i) \in L'_n \times H_n;\ \exists T \in \mathscr{T},\ A_t = F_{f,i} \text{ for } t = (n, f, i, T)\right\}.$$

Let $\beta > 0$. Let

$$W_n(\beta) = \left\{f \in L'_n;\ \operatorname{card}\{i \in H_n;\ (f, i) \in \Gamma_n\} \geqslant \beta k_n\right\}.$$

Let $Z_n(\beta) = \bigcup\{F_{f,i};\ (f, i) \in W_n(\beta)\}$. Let $Z^*(\beta) = \limsup Z_n(\beta)$. When β decreases, $Z^*(\beta)$ increases. We show that there is a $\beta > 0$ such that $P(A^* \setminus Z^*(\beta)) < \varepsilon/2$.

Otherwise, if $Z^* = \bigcup_{\beta>0} Z^*(\beta)$, we have $P(A^* \setminus Z^*) \geqslant \varepsilon/2$. For each $k > 0$, let n_k be such that

$$P\left(\bigcup_{n \geqslant n_k} Z_n(2^{-k}) \setminus Z^*(2^{-k})\right) \leqslant \varepsilon 2^{-k-2}.$$

Let

$$Z' = \bigcup_k \bigcup_{n_k \leqslant n < n_{k+1}} Z_n(2^{-k}).$$

We have $P(A^* \setminus Z') \geqslant \varepsilon/4 > 0$. For each n, let $k(n)$ be the unique k with $n_k \leqslant n < n_{k+1}$. Let

$$T_n = \left\{(f, i) \in L'_n \times H_n;\ (f, i) \in \Gamma_n,\ f \notin W_n(2^{-k(n)})\right\}.$$

Then $S_n(T_n) \leqslant 2^{-k(n)}$, so $S_n(T_n) \to 0$ and $T = (T_n) \in \mathscr{T}$.

Consider

$$A' = \bigcup\{A_t;\ t = (n, f, i, T');\ T' \geqslant T\}.$$

Let $(f, i) \in L'_n \times H_n$ such that there exists $T' \geqslant T$ with $A_t = F_{f,i}$, for $t = (n, f, i, T')$. Since $(f, i) \in T'_n$, we have $(f, i) \in T_n$. Since $(f, i) \in \Gamma_n$, we have $f \in W_n(2^{-k(n)})$, so $F_{f,i} \subset Z_n(2^{-k(n)})$. It follows that $A' \subset Z'$. This contradicts the fact that $P(A^* \setminus Z') > 0$, and proves the claim. We fix β with $P(A^* \setminus Z^*(\beta)) < \varepsilon/2$.

Second step. Consider a set $V \subset H_n$ with $l = \operatorname{card} V \geqslant \beta k_n$. Consider the function $\eta = (1/nm)\Sigma_{i \in V} 1_{D_i}$. We have $\|\eta\|_1 = (l/nm)P_n(D_1)$, so (32) shows that there is n_0 such that, if $n > n_0$, we have $\|\eta\|_1 \geqslant \beta/2m$. Since any collection of $n+1$ sets B_i has empty intersection, we have $\eta \leqslant 1/m \leqslant 1$ on M_n. Similarly, $\eta \leqslant na_n/m$, so we have

$$\int_{\{\eta > 1\}} \Phi(m\eta)\, dP \leqslant b_n \Phi(na_n) \leqslant 2^{-n}.$$

Third step. Let us fix $p > n_0$ and let K be an atom of Σ_p. Let $\gamma > 0$. We show that we can find a finite set $I \subset J$ and for $t \in I$ numbers γ_t such that, if $\eta = \Sigma_I \gamma_t 1_{A_t \cap K}$, we have $\|\eta\|_1 \geqslant (\beta/4m)P(Z^*(\beta) \cap K)$ and $\int_{\{\eta>1\}} \Phi(m\eta)\, dP \leqslant \gamma$.

Let $p_0 > p$ with $2^{-p_0} \leqslant \gamma$. For $n \geqslant p_0$, we define

$$Z_n'' = \bigcup \{\{f\} \times L_{n-1}''; f \in W_n(\beta)\} \cap K.$$

Z_n'' belongs to Σ_n, and $Z_n'' \supset Z_n(\beta) \cap K$,

$$\bigcup_{n \geqslant p_0} Z_n'' \supset \bigcup_{n \geqslant p_0} Z_n(\beta) \cap K \supset Z^*(\beta) \cap K. \tag{33}$$

Let $Z_n' = Z_n'' \setminus \bigcup_{q<n} Z_q''$. The sets Z_n' are disjoint. Let π_n be the natural projection of L on L_n', and let $V_n = \pi_n(Z_n')$.

Let p' be such that

$$P\left(\bigcup_{p_0 \leqslant n \leqslant p'} Z_n''\right) \geqslant P(Z^*(\beta) \cap K)/2.$$

Define $\eta = \Sigma(1/nm)1_{F_{f,i}}$ where the summation is for $p_0 \leqslant n \leqslant p'$, $(f, i) \in \Gamma_n$. For $f \in V_n$, if $K' = \pi_n^{-1}(\{f\})$, the second step shows that

$$\|\eta 1_{K'}\|_1 \geqslant \beta P(K')/2m, \qquad \int_{K' \cap \{\eta > 1\}} \Phi(m\eta)\, dP \leqslant \gamma P(K'),$$

and the result follows by summation over $f \in V_n, p \leqslant n \leqslant p'$.

Fourth step. By induction over k we show that if K is an atom of Σ_p, $p \geqslant n_0$, there is a finite set i, and for $t \in I$ a γ_t such that if $\eta = \Sigma_I \gamma_t 1_{A_t}$, we have

$$\|\eta\|_1 \geqslant \left(1 - (1 - \beta/4m)^k\right) P(Z^*(\beta) \cap K), \tag{34}$$

$$\int_{\{n>1\}} \Phi(m\eta)\, dP \leqslant \left(1 - (1 - \beta/4m)^k\right) P(Z^*(\beta) \cap K). \tag{35}$$

The proof of the proposition will follow by taking k large enough that

$$(1 - \varepsilon/2)(1 - \beta/4m)^k > 1 - \varepsilon$$

and by summation over the atoms of Σ_{n_0}.

The case $k = 1$ has been proved in the third step. Suppose the result has been proved up to k. Let K be an atom of Σ_p, and let η satisfy (34) and (35). We can suppose $\|\eta\|_1 < P(Z^*(\beta) \cap K)$. Let q be such that η is Σ_q-measurable. Let us fix an atom F of Σ_q. Let

$$d_F = (\beta/4m)(1 - \beta/4m)^k P(Z^*(\beta) \cap F).$$

The third step gives a function η' such that

$$\|\eta'\|_1 \geqslant (\beta/4m)P(Z^*(\beta)\cap F), \tag{36}$$

$$\int_{\{\eta'>1\}} \Phi(m\eta')\,dP \leqslant d_F. \tag{37}$$

Define $\eta_F = \eta 1_F$ if $\eta \geqslant 1$ on F and $\eta_F = (\eta + \eta'(1-\eta))1_F$ if $\eta < 1$ on F. If a_F denotes the (constant) value of η on F, we have

$$\|\eta_F\|_1 = a_F P(F) + (1-a_F)\|\eta\|_1 \geqslant a_F P(F) + (1-a_F)(\beta/4m)P(Z^*(\beta)\cap F),$$

so

$$P(Z^*(\beta)\cap F) - \|\eta_F\|_1 \leqslant (1-\beta/4m)\big(P(Z^*(\beta)\cap F) - a_F P(F)\big). \tag{38}$$

If $\eta \leqslant 1$ we have $\eta_F > 1 \Rightarrow \eta' > 1$, so

(39)

$$\int_{\{\eta_F>1\}} \Phi(m\eta_F)\,dP \leqslant \int_{\{\eta'>1\}} \Phi\big(m(a_F + \eta'(1-a_F))\big)\,dP \leqslant \int_{\{\eta'>1\}} \Phi(m\eta')\,dP.$$

Define $\eta_1 = \sum \eta_F$, where the summation is taken for the atoms F on Σ_{ς} contained in F. Summation of (39) gives

$$\begin{aligned}\int_{\{\eta_1>1\}} \Phi(m\eta_1)\,dP &\leqslant \int_{\{\eta>1\}} \Phi(m\eta)\,dP + \sum d_F\\ &\leqslant \big(1-(1-\beta/4m)^{k+1}\big)P(Z^*(\beta)\cap K).\end{aligned}$$

We note that (38) remains true if $\eta \geqslant 1$ on F, since in that case $\|\eta_F\|_1 = a_F P(F) > P(Z^*(\beta)\cap F)$. Summation over the $F \subset K$ gives

$$P(Z^*(\beta)\cap F) - \|\eta_1\|_1 \leqslant (1-\beta/4m)\big(P(Z^*(\beta)\cap F) - \|\eta\|_1\big)$$

and this implies

$$\|\eta_1\|_1 \geqslant \big(1-(1-\beta/4m)^{k+1}\big)P(Z^*(\beta)\cap F).$$

The proposition is proved.

We now complete the proof that condition FV_Φ holds. Let (A_t) be an adapted family of sets. If $t = (n, f, i, T)$, let

$$A_t^1 = A_t \setminus \big(\pi_n^{-1}(f)\big),\quad A_t^2 = A_t \cap \big(\{f\} \times D_i^c \times L_n''\big),\quad A_t^3 = A_t \cap F_{n,i}.$$

Let $\varepsilon > 0$. It follows from Lemma 42 that there exists $\tau_1 \in IM$ with $\{\tau_1 = t\} \subset A_t$ for each t, $P(A^{1*} \setminus A^1(\tau_1)) \leqslant \varepsilon/3$ and $e_{\tau_1} = 0$. Let t_1 be such that $\tau_1 \leqslant t_1$. For $t \geqslant t_1$, let $A_t' = A_t^2 \setminus A^1(\tau_1)$ and let $A_t' = \varnothing$ otherwise. From Lemma 43 there is $\tau_2 \in IM$, with $\{\tau_2 = t\} \subset A_t$ for each t, $P(A'^* \setminus A'(\tau_2)) \leqslant \varepsilon/3$ and $e_{\tau_2} = 0$. Let t_2 with $\tau_2 \leqslant t_2$. For $t \geqslant t_2$, let $A_t'' = A_t^3 \setminus (A^1(\tau_1) \cup A^2(\tau_2))$ and let $A_t'' = \varnothing$ otherwise. From Lemma 44 there are numbers γ_t such that, if $\eta = \sum \gamma_t A_t''$, then $\|\eta\|_1 \geqslant P(A''^*) - \varepsilon/3$ and $\int_{\{\eta>1\}} \Phi(\eta/\varepsilon)\,dP \leqslant 1$. Define now $\eta' = \eta + \sum 1_{\{\tau_1 = t\}} + \sum 1_{\{\tau_2 = t\}}$. Since $A^* \subset A^{1*} \cup A^{2*} \cup A^{3*}$, we have $\|\eta'\|_1 \geqslant P(A^*) - \varepsilon$ and $\int_{\{\eta'>1\}} \Phi(\eta'/\varepsilon)\,dP \leqslant 1$, so $\|\eta' - \eta' \wedge 1\|_\Phi \leqslant \varepsilon$. This completes the proof of Theorem 12.

9. Proof of Theorem 13. Let $k_n = 2^n$ and $L_n = \{0,\ldots,k_n\}$. Let P_n be the uniform probability on L_n. Let $L = \prod L_n$ and let P be the product probability on L. Let $L'_n = \prod_{i<n} L_i$. Let π_n be the natural projection of L on L'_n. For $x \in L'_n$, let

$$H_n(x) = \{\, y = (y_n) \in L;\ \pi_n(y) = x,\ y_n = 0\}$$

and let $H_n = \bigcup_{x \in L'_n} H_n(x)$, so $H_n = \{\, y \in L;\ y_n = 0\}$. Let $G_n = \bigcup_{n \leqslant m} H_m$. We note that $P(H_n) = 1/(k_n + 1)$, so $P(G_n) \to 0$. For $x \in L'_n$, let $\tilde{x} = \pi_n^{-1}(x)$.

Let ω_1 denote the first uncountable ordinal. Continuum Hypothesis means that ω_1 has the power of continuum. So, we can find an enumeration $(f_\alpha, \eta_\alpha)_{\alpha<\omega_1}$ of all the couples (f, n), where $f \in L^1(P)$, $f \geqslant 0$, and $n \in \mathbf{N}$. By induction over $\alpha < \omega_1$, we construct a sequence of Borel sets B_α such that the following hold:

(40) $B_\alpha \supset L \setminus G_{n_\alpha}$;

(41) for each n, each $x \in L'_{n+1}$, either $B_\alpha \supset \tilde{x} \setminus G_n$ or $B_\alpha \cap \tilde{x} \subset G_n$;

(42) whenever R is a finite subset of $[0, \omega_1[$, the set $T(R) = \bigcap_{\alpha \in R} B_\alpha$ meets every set $H_n(x)$ for $n \in \mathbf{N}$, $x \in L'_n$, in a set with nonempty interior;

(43) if $n > n_\alpha$, $x \in L'_n$, $x_i \neq 0$ for $i < n - 1$, $x_{n-1} = 0$, then

$$\int_{\tilde{x} \cap B_\alpha} f_\alpha \, dP \leqslant 2^{-n} P(\tilde{x}).$$

We first construct B_0. We can enumerate the sets $H_n(x)$ as $Z_l = H_{n(l)}(x(l))$. By induction over l, we construct an increasing sequence $(p(l))$ with $p(1) > n_0$, and a sequence $y(l) \in L'_{p(l)}$ such that the following properties hold:

(44) $p(l) > n(l) + 2$;

(45) the component of $y(l)$ of index $p(l) - 1$ is zero and $\tilde{y}(l) \subset Z_l$;

(46) $\int_{\tilde{y}(l)} f_0 \, dP \leqslant 2^{-l-n(l)} P(Z_l)$.

The construction is easy. There are $K = \prod(k_i + 1)$ (where the product is for $n(l) < i < p(l) - 1$) possible choices of $y(l)$ satisyfing (45), and the corresponding sets $\tilde{y}(l)$ are disjoint. So if $p(l)$ is large enough that $K 2^{-l-n(l)} P(Z_l) \geqslant \|f_0\|_1$, we can find one of them for which (46) holds.

We now prove (40). Let $B_0 = L \setminus (G_{n_0} \setminus \bigcup_l \tilde{y}(l))$. Since the component of $y(l)$ of index $p(l) - 1$ is zero, we have $\tilde{y}(l) \subset G_{n_0}$ for each l, so (40) holds, and $B_0 = (L \setminus G_{n_0}) \cup \bigcup_l \tilde{y}(l)$.

We now prove (41). Let us fix n and $x \in L'_{n+1}$. Suppose that we have $\tilde{x} \cap B_0 \neq \varnothing$. If $\tilde{x} \not\subset G_{n_0}$, the components of $\tilde{x}$ of index between n_0 and n are nonzero, so $\tilde{x} \cap G_{n_0} = \tilde{x} \cap G_{n_0}$, so $\tilde{x} \setminus G_n = \tilde{x} \setminus G_{n_0}$, and $B_\alpha \cap \tilde{x} \supset \tilde{x} \setminus G_n$ from (40). If $\tilde{x} \subset G_{n_0}$, then $\tilde{x} \cap B_0 = \tilde{x} \cap \bigcup_l \tilde{y}(l)$. If for some l we have $\tilde{x} \subset \tilde{y}(l)$, then $\tilde{x} \setminus G_n \subset \tilde{x} \subset B_0$. Otherwise let us fix l with $\tilde{y}(l) \cap \tilde{x} \neq \varnothing$. Then $\tilde{y}(l) \subset \tilde{x}$ and $p(l) > n$. Since the last component of y is zero, we have $\tilde{y}(l) \subset G_n$, so $B \cap \tilde{x} \subset G_n$ in that case. This proves (41).

We now prove (43). Let $n > n_0$, $x \in L'_n$ with $x_i \neq 0$ for $i < n - 1$ and $x_{n-1} = 0$. We have $\tilde{x} \subset G_{n_0}$. We have $\tilde{x} \subset G_{n_0}$. Fix l. Suppose $\tilde{x} \cap \tilde{y}(l) \neq \varnothing$. We cannot have $p(l) \leqslant n$, since $y(l)$ has at least two components of index $\leqslant p(l) - 1$ which are zero (i.e. those of index $n(l)$ and $p(l)$) while x has at most one. So if $\tilde{y}(l) \cap \tilde{x} \neq \varnothing$, we have $p(l) > n$. Since the component of $\tilde{y}$ of rank $n(l)$ is zero, we have $n(l) \geqslant n - 1$, so $P(Z_l) \leqslant P(\tilde{x})$. Now (46) implies that

$$\int_{\tilde{x} \cap \tilde{y}(l)} f_0 \, dP \leqslant 2^{-n-l} P(Z_l) \leqslant 2^{-n-l} P(\tilde{x})$$

and this implies (43).

We now construct the sets B_α by induction. Suppose that the sets B_α have been constructed whenever $\alpha < \beta$. We can enumerate as

$$U_l = T(R_l) \cap H_{n(l)}(x(l))$$

the sets $T(R) \cap H_n(x)$ where R is a finite subset of $[0, \beta[$, and $x \in L'_n$. Let $Z_l = H_{n(l)}(x(l))$. By induction over l, we construct an increasing sequence $(p(l))$ with $p(1) > n_\beta$ and a sequence $y(l) \in L'_{p(l)}$ such that (40) holds, together with the following:

(47) the component of $y(l)$ of index $p(l) - 1$ is zero and $\tilde{y}(l) \subset U_l$, $P(\tilde{y}(l)) \leqslant 2^{-l-1} P(U_l)$;

(48) $\int_{\tilde{y}(l)} f_\beta \, dP \leqslant 2^{-l-n(l)} P(Z_l)$.

The construction is very similar to the construction in the case $\alpha = 0$, so we leave it to the reader. We set $B_\beta = L \setminus (G_{n_\beta} \setminus \bigcup \tilde{y}(l))$. To check that (44)–(46) hold, one proceeds as in the case $\alpha = 0$. We now check (42). Let R be a finite subset of $[0, \beta[$, and fix $n \in \mathbf{N}$, $x \in L'_n$. We now show that $T(R) \cap H_n(x) \cap B_\beta$ has nonempty interior. There exists $n' \geqslant n_\beta$, and $x' \in L'_{n'}$ with $H_{n'}(x') \subset H_n(x)$. So we can actually suppose $n \geqslant n_\beta$, so we have $H_n(x) \subset G_{n_\beta}$. By construction there is l with $\tilde{y}(l) \subset T(R) \cap H_n(x)$, so $\tilde{y}(l) \subset T(R) \cap H_n(x) \cap B_\beta$. This completes the construction.

Let $M_n = \{1, \ldots, k_n\}$. For $i \in M_n$, let $C_{n,i} = \{y \in L;\, y_n \in \{0, i\}\}$. Let Σ_n be the algebra of subsets of L that depend only on the coordinates of rank $\leqslant n - 1$. Let S be the set of finite subsets of $[0, \omega_1[$, ordered by inclusion. The index set J is given by

$$J = \{(n, i, R) \colon i \in M_n,\, R \in S\}.$$

We order it by $(n, i, R) \leqslant (n', i', R')$ if either $n < n'$, $R \subset R'$ *or* $n = n'$, $i = i'$, $R \subset R'$. This order makes J a directed set. For $t \in J$, $t = (n, i, R)$, let Σ_t be the algebra generated by Σ_n, $C_{n,i}$, and the sets B_α for $\alpha \in R$. The map $t \to \Sigma_t$ is increasing. Denote by Ξ the union of the algebras Σ_t. For $x \in L'_n$, $q \in \mathbf{N}$, let

$$F_{n,q}(x) = \{y \in L;\, \pi_n(y) = x,\, y_i = 0, \forall n \leqslant i \leqslant n + q\}.$$

Let $a_n = 2^{-n}/\operatorname{card} L'_n$. From (42) there exists a unique positive finitely additive measure $\mu_{n,x}$ such that $\mu_{n,x}(L) = a_n$ and

$$\forall \alpha < \omega_1, \quad \mu_{n,x}(B_\alpha) = a_n, \quad \forall q, \mu_{n,x}\big(F_{n,q}(x)\big) = a_n.$$

Let $\mu = \sum \mu_{n,x}$, where the summation is over n and $x \in L'_n$. Note that $\mu(L) = 1$. For $t \in J$, we define $X = (X_t)$ by $X_t = \mu(A)/P(A)$ on A for each atom A of Σ_t. This is a positive martingale, and $\|X_t\|_1 = 1$ for each t. We show that it does not converge. It is clear that μ is singular; that is, for each $\varepsilon > 0$ there is $t \in J$ and $A \in \Sigma_t$ with $P(A) \geqslant 1 - \varepsilon$, $\mu(A) \leqslant \varepsilon$. This shows that $X_* = 0$.

Now let $n \in \mathbf{N}$, and R be a finite subset of $[0, \omega_1[$. Let $y \in L$ with $y \notin G_1$. Then from (40) y belongs to $T(R)$. Let $z = \pi_n(y) \in L'_n$, and let $i = y_n$. Then $y \in A = \tilde{z} \cap C_{n,i} \cap T(R)$ and A is an atom of Σ_t, for $t = (n, i, R)$. We have

$$\mu(A) \geqslant \mu_{n,z}(A) \geqslant \mu_{n,z}(T(R) \cap H_n(z)) = a_n.$$

On the other hand,

$$m(A) \leqslant m(C_{n,i}) \leqslant 2/k_n \text{ card } L'_n = 2^{-n+1}/\text{card } L'_n = 2a_n,$$

so $X_t(y) \geqslant \frac{1}{2}$. This shows that $X^* \geqslant \frac{1}{2}$ on $L \setminus G_1$. Since $P(L \setminus G_1) > 0$, we have $X^* \neq X_*$, so (X_t) does not converge essentially.

Now let $f \in L^1(P)$. We show that $X_t = (E^t(f))$ converges. By approximation, it is enough to show that $P(X^* \geqslant \lambda) \leqslant \| f \|_1/\lambda$ when $f \geqslant 0, \lambda > 0$.

Let $g = \operatorname{Sup}_n(E(f \mid \Sigma_n))$. We know that $P(g > \lambda) \leqslant \| f \|_1/\lambda$ since the sequence Σ_n is increasing. Let $m > 0$. Let $\alpha < \omega_1$ with $(f, m) = (f_\alpha, n_\alpha)$. We show that if $(m, 1, \{\alpha\}) \leqslant t = (n, i, R)$, then $X_t \leqslant c_m g + d_m$ on $L \setminus G_m$, where c_m, d_m depend only on m and $c_m \to 1$, $d_m \to 0$. This implies $X^* \leqslant g$ on L, and concludes the proof. Let $y \in L \setminus G_m$. Let $x = \pi_m(y)$. It follows from (41) that, for each $\beta \in R$, either $\tilde{x} \setminus G_m \subset B_\beta$ or $(\tilde{x} \setminus G_m) \cap B_\beta = \varnothing$. Let $i = y_m$. It follows that the atom A of Σ_t that contains y contains $(\tilde{x} \setminus G_m) \cap C_{m,i}$, so if $z = \pi_{m+1}(y)$, A contains $\tilde{z} \setminus G_m$. Easy computation shows $P(\tilde{z} \setminus G_m) \geqslant P(\tilde{z}) r_m$, where $r_m = \prod_{i \geqslant m}(1 - 2^{-i})$, so $P(A) \geqslant P(\tilde{z}) r_n$.

Since $y \notin G_m$, we have $y \in B_\alpha$ from (40), so $A \subset \tilde{z} \cup (H_m(x) \cap B_\alpha)$. From (43) it follows that

$$\int_{H_n(x) \cap B_\alpha} f_\alpha \, dP \leqslant 2^{-m} P(H_m(x)) = 2^{-m} P(\tilde{z}).$$

So we have

$$\begin{aligned} X_t(y) &\leqslant \int_A f_\alpha \, dP/P(A) \leqslant \left(\int_{\tilde{z}} f_\alpha \, dP \right) / (r_m P(\tilde{z})) + 2^{-m}/r_m \\ &\leqslant g(y)/r_m + 2^{-m/r_m}. \end{aligned}$$

The proof is complete.

10. Proof of Theorem 21. We shall perform the construction in the case $L^\Psi \neq L^1$; that is, $\Phi < \infty$ everywhere. The details in the case $L^\Psi = L^1$ are somewhat simpler.

Since Φ fails condition Exp, Lemma 40 shows that there is a sequence (l_n) of integers such that

$$\Phi(2 l_n) \geqslant 2^{2n} (2^{2n+1})^{n l_n} \Phi(l_n). \tag{49}$$

We can assume $\Phi(l_n) \geqslant 2$. Let $b_n = 2^{-n}/\Phi(l_n)$, $k_n = n 2^{n+1} l_n$. Let

$$H_n = \{1, \ldots, k_n\} \times \{1, \ldots, 2^n\}.$$

Let U_n be the uniform probability on H_n. Let $M_n = H_n^{n l_n}$, and let Q_n be the power of U_n on M_n. Let $L_n = H_n \cup M_n$, and on L_n, let P_n be given by

$$P_n(A) = (1 - b_n) U_n(A \cap H_n) + b_n Q_n(A \cap M_n).$$

Let $L = \prod L_n$, and μ be the product measure. Let $F_n = \{x \in L; x_n \in H_n\}$. For $x = (x_m) \in F_n$, write $x_n = (p_n(x), q_n(x))$ where $P_n(x) \in \{1, \dots, k_n\}$, $q_n(x) \in \{1, \dots, 2^n\}$. For $t \in H_n$, let

$$C_t = \left\{ z = (z_i) \in H_n^{nl_n}; \exists i \leqslant nl_n; z_i = t \right\}.$$

Let $Z \subset H_n$. A point x in $H_n^{nl_n}$ which has all its coordinates distinct and in Z satisfies $\sum_{t \in Z} 1_{C_t}(x) \geqslant nl_n$, so if card $Z \geqslant 2nl_n$, we get

$$Q_n\left(\left\{ \sum_{t \in Z} 1_{C_t} \geqslant nl_n \right\}\right) \geqslant (\text{card } Z/2 \,\text{card } H_n)^{nl_n}. \tag{50}$$

Also, we note the following:

(51) Any family of $nl_n + 1$ sets C_t has empty intersection. For $x \notin F_n$, let

$$W_n(x) = \left\{ y \in L; \forall q \leqslant n, y_q = x_q \right\}.$$

For $x \in F_n$, let $W_n(x) = \{ y \in L;\ \forall q < n, y_q = x_q$, and either $y_n \in H$ and $p_n(y) = p_n(x)$ or $y_n \in M_n$ and $y_n \in C_{x_n}\}$. For $x \in L$, define $\mathscr{B}(x)$ as the family of subsequences of $(W_n(x))$.

We first show that condition C_Φ fails. Let $F = \bigcap F_n$, so $\mu(F) \geqslant \frac{1}{2}$. Fix $m \geqslant 1$. Define a Vitali cover $\mathscr{V}$ of F by associating to $x \in F$ the sequence $(W_n(x))_{n \geqslant 2m+1}$. Let $\mathscr{F}$ be a finite family of $\mathscr{V}$-sets with $\mu(F \cap d_{\mathscr{F}}) \geqslant \frac{1}{4}$. Fix each n, let

$$D_n = \bigcup \left\{ A \in \mathscr{F};\ A \text{ is of the type } W_n(x) \right\}.$$

Since $d_{\mathscr{F}} = \bigcup_{n \geqslant m} D_n$, there is $n \geqslant 2m + 1$ with $\mu(F \cap D_n) \geqslant 2^{-n+1}$. Denote by Σ_n the algebra of sets that depend only on the coordinates of rank $< n$. Let G be the collection of the atoms Y of Σ_n such that

$$\mu(F \cap D_n \cap Y) \geqslant 2^{-n}\mu(Y).$$

Then the union of G has measure $\geqslant 2^{-n}$. Fix Y in G so that $\mu(F_n \cap D_n \cap Y) \geqslant 2^{-n}\mu(Y)$. For x in $F \cap Y$, we have easily $\mu(W_n(x) \cap F_n \cap Y) \leqslant \mu(Y)/k_n$. It follows that there are at least $q = k_n 2^{-n}$ points $x^1, \dots, x^q$ of $F \cap Y$ such that the sets $W_n(x^1), \dots, W_n(x^q)$ are distinct and belong to G. Note that $q \geqslant 2nl_n$, so (50) shows that

$$\mu\left(\left\{ \sum_{i \leqslant q} 1_{W_n(x^j)} \geqslant nl_n \right\} \cap Y\right) \geqslant b_n (2^{-2n-1})^{nl_n} \mu(Y),$$

so

$$\int_Y \Phi(e_{\mathscr{F}}/m)\, dP \geqslant \Phi((nl_n - 1)/m) b_n (2^{-2n-1})^{nl_n} \mu(Y).$$

By summation over $Y \in G$, from (49) and since $(nl_n - 1)/m \geqslant 2l_n$, we get $\|e_{\mathscr{F}}\|_\Phi \geqslant 1$.

We prove now that condition FV_Φ holds. Let $X \subset L$, $\mathscr{V}$ be a Vitali cover of X, and $\varepsilon > 0$. To each $x \in X$ is associated a sequence $(W_{n_q}(x))$ in $\mathscr{B}(x)$. For each n, let

$$T_n = \left\{ x \in X;\ n \text{ is of the type } n_q(x) \text{ for some } q \right\}.$$

Then for $x \in T_n$, $W_n(x)$ is a $\mathscr{V}$ set. Also, since $\mathscr{V}$ is a Vitali cover of X, we have $X \subset \limsup T_n$. For an atom Y of Σ_n and $j \leqslant k_n$, let us write

$$Y_j = \{x \in Y \cap F_n;\ p_n(x) = j\}.$$

For each n, let G_n be the family of the sets Y_j for Y an atom of Σ_n, $j \leqslant k_n$, such that

$$\operatorname{card}\{r \leqslant 2^n, \exists x \in Y_j \cap T_n, q_n(x) = r\} \geqslant n.$$

If $Y_n \notin G_n$, we have $\mu^*(Y_j \cap T_n) \leqslant n2^{-n}\mu(Y_j)$. So, if D_n is the union of G_n, we have

$$\mu^*(T_n \setminus D_n) \leqslant \mu(L \setminus F_n) + n2^{-n} \leqslant (n+1)2^{-n}.$$

This implies that $X \subset \limsup D_n$. Let us fix $\varepsilon > 0$ and let m large enough that $2^{-m+1} < \varepsilon$. For $n \geqslant m$, let

$$S_n = \bigcup_{m-1 \leqslant r < n} F_r \setminus \bigcup_{m \leqslant r < n} D_r.$$

Let $D_n' = D_n \cap S_n$. Since $S_n \in \Sigma_n$, D_n' is the union of the sets Y_j that it contains. Moreover, $P^*(X \cap \bigcap_{r \geqslant m-1} F_r \setminus \bigcup_{n \geqslant m} D_n') = 0$, so $P^*(X \setminus \bigcup_{n \geqslant m} D_n') < \varepsilon$.

For Y_j contained in D_n', let us pick, n points $x^1, \ldots, x^n \in Y_j \cap T_n$ such that the numbers $q_n(x^i)$ are all different for $i \leqslant n$. Let $\xi_{Y,j} = (1/n)\sum_{i \leqslant n} 1_{W_n(x^i)}$. This function is one on Y_j. Let ξ_n denote the sum of the $\xi_{Y,j}$ for $Y_j \subset D_n'$. Then $\xi_n = 1$ on D_n'. The point is that (51) implies that $\xi_n \leqslant l_n$, so $\int \Phi(\xi_n - \xi_n \wedge 1)\, dP \leqslant b_n \Phi(l_n) \leqslant 2^{-n}$. On the other hand, $\xi_n = 1$ on D_n'. Given $\varepsilon > 0$, one can find m' with

$$P^*\Big(X \setminus \bigcup_{m \leqslant n \leqslant m'} D_n'\Big) \leqslant \varepsilon.$$

Since ξ_n is zero outside $D_n' \cup (F_{n-1} \setminus F_n)$, the functions ξ_n have disjoint support, so if $\xi = \sum_{m \leqslant n \leqslant m'} \xi_n$, we have $\xi = 1$ on $\bigcup_{m \leqslant n \leqslant m'} D_n'$ while $\int \Phi(\xi - \xi \wedge 1)\, dP \leqslant 2^{-m+1} \leqslant \varepsilon$. The proof is complete.

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