TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 293, Number 1, January 1986

# THE NORMAL CLOSURE OF THE COPRODUCT OF RINGS OVER A DIVISION RING

BY

WALLACE S. MARTINDALE III

ABSTRACT. Let $R = R_1 \coprod R_2$ be the coproduct of $\Delta$-rings $R_1$ and $R_2$ with 1 over a division ring $\Delta$, $R_1 \neq \Delta$, $R_2 \neq \Delta$, with at least one of the dimensions $(R_i:\Delta)_r, (R_i:\Delta)_l$, $i = 1,2$, greater than 2. If $R_1$ and $R_2$ are weakly 1-finite (i.e., one-sided inverses are two-sided) then it is shown that every $X$-inner automorphism of $R$ (in the sense of Kharchenko) is inner, unless $R_1$, $R_2$ satisfy one of the following conditions: (I) each $R_i$ is primary (i.e., $R_i = \Delta + T$, $T^2 = 0$), (II) one $R_i$ is primary and the other is 2-dimensional, (III) char. $\Delta = 2$, one $R_i$ is not a domain, and one $R_i$ is 2-dimensional. This generalizes a recent joint result with Lichtman (where each $R_i$ was a domain) and an earlier joint result with Montgomery (where each $R_i$ was a domain and $\Delta$ was a field).

**0. Introduction.** Let $R_1$ and $R_2$ be $\Delta$-rings with 1 over a division ring $\Delta$, $R_1 \neq \Delta$, $R_2 \neq \Delta$, and let $R = R_1 \coprod R_2$ be the coproduct of $R_1$ and $R_2$ over $\Delta$. It is known (and easy to see) that $R$ is a prime ring. Throughout this paper we shall be assuming that at least one of the four dimensions $(R_i:\Delta)_r, (R_i:\Delta)_l$, $i = 1,2$, is greater than 2. Denoting by $Z = Z(R)$ the center of $R$ we know [**5**, Theorem 1] that $Z = Z(R_1) \cap Z(R_2) \subseteq Z(\Delta)$. Let $R_{\mathscr{F}}$ be the left quotient ring of $R$ relative to the filter $\mathscr{F}$ of all nonzero two-sided ideals of $R$. Then the set $N^* = N^*(R)$ of all units of $R_{\mathscr{F}}$ such that $u^{-1}Ru = R$ is called the set of normalizing elements for $R$ [**9**, p. 240] and the automorphisms thus induced on $R$ are just the $X$-inner automorphisms of Kharachenko [**9**, p. 3]. An important subset of $N = N^* \cup \{0\}$ is the extended center $C$ of $R$. The main goal of this paper is to show that under proper restrictions every $X$-inner automorphism of $R$ is inner; a precise statement of this result will be given at the end of this introductory section.

One restriction of a general nature which we shall make is that the rings $R_1$ and $R_2$ be weakly 1-finite (i.e., any one-sided inverse is two-sided). Before describing restrictions of a more specialized nature we must recall a couple of definitions from [**5**]. A $\Delta$-ring $S$ is said to be primary in case $S = \Delta \oplus T$, where $T$ is a $\Delta$-bimodule such that $T^2 = 0$. A $\Delta$-ring $S$ is said to be quadratic in case $S = \Delta \oplus \Delta x = \Delta \oplus x\Delta$ for some $x \in S$. Situations which we shall eventually want to avoid are of the following three (seemingly artificial) types:

(I) Both $R_1$ and $R_2$ are primary.

(II) One $R_i$ is primary and the other is quadratic.

(III) The characteristic of $\Delta$ is 2, at least one of the $R_i$ is not a domain, and one of the $R_i$ is quadratic.

Received by the editors February 22, 1985.

1980 *Mathematics Subject Classification*. Primary 16A06; Secondary 16A08, 16A03.

In a corrected version of [**5**, Theorems 4, 5] the following result was established: $C = Z$ unless $R_1$, $R_2$ satisfy (I), (II), or (III).

In §1 we shall begin by reviewing the definition and basic properties of the key notion of height $|r|$ of an element $r \in R$. These are due primarily to Cohn [**3**]. Next we shall recall how left and right $\Delta$-bases for $R_1$ and $R_2$ lead to various types of "monomial" bases for $R$ regarded variously as a left or right $\Delta$, $R_1$, or $R_2$ module. We shall sometimes require a more refined notion than that of height; to this end we show that a lexicographic ordering of the "basis" monomials induces a partial well-ordering $<$ on the elements of $R$.

In §2 the units of $R$ are discussed and it seems natural here to make the assumption that $R_1$ and $R_2$ are weakly 1-finite. The obvious units of $R$ are those generated by the units of $R_1$ and $R_2$ (Cohn [**4**] calls these monomial units) and by units of the form $1 + afb$, $a, b \in R_i$, $ba = 0$, $f \in R$ (these are the transvections of Bergman [**1**]). We shall refer to the totality of all such units as standard units. We then develop a result (Theorem 2) which is crucial to our arguments later on. It is too technical to state here with any precision but in a very loose sense it says the following: if $|fg|$ is not what one would normally expect it to be then there exists a standard unit $u$ such that $fu < f$. Among various corollaries is Bergman's result [**1**, p. 10]: if each $R_i$ is weakly 1-finite then every unit of $R$ is a standard unit.

The study of $X$-inner automorphisms is taken up in §3. If $\sigma$ is an $X$-inner automorphism it is easily seen that there exist nonzero elements $f$, $g$, $h \in R$ such that

$$frg = hr^{\sigma}f \tag{1}$$

for all $r \in R$. The previously mentioned results on height and partial ordering are then persistently applied to (1) in preparation for the main result of the paper, whose statement and proof occupy §4:

THEOREM 6. *Let $R = R_1 \amalg R_2$ be the coproduct of weakly* 1*-finite $\Delta$-rings $R_1$ and $R_2$ with* 1 *over a division ring $\Delta$, with $R_1 \neq \Delta$, $R_2 \neq \Delta$, and at least one of the four dimensions $(R_i : \Delta)_r, (R_i : \Delta)_l$, $i = 1, 2$, greater than* 2. *If $R_1$, $R_2$ do not satisfy* (I), (II), *or* (III) *then every $X$-inner automorphism of $R$ is inner.*

Examples are given to illustrate the need to avoid the situations (I)–(III).

This paper is the latest in a sequence of papers [**7**, **8**, **5**, **6**] studying the central and normal closure of the coproducts of rings. Theorem 6 generalizes a recent joint result with Lichtman [**6**, Theorem 4] in which $R$ was the coproduct of domains $R_1$ and $R_2$ over a division ring $\Delta$. This latter result in turn was a generalization of a joint result with Montgomery [**8**, Theorem 5] in which $R$ was the coproduct of domains $R_1$ and $R_2$ over a field $F$. We take this opportunity to express our appreciation to both of our coauthors—to Susan Montgomery for originally suggesting that we look at the normal closure of coproducts and to Alexander Lichtman for suggesting that fields could be replaced by division rings. Hopefully the statute of limitations on a referee's anonymity has by now expired and we can indicate our debt to Warren Dicks for his incisive suggestions for improving our original paper (with Montgomery [**8**]), as well as pointing out to us George Bergman's paper [**1**]. We suspect that many of our arguments in §2 are really equivalent to some of those found in Bergman's paper but simply phrased in a way more congenial to this author. Finally, helpful earlier comments were made by Paul Cohn and his fundamental papers on coproducts, especially [3], have served as a framework for the present paper.

**1. Height and partial ordering.** Throughout this paper we will be considering the coproduct $R = R_1 \amalg R_2$ of $\Delta$-rings $R_1$ and $R_2$ with 1 over a division ring $\Delta$, $R_1 \neq \Delta$, $R_2 \neq \Delta$. For questions concerning the existence and foundations of coproducts (also called free products) we refer the reader to Cohn's papers [**2**, **3**]. Our starting point will be to recall the definition and basic properties of the notion of height as developed by Cohn in [**3**]. $R$ has a filtration given by

$$H^{-1} = 0, \qquad H^0 = \Delta, \qquad H^1 = R_1 + R_2,$$
$$H^n = \Sigma R_{i_1} R_{i_2} \cdots R_{i_n}, \qquad n = 2, 3, \ldots,$$

and so we may define the height $|r|$ of an element $r \in R$ by

$$|r| = \begin{cases} n & \text{if } r \neq 0, r \in H^n, r \notin H^{n-1}, \\ -\infty & \text{if } r = 0. \end{cases}$$

We will sometimes express the fact that an element $r$ of $H^n$ actually lies in $H^{n-1}$ by saying that $r \equiv 0 \pmod{H^{n-1}}$ or simply $r \equiv 0$ if the context is clear. Every $H^n$ is a $\Delta$-bimodule and the bimodule $H^n/H^{n-1} = (\overline{R}_1 \otimes \overline{R}_2 \otimes \cdots \otimes \overline{R}_n) \oplus (\overline{R}_2 \otimes \overline{R}_3 \otimes \cdots \otimes \overline{R}_{n+1})$, where $\overline{R}_i = R_i/\Delta$, $i = 1, 2$, $R_j = R_1$ if $j$ is odd and $R_j = R_2$ if $j$ is even. If $n$ is even, the submodule of $H^n$ corresponding to the first summand, namely $R_1 R_2 R_1 \cdots R_2$, is denoted by $H^n_{12}$ and that corresponding to the second summand, namely $R_2 R_1 R_2 \cdots R_1$, is denoted by $H^n_{21}$. Thus $H^n = H^n_{12} + H^n_{21}$, with uniqueness of representation modulo $H^{n-1}$. Similarly, if $n$ is odd, we write $H^n_{11} = R_1 R_2 R_1 \cdots R_1$, $H^n_{22} = R_2 R_1 R_2 \cdots R_2$ and note that $H^n = H^n_{11} + H^n_{22}$. Depending on whether $n$ is even or odd we will sometimes write $H^n_{1\cdot}$ to signify either $H^n_{12}$ or $H^n_{11}$ and $H^n_{2\cdot}$ to signify either $H^n_{21}$ or $H^n_{22}$. The elements of $H^n_{ij}$ which are of height $n$ are called $(i, j)$-pure, or simple $(\cdot, j)$-pure or $(i, \cdot)$-pure. All elements of height $\geqslant 1$ which are not $(i, j)$-pure for some $i$, $j$ are called 0-pure. One always has $|ab| \leqslant |a| + |b|$. Strict inequality can occur only when, for some $i$, $a$ is $(\cdot, i)$-pure and $b$ is $(i, \cdot)$-pure, in which case we say that $a$, $b$ is an interacting pair. When equality holds, or more generally when $|a_1 a_2 \cdots a_k| = \Sigma_{j=1}^k |a_j|$, we say that $a_1, a_2, \ldots, a_k$ is a noninteracting sequence. Given any two nonzero elements $f$, $g \in R$ it is easy to see that there exists $r \in R$ such that $f$, $r$, $g$ is a noninteracting triple. An immediate corollary of this observation is the fact that $R$ must be a prime ring. The following result of Cohn concerning sums of products of noninteracting elements will be used many times (often tacitly) in the course of this paper.

LEMMA 1 [**3**, p. 438]. *Let* $u_1, u_2, \ldots, u_n \in H^p_{\tau\sigma}$ *be right $\Delta$-independent* $\pmod{H^{p-1}}$, *let* $v_1, v_2, \ldots, v_n \in H^q_{\lambda\mu}$, $\sigma \neq \lambda$. *If* $\Sigma_{i=1}^n u_i v_i \equiv 0 \pmod{H^{p+q-1}}$, *then each* $v_i \equiv 0 \pmod{H^{q-1}}$. (*Equivalently*: *if some* $|v_i| = q$, *then* $\Sigma_{i=1}^n u_i v_i$ *is* $(\tau, \mu)$*-pure of height* $p + q$.) *Similarly, if* $u_1, u_2, \ldots, u_n \in H^p_{\tau\sigma}$, $v_1, v_2, \ldots, v_n \in H^q_{\lambda\mu}$ *and are left $\Delta$-independent* $\pmod{H^{q-1}}$, *and* $\Sigma_{i=1}^n u_i v_i \equiv 0 \pmod{H^{p+q-1}}$, *then each* $u_i \equiv 0 \pmod{H^{p-1}}$.

Not only is $R$ both a left and right $\Delta$-space but it will also be very useful to consider $R$ as either a left or right $R_i$-module, $i = 1, 2$. To this end we fix a right $\Delta$-basis $\{x_i\} \cup 1$ for $R_1$, a right $\Delta$-basis $\{y_j\} \cup 1$ for $R_2$, a left $\Delta$-basis $\{s_p\} \cup 1$ for $R_1$, and a left $\Delta$-basis $\{t_q\} \cup 1$ for $R_2$. We let $\{M_k\}$ denote the set of all finite products of alternating $x_i$'s and $y_j$'s and $\{N_l\}$ the set of all finite products of alternating $s_p$'s and $t_q$'s. It is straightforward to show that $\{M_k\} \cup 1$ is a right $\Delta$-basis for $R$ and $\{N_l\} \cup 1$ is a left $\Delta$-basis for $R$. We shall refer to the $M_k$'s as right $\Delta$-basis monomials and the $N_l$'s as left $\Delta$-basis monomials. It is also easy to show that those right $\Delta$-basis monomials which end in a $y_j$ (together with 1) form a basis for $R$ considered as a right $R_1$-module. Likewise those left $\Delta$-basis monomials beginning with a $t_q$ (together with 1) form a left $R_1$-basis for $R$. Right and left $R_2$-bases for $R$ are similarly described.

Now let $f$ be a $(\tau, \lambda)$-pure element of height $n + 1$. In view of the above framework we may then write

$$f = \sum_{i=0}^{n} U_{ik} a_{ik} \tag{2}$$

where the $U_{ik}$ are right $R_\lambda$-basis monomials of height $i$, $a_{ik} \in R$, and some $a_{nk} \notin \Delta$. Such $a_{nk} \notin \Delta$ are called leading coefficients of $f$. Similarly, we may write any $(\lambda, \mu)$-pure element $g$ of height $m + 1$ as

$$g = \sum_{j=0}^{m} b_{lj} V_{lj} \tag{3}$$

where the $V_{lj}$ are left $R_\lambda$-basis monomials of height $j$, $b_{lj} \in R_\lambda$, and some $b_m \notin \Delta$.

A partially ordered set $S$ under $<$ is said to be partially well-ordered if every nonempty subset $T$ of $S$ has a minimal element. Here we mean by a minimal element of $T$ an element $t_0 \in T$ such that there exists no element $t \in T$ with $t < t_0$. In our situation height yields a partial well-ordering on $R$: $f < g$ means $|f| < |g|$. But, as shall become apparent in the next section, we will need a more refined partial well-ordering than that given by height. We proceed now to define such a partial ordering.

We start by well-ordering the right $\Delta$-basis $\{x_i\} \cup 1$ of $R_1$ and the right $\Delta$-basis $\{y_j\} \cup 1$ of $R_2$, with 1 the minimal element in each case. We next decree that any $x_i <$ any $y_j$. We now proceed to well-order the right $\Delta$-basis monomials. If $|M| < |M'|$ then $M < M'$. Basis monomials of the same height are ordered lexicographically. The following diagram will help fix in the reader's mind the way the order works:

$$1 < \{x_i\} < \{y_j\} < \{x_i y_j\} < \{y_j x_i\} < \{x_i y_j x_k\} < \cdots .$$

For instance, $x_1 y_7 x_9 < x_2 y_1 x_4$. We are now ready to define a partial ordering on the elements of $R$. We write $f = \sum_{k=1}^{n} M_k \alpha_k$ and $g = \sum_{k=1}^{n} M_k \beta_k$, $\alpha_k$, $\beta_k \in \Delta$, with the $M_k$'s listed in descending order $M_1 > M_2 > \cdots > M_n$. We say that $f > g$ means that $f \neq g$ and at the first $k$ at which $\alpha_k \neq \beta_k$ we have $\alpha_k \neq 0$ and $\beta_k = 0$. For example, if $f = y_3 x_2 \alpha_0 + x_2 y_1 \alpha_1 + x_1 \alpha_3$ and $g = y_3 x_2 \alpha_0 + x_1 y_5 \beta_2$ then $f > g$. On the other hand $y_3 x_2 + y\alpha$ and $y_3 x_2 + y\beta$, $\alpha \neq \beta$, $\alpha \neq 0$, $\beta \neq 0$ are not comparable. Of course, if $|f| > |g|$ it is clear that $f > g$.

LEMMA 2. *R is partially well-ordered under* $<$.

PROOF. To show that $>$ yields a partial ordering on $R$ we need to establish transitivity. Thus suppose $f > g > h$ and write $f = \Sigma M_k\alpha_k$, $g = \Sigma M_k\beta_k$, $h = \Sigma M_k\gamma_k$, with the $M_k$'s given in descending order. Let $p$ be the first subscript where $\alpha_p \neq \beta_p$ (hence $\alpha_p \neq 0$, $\beta_p = 0$) and let $q$ be the first subscript where $\beta_q \neq \gamma_q$ (hence $\beta_q \neq 0$, $\gamma_q = 0$). If $M_q \geqslant M_p$ it is clear that $q$ is the first subscript where $\alpha_q \neq \gamma_q$ and furthermore $\alpha_q \neq 0$ (otherwise $f < g$). Likewise if $M_q < M_p$ then $p$ is the first possible subscript where $\alpha_p \neq \gamma_p$ and here we have $\gamma_p = 0$ (since $\beta_p = 0$). We have thereby verified that $R$ is partially ordered under $<$.

We now claim that $R$ is partially well-ordered under $<$. Indeed, let $S$ be any nonempty subset of $R$. If $0 \in S$, set $f_1 = 0$; otherwise choose $f_1 = M_1\alpha_1$ + lower terms, with $M_1$ minimal, $\alpha_1 \neq 0$, $f_1 \in S$. If $M_1\alpha_1 \in S$, set $f_2 = M_1\alpha_1$; otherwise choose $f_2 = M_1\alpha_1 + M_2\alpha_2$ + lower terms with $M_2$ minimal, $\alpha_2 \neq 0$, $f_2 \in S$. At the $n$th stage we either have $f_n = \Sigma_{i=1}^{n-1}M_i\alpha_i \in S$ or $f_n = \Sigma_{i=1}^{n}M_i\alpha_i$ + lower terms, with $M_n$ minimal, $\alpha_n \neq 0$, $f_n \in S$, and $M_1 > M_2 > \cdots > M_n$. Since the right basis monomials are well-ordered there cannot exist an infinite sequence $M_1 > M_2 > \cdots > M_n > \cdots$, and for some $n$, we have $f_{n+1} = \Sigma_{i=1}^{n}M_i\alpha_i \in S$. Suppose $f \in S$ and $f < f_{n+1}$. Let $M_k$, $1 \leqslant k \leqslant n$, be the first monomial at which $f$ does not agree with $f_{n+1}$. If $f = \Sigma_{i=1}^{k-1}M_i\alpha_i$, then we should have chosen $f_k = \Sigma_{i=1}^{k-1}M_i\alpha_i$. If $f = \Sigma_{i=1}^{k-1}M_i\alpha_i + M\beta + \ldots$, $\beta \neq 0$, then $M < M_k$, a contradiction to the minimality of $M_k$ in the choice of $f_k$. We may therefore conclude that $f_{n+1}$ is a minimal member of $S$ and the proof is complete.

**2. Interacting elements and the units of** $R$. During this section we impose the additional condition that $R_1$ and $R_2$ are weakly 1-finite, that is, any one-sided inverse in $R_i$ is in fact two-sided.

In the present section we analyze in depth the situation where we have an interacting pair $f, q$, i.e., where $|fg| < |f| + |g|$. As pointed out in the previous section this means that we may assume there is a $\lambda$ such that $f$ is $(\tau, \lambda)$-pure and $g$ is $(\lambda, \mu)$-pure, for some $\tau$ and $\mu$. Therefore, setting $|f| = n + 1$ and $|g| = m + 1$, we may write

$$f = \sum_{i=0}^{n} U_{ik}a_{ik} \tag{2}$$

where the $U_{ik}$ are right $R_\lambda$-basis monomials of height $i$, $a_{ik} \in R$, some $a_{nk} \notin \Delta$, and

$$g = \sum_{j=0}^{m} b_{lj}V_{lj} \tag{3}$$

where the $V_{lj}$ are left $R_\lambda$-basis monomials of height $j$, $b_{lj} \in R$, some $b_{lm} \notin \Delta$. Using the representations (2) and (3) we see that

$$fg = h_1 + h_2$$

where

$$h_1 = \sum_{k,l} U_{nk}a_{nk}b_{lm}V_{lm}, \qquad |h_2| \leqslant n + m.$$

The case where $|fg| = |f| + |g| - 1 = n + m + 1$, which one thinks of as the "usual" case, says that $h_1 \not\equiv 0 \pmod{H^{n+m}}$, and so $fg$ is $(\tau, \mu)$-pure. The remainder of this section is devoted to analyzing the case where $|fg| \leqslant |f| + |g| - 2 = n + m$ and we break the argument into two cases, depending on whether $a_{nk}b_{lm} \neq 0$ for some $k$, $l$ or $a_{nk}b_{lm} = 0$ for all $k$, $l$.

THEOREM 1. *Let $f = \sum_{i=0}^{n} U_{ik}a_{ik}$ be $(\tau, \lambda)$-pure of height $n + 1$, let $g = \sum_{j=0}^{m} b_{lj}V_{lj}$ be $(\lambda, \mu)$-pure of height $m + 1$, and suppose $|fg| \leqslant n + m$. If $a_{nk}b_{lj} \neq 0$ for some $k$, then there exists a unit $b = b_{lm} \in R_\lambda$ such that $|fb| = n$, $fb$ not $(\cdot, \lambda)$-pure, $|b^{-1}g| = m$, $b^{-1}g$ not $(\lambda, \cdot)$-pure.*

PROOF. Since $|fg| \leqslant n + m$ we have $h_1 = \sum U_{nk}(a_{nk}b_{lj}V_{lj}) \equiv 0 \pmod{H^{n+m}}$. By Lemma 1 we conclude that for each $k$, $\sum_l a_{nk}b_{lm}V_{lm} \equiv 0 \pmod{H^m}$. Another application of Lemma 1 forces each $a_{nk}b_{lm} \equiv 0 \pmod{\Delta}$, i.e., each $a_{nk}b_{lm} = \alpha_{kl} \in \Delta$. We may assume that $a_{n1}b_{1m} = \alpha_{11} \neq 0$. Thus $b = b_{1m}$ has a left inverse $\alpha_{11}^{-1}a_{n1}$, and from our underlying assumption that the $R_i$ are weakly 1-finite we see that $b$ is in fact a unit in $R_\lambda$. Then

$$fb = f_1 + f_2 + f_3$$

where

$$f_1 = \sum_k U_{nk}\alpha_{k1} \in H_{\tau\rho}^n, \qquad \rho \neq \lambda,$$

$$f_2 = \sum U_{n-1,k}(a_{n-1,k}b) \in H_{\gamma\lambda}^n, \qquad \gamma \neq \tau,$$

$$|f_3| < n.$$

Furthermore $f_1 \not\equiv 0 \pmod{H^{n-1}}$ since $\alpha_{11} \neq 0$, and so $|fb| = n$ and $fb$ is not $(\cdot, \lambda)$-pure. Similarly we have $|b^{-1}g| = m$ and $b^{-1}g$ is not $(\lambda, \cdot)$-pure.

At this point let us recall two basic types of units of $R$ mentioned in the Introduction. One type arises from the individual units of $R_1$ and $R_2$; products of these are called monomial units. Another type is of the form $1 + arb$, $a, b \in R_i$, $ba = 0$, $r \in R$; following Bergman [1] we call these transvections. Letting $U = U(R)$, denote the group of units of $R$. We let $S(R)$ denote the subgroup of $U$ generated by the monomial units and the transvections. We shall refer to the elements of $S(R)$ as standard units.

We are now ready to look at the more complicated situation in which the pair $f$, $g$ is interacting (using the same terminology as in (2) and (3)) with $|fg| \leqslant |f| + |g| - 2$ and $a_{nk}b_{lm} = 0$ for all $k$, $l$. Let $0 \leqslant r < n + m$ (or $r = -\infty$) be such that $a_{ik}b_j = 0$ for all $k$, $l$ whenever $i + j > r$. Because of this we may write

$$fg = h_\lambda + h_\gamma + h'$$

where

$$h_\lambda = \sum U_{ik}a_{ik}b_{lj}V_{lj} \in H_{\lambda\cdot}^{r+1}, \qquad i + j = r, \quad i \text{ even},$$

$$h_\gamma = \sum U_{ik}a_{ik}b_{lj}V_{lj} \in H_{\lambda\cdot}^{r+1}, \qquad i + j = r, \quad i \text{ odd}, \gamma \neq \lambda,$$

$$|h'| \leqslant r.$$

THEOREM 2. *Let $f$, $g$ be an interacting pair (as indicated above) with $r < |f| + |g| - 2 = n + m$ such that $a_{ik}b_{lj} = 0$ for all $k$, $l$ whenever $i + j > r$. If $h_\lambda \equiv 0 \pmod{H^r}$, then either $a_{ik}b_{lj} = 0$ for all $k$, $l$, $i + j = r$, $i$ even, or there exists a standard unit $u$ such that $fu < f$. Similarly, if $h_\gamma \equiv 0 \pmod{H^r}$, then either $a_{ik}b_{lj} = 0$ for all $k$, $l$, $i + j = r$, $i$ odd, or there exists a standard unit $u$ such that $fu < f$.*

PROOF. We will prove the theorem for the case where $h_\lambda \equiv 0$ (a similar proof will prevail if $h_\gamma \equiv 0$). Suppose therefore that for some $i$, $j$, $k$, $l$, $i$ even, $i + j = r$, we have $a_{ik}b_{lj} \neq 0$. Among all such choices pick $i = i_0$ maximal such that $a_{i_0k}b_{lj_0} \neq 0$ for some $k$, $l$, $i_0 + j_0 = r$, $i_0$ even. Set $A = \{k | a_{i_0k}b_{lj_0} = 0 \text{ for all } l\}$ and $B = \{k | a_{i_0k}b_{lj_0} \neq 0 \text{ for some } l\}$. Next choose $k_0 \in B$ for which $U_{i_0k_0} > U_{i_0k}$ for all other $k \in B$ (this is possible since $B$ is a finite set). Since $k_0 \in B$ we have $a_{i_0k_0}b_{l_0j_0} \neq 0$ for some $l_0$.

We claim that $a_{i_0k_0}b_{l_0j_0} \notin \Delta$. Indeed, suppose $a_{i_0k_0}b_{l_0j_0} = \beta \neq 0 \in \Delta$. Then $a_{i_0k_0}$ has a right inverse and $b_{l_0j_0}$ has a left inverse, whence $a_{i_0k_0}$ and $b_{l_0j_0}$ are both units in $R_\lambda$ (since $R_\lambda$ is weakly 1-finite). Since $r < n + m$ either $i_0 < n$ or $j_0 < m$. Without loss of generality we may assume $i_0 < n$. Then $a_{n1}b_{l_0j_0} = 0$ since $n + j_0 > i_0 + j_0 = r$, thus contradicting $b_{l_0j_0}$ being a unit.

Reviewing our situation so far we have

$$\sum_{k\in B} U_{i_0k}a_{i_0k}b_{lj_0}V_{lj_0} + \sum U_{ik}a_{ik}b_{lj}V_{lj} \equiv 0 \pmod{H^r} \tag{4}$$

where in the first summand $k \in B$, $U_{i_0k_0} > U_{i_0k}$ if $k \neq k_0$, $a_{i_0k_0}b_{l_0j_0} \notin \Delta$, and in the second summand $i < i_0$, $i + j = r$, $i$ even. We set $V = V_{l_0j_0}$ and consider all $V_{lj}$ appearing in the second summand of (4) which end in $V$, i.e., $V_{lj} = W_{lj}z_{lj}V$, where $z_{lj}$ belongs to the given left $\Delta$-basis $\{u_q\}$ of $R_\lambda$. Writing the left-hand side of (4) in terms of left basis monomials of height $j_0$ we may then apply Lemma 1 to obtain in particular

$$\sum_{k\in B} U_{i_0k}a_{i_0k}b_{l_0j_0} + \sum_{i<i_0} U_{ik}a_{ik}b_{lj}W_{lj}z_{lj} \in H^{r-j_0} \tag{5}$$

where $i + j = r$ and $i$ is even. Now write $a_{i_0k}b_{l_0j_0} = \sum_q \beta_{kq}u_q$, $\beta_{kq} \in \Delta$, and fix $q$ such that $\beta_{k_0q} = \beta \neq 0$ (this is possible in view of our claim established earlier that $a_{i_0k_0}b_{l_0j_0} \notin \Delta$). In the second summand of (5) we focus on those $l$, $j$ for which $z_{lj} = u_q$, and by another application of Lemma 1 we obtain

$$\sum_{k\in B} U_{i_0k}\beta_{kq} + \sum_{i<i_0} U_{ik}a_{ik}b_{lj}W_{lj} \in H^{r-j_0-1} = H^{i_0-1} \tag{6}$$

with $i + j = r$, $i$ even, $W_{lj}u_qV = V_{lj}$.

We set

$$h = \Sigma b_{lj}W_{lj}\beta^{-1}a_{i_0k_0}, \quad \text{where } W_{lj}u_qV = V_{lj}, j > j_0,$$

and note immediately that $h^2 = 0$, since $a_{i_0k_0}b_{lj} = 0$ in view of $i_0 + j > i_0 + j_0 = r$. Now $1 + h$ is in fact a standard unit since it can be written as a product of transvections $\prod_{l,j}(1 + b_{lj}W_{lj}\beta^{-1}a_{i_0k_0})$. Our aim is now to show that $f(1 + h) < f$.

Since $a_{ik}b_{lj} = 0$ for $i + j > r$ and $|W_{lj}| = j - j_0 - 1$ (where $V_{lj} = W_{lj}u_qV$) it is clear that $U_{ik}a_{ik}b_{lj}W_{lj}\beta^{-1}a_{i_0k_0} \in H^{i_0+1}$ (since $i + 1 + (j - j_0 - 1) + 1 \leqslant r - j_0 + 1 = i_0 + 1$). But for $i$ odd we may assume $i + j < r$, for otherwise $i + j = r = i_0 + j_0$ implies $i + (j - j_0 - 1) = i_0 - 1$, a contradiction since the left-hand side is even and the right-hand side is odd. Therefore,

$$fh = \sum U_{ik}a_{ik}b_{lj}W_{lj}\beta^{-1}a_{i_0k_0} + d, \tag{7}$$

where $i + j = r$, $i$ even, $i > i_0$, and $d \in H^{i_0}$. But in view of (6) we may rewrite (7) as

$$fh = -\sum_{k \in B} U_{i_0k}\beta_{kq}\beta^{-1}a_{i_0k_0} + c\beta^{-1}a_{i_0k_0} + d, \tag{8}$$

where $c \in H^{i_0-1}$, $d \in H^{i_0}$. Using (2) we now have

$$\begin{aligned} f(1 + h) = f + fh = \sum_{i > i_0} U_{ik}a_{ik} + \sum_{k \in A} U_{i_0k}a_{i_0k} \\ + \sum_{\substack{k \in B \\ k \neq k_0}} U_{i_0k}\left(a_{i_0k} - \beta_{kq}\beta^{-1}a_{i_0k_0}\right) \\ + \left[\sum_{k < i_0} U_{ik}a_{ik} + c\beta^{-1}a_{i_0k_0} + d\right], \end{aligned} \tag{9}$$

where the terms in the last square bracket of (9) lie in $H^{i_0}$. By writing $a_{i_0k_0}$ in terms of the given right $\Delta$-basis for $R_\lambda$ and recalling that $U_{i_0k_0} > U_{i_0k}$ for $k \neq k_0$ in $B$ it is clear from (9) that $f(1 + h) < f$.

As an immediate corollary to Theorem 2 we have

THEOREM 3. *Let $f, g \in R$ be an interacting pair (with $f$ and $g$ written according to* (2) *and* (3)) *such that $|fg| \leqslant |f| + |g| - 2$ and $a_{nk}b_{lm} = 0$ for all $k, l$. Let $r$ be minimal with respect to the property that $a_{ik}b_{lj} = 0$ for all $k, l$ whenever $i + j > r$. Then either $|fg| = r + 1$ or there is a standard unit $u$ such that $fu < f$.*

PROOF. If $|fg| \leqslant r$ then both $h_\lambda \equiv 0 \pmod{H^r}$ and $h_\gamma \equiv 0 \pmod{H^r}$. By assumption some $a_{ik}b_{lj} \neq 0$ for $i + j = r$, whence Theorem 2 assures us there is a standard unit such that $fu < f$.

As a joint corollary to Theorems 1 and 3 we are able to establish the following result due to Bergman.

THEOREM 4 [**1**, COROLLARY 2.16(i)]. *If $R = R_1 \amalg R_2$ is the coproduct of weakly* 1*-finite $\Delta$-rings $R_1$ and $R_2$ with* 1 *over a division ring $\Delta$, $R_1 \neq \Delta$, $R_2 = \Delta$, then every unit of $R$ is a standard unit.*

PROOF. Suppose the set $E$ of nonstandard units is nonempty. By Lemma 2 $E$ has a minimal member $f$. We write $fg = 1$. Clearly $|f| = n + 1 \geqslant 1$ and $|g| = m + 1 \geqslant 1$, and so $f, g$ is an interacting pair with $|fg| \leqslant |f| + |g| - 2$. Writing $f = \sum_{i=0}^{n} U_{ik}a_{ik}$ and $g = \sum_{j=0}^{m} b_{lj}V_{lj}$ in accordance with (2) and (3), we consider two possibilities. If $a_{nk}b_{lm} \neq 0$ for some $k, l$ Theorem 1 says there is a monomial unit $u$ such that $|fu| < |f|$. By the minimality of $f$ this makes $fu$ a standard unit, whence the contradiction that $f$ must be a standard unit. If $a_{nk}b_{lm} = 0$ for all $k$, then since $f$ is a minimal nonstandard unit Theorem 3 says that $r < 0$ (otherwise $|fg| = r + 1 \geqslant 1$). This leaves $r = -\infty$ which means that $a_{ik}b_{lj} = 0$ for all $i, j, k, l$, i.e., $fg = 0$.

Another result of Bergman follows as a corollary of Theorems 1 and 3.

THEOREM 5 [**1**, COROLLARY 2.16(ii)]. *Under the same conditions as Theorem* 4 *suppose* $fg = 0$ *with* $f \neq 0$, $g \neq 0$. *Then there exists a unit* $u$ *such that if* $f' = fu$ *and* $g' = u^{-1}g$ *are written as* $f' = \sum_{i=0}^{n} U'_{ik}a'_{ik}$ *and* $g' = \sum_{j=0}^{n} b'_{lj}V'_{lj}$ *according to* (2) *and* (3) *then* $a'_{ik}b'_{lj} = 0$ *for all* $i, j, k, l$.

PROOF. Let $E = \{fv \mid v \text{ a unit}\}$ and by Lemma 2 we let $fu$ be a minimal member of $E$. Set $f' = fu$, $g' = u^{-1}g$ and note that $f'g' = 0$. Clearly $|f'| \geqslant 1$ and $|g'| \geqslant 1$, and so $f', g'$ is an interacting pair with $-\infty = |f'g'| < |f'| + |g'| - 2$. If $a_{nk}b_{lm} \neq 0$ for some $k, l$ a contradiction to the minimality of $f'$ is immediately reached through Theorem 1. Therefore the conditions of Theorem 3 must prevail and we conclude by the minimality of $f'$ that $r$ (as defined in Theorem 3) must be $-\infty$ since $r + 1 = |f'g'|$. But this just says that $a'_{ik}b'_{lj} = 0$ for all $i, j, k, l$.

An immediate implication of Theorem 5 is the following result due to Cohn:

COROLLARY 1 [**4**]. *The coproduct* $R = R_1 \amalg R_2$ *of* $\Delta$*-domains* $R_1$ *and* $R_2$ *is a domain.*

**3.** ***X*-inner automorphisms.** We begin with a brief review of the basic definitions and properties of the normalizing elements of any prime ring $R$ [**9**]. Let $\mathscr{F}$ be the filter of all nonzero two-sided ideals of $R$ and let $R_{\mathscr{F}}$ be the left quotient ring of $R$ relative to $\mathscr{F}$. $R$ is embeddable in $Q$ (via right multiplications) and, given any element $q \in Q$, there exists an ideal $I$ in $\mathscr{F}$ such that $Iq \subseteq R$. The set $N^* = N^*(R)$ (the main object of study in this paper) is the set of all units $u$ of $R_{\mathscr{F}}$ such that $u^{-1}Ru = R$ [**9**, p. 239]. The automorphisms thus induced on $R$ are just the $X$-inner automorphisms of Kharchenko [**9**, p. 239]. The subring $RN$ of $R_{\mathscr{F}}$ generated by $R$ and $N = N^*(R) \cup \{0\}$ is called the normal closure of $R$. In case $R$ is a domain then $RN$ is again a domain [**9**]. The set $N$ may equivalently be described as the set $\{x \in R_{\mathscr{F}} \mid xR = Rx\}$. An important subset of $N$ is the extended center $C$ of $R$, which is simply the center of $R_{\mathscr{F}}$. Given $s \in N^*$ we let $I \in \mathscr{F}$ be such that $Is \subseteq R$, and we set $J = I \cap s^{-1}Is \in \mathscr{F}$. For $a \in J$ we write $a = s^{-1}bs$, $b \in I$, and note that $sa = s(s^{-1}bs) = bs \in R$. Thus we have the key property of $N$ that for any $s \in N^*$ there exists an ideal $J \neq 0$ of $R$ such that $0 \neq Js \subseteq R$ and $0 \neq sJ \subseteq R$. Let $\sigma$ denote the $X$-inner automorphism of $R$ determined by $s$, i.e., $r^\sigma = s^{-1}rs$ for all $r \in R$. It is straightforward to establish

(10) $$frg = hr^\sigma f, \qquad f \in J, r \in R,$$

where $g = sf$ and $h = fs$. Conversely, if $\sigma$ is an automorphism of $R$ satisfying (1) for certain nonzero elements $f, g, h$, then it is known that $\sigma$ is $X$-inner (just define $s$ by $s\colon \sum x_i f y_i \to \sum x_i^\tau g y_i$, $x_i, y_i \in R$, $\tau = \sigma^{-1}$). The inverse automorphism $\tau = \sigma^{-1}$ is then the $X$-inner automorphism of $R$ determined by $s^{-1}$, and we have the analogous relation

$$grf = er^\tau g, \qquad g \in sJ, r \in R,$$

where $f = s^{-1}g$ and $e = gs^{-1}$.

We return now to our study of $R = R_1 \amalg R_2$, for the moment only assuming that $R_1 \neq \Delta$, $R_2 \neq \Delta$. Our aim will be to show that any $s \in N^*(R)$ must actually be a unit of $R$. Along the way, of course, we may replace $s$ by $s^{-1}$ or $us$ or $su$ ($u$ a unit of $R$) without altering the problem.

We recall a concept from [**6**]: An automorphism $\sigma$ of $R$ is strongly bounded if there exists an integer $k \geqslant 0$ such that $|r^\sigma| \leqslant |r| + k$ for all $r \in R$.

LEMMA 3. *Every X-inner automorphism $\sigma$ is strongly bounded.*

PROOF. We first fix $f \neq 0 \in J$ in (10); hence $g$ and $h$ are also fixed. Also fix $x_1 \notin \Delta$ in $R_1$ and $y_1 \notin \Delta$ in $R_2$. Now pick $r \in R$. We may choose $z = 1, x_1$, or $y_1$ so that $r^\sigma$, $z$, $f$ is a noninteracting triple. It follows that $zh \neq 0$, since otherwise $zfyg = zhy^\sigma f = 0$ for all $y \in R$, a contradiction to the primeness of $R$. Now pick $w = 1, x_1$, or $y_1$ so that $zh$, $w$, $r^\sigma zf$ is a noninteracting triple. Replacing $r$ by $w^\tau r z^\tau$ in (1) we see that $zf(w^\tau r z^\tau)g = zhwr^\sigma zf$. Therefore $|r^\sigma| \leqslant |r^\sigma zf| \leqslant |zhwr^\sigma zf| = |zfw^\tau rz^\tau g| \leqslant |zfw^\tau| + |r| + |z^\tau g| \leqslant |r| + k$, where $k$ is independent of $r$.

From Lemma 3 we have the immediate

COROLLARY 2. *If $\sigma$ is strongly bounded then for any $z \in R_i$ either $z^\sigma \in \Delta$ or $z^\sigma$ is $(\lambda, \lambda)$-pure for some $\lambda$.*

It is now time to make two further assumptions on $R_1$ and $R_2$, which will hold for the remainder of this paper. The first is that $R_1$ and $R_2$ are weakly 1-finite (so that the results of §2 may be invoked), and the second is that at least one of the four dimensions $(R_i : \Delta)_r, (R_i : \Delta)_l$, $i = 1, 2$, is greater than 2. Without loss of generality we may suppose that $(R_1 : \Delta)_r > 2$. This is possible because of the symmetry afforded by the fact that a partial well-ordering of $R$ could just as well have been defined in terms of left bases (see §1) and the fact that $s \in N^*$ could be replaced by $s^{-1}$ along with the equation $hrf = fr^\tau h^\tau, f \in J \cap sJ, h = fs, \tau = \sigma^{-1}$.

Under this framework let us now take $s \in N^*$ and endeavor to show (in the course of this section and the next) that $s$ must in fact be a unit of $R$. In the accompanying equation (10) we select and fix $f$ to be 0-pure of even height $n > 0$. This is possible by pre-multiplication and post-multiplication of $f$ (if necessary) by appropriate elements of $R_1$ and $R_2$. Of course $g$ and $h$ in (10) are also fixed, whatever they may be.

We digress momentarily to indicate some notation for writing elements which we shall use in the arguments to come. Suppose $r \in H^m$ where $m$ is even. Then we will frequently write $r \equiv r_{12} + r_{21}$, where $r_{12} \in H^m_{12}$, $r_{21} \in H^m_{21}$, it being understood that $r \equiv r_{12} + r_{21} \pmod{H^{m-1}}$. If furthermore $r_{12} \notin H^{m-1}$ and/or $r_{21} \notin H^{m-1}$ we indicate this by simply writing $r_{12} \not\equiv 0$ and/or $r_{21} \not\equiv 0$. Similarly, in case $m$ is odd, we will be writing $r = r_{11} + r_{22}$. We also caution the reader to be on the lookout for Lemma 1 to be frequently used without specific indication being made.

LEMMA 4. (a) $|g| = |h|$ *and* (b) *$g$ and $h$ are of the same type.*

PROOF. Setting $r = 1$ in (10) we have $fg = hg$. Computing the height of both sides we see that $|f| + |g| = |fg| = |hf| = |h| + |f|$ since $f$ is 0-pure. Part (a) is thereby proved. To show (b) we first note that $g$ is 0-pure if and only if $h$ is 0-pure, for otherwise one side of $fg = hf$ would be 0-pure but not the other side. Next suppose that $g$ is (1,1)-pure but $h$ is (2,2)-pure, that is $g \equiv g_{11}$ but $h \equiv h_{22}$. Then $fg \equiv f_{12}g_{11}$ whereas $hf \equiv h_{22}f_{12}$. A similar contradiction occurs if $g = g_{12}$ but $h \equiv h_{21}$, since $fg \equiv f_{12}g_{12}$, whereas $hf \equiv h_{21}f_{21}$.

LEMMA 5. *Suppose g is 0-pure of height m (and hence by Lemma 4 h is 0-pure of height m). Then*
(a) $|r| = |r^\sigma| = |r^\tau|$ *for all* $r \in R$,
(b) $n = m$,
(c) $R_\lambda^\sigma \subseteq R_\lambda$, $\lambda = 1, 2$.

PROOF. Using the fact that $f$, $g$, $h$ are all 0-pure we see that $|frg| = |f| + |rg| = |f| + |r| + |g|$ and $|hr^\sigma f| = |h| + |r^\sigma f| = |h| + |r^\sigma| + |f|$, thereby establishing (a). To show (b) we first suppose $n < m$. We write $h \equiv h_{1\rho} + h_{2\tau}$ and $g \equiv g_{1\rho} + h_{2\tau}$, $\rho \neq \tau$. We next express $f_{12}$ and $h_{1\rho}$ in terms of the given right $\Delta$-basis monomials $\{U_k\}$ of height $n$:

$$f_{12} \equiv \sum U_k \alpha_k, \quad \alpha_k \in \Delta; \qquad h_{1\rho} \equiv \sum U_k w_k, \quad w_k \in H_{1\rho}^{m-n}.$$

Let $U_1$ be such that $\alpha_1 \neq 0$ and set $r = \alpha_1^{-1}x_1$ where $x_1$ belongs to the given right $\Delta$-basis of $R_1$. Then $|r^\sigma| = 1$ by (a) and so we may write $r^\sigma = z_\tau + z_\rho$, $z_\tau \in R_\tau$, $z_\rho \in R_\rho$, $z_\tau$, $z_\rho$, depending on $r$. A glance at $frg = hr^\sigma f$ then shows that $f_{12}rg_{2\tau} \equiv h_{1\rho}z_\tau f_{\rho\tau}$, that is, $\sum_k U_k\alpha_k\alpha_1^{-1}x_1g_{2\tau} \equiv \sum_k U_k w_k z_\tau f_{\rho\tau}$. By viewing each side in terms of the right $\Delta$-basis monomials $\{U_k\}$ and applying Lemma 1 we are forced to conclude that $x_1 g_{2\tau} \equiv w_1 z_\tau f_{\rho\tau}$ and, in particular, that $w_1 \not\equiv 0$ and $z_\tau \not\equiv 0$. Now we write $w_1 \equiv \sum_i x_i v_i$, $x_i$ a right $\Delta$-basis element of $R_1$ and $v_i \in H_{2\rho}^{m-n-1}$. Applying Lemma 1 again we see that $g_{2\tau} \equiv v_1 z_\tau f_{\rho\tau}$ and $v_i z_\tau f_{\rho\tau} \equiv 0$, which forces $v_i \equiv 0$ for $i \neq 1$. The upshot is that $w_1 = x_1 v_1$. But since $(R_1 : \Delta)_r > 1$ we may also choose $x_2 \in R_1$ independent of $x_1$ (mod $\Delta$), and a repetition of the preceding argument yields the contradiction that $w_1 \equiv x_2 u_2$. In case $n > m$, in a similar manner as before we write $h_{\rho 2} = \sum U_k \beta_k$, $\beta_k \in \Delta$, $\beta_1 \neq 0$, set $r = \beta_1^{-1}x_1$ (with $r^\tau = z_\rho + z_\tau$), and from $f_{\rho\tau}z_\rho g_{\tau 1} \equiv h_{\rho 2} r f_{21}$ we reach a similar contradiction as in the case $n < m$. Therefore $n = m$ and (b) has been established. Since $n = m$ we write $g = g_{12} + g_{21}$ and $h = h_{12} + h_{21}$. First setting $r \in R_1$, $r \notin \Delta$ and writing $r^\sigma = x + y$, $x \in R_1$, $y \in R_2$, we claim that $y \in \Delta$, i.e., $r^\sigma \in R_1$. Indeed, comparing the terms of height $2n + 1$ in (10), we now have

$$f_{12}rg_{21} \equiv h_{12}xf_{21} + h_{21}yf_{12}$$

which forces $y \in \Delta$. In a similar fashion the case $r \in R_2$ is handled and thereby (c), and with it Lemma 5, has been proved.

In (10) the element $f \equiv f_{12} + f_{21}$ will remain fixed, of course, but we still have some leeway in picking $g$. To be specific if we replace $s$ by $su$, $u$ a unit of $R$, then, setting $g' = u^\tau g = (sus^{-1})sf = (su)f$ and $h' = hu = f(su)$, we have the relation $frg' = h'r^{\sigma'}f$, where $\sigma'$ is the $X$-inner automorphism determined by $su$. By Lemma 2 the set $S_f = \{h' = hu \mid u \text{ unit in } R\}$ has a minimal member, and so we may select a unit $u$ so that $hu$ is minimal. Without loss of generality, therefore, we may assume to begin with in (10) that there is no unit $u$ in $R$ for which $hu < h$. Under these conditions our immediate aim is to show that the conditions of Lemma 5 must prevail.

LEMMA 6. $g$ (*and hence* $h$) *must be* 0-*pure.*

PROOF. Since the condition $(R_1 : \Delta)_r > 2$ is not involved in the proof of this lemma there is no loss of generality in assuming that $g$ (and hence $h$ by Lemma 4) is $(1, \lambda)$-pure of height $m$. We consider $fyg = hy^\sigma f$ for $y \in R_2$, $y \notin \Delta$. Suppose $y^\sigma$ is $(\rho, \rho)$-pure, $\rho \neq \lambda$. Then $fyg \equiv f_{21}yg_{1\lambda}$ is $(2, \lambda)$-pure, whereas $y^\sigma f_{\lambda\rho}$ is $(1, \rho)$-pure. If $y^\sigma \in R_\lambda$ then $|fyg| = n + m + 1$, whereas $|hy^\sigma f| \leqslant n + m$. Hence by Corollary 3 we are left with $y^\sigma$ being $(\lambda, \lambda)$-pure of height $\geqslant 3$. Next we consider $fz^\tau g = hzf$ for $z \in R_\rho$, $z \notin \Delta$, $\rho \neq \lambda$. Suppose $z^\tau$ is $(2, 2)$-pure. Then $hzf \equiv h_{1\lambda}zf_{\lambda\rho}$ is $(1, \rho)$-pure, whereas $fz^\tau g \equiv f_{21}z^\tau g_{1\lambda}$ is $(2, \lambda)$-pure. If $z^\tau \in R_1$ then $|hzf| = m + n + 1$, whereas $|fz^\tau g| \leqslant n + m$. Hence by Corollary 3, again, we see that $z^\tau$ is $(1, 1)$-pure of height $\geqslant 3$. Now let $w \in R_\lambda$ and suppose $w^\tau$ is $(2, 2)$-pure. Since $\tau$ is strongly bounded there exists an integer $k \geqslant 0$ such that $|r^\tau| \leqslant |r| + k$ for all $r \in R$. Choose $m$ such that $2m > k$ and set $r = (wz)^m$. Clearly $|r| = 2m$ and $|r^\tau| = |(w^\tau z^\tau)^m| \geqslant m(1 + 3) = 2m + 2m > |r| + k$. Therefore, by Corollary 3, either $w^\tau \in \Delta$ or $w^\tau$ is $(1, 1)$-pure. Thus far we have established that, for $y \in R_2, y \notin \Delta, y^\sigma$ is $(\lambda, \lambda)$-pure of height $\geqslant 3$ and, for $w \in R_\lambda$, either $w^\tau \in \Delta$ or $w^\tau$ is $(1, 1)$-pure.

We write $h = \sum_{i=0}^{m-1} U_{ik}a_{ik}$ (with $h$ playing the role of $f$ in (2)), where $a_{ik} \in R_\lambda$ and some $a_{m-1,k} \notin \Delta$. Choosing $y \in R_2$, $y \notin \Delta$, we write $y^\sigma = \sum_{j=0}^{p} b_{lj}V_{lj}$ (with $y^\sigma$ playing the role of $g$ in (3)), where $b_{lj} \in R_\lambda$ and with the single term corresponding to $j = 0$ simply written $b_0$. From $fyg = hy^\sigma f$ we see that $|hy^\sigma| = |yg| = m + 1$. On the other hand $|h| + |y^\sigma| \geqslant m + 3$, and so by Theorem 1 we conclude that $a_{m-1,k}b_{lq} = 0$ for all $k, l$, in view of the minimality of $h$. Now by Theorem 3 we see that $a_{ik}b_{lj} = 0$ for all $k, l$ whenever $i + j > m$, again using the minimality of $h$. We can then write

$$hy^\sigma = q_1 + q_2 + q'$$

where

$$q_1 = \sum U_{ik}a_{ik}b_{lj}V_{lj} \in H^{m+1}_{\cdot\lambda}, \qquad i + j = m, j \text{ even},$$

$$q_2 = \sum U_{ik}a_{ik}b_{lj}V_{lj} \in H^{m+1}_{\cdot\rho}, \qquad i + j = m, j \text{ odd},$$

$$|q'| \leqslant m.$$

Since $fyg \equiv f_{21}yg_{1\lambda}$ is $(2, \lambda)$-pure it follows from $hy^\sigma f \equiv q_1 f_{\rho\lambda} + q_2 f_{\lambda\rho}$, $\lambda \neq \rho$, that $q_2 f_{\lambda\rho} \equiv 0 \pmod{H^{m+n}}$, whence $q_2 \equiv 0 \pmod{H^m}$. Therefore by Theorem 2 we have $a_{ik}b_{lj} = 0$ for all $k$ whenever $i + j = m$, $j$ odd, using again the minimality of $h$. In particular, $a_{m-1,k}b_{l1} = 0$ for all $k, l$, and so we have $a_{m-1,k}b_{lj} = 0$ for all $k$, whenever $j \geqslant 1$. Hence

$$a_{m-1,k}y = a_{m-1,k}\sum_{j=0}^{p} b_{lj}V_{lj} = a_{m-1,k}b_0. \tag{11}$$

Applying $\tau$ to (11) we see that

$$(12) \qquad a^{\tau}_{m-1,k}y = (a_{m-1,k}b_0)^{\tau}.$$

But we have previously shown that for $w \in R_\lambda$ either $w^\tau \in \Delta$ or $w^\tau$ is $(1,1)$-pure. In any case we reach a contradiction in (12) since $a_{m-1,k}$ and $a_{m-1,k}b_0$ lie in $R_\lambda$, whereas $y \in R_2$, $y \notin \Delta$. Lemma 6 is thereby proved.

**4. The main theorem.** The import of Lemma 6 is that the conditions of Lemma 5 must hold, which puts us in a position to prove the main result of this paper.

THEOREM 6. *Let $R = R_1 \amalg R_2$ be the coproduct of weakly 1-finite $\Delta$-rings $R_1$ and $R_2$ with 1 over a division ring $\Delta$, with $R_1 \neq \Delta$, $R_2 \neq \Delta$, and at least one of the four dimensions $(R_i : \Delta)_r, (R_i : \Delta)_l$, $i = 1, 2$, greater than 2. Then every X-inner automorphism of R is inner (i.e., every normalizing element of R is a unit of R) unless $R_1$, $R_2$ satisfy one of the following three conditions:*

(I) *Both $R_1$ and $R_2$ are primary.*

(II) *One $R_i$ is primary and the other is quadratic.*

(III) *The characteristic of $\Delta$ is 2, at least one $R_i$ is not a domain, and one $R_i$ is quadratic.*

PROOF. We choose $s \in N^*$, let $\sigma$ be the $X$-inner automorphism determined by $s$, and recall the fundamental equation

$$(10) \qquad frg = hr^{\sigma}f, \qquad r \in R,$$

where $g = sf$ and $h = fs$. In view of Lemmas 5 and 6 we may assume without loss of generality that $f \equiv f_{12} + f_{21}$, $g \equiv g_{12} + g_{21}$, $h \equiv h_{12} + h_{21}$ are each 0-pure of the same even height $n$, that $|r^\sigma| = |r^\tau| = |r|$ for all $r \in R$ (hence $\Delta^\sigma = \Delta^\tau = \Delta$), and that $R_i^\sigma \subseteq R_i$, $i = 1, 2$. Setting $r = 1$ in (10) we see in particular that $f_{12}g_{12} \equiv h_{12}f_{12}$, whence, by Lemma 1, $g_{12} \equiv \alpha f_{12}$ for some $\alpha \in \Delta$. It follows that $h_{12} \equiv f_{12}\alpha$. Replacing $s$ by $\alpha^{-1}s$ (this does not disturb any of our assumptions above) we may assume without loss of generality that $\alpha = 1$, i.e.,

$$(13) \qquad g_{12} \equiv f_{12} \equiv h_{12}.$$

Of course, we also conclude from $fg = hf$ that

$$(14) \qquad g_{21} \equiv \lambda f_{21}, \qquad h_{21} \equiv f_{21}\lambda$$

for appropriate $\lambda \in \Delta$. We next set $r = \gamma$ in (10) and see from (13) that $f_{12}\gamma f_{12} \equiv f_{12}\gamma^\sigma f_{12}$, whence $\gamma^\sigma = \gamma$, i.e., $\sigma$ acts as the identity on $\Delta$. From (14) we note that $f_{21}\gamma\lambda f_{21} \equiv f_{21}\lambda\gamma f_{21}$, whence $\gamma\lambda = \lambda\gamma$ for all $\gamma \in \Delta$, i.e., $\lambda \in Z(\Delta)$.

Next, setting $r = x \in R_1$, $x \notin \Delta$, in (10) we see from $f_{12}x\lambda f_{21} \equiv f_{12}x^\sigma f_{21}$ that $x^\sigma \equiv x\lambda$, i.e.,

$$(15) \qquad x^\sigma = x\lambda + \alpha, \qquad \alpha = \alpha(x) \in \Delta$$

for all $x \in R_1$. Similarly, setting $r = y \in R_2$, $y \notin \Delta$, in (10) we have $f_{21}yf_{12} \equiv f_{21}\lambda y^\sigma f_{12}$, whence $y \equiv \lambda y^\sigma$, i.e.,

$$(16) \qquad y^\sigma = \lambda^{-1}y + \beta, \qquad \beta = \beta(y) \in \Delta,$$

for all $y \in R_2$.

We now dispose of the case where $\lambda \neq 1$ by showing that in this situation each $R_i$ must be either primary or quadratic. We present the argument for $R_1$; a nearly identical one works for $R_2$. For $x \in R_1$ we see from (15) that $x\lambda - x^\sigma = \gamma(\lambda - 1)$, $\gamma \in \Delta$, using the fact that $\lambda - 1 \neq 0$. Rearranging these terms and using $\gamma^\sigma = \gamma$ we obtain $(x - \gamma)\lambda = (x - \gamma)^\sigma$, which shows that $x - \gamma \in T = \{t \in R_1 | t^\sigma = t\lambda\}$. Thus $R_1 = \Delta + T$. In view of $\lambda \in Z(\Delta)$, it is straightforward to check that $T$ is a $\Delta$-bimodule and that $T^\sigma = T$. Now let $x, u \in T$ and write $u = (1 - \lambda)t^\sigma$ for suitable $t \in T$. Then $xu = x(1 - \lambda)t^\sigma = xt^\sigma - x\lambda t^\sigma = xt\lambda - x^\sigma t^\sigma = (xt)\lambda - (xt)^\sigma \in \Delta$ in view of (15). This shows that $T^2 \subseteq \Delta$. If $T^2 = 0$ then $R_1$ is by definition primary. If $T^2 \neq 0$ then $xu = \rho \neq 0 \in \Delta$ for some $x_1 u \in T$ which forces $x^2 = \gamma \neq 0$, whence $x^{-1} = \gamma^{-1}x \in T$. For any $t \in T$ we have $x^{-1}t \in \Delta$, i.e. $t \in x\Delta$, and $tx^{-1} \in \Delta$, i.e. $t \in \Delta x$. Therefore $T = x\Delta = \Delta x$ and $R_1$ is quadratic. Since at least one of $(R_i : \Delta)_r, (R_i : \Delta)_l$, $i = 1, 2$, is greater than 2, we have shown that in case $\lambda = 1$ one of the best situations (I) and (II) must prevail.

We now assume that $\lambda = 1$, and so (15) and (16) now read

$$x^\sigma = x + \alpha(x), \quad y^\sigma = y + \beta(y), \qquad x \in R_1, y \in R_2. \tag{17}$$

It is an easy consequence of (17) that $r^\sigma \equiv r$ for all $r \in R$. We also may assume that $(R_1 : \Delta)_r > 2$. Suppose that $\sigma \neq 1$ on $R_1$, in other words, for some $x \in R_1$, $x \notin \Delta$, we have $x^\sigma = x + \delta$, $0 \neq \delta \in \Delta$. Setting $x_1 = x\delta^{-1}$ we see that $x_1^\sigma = (x + \delta)\delta^{-1} = x_1 + 1$. Now for any $x_2 \in R_1$ we write $x_2^\sigma = x_2 + \delta_2$, $\delta_2 \in \Delta$, and compute $(x_1x_2)^\sigma$ in two ways:

$$x_1x_2 + \delta_3 = (x_1x_2)^\sigma = (x_1 + 1)(x_2 + \delta_2) = x_1x_2 + x_1\delta_2 + x_2 + \delta_2.$$

It follows that $x_2 = -x_1\delta_2 + \delta_4$, which implies that $R_1 = x_1\Delta + \Delta$. Similarly, computation of $(x_2x_1)^\sigma$ in two ways leads to $R_1 = \Delta x_1 + \Delta$. What we have now shown is that if $\sigma \neq 1$ on $R_i$ then $R_i$ is quadratic. If neither $R_1$ nor $R_2$ is quadratic we conclude that $\sigma$ is the identity, i.e., $s$ lies in the extended centroid $C$. But, as we have stated in the Introduction, it is known from [5] that if we avoid the situations (I)–(III) then $C = Z(R)$, and we are finished. Otherwise we are left with the situation in which one of the $R_i$, say $R_1$, is not quadratic (and hence $\sigma = 1$ on $R_1$) and $R_2 = \Delta + y\Delta = \Delta + \Delta y$ is quadratic, with $y^\sigma = y + 1$. On the one hand,

$$(y^2)^\sigma = (y\alpha + \beta)^\sigma = (y + 1)\alpha + \beta = y\alpha + \alpha + \beta, \qquad \alpha, \beta \in \Delta, \tag{18}$$

whereas on the other hand,

$$\begin{aligned}(y^2)^\sigma &= (y + 1)^2 = y^2 + 2y + 1 \\ &= y\alpha + \beta + 2y + 1 = y(\alpha + 2) + \beta + 1.\end{aligned} \tag{19}$$

From (18) and (19) we see in particular that $\alpha = \alpha + 2$, that is, char. $\Delta = 2$. Then $y^{\sigma^2} = (y + 1)^\sigma = y + 2 = y$ which implies that $\sigma^2 = 1$ on $R_2$. Since we already know that $\sigma = 1$ on $R_1$ we then have $\sigma^2 = 1$ on $R$. This fact, when translated in terms of $s$, says that $s^2 \in C$. We have already pointed out that $C = Z \subseteq Z(\Delta)$ since we are avoiding the situations (I)–(III). Therefore $s^2$ is an element of $Z(\Delta)$. Using the fact that $r^\sigma \equiv r$ for all $r \in R$ we see that $f^2 \equiv g^2 = sfg = shf = sfsf = s^2(s^{-1}fs)f = s^2 f^\sigma f \equiv s^2 f^2$, and from this equation it follows that $s^2 = 1$. If either $R_1$ or $R_2$ is not a domain then situation (III) prevails. Therefore we may assume that both $R_1$ and $R_2$ are domains. By Corollary 1 we then know that $R$ is a domain, in which case it is known [9] that the normal closure $RN$ is a domain. Thus $s^2 = 1$ implies $(s-1)^2 = 0$ (char. $\Delta = 2$), whence $s = 1$. In other words $\sigma = 1$, contrary to our present assumption that $y^\sigma = y + 1$. This completes the proof of Theorem 6.

We illustrate Theorem 6 by exhibiting examples satisfying conditions (I) and (II) in which not every $X$-inner automorphism is inner. We have not yet succeeded in producing a similar example satisfying (III) which does not already satisfy (II).

Let $R_1$ and $R_2$ be any two primary algebras over a field $F$, say $R_i = F \oplus T_i$, $T_i^2 = 0$, $i = 1,2$. Fix any $\lambda \neq 1 \in F$. We define an automorphism of $R_1$ via $\alpha + x \to \alpha + \lambda x$, $\alpha \in F$, $x \in T_1$ and an automorphism of $R_2$ via $\alpha + y \to \alpha + \lambda^{-1}y$, $\alpha \in F$, $y \in T_2$. One easily checks that these automorphisms extend to an automorphism $\sigma$ of $R = R_1 \amalg_F R_2$. We now fix $0 \neq x_1 \in T_1$ and $0 \neq y_1 \in T_2$ and set $f = x_1 y_1 + y_1 x_1$, $g = x_1 y_1 + \lambda y_1 x_1 = h$. It can be verified that $frg = hr^\sigma f$ for all $r \in R$, which we know implies that $\sigma$ is $X$-inner. It is easy to show, however, that $\sigma$ cannot be inner. Indeed, suppose there exists a unit $u$ in $R$ such that $r^\sigma = u^{-1}ru$ for all $r \in R$. In particular we have $\lambda x = u^{-1}xu$, $x \in T_1$, $\lambda^{-1}y = u^{-1}yu$, $y \in T_2$, that is

$$xu = \lambda ux, \quad x \in T_1; \qquad yu = \lambda^{-1}uy, \quad y \in T_2. \tag{20}$$

Contradictions are easily seen to result in analyzing (20) when $u$ is either 0-pure or $(i, j)$-pure, $i, j = 1,2$. This forces $u \in F$, whence $\sigma = 1$. But this says that $\lambda x = x$, $x \in T_1$, whence $\lambda = 1$, contrary to our own original assumption.

Finally, let $R_1 = \mathbf{R} + T$ be any primary algebra over the reals $\mathbf{R}$, let $R_2$ be the quadratic algebra $\mathbf{C} = \mathbf{R} + \mathbf{R}i$, and let $\sigma$ be the automorphism of $R = R_1 \amalg R_2$ over $\mathbf{R}$ which sends $\alpha + x \to \alpha - x$ on $R_1$ and is conjugation on $R_2$. We fix $x_1 \neq 0 \in T$ and set $f = x_1 i + i x_1$, $g = h = x_1 i - i x_1$. One can then verify that $frg = hr^\sigma f$ for all $r \in R$ and so $\sigma$ must be $X$-inner. If $\sigma$ is inner, that is, there is a unit $u$ in $R$ such that $r^\sigma = u^{-1}ru$ for all $r \in R$, we obtain in particular

$$xu = -ux, \quad x \in T; \qquad iu = -ui. \tag{21}$$

As in the preceding example, an analysis of (21) forces $u \in \mathbf{R}$, that is, $\sigma = 1$, contrary to $i^\sigma = -i$. Therefore in situation (II), as well as in situation (I), we have exhibited $X$-inner automorphisms which are not inner.

## References


1. G. Bergman, *Modules over coproducts of rings*, Trans. Amer. Math. Soc. **200** (1974), 1–32.
2. P. M. Cohn, *On the free product of associative rings*, Math. Z. **71** (1959), 380–398.
3. ______, *On the free product of associative rings*. II, Math. Z. **73** (1960), 433–456.
4. ______, *On the free product of associative rings*. III, J. Algebra **8** (1968), 376–383.
5. A. Lichtman and W. S. Martindale, *The extended center of the coproduct of rings over a division ring*, Comm. Algebra **12** (1984), 2067–2080.
6. ______, *The normal closure of the coproduct of domains over a division ring*, Comm. Algebra **13** (1985), 1643–1664.
7. W. S. Martindale, *The extended center of coproducts*, Canad. Math. Bull. **25** (1982), 245–248.
8. W. S. Martindale and S. Montgomery, *The normal closure of coproducts of domains*, J. Algebra **82** (1983), 1–17.
9. S. Montgomery, *Automorphism groups of rings with no nilpotent elements*, J. Algebra **60** (1979), 238–248.


Department of Mathematics, University of Massachusetts, Amherst, Massachusetts 01003