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APPROXIMATION THEOREMS FOR NASH MAPPINGS AND NASH MANIFOLDS

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Dedicated to Professor Nobuo Shimada on his 60th birthday

ABSTRACT. Let $0 \leqslant r < \infty$. A C^r Nash function on $\mathbf{R}^n$ is a C^r function whose graph is semialgebraic. It is shown that a C^r Nash function is approximated by a C^ω Nash one in a strong topology defined in the same way as the usual topology on the space $\mathcal{S}$ of rapidly decreasing C^∞ functions. A C^r Nash manifold in $\mathbf{R}^n$ is a semialgebraic C^r manifold. We also prove that a C^r Nash manifold for $r \geqslant 1$ is approximated by a C^ω Nash manifold, from which we can classify all C^r Nash manifolds by C^r Nash diffeomorphisms.

1. Introduction. Let $r = 0,\ldots,\infty$ or ω. A submanifold of $\mathbf{R}^n$ is called a *C^r Nash manifold* if it is semialgebraic and of class C^r. A C^r map from one C^r Nash manifold to another is called a *C^r Nash map* if the graph is semialgebraic. We define similarly a *C^r Nash vector field*. For a C^r Nash manifold M, let $N^r(M)$ denote the ring of all C^r Nash functions on M. As a C^∞ Nash manifold and a C^∞ Nash map are automatically of class C^ω (Proposition 3.11, [**14**]), we assume $r \neq \infty$.

Our purpose is to approximate a C^r Nash manifold and a C^r Nash map between C^ω Nash manifolds by C^ω Nash ones. If the manifolds are compact, the problem is easy (see [**7** and **8**]). In fact a C^r map between C^ω compact Nash manifolds is approximated by a C^ω Nash map in the C^r topology, and for a compact C^r Nash manifold $M \subset \mathbf{R}^n$ $(r \geqslant 1)$ there exists a C^r Nash imbedding τ of M in $\mathbf{R}^n$ arbitrarily close to the identity in the C^r topology such that $\tau(M)$ is a C^ω Nash manifold.

When we consider the noncompact case, the compact-open or uniform C^r topology on $N^r(M)$ is too weak to apply approximation theorems (indeed, for example, polynomial approximation works only in the compact-open C^r topology and is not useful to investigate noncompact manifolds). So we use a stronger topology defined as follows. Let $f_k \in N^r(M)$, $k = 1,2,\ldots$. We define $f_k \to 0$ when $v_1 \cdots v_{r'} f$ uniformly converges to 0 for any C^r Nash vector fields $v_1,\ldots,v_{r'}$ with $\infty > r' \leqslant r$. When $M = \mathbf{R}^n$ and $r = \infty$ this coincides with the usual topology on $\mathcal{S}$ (the space of rapidly decreasing C^∞ functions [**3**]). Namely $f_k \to 0$ if and only if $x^\alpha D^\beta f_k(x)$ uniformly converges to 0 for any multi-indices α and β. We remark that $N^r(M)$ in this topology is not a linear topological space since af does not converge

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to 0 as $a \in \mathbf{R} \to 0$ unless the support of f is compact. In this paper we always treat this topology and we call it simply the C^r topology.

THEOREM 1. *Let M_1 and M_2 be C^ω Nash manifolds and $f: M_1 \to M_2$ a C^r Nash map. Then f can be approximated by a C^ω Nash map in the C^r topology. Moreover assume the restriction of f to a given compact C^ω Nash submanifold M_3 of M_1 to be of class C^ω. Then we can approximate f fixing on M_3.*

COROLLARY 2. *Let U_1 and U_2 be open semialgebraic sets in $\mathbf{R}^n$ with $\overline{U}_1 \subset U_2$ and let f be a C^ω Nash function on U_2. Then there exists a C^ω Nash function g on $\mathbf{R}^n$ such that $g|_{U_1}$ is an approximation of $f|_{U_1}$ in the C^ω topology.*

Efroymson [2] stated Corollary 2 in the C^0 topology and the author is greatly inspired by that paper. To prove Theorem 1 we use a partition of unity by C^r Nash functions, whose existence is shown in §2 and it clarifies a difference between Nash functions and polynomials or rational functions.

THEOREM 3. *Let $M \subset \mathbf{R}^n$ be a C^r Nash manifold with $1 \leqslant r < \infty$. Then there exists a C^r Nash imbedding τ of M in $\mathbf{R}^n$ arbitrarily close to the identity in the C^r topology such that $\tau(M)$ is a C^ω Nash manifold. Moreover for any compact C^ω Nash manifold M_1 contained in M we can choose τ so that $\tau|_{M_1} =$* ident. *Another additional property is: Given a C^r Nash manifold M_2 contained in M but not necessarily closed in M, we can require that $\tau(M_2)$ is of class C^ω Nash.*

If $r = 0$, then $\tau(M)$ becomes a PL manifold [12] and the interior of a compact PL manifold possibly with boundary is of class C^0 Nash.

Let N_1, N_2 and N_3 be the C^r Nash diffeomorphism classes of all compact C^r Nash manifolds possibly with boundary, the C^r diffeomorphism classes of the same ones and the C^r Nash diffeomorphism classes of all C^r Nash manifolds respectively. Then we have natural maps $i_1: N_1 \to N_2$ and $i_2: N_1 \to N_3$ defined by $i_2(M) = M - \partial M$.

COROLLARY 4. *i_1 and i_2 are bijective.*

This in the case of $r = \omega$ or $r = 0$ is proved in [9, 12], respectively, from which, along with Theorem 3, the case $0 < r < \omega$ follows immediately.

§3 proves Theorem 1 and Corollary 2, and applying Theorem 1 we study C^r Nash manifold structure in §4. In §4 we also show the unique existence of C^r Nash vector bundle structure on a C^0 vector bundle over a C^r Nash manifold, which we see without any trouble in the case of compact base space.

2. Partition of unity. Let $X \subset \mathbf{R}^n$ be an algebraic set. We shall construct a C^ω Nash function on $\mathbf{R}^n$ which is an approximation of 0 outside a small semialgebraic neighborhood of X and of 1 in another one. The function is required to hold a useful well-known property of a C^∞ partition of unity (Proposition 2.5). Let $f \in N^r(\mathbf{R}^n)$ and $e(x) = 1/(C + |x|^{2k})$, where C is a positive number, k is a positive integer and $|x|^2 = x_1^2 + \cdots + x_n^2$. We write e as $e_{C,k}$ when we need to emphasize C and k. Let $U \subset \mathbf{R}^n$ be an open semialgebraic neighborhood of $f^{-1}(0)$. Put

$$V_1 = \{x \notin U: f(x) > 0\}, \qquad V_2 = \{x \notin U: f(x) < 0\}.$$

LEMMA 2.1.

$$F = \left(\left(f^2 + e\right)^{1/2} + f\right)/2 \to \begin{cases} 0 & \text{on } V_2, \\ f & \text{on } V_1, \end{cases}$$

in the C^r topology as C and $k \to \infty$ satisfying $k^{2k} \leqslant C$.

PROOF. We can assume $r < \infty$. As the problem is to prove $(f^2 + e)^{1/2} \to |f|$ on $V_1 \cup V_2$, it is sufficient to consider the convergence on V_1. We prove it by induction on r.

Case $r = 0$. Let ε be a Nash function of the same form as the above e. Put

$$\psi(t) = \inf\left\{f(x): |x| = t \text{ and } x \in V_1\right\}.$$

Then by the Tarski-Seidenberg Theorem, ψ is a positive upper semicontinuous semialgebraic function on the closed semialgebraic set $W = \{|x|: x \in V_1\}$ (here a semialgebraic function means that the graph is semialgebraic). Hence it follows from Łojasiewicz' inequality **[5]** and the stereographic projection that there exist C_1, $k_1 > 1$ such that for any $C \geqslant C_1$ and $k \geqslant k_1$

$$\varepsilon(t)\psi(t) \geqslant 1/(C + t^{2k}) \quad \text{for } t \in W,$$

where $\varepsilon(t)$ is defined so that $\varepsilon(|x|) = \varepsilon(x)$, in other words

$$\varepsilon(x)f(x) \geqslant e_{C,k}(x) \quad \text{for } x \in V_1.$$

Hence we have

$$0 < \left(f^2 + e\right)^{1/2} - f = e/\left(\left(f^2 + e\right)^{1/2} + f\right) < e/2f \leqslant \varepsilon/2 \quad \text{on } V_1,$$

which proves Case $r = 0$.

Assume $(f^2 + e)^{1/2} \to f$ in the C^{r-1} topology, to be precise, for any ε as above there exist C_2, $k_2 \geqslant 1$ such that for any $C \geqslant C_2$, $k \geqslant k_2$ with $k^{2k} \leqslant C$ and a multi-index α with $|\alpha| \leqslant r - 1$

$$\left|D^\alpha\left(f^2 + e_{C,k}\right)^{1/2} - D^\alpha f\right| \leqslant \varepsilon \quad \text{on } V_1.$$

We need to prove this inequality for all α with $|\alpha| = r$. Let α be such a one. Obviously

$$\begin{aligned} D^\alpha e &= D^\alpha\left\{\left(\left(f^2 + e\right)^{1/2} - f\right)\left(\left(f^2 + e\right)^{1/2} + f\right)\right\} \\ &= \left\{D^\alpha\left(\left(f^2 + e\right)^{1/2} - f\right)\right\}\left\{\left(f^2 + e\right)^{1/2} + f\right\} \\ &\quad + \sum_{\substack{\beta+\gamma=\alpha \\ \gamma \neq 0}} \left\{D^\beta\left(\left(f^2 + e\right)^{1/2} - f\right)\right\}\left\{D^\gamma\left(\left(f^2 + e\right)^{1/2} + f\right)\right\} \end{aligned}$$

and we have constants $d_0, \ldots, d_{r-1}$ which depend on r but not on C nor k such that

$$|D^\alpha e(x)| \leqslant \sum_{0 \leqslant i < r} d_i k^r |x|^{(r-i)(2k-1)-i} / \left(C + |x|^{2k}\right)^{r-i+1}.$$

Easy calculations show

$$k^r |x|^{(r-i)(2k-1)-i} / \left(C + |x|^{2k}\right)^{r-i} \leqslant 1$$

by $k \leqslant k^{2k} \leqslant C$. Hence we have

$$|D^\alpha e| \leqslant \sum_{i=0}^{r-1} d_i e,$$

which, together with the induction hypothesis, implies

$$\left|D^\alpha\left((f^2+e)^{1/2}-f\right)\right|$$
$$\leqslant \left\{de + \varepsilon_1 \sum_{0<\gamma\leqslant\alpha} \left|D^\alpha\left((f^2+e)^{1/2}+f\right)\right|\right\} \Big/ \left((f^2+e)^{1/2}+f\right) \quad \text{on } V_1$$

for any ε_1 as ε and sufficiently small e with $k^{2k} \leqslant C$, where $d = \sum_{i=0}^{r-1} d_i$. Therefore by the same argument as the case $r = 0$, choosing small ε_1 we obtain C_3, $k_3 \geqslant 1$ such that for any $C \geqslant C_3$ and $k \geqslant k_3$ with $k^{2k} \leqslant C$

$$\left|D^\alpha\left((f^2+e)^{1/2}-f\right)\right| \leqslant \varepsilon \quad \text{on } V_1,$$

which proves the lemma.

DEFINITION. We call the argument for $r = 0$ in the above proof *Argument* 2.1.

Let $\infty > r' \leqslant r$ and let φ be a polynomial on $\mathbf{R}$ such that $\varphi(0) = \cdots = \varphi^{(r')}(0) = 0$. Then

$$\varphi\{(|f|+f)/2\}$$

is a $C^{r'}$ Nash function r'-flat at $f^{-1}(0)$ (i.e. $D^\alpha\varphi\{(|f|+f)/2\} \equiv 0$ on $f^{-1}(0)$ when $|\alpha| \leqslant r'$).

LEMMA 2.2. *$\varphi(F) \to \varphi\{(|f|+f)/2\}$ in the $C^{r'}$ topology as C and $k \to \infty$ satisfying $k^{2k} \leqslant C$, where F is defined in Lemma* 2.1.

PROOF. Put $f_1 = (|f|+f)/2$. Let α be a multi-index with $|\alpha| \leqslant r'$. Obviously if $|\alpha| > 0$

$$D^\alpha\varphi(F) = \varphi'(F)D^\alpha F + \varphi''(F) \sum_{\substack{\beta+\gamma=\alpha\\ \beta,\gamma\neq 0}} D^\beta F D^\gamma F + \cdots,$$

$$D^\alpha\varphi(f_1) = \begin{cases} 0 & \text{on } f^{-1}(0),\\ \varphi'(f_1)D^\alpha f_1 + \varphi''(f_1) \displaystyle\sum_{\substack{\beta+\gamma=\alpha\\ \beta,\gamma\neq 0}} D^\beta f_1 D^\gamma f_1 + \cdots & \text{outside } f^{-1}(0). \end{cases}$$

Hence, for any open semialgebraic neighborhood U of $f^{-1}(0)$, the convergence $D^\alpha\varphi(F) \to D^\alpha\varphi(f_1)$ on $\mathbf{R}^n - U$ in the C^0 topology follows from Lemma 2.1. So it suffices to prove the following.

Let $\alpha_1, \ldots, \alpha_l > 0$ be multi-indices with $|\alpha_1| + \cdots + |\alpha_l| = r'' \leqslant r'$, and let ε be a Nash function of the same form as e. Then there exist C_1, $k_1 \geqslant 1$ and an open semialgebraic neighborhood U of $f^{-1}(0)$ such that for any $C \geqslant C_1$ and $k \geqslant k_1$ with $k^{2k} < C$

$$\left|\varphi^{(l)}(F)D^{\alpha_1}F \cdots D^{\alpha_l}F\right| < \varepsilon \quad \text{on } U,$$
$$\left|\varphi^{(l)}(f_1)D^{\alpha_1}f_1 \cdots D^{\alpha_l}f_1\right| < \varepsilon \quad \text{on } U - f^{-1}(0).$$

As the existence of U which satisfies the second inequality is trivial, we consider only the first one. By assumption, $\varphi^{(l)}(F) = F^{r''-l+1}\psi(F)$ for some polynomial ψ. Hence, by Argument 2.1 it suffices to prove

$$|F^{r''-l+1}D^{\alpha_1}F \cdots D^{\alpha_l}F| < \varepsilon \quad \text{on } U,$$

which is equivalent to

$$\left|f_2^{r''-l+1}D^{\alpha_1}(f_2+f)\cdots D^{\alpha_l}(f_2+f)\right| < \varepsilon \quad \text{on } U,$$

where $f_2 = (f^2+e)^{1/2}$, because of $f_2/2 < |F| < f_2$. Moreover, by induction on r'' (see below) and Argument 2.1, this inequality is reduced to

$$\left|f_2^{r''-l+1}D^{\alpha_1}f_2 \cdots D^{\alpha_l}f_2\right| < \varepsilon \quad \text{on } U.$$

Case $r''=0$. Put

$$U = \left\{x \in \mathbf{R}^n\colon |f(x)| < \varepsilon(x)/2\right\},$$
$$U' = \left\{(x,t) \in U \times \mathbf{R}\colon \left(f^2(x)+t^2\right)^{1/2} < \varepsilon(x)\right\}.$$

Then U' is an open semialgebraic set containing $f^{-1}(0)\times 0$. Hence, by Argument 2.1, for arbitrarily small e, U' contains the graph of $e(x)^{1/2}$ on U, consequently

$$\left(f^2(x)+e(x)\right)^{1/2} < \varepsilon(x) \quad \text{on } U.$$

Thus Case $r''=0$ is proved.

Case $r''>0$. By Case $r''=0$ and the equality

$$f_2^{r''-l+1}D^{\alpha_1}f_2 \cdots D^{\alpha_l}f_2 = f_2\prod_{i=1}^{l} f_2^{|\alpha_i|-1}D^{\alpha_i}f_2$$

it is enough to prove globally

$$\left|f_2^{|\alpha_i|-1}D^{\alpha_i}f_2\right| \leqslant C_2 + |x|^{2k_2}$$

for each i, some C_2, k_2 and arbitrarily small e with $k^{2k} \leqslant C$. Now

$$|D^{\alpha_i}f_2| \leqslant C \sum_{\substack{\beta_1+\cdots+\beta_k=\alpha_i\\ \beta_j>0}} \left|(f^2+e)^{1/2-k}D^{\beta_1}(f^2+e)\cdots D^{\beta_k}(f^2+e)\right|$$

for some constant $C>0$. Hence we only need

$$\left|(f^2+e)^{|\beta_j|/2-1}D^{\beta_j}(f^2+e)\right| < C_2 + |x|^{2k_2}.$$

But this is trivial if $|\beta_j|>1$, and if $|\beta_j|=1$ this follows from the inequality $|D^{\beta_j}e| \leqslant d_0 e$ in the proof of Lemma 2.1. Hence the proof is completed.

Given C_1, k_1, C_2 and k_2 put $e_1 = e_{C_1,k_1}$ and $e_2 = e_{C_2,k_2}$. Let $\infty > r' \leqslant r$ and let φ be a polynomial on $\mathbf{R}$ such that $\varphi(0)=0$, $\varphi(1)=1$ and $\varphi' = \cdots = \varphi^{(r')} = 0$ at 0 and 1 if $r' \geqslant 1$. Put

$$F_1 = \left(|3-f-|f-1||+3-f-|f-1|\right)/4,$$

$$F_2 = \left(\left\{\left(3-f-\left\{(f-1)^2+e_1\right\}^{1/2}\right)^2+e_2\right\}^{1/2}+3-f-\left\{(f-1)^2+e_1\right\}^{1/2}\right)/4.$$

Then $\varphi(F_1)$ is a $C^{r'}$ Nash function such that $\varphi=0$ on $\{f \geqslant 2\}$ and $=1$ on $\{f \leqslant 1\}$.

LEMMA 2.3. *Choosing small e_1 and e_2 we can approximate $\varphi(F_1)$ by $\varphi(F_2)$ in the $C^{r'}$ topology.*

PROOF. At present fix e_1. Put

$$F_3 = \left\{\left|3 - f - \left((f-1)^2 + e_1\right)^{1/2}\right| + 3 - f - \left((f-1)^2 + e_1\right)^{1/2}\right\}/4.$$

Then $\varphi(F_3)$ is a $C^{r'}$ Nash function and Lemma 2.2 tells us $\varphi(F_2) \to \varphi(F_2)$ in the $C^{r'}$ topology as C_2 and $k_2 \to \infty$ satisfying $k_2^{2k_2} \leqslant C_2$. Hence it suffices to prove $\varphi(F_3) \to \varphi(F_1)$ in the $C^{r'}$ topology as C_1 and $k_1 \to \infty$ satisfying $k_1^{2k_1} \leqslant C_1$.

The case $r' = 0$ follows from Lemma 2.2. So assume $r' > 0$. We want an inequality

$$|D^\alpha\varphi(F_3) - D^\alpha\varphi(F_1)| \leqslant \varepsilon \tag{1}$$

for small e_1, where α is a multi-index with $0 < |\alpha| \leqslant r'$ and ε is a given function of the same form as e. If $|\alpha| > 0$ we have, like in the proof of Lemma 2.3,

$$D^\alpha\varphi(F_3) = \begin{cases} 0 \quad \text{on } Y_3, \\ \varphi'(F_3)D^\alpha F_3 + \varphi''(F_3) \sum\limits_{\substack{\beta+\gamma=\alpha \\ \beta,\gamma\neq 0}} D^\beta F_3 D^\gamma F_3 + \cdots \quad \text{outside } Y_3, \end{cases}$$

$$D^\alpha\varphi(F_1) = \begin{cases} 0 \quad \text{on } Y_1 \cup Y_2, \\ \varphi'(F_1)D^\alpha F_1 + \varphi''(F_1) \sum\limits_{\substack{\beta+\gamma=\alpha \\ \beta,\gamma\neq 0}} D^\beta F_1 D^\gamma F_1 + \cdots \quad \text{outside } Y_1 \cup Y_2, \end{cases}$$

where $Y_1 = \{f = 1\}$, $Y_2 = \{f = 2\}$, and $Y_3 = \{3 = f + ((f-1)^2 + e_1)^{1/2}\} = \{f = 2 - e_1/4\}$. Now Argument 2.1 says that for sufficiently small e_1, Y_3 is contained in a given semialgebraic neighborhood of Y_2. Hence for (1) we only need to find open semialgebraic neighborhoods U_1 and U_2 of Y_1 and Y_2, respectively, and C_{10}, k_{10} such that for each $\alpha_1, \ldots, \alpha_l > 0$ with $|\alpha_1| + \cdots + |\alpha_l| \leqslant r'$ and any $C_1 > C_{10}$ and $k_1 > k_{10}$ with $k_1^{2k_1} \leqslant C_1$,

$$|\varphi^{(l)}(F_3)D^{\alpha_1}F_3 \cdots D^{\alpha_l}F_3| < \varepsilon \quad \text{on } U_1 \cup U_2 - Y_3, \tag{2}$$

$$|\varphi^{(l)}(F_1)D^{\alpha_1}F_1 \cdots D^{\alpha_l}F_1| < \varepsilon \quad \text{on } U_1 \cup U_2 - Y_1 \cup Y_2, \tag{3}$$

$$|\varphi^{(l)}(F_3)D^{\alpha_1}F_3 \cdots D^{\alpha_l}F_3 - \varphi^{(l)}(F_1)D^{\alpha_1}F_1 \cdots D^{\alpha_l}F_1| < \varepsilon \quad \text{on } \mathbf{R}^n - U_1 \cup U_2. \tag{4}$$

Here we can replace F_1 and F_3 by

$$F_{10} = (3 - f - |f-1|)/2 \quad \text{and} \quad F_{30} = \left(3 - f - \left((f-1)^2 + e_1\right)^{1/2}\right)/2,$$

respectively, because

$$F_1(x) = \begin{cases} F_{10}(x) & \text{if } F_{10}(x) > 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$F_3(x) = \begin{cases} F_{30}(x) & \text{if } F_{30}(x) > 0, \\ 0 & \text{otherwise}, \end{cases}$$

and because $F_{10}(x) > 0$ if and only if $F_{30}(x) > 0$ for $x \in \mathbf{R}^n - U_1 \cup U_2$ and small e_1. Let $(2)_0$, $(3)_0$ and $(4)_0$ denote the respective replaced inequalities.

At first $(3)_0$ is trivial for some small U_1 and U_2 because

$$\varphi^{(l)}(t) = t^{r'-l+1}(t-1)^{r'-l+1}\psi(t)$$

for some polynomial ψ. Next Lemma 2.1 implies $F_{30} \to F_{10}$ on $\mathbf{R}^n - U_1$ in the $C^{r'}$ topology as C_1 and $k_1 \to \infty$ satisfying $k_1^{2k_1} \leqslant C_1$, which, together with $(3)_0$, proves $(2)_0$ on U_2 and $(4)_0$. $(2)_0$ on U_1 is reduced, in the same way as the proof of Lemma 2.2, to

$$\left|f^{r'-l+1}D^{\alpha_1}f_1 \cdots D^{\alpha_l}f_1\right| < \varepsilon \quad \text{on } U_1,$$

where $f_1 = ((f-1)^2 + e_1)^{1/2}$. But this is the same as an inequality desired in the proof of Lemma 2.2. Hence the lemma is proved.

Let $X \subset \mathbf{R}^n$ be an algebraic set, I the ideal of $\mathbf{R}[x_1, \ldots, x_n]$ defined by X, namely, consisting of polynomials vanishing on X, and h the square sum of finite generators of I. Put $f = h^{r'}/e_3$, where $C_3, k_3 > 1$ and $e_3 = e_{C_3,k_3}$. Let e_1, e_2, r', φ, F_1 and F_2 be the same as stated just before Lemma 2.3.

PROPOSITION 2.4. *$\varphi(F_1)$ and $\varphi(F_2)$ are $C^{r'}$ and C^ω Nash functions respectively. Let U be a semialgebraic neighborhood of X. Then, for small e_3, $\varphi(F_1) = 0$ outside U and $= 1$ in another neighborhood. Fix e_3. Then $\varphi(F_2)$ is an approximation of $\varphi(F_1)$ in the $C^{r'}$ topology for small e_1 and e_1.*

PROOF. The first statement is clear by definition; the second follows if we can choose e_3 so that $U \supset \{h < 2e_3\}$, but this is possible by Argument 2.1; and the last is Lemma 2.3.

PROPOSITION 2.5. *Let $Y \subset X$ be a connected component of $X - \operatorname{Sing} X$ (= the set of singular points of X) and let V be a semialgebraic neighborhood of $X - Y$ in $\mathbf{R}^n$. Let g be a C^r Nash function on $\mathbf{R}^n$ r'-flat on Y. Then $g\varphi(F_1) \to 0$ on $\mathbf{R}^n - V$ in the $C^{r'}$ topology as $C_3, k_3 \to \infty$ satisfying $k_3^{2k_3} \leqslant C_3$.*

PROOF. Let $\varepsilon \in N^\omega(\mathbf{R}^n)$ be of the same form as e and α a multi-index with $|\alpha| \leqslant r'$. Then we have to see

$$|D^\alpha(g\varphi(F_1))| \leqslant \varepsilon \quad \text{on } \mathbf{R}^n - V \tag{1}$$

for large C_3 and k_3 with $k_3^{2k_3} \leqslant C_3$. We will reduce (1) to plainer inequalities. As $\varphi(F_1) = 0$ outside $W = \{f \leqslant 2\}$, it suffices to consider (1) on $W - V$. Moreover we can replace (1) by

$$|D^\alpha(g\varphi(2-f))| \leqslant \varepsilon \quad \text{on } W - V \tag{2}$$

because of

$$|D^\alpha(g\varphi(2-f))| \geqslant |D^\alpha(g\varphi(F_1))| \quad \text{globally.}$$

Now we have

$$D^{\alpha}(g\varphi(2-f)) = \sum_{\beta+\gamma=\alpha} D^{\beta}gD^{\gamma}\varphi(2-f),$$

$$D^{\gamma}\varphi(2-f) = \varphi'(2-f)D^{\gamma}(2-f) + \varphi''(2-f) \times \sum_{\substack{\delta+\zeta=\gamma \\ \delta,\zeta\neq 0}} D^{\delta}(2-f)D^{\zeta}(2-f) + \cdots$$

if $|\gamma| > 0$. Hence (2) follows from

$$(3) \qquad |D^{\beta}gD^{\alpha_1}f \cdots D^{\alpha_l}f| \leqslant \varepsilon \quad \text{on } W - V,$$

where $|\beta| + |\alpha_1| + \cdots + |\alpha_l| \leqslant r'$, because $\varphi'(2-f),\ldots,\varphi^{(r')}(2-f)$ are bounded on W. Furthermore (3) can be reduced to

$$(4) \qquad |D^{\beta}gD^{\alpha_1}H \cdots D^{\alpha_l}HD^{\gamma_1}e_3 \cdots D^{\gamma_k}e_3/e_3^{k+1}| \leqslant \varepsilon \quad \text{on } W - V$$

by an easy calculation, where $|\beta| + |\alpha_1| + \cdots + |\alpha_l| + |\gamma_1| + \cdots + |\gamma_k| \leqslant r'$ and $H = h^{r'}$. Here if $l = 0$, then $k = -1$. Recall the inequality $|D^{\gamma_i}e_3| \leqslant de$ for some constant d in the proof of Lemma 2.1, by which we only need to prove

$$(5) \qquad |D^{\beta}gD^{\alpha_1}H \cdots D^{\alpha_l}H/H| \leqslant \varepsilon \quad \text{on } W - V - Y,$$

where $|\beta| + |\alpha_1| + \cdots + |\alpha_l| \leqslant r'$ and $l \geqslant 1$, and

$$(5)' \qquad |D^{\beta}g| \leqslant \varepsilon \quad \text{on } W - V,$$

where $|\beta| \leqslant r'$. Here we used the inequality $H \leqslant 2e_3$ on W and the hypothesis $D^{\beta}g = 0$ on Y.

Consider the sets

$$Z = \{x \in \mathbf{R}^n : |D^{\beta}gD^{\gamma_1}H \cdots D^{\gamma_l}H(x)| \leqslant \varepsilon(x)H(x)\},$$

$$Z' = \{x \in \mathbf{R}^n : |D^{\beta}g(x)| \leqslant \varepsilon(x)\}.$$

Then they are semialgebraic and contain Y. Hence Argument 2.1 tells us that it suffices to prove Z and Z' are neighborhoods of Y. That is trivial for Z'. For Z, let x_0 be a point of Y and consider a small neighborhood of x_0. We can obtain a C^{∞} local coordinate system $(y, z) = (y_1,\ldots,y_m, z_{m+1},\ldots,z_n)$ around x_0 such that $(y, z) = 0$ at x_0 and $h(y, z) = y_1^2 + \cdots + y_m^2$ and $Y = \{y_1 = \cdots = y_m = 0\}$ (see the proof of Lemma 4.11 in **[10]**). By hypothesis we have

$$|D_x^{\beta}g(y, z)| \leqslant d'|y|^{r'+1-|\beta|}$$

in a neighborhood of 0 for some constant d'. Hence it follows

$$|D_x^{\beta}gD_x^{\alpha_1}H \cdots D_x^{\alpha_l}H(y, z)| \leqslant d''|y|^{r'+1-|\beta|+2r'-|\alpha_1|+\cdots+2r'-|\alpha_l|} \leqslant d''|y|^{2r'+1}$$

in a neighborhood of 0 for some constant d''. Thus Z contains a neighborhod of x_0, which completes the proof.

3. The approximation theorem for Nash mappings. Assume $r < \infty$ in this section. In **[2]** Efroymson stated Corollary 2 in the C^0 topology, and he gave a key lemma to it. For the proof of Theorem 1 we shall need the lemma in a more general form (Lemma 3.1). But, unfortunately, **[2]** has several mistakes in the proofs. So we shall give a complete proof.

Let $h \in N^\omega(\mathbf{R}^n)$, $X = h^{-1}(0)$, $U \subset X$ a connected C^ω Nash manifold open in X and $g \in N^\omega(U)$. A *minimal polynomial* $P(z, x)$ for g means a polynomial in $n+1$ variables such that $P(z, x)|_{\mathbf{R}\times U} \not\equiv 0$, $P(g(x), x) \equiv 0$ on U and the degree in z is minimal. We say the pair (g, P) has *Property* (A1) if $P^{-1}(0) \cap (\partial P^{-1}/\partial z)(0) \cap \mathbf{R} \times U = \varnothing$, P is of constant degree in z at every point of U and $\{P^{-1}(0) \cup (\partial P^{-1}/\partial z)(0)\} \cap \mathbf{R} \times U$ is the disjoint union of the graphs of C^ω Nash functions on U. Moreover, by induction, if the pair of each C^ω Nash function on U whose graph is contained in $(\partial P^{-1}/\partial z)(0)$ and some minimal polynomial for it has Property (A $k-1$), then we say (g, P) has *Property* (Ak) for $k > 1$. Write simply Property (Am) as *Property* (A) for m = degree in z of P. Let (g, P) have Property (A). Then we say (g, P) is of *height* 0 if $(\partial P^{-1}/\partial z)(0) \cap \mathbf{R} \times U = \varnothing$ and, inductively, (g, P) is of *height* l if it is not of height $l-1$ and the pair of each C^ω Nash function on U defined by $(\partial P^{-1}/\partial z)(0)$ and some minimal polynomial for it is of height $\leqslant l-1$. It is clear that if (g, P) has Property (A1) we can extend g uniquely to some semialgebraic neighborhood of U in $\mathbf{R}^n$ satisfying $P(g(x), x) \equiv 0$. We write the extension as $\tilde{g}_P$. We will not specify the domain of $\tilde{g}_P$.

LEMMA 3.1. *Let $D \subset \mathbf{R}^n$ be a closed semialgebraic set contained in U. Assume the pair of $g \in N^\omega(U)$ and a polynomial P has Property* (A). *Then there exists a closed semialgebraic neighborhood $\tilde{D}$ of D in $\mathbf{R}^n$ such that $\tilde{g}_P$ is defined on $\tilde{D}$ and $\tilde{g}_P|_{\tilde{D}}$ can be approximated in the C^ω topology by the restriction to $\tilde{D}$ of a C^ω Nash function on $\mathbf{R}^n$.*

PROOF. At first we can assume D is connected for the following reason. By Theorem 1 in [9] U is C^ω Nash diffeomorphic to the interior of a compact C^ω Nash manifold possibly with boundary. Hence consider in place of D the compact manifold—an open collar. Next applying the Mostowski separation theorem [6] to D and $X - U$, we have $h_1 \in N^\omega(\mathbf{R}^n)$ such that $h_1 > 0$ on D and $h_1 < 0$ on $X - U$. Put $D_1 = \{x \in X: h_1(x) \geqslant 0\}$ and let D_2 be the connected component of D_1 containing D. Once more by the separation theorem there exists $h_2 \in N^\omega(\mathbf{R}^n)$ such that $h_2 > 0$ on D_2 and $h_2 < 0$ on $D_1 - D_2$. Hence

$$D_2 = \{x \in X: h_1(x) \geqslant 0 \text{ and } h_2(x) \geqslant 0\}.$$

For C and $k > 1$, put

$$D_{C,k} = \{x \in \mathbf{R}^n: h_0(x) = e_{C,k}(x) - h^2(x) \geqslant 0, h_1(x) \geqslant 0, h_2(x) \geqslant 0\}.$$

We shall define $\tilde{D}$ so that it will be contained in the interior of this $D_{C,k}$ for some large C and k. Here we remark every semialgebraic neighborhood of D_2 contains $D_{C,k}$ for some C and k by Argument 2.1.

The present subject is to prove the lemma in the C^0 topology by induction on the height of (g, P). Let $P(z, x) = a_m z^m + \cdots + a_0$, $a_m \neq 0$. Then the minimality and the constancy of degree of P show $a_m > 0$ on U or $a_m < 0$ on U, hence assume $a_m > 0$ on U.

Case height $= 0$. By definition and by assumption, $\partial P/\partial z$ and a_m are positive on $\mathbf{R} \times U$ and $D_{C,k}$ for some C, k respectively. Hence we can choose $D_{C,k}$ so that $\partial P/\partial z > 0$ on $\mathbf{R} \times D_{C,k}$ and, consequently, $\tilde{g}_P$ is defined on $D_{C,k}$. Fix such C and k. By Argument 2.1 we have polynomials $\varphi_1(x)$ and $\varphi_2(x)$ on $\mathbf{R}^n$ such that $\varphi_2 > \tilde{g}_P > \varphi_1$

on $D_{C,k}$ and $\varphi_2 > \varphi_1$ on $\mathbf{R}^n$. Clearly $P(\varphi_1(x), x) < 0$ and $P(\varphi_1(x), x) > 0$ on $D_{C,k}$. Put

$$H_{C',k'} = \sum_{i=0}^{2} \left(\left(h_i^2 + e_{C',k'} \right)^{1/2} - h_i \right),$$

$$P_{C',k'}(z, x) = P(z, x) + (u_2 + u_1)(z - \varphi_1) H_{C',k'}/(\varphi_2 - \varphi_1) - u_1 H_{C',k'}$$

for polynomials u_1 and u_2 on $\mathbf{R}^n$. Then for some large u_1, u_2 and for any C', k' we have

$$P_{C',k'}(\varphi_1(x), x) = P(\varphi_1(x), x) - u_1 H_{C',k'} < 0 \quad \text{on } \mathbf{R}^n,$$
$$P_{C',k'}(\varphi_2(x), x) = P(\varphi_2(x), x) + u_2 H_{C',k'} > 0 \quad \text{on } \mathbf{R}^n,$$
$$\frac{\partial P_{C',k'}}{\partial z}(z, x) = \frac{\partial P}{\partial z}(z, x) + \frac{(u_2 + u_1) H_{C',k'}}{\varphi_2 - \varphi_1} > 0 \quad \text{for } \varphi_1(x) \leqslant z \leqslant \varphi_2(x).$$

Fix such u_1 and u_2. Then $P_{C',k'}(z, x) = 0$ has the unique solution $g_{C',k'}(x)$ on $\mathbf{R}^n$ such that $\varphi_2 > g_{C',k'} > \varphi_1$. Trivially $g_{C',k'}$ is of class C^ω Nash; and Proposition 3 in [**2**] whose proof is easy and complete implies $g_{C',k'}|_{D_{C,k}} \to \tilde{g}_P|_{D_{C,k}}$ in the C^0 topology as C', $k' \to \infty$, because the coefficients of the z-polynomial $P - P_{C',k'}$ converge to 0 on $D_{C,k}$ in the C^0 topology as C', $k' \to \infty$ by Lemma 2.2. Thus we have proved the case height $= 0$.

By induction assume height$(g, P) = l > 0$ and the lemma in the C^0 topology in the case height $\leqslant l - 1$. We define $\varphi_1 \in N^\omega(\mathbf{R}^n)$ like in the case height $= 0$ as follows. If every root of $(\partial P/\partial z)(z, x)$ on U is larger than $g(x)$, then let φ_1 be a polynomial on $\mathbf{R}^n$ such that $g > \varphi_1$ on D_2. In the other case let $g' \in N^\omega(U)$ be the largest root of $(\partial P/\partial z)(z, x)$ with $g > g'$ on U. Then, by definitions of Property (A) and height we have a minimal polynomial P' for g' such that (g', P') has Property (A) and is of height $< l$. Hence, by the induction hypothesis there exist $\varphi_1' \in N^\omega(\mathbf{R}^n)$ such that $\varphi_1'|_{D_2}$ is an approximation of $g'|_{D_2}$ in the C^0 topology. Choose a positive C^ω Nash function φ_1'' on $\mathbf{R}^n$ so small that $2\varphi_1'' < g - g'$ on D_2, and take the above approximation closely enough. Then $\varphi_1 = \varphi_1' + \varphi_1''$ satisfies

$$(1) \quad \varphi_1 < g \quad \text{on } D_2, \qquad \frac{\partial P}{\partial z}(z, x) > 0 \quad \text{for } \varphi_1(x) \leqslant z \leqslant g(x) \text{ and } x \in D_2.$$

We remark (1) holds true in the previous case too. We also define $\varphi_2 \in N^\omega(\mathbf{R}^n)$ so that

$$(2) \quad \varphi_2 > g \quad \text{on } D_2, \qquad \frac{\partial P}{\partial z}(z, x) > 0 \quad \text{for } g(x) \leqslant z \leqslant \varphi_2(x) \text{ and } x \in D_2.$$

If necessary, adding to φ_2 a C^ω Nash function of the form $uH_{C',k'}$ for large $u \in N^\omega(\mathbf{R}^n)$, C' and k', we can assume, moreover, $\varphi_1 < \varphi_2$ on $\mathbf{R}^n$. Hence gathering (1) and (2) we have

$$(3) \quad \begin{gathered} \varphi_1 < \varphi_2 \quad \text{on } \mathbf{R}^n, \qquad \varphi_1 < g < \varphi_2 \quad \text{on } D_2, \\ \frac{\partial P}{\partial z}(z, x) > 0 \quad \text{for } \varphi_1(x) \leqslant z \leqslant \varphi_2(x) \text{ and } x \in D_2. \end{gathered}$$

As the second and last inequalities in (3) hold in a semialgebraic neighborhood of D_2 in $\mathbf{R}^n$, (3) implies for arbitrarily large C and k

$$(3)' \qquad \begin{gathered} \varphi_1 < \varphi_2 \quad \text{on } \mathbf{R}^n, \qquad \varphi_1 < \tilde{g}_P < \varphi_2 \quad \text{on } D_{C,k}, \\ \frac{\partial P}{\partial z}(z, x) > 0 \quad \text{for } \varphi_1(x) \leqslant z \leqslant \varphi_2(x) \text{ and } x \in D_{C,k}. \end{gathered}$$

Hence the case height $= l$ follows in exactly the same way as the case height $= 0$. Thus, in any case, $\tilde{g}_P|_{D_{C,k}}$ is approximated by $\tilde{g}_{C',k'}|_{D_{C,k}}$ in the C^0 topology where $g_{C',k'}$ is a root of $P_{C',k'}$ which is of the form

$$P_{C',k'} = P + a_1' H_{C',k'} z + a_0' H_{C',k'},$$

where $a_0', a_1' \in N^\omega(\mathbf{R}^n)$.

Let $0 < r$ and $\tilde{D}$ be a closed semialgebraic neighborhood of D in $\mathbf{R}^n$ contained in the interior of $D_{C,k}$. What is left is to show $g_{C',k'}|_{\tilde{D}} \to \tilde{g}_P|_{\tilde{D}}$ in the C^r topology as C' and $k' \to \infty$ satisfying $k'^{2k'} \leqslant C'$. We will work by induction on r. So assume this convergence in the C^{r-1} topology. By Lemma 2.1 we already know $H_{C',k'} \to 0$ on $\tilde{D}$ in the C^r topology as C' and $k' \to \infty$ satisfying $k'^{2k'} \leqslant C'$.

Let α be a multi-index with $|\alpha| = r$. Trivially we have

$$\begin{aligned} 0 &= D^\alpha\{P(\tilde{g}_P(x), x)\} \\ &= (D^\alpha P)(\tilde{g}_P, x) + \frac{\partial P}{\partial z}(\tilde{g}_P, x) D^\alpha \tilde{g}_P \\ &\quad + \sum_{\substack{\beta+\gamma=\alpha \\ \beta,\gamma>0}} \frac{\partial^2 P}{\partial z^2}(\tilde{g}_P, x) D^\beta \tilde{g}_P D^\gamma \tilde{g}_P + \cdots, \\ 0 &= D^\alpha\{P_{C',k'}(g_{C',k'}(x), x)\} \\ &= (D^\alpha P_{C',k'})(g_{C',k'}, x) + \frac{\partial P_{C',k'}}{\partial z}(g_{C',k'}, x) D^\alpha g_{C',k'} \\ &\quad + \sum_{\substack{\beta+\gamma=\alpha \\ \beta,\gamma>0}} \frac{\partial^2 P_{C',k'}}{\partial z^2}(g_{C',k'}, x) D^\beta g_{C',k'} D^\gamma g_{C',k'} + \cdots. \end{aligned}$$

By (3)′,

$$\frac{\partial P}{\partial z}(\tilde{g}_P(x), x) > 0 \quad \text{for } x \in \tilde{D}.$$

Hence, by Argument 2.1, $1/(\partial P_{C',k'}/\partial z)(g_{C',k'}(x), x)$ is a C^ω Nash function on an open semialgebraic neighborhood of $\tilde{D}$ for arbitrarily large C' and k', and $1/(\partial P_{C',k'}/\partial z)(g_{C',k'}(x), x) \to 1(\partial P/\partial z)(\tilde{g}_P(x), x)$ on $\tilde{D}$ in the C^0 topology as C' and $k' \to \infty$. Therefore the above equalities and the induction hypothesis imply $D^\alpha g_{C',k'} \to D^\alpha \tilde{g}_P$ on $\tilde{D}$ in the C^0 topology as C' and $k' \to \infty$ satisfying $k'^{2k'} \leqslant C'$, which completes the proof.

3.2. *Reduction of Theorem* 1 *to the case* $M_1 = \mathbf{R}$ *and* $M_2 = \mathbf{R}$. Let M_2 be contained in $\mathbf{R}^{n_2}$. By Lemma 7 in **[9]** there exists a C^ω Nash tubular neighborhood V_2 of M_2 in $\mathbf{R}^{n_2}$ (i.e. V_2 is a C^ω Nash manifold and the projection p_2: $V_2 \to M_2$ is of class C^ω Nash). If F: $M_1 \to \mathbf{R}^{n_2}$ is a C^ω Nash approximation of f in the C^r topology such that $F(M_1) \subset V_2$, then $p_2 \circ F$ is a required approximation. Hence we can assume $M_2 = \mathbf{R}^{n_2}$ and, hence, $M_2 = \mathbf{R}$.

As **[9]** pointed out, we can assume M_1 is closed in $\mathbf{R}^n$. Let p_1: $V_1 \to M_1$ be a C^ω Nash tubular neighborhood of M_1 in $\mathbf{R}^n$. By the separation theorem and the argument just before Lemma 2.3 we have $\varphi \in N^r(\mathbf{R}^n)$ such that $\varphi = 1$ on a neighborhood of M_1 and $\varphi = 0$ on a neighborhood of $\mathbf{R}^n - V_1$. Consider $(f \circ p_1)\varphi \in N^r(V_1)$. This is extensible to $\mathbf{R}^n$ and the restriction to M_1 is equal to f. Hence it suffices to approximate the extension by a C^ω Nash function on $\mathbf{R}^n$.

3.3. *There exist a stratification* $\{U_{ij}\}_{0 \leqslant i \leqslant n;\, 1 \leqslant j \leqslant m_i}$ *of* $\mathbf{R}^n$, *polynomials* h_{ik} *on* $\mathbf{R}^n$ *for all* i *and* k *with* $1 \leqslant k \leqslant i$, $f_{ij\alpha} \in N^\omega(U_{ij})$ *for all* i, j *and* $\alpha \in \mathbf{N}^i$ *with* $|\alpha| \leqslant r$, *and minimal polynomials* $P_{ij\alpha}$ *for* $f_{ij\alpha}$ *such that the following is satisfied.* (*Here* $\mathbf{N}^0$ *means* $\{0\}$.)

(3.3.1) *Each* U_{ij} *is a connected* C^ω *Nash manifold of dimension* $n - i$.

(3.3.2) *For each* i, $B_i = \bigcup_{i' \geqslant i;\, 1 \leqslant j \leqslant m_{i'}} U_{i'j}$ *is an algebraic set and each* U_{ij} *consists of nonsingular points of* B_i.

(3.3.3) *For each* i, j *and all* k, h_{ik} *vanishes on* U_{ij} *and* grad h_{ik} *span the normal vector bundle of* U_{ij} *in* $\mathbf{R}^n$.

(3.3.4) $(f_{ij\alpha}, P_{ij\alpha})$ *have Property* (A).

(3.3.5) *For each* i *and* j, $f - \sum_{|\alpha| \leqslant r} \tilde{f}_{ij\alpha P_{ij\alpha}} h_i^\alpha$ *is r-flat on* U_{ij}, *where* h_i^α *means* $\prod_{k=1}^i h_{ik}^{\alpha_k}$ *for* $\alpha = (\alpha_1, \ldots, \alpha_i)$.

PROOF OF 3.3. As (3.3.5) requires $f_{ij0} = f|_{U_{ij}}$ for any i and j, we define it so. We construct U_{ij}, h_{ik}, $f_{ij\alpha}$ and $P_{ij\alpha}$ by double induction on i and $|\alpha|$. Let $0 \leqslant s \leqslant n$. Assume we already have an algebraic set $B_s \subset \mathbf{R}^n$ of codimension s, a stratification $\{U_{ij}\}_{0 \leqslant i < s;\, 1 \leqslant j \leqslant m_i}$ of $\mathbf{R}^n - B_s$, h_{ik}, f_{ij} and $P_{ij\alpha}$ for $0 \leqslant i < s$ such that (3.3.1)–(3.3.5) are satisfied (here B_i in (3.3.2) is modified to be $\bigcup_{i' \geqslant i;\, 1 \leqslant j \leqslant m_{i'}} U_{i'j} \cup B_s$ and we put $B_0 = \mathbf{R}^n$). We shall define an algebraic set B_{s+1} ($\subset B_s$) of codimension $< n - s$ as the Zariski closure of $B_{s+1}^1 \cup B_{s+1}^2 \cup \cdots$, where B_{s+1}^t will be semialgebraic sets defined one after another; and we shall let U_{sj}, $j = 1, \ldots, m_s$, be the connected components of $B_s - B_{s+1}$ so that we shall be able to define h_{sk}, $f_{sj\alpha}$ and $P_{sj\alpha}$.

It is elementary to find polynomials h_{sk}, $1 \leqslant k \leqslant s$, so that they vanish on B_s and the set of points of B_s where grad h_{sk}, $1 \leqslant k \leqslant s$, are linearly dependent is of dimension $< n - s$. Denote the set by B_{s+1}^1. Then (3.3.3) will be satisfied for whatever U_{sj}.

Let E be the graph of f on B_s. As E is semialgebraic, it admits a semialgebraic stratification **[5]**. Hence we have a semialgebraic set E' ($\subset E$) of dimension $< n - s$ such that $E - E'$ is a C^ω Nash manifold of dimension $n - s$. Let E'' ($\subset E - E'$) be the set of points where the projection p: $E - E' \to \mathbf{R}^n$ is not C^∞ regular ($E'' = \varnothing$ if $r \geqslant 1$). Then it is easy to see by the Tarski-Seidenberg Theorem that E'' is a semialgebraic set of dimension $< n - s$, and $B_s - p(E') - p(E'')$ is a C^ω Nash manifold on which f is of class C^ω. Put $B_{s+1}^2 = p(E') \cup p(E'')$.

We will define $f_{sj\alpha}$ and $P_{sj\alpha}$ by induction on $|\alpha|$.

Case $\alpha = 0$. Let W be a connected component of $B_s - B^1_{s+1} - B^2_{s+1}$ and P_W a minimal polynomial for $f_W = f|_W$. Apply the above argument to $P_W^{-1}(0)$. Then we have a semialgebraic set $W' \subset W$ of dimension $< n - s$ such that $P_W^{-1}(0) \cap \mathbf{R} \times$ (each connected component of $W - W'$) is the disjoint union of the graphs of C^ω Nash functions. Here we remark the minimality of P_W implies

$$\dim\left(P_W^{-1}(0) \cap \frac{\partial P^{-1}}{\partial z}(0) \cap \mathbf{R} \times W \right) < n - s.$$

Hence, in the same way as above, we can choose a semialgebraic set $W'' \subset W$ of dimension $< n - s$ so that

$$P_W^{-1}(0) \cap \frac{\partial P^{-1}}{\partial z}(0) \cap \mathbf{R} \times (W - W' - W'') = \varnothing$$

and $(\partial P^{-1}/\partial z)(0) \cap \mathbf{R} \times$ (each connected component of $W - W' - W''$) is the disjoint union of the graphs of C^ω Nash functions. Considering minimal polynomials for these C^ω Nash functions and repeating this argument, we obtain a finite number of semialgebraic sets $W', W'', \ldots$ such that, for each connected component U of $W - W' - W'' - \cdots$, $(f|_U, P_W)$ has Property (A). Hence we define $P_{sj0} = P_W$ if $U_{sj} \subset W - W' - \cdots$ and we put

$$B^3_{s+1W} = W' \cup W'' \cup \cdots \quad \text{and} \quad B^3_{s+1} = \bigcup_W B^3_{s+1W}.$$

Case $|\alpha| = l > 0$. By induction assume we already have semialgebraic sets $B^3_{s+1}, \ldots, B^{l+2}_{s+1}$ of dimension $< n - s$, the connected components O_j, $j = 1, 2, \ldots,$ of $B_s - B^1_{s+1} - \cdots - B^{l+2}_{s+1}$, $f_{j\beta} \in N^\omega(O_j)$ for all j and $\beta \in \mathbf{N}^s$ with $|\beta| < l$ and minimal polynomials $P_{j\beta}$ for $f_{j\beta}$ such that $(f_{j\beta}, P_{j\beta})$ have Property (A) and, for each j, $F_{jl-1} = f - \sum_{|\beta|<l} \tilde{f}_{j\beta P_{j\beta}} h_s^\beta$ is $(l-1)$-flat on O_j. Then there exist $f_{j\alpha} \in N^\omega(O_j)$ for each j and all α with $|\alpha| = l$ such that $F_{jl-1} - \sum_{|\alpha|=l} \tilde{f}_{j\alpha} h_s^\alpha$ is l-flat on O_j where $\tilde{f}_{j\alpha}$ are any C^∞ extensions of $f_{i\alpha}$ for the following reason. Consider a local coordinate system of $\mathbf{R}^m$ of class C^ω Nash around each point of O_j such that the system contains $h_{s1}, \ldots, h_{ss}$. Then O_j is a linear subspace in this coordinate system. Hence the existence of $\tilde{f}_{j\alpha}$ of class C^∞ follows immediately, moreover the uniqueness of $f_{j\alpha}$ follows. This uniqueness, together with the fact that a derivative of a C^ω Nash function is of class C^ω Nash, implies that $f_{j\alpha}$ are C^ω Nash functions.

For all j and α with $|\alpha| = l$, apply to $f_{j\alpha}$ the argument in the case $\alpha = 0$. Then we obtain a semialgebraic set B^{l+3}_{s+1} of dimension $< n - s$, the connected components $O'_{j'}$, $j' = 1, 2, \ldots,$ of $B_s - B^1_{s+1} - \cdots - B^{l+3}_{s+1}$, $f_{j'\alpha} \in N^\omega(O'_{j'})$ for all j' and α with $|\alpha| = l$ and minimal polynomials $P_{j'\alpha}$ for $f_{j'\alpha}$ such that $(f_{j'\alpha}, P_{j'\alpha})$ have Property (A) and, for each j', $F_{jl-1}|_{O'_{j'}} - \sum_{|\alpha|=l} \tilde{f}_{j'\alpha P_{j'\alpha}} h_s^\alpha$ is l-flat on $O'_{j'}$ where j is such that $O_j \supset O'_{j'}$. If $|\alpha| < l$ we put $f_{j'\alpha} = f_{j\alpha}|_{O'_{j'}}$, $P_{j'\alpha} = P_{j\alpha}$ for some j with $O_j \supset O'_{j'}$. Hence we can define by induction $f_{sj\alpha}$ and $P_{sj\alpha}$ for all α with $|\alpha| \leqslant r$. Thus we have proved the statement for $s + 1$ at the beginning of the proof. Therefore the proof is completed by induction on s.

Let $0 \leqslant s \leqslant n$ and ε be of the same form as e. Under the same notations as 3.3

3.4.s. *For all i and j with $i \geqslant s$ there exist $H_{ij} \in N^r(\mathbf{R}^n)$ and arbitrarily small open semialgebraic neighborhoods $V_{ij} \subset V'_{ij}$ of U_{ij} such that*

(3.4.s.1)

$$\overline{V}_{ij} \subset V'_{ij} \cup \overline{U}_{ij}, \qquad H_{ij} = \begin{cases} 1 & \text{on } V_{ij} - W'_{i+1}, \\ 0 & \text{on } \mathbf{R}^n - \left(V'_{ij} - W_{i+1}\right), \end{cases}$$

where $W_{i+1} = \bigcup_{i<i';1\leqslant j'\leqslant m_{i'}} V_{i'j'}$, $W'_{i+1} = \bigcup_{i<i';1\leqslant j'\leqslant m_{i'}} V'_{i'j'}$;

(3.4.s.2) *we can approximate H_{ij} by C^ω Nash functions on $\mathbf{R}^n$ in the C^r topology*;

(3.4.s.3) $\sum_{i\geqslant s;1\leqslant j\leqslant m_i} H_{ij} = 1$ *on W_s; and*

(3.4.s.4) *for each i and j, $\sum_{|\alpha|\leqslant r} \tilde{f}_{ij\alpha P_{ij\alpha}} h_i^\alpha H_{ij}$ is an ε-approximation of fH_{ij} of order r, i.e.*

$$\left| D^\beta \Big\{ fH_{ij} - \sum_{|\alpha|\leqslant r} \tilde{f}_{ij\alpha P_{ij\alpha}} h_i^\alpha H_{ij} \Big\} \right| \leqslant \varepsilon \quad \text{on } \mathbf{R}^n$$

for all β with $|\beta| \leqslant r$.

Proof of 3.4.s. We work by downward induction on s. If $s = n$, 3.4.s follows from Propositions 2.4 and 2.5 and (3.3.5). Hence assume 3.4.$s+1$. It suffices to consider 3.4.s on one U_{sj} because we require, moreover, $V'_{sj} \cap V'_{sj'} = \varnothing$ for $j \neq j'$. Put

$$G = 1 - \sum_{\substack{i>s \\ 1\leqslant j'\leqslant m_i}} H_{ij'} \quad \text{and} \quad \rho = \sum_{|\alpha|\leqslant r} \tilde{f}_{sj\alpha P_{sj\alpha}} h_s^\alpha.$$

Then $G = 1$ on $\mathbf{R}^n - W'_{s+1}$ by (3.4.$s+1$.1), $G = 0$ on W_{s+1} by (3.4.$s+1$.3) and $\rho - f$ is r-flat on U_{sj} by (3.3.5). Apply the separation theorem to $U_{sj} - W_{s+1}$ and the complement of its small open semialgebraic neighborhood and apply Lemma 2.3 to the resultant separation function. Then there exists $H'_{sj} \in N^r(\mathbf{R}^n)$ such that $H'_{sj} = 0$ outside a small semialgebraic neighborhood of $U_{sj} - W_{s+1}$, $H'_{sj} = 1$ on another one and we can approximate H'_{sj} by a C^ω Nash function on $\mathbf{R}^n$ in the C^r topology. Hence, by Propositions 2.4 and 2.5 we have $H''_{sj} \in N^r(\mathbf{R}^n)$ such that $H''_{sj} = 0$ outside an arbitrarily small open semialgebraic neighborhood of B_s, $H''_{sj} = 1$ on another one, H''_{sj} can be approximated by a C^ω Nash function on $\mathbf{R}^n$ in the C^r topology, and $\rho G H'_{sj} H''_{sj}$ is an ε-approximation of $fGH'_{sj}H''_{sj}$ of order r. Therefore $H_{sj} = GH'_{sj}H''_{sj}$, together with some arbitrarily small V_{sj} and V'_{sj}, satisfies 3.4.s. Clearly we can choose V'_{sj}'s so that $V'_{sj} \cap V'_{sj'} = \varnothing$ for $j \neq j'$ when we repeat this argument for every U_{sj}. Hence 3.4.s follows for all s.

3.5. Proof of the first half of Theorem 1. Keep the same notations as 3.3 and 3.4.0. For each i, j and α choose V'_{ij} small enough, then by (3.4.0.1), Lemma 3.1 and Argument 2.1 there exists an ε-approximation of $\tilde{f}_{ij\alpha P_{ij\alpha}} H_{ij}$ of order r of the form $f'_{ij\alpha} H_{ij}$, where $f'_{ij\alpha}$ is a C^ω Nash function on $\mathbf{R}^n$ (to be precise, we have to construct $f'_{ij\alpha}$ by downward induction on i because V'_{ij} depends on $U_{ij} - W_{i+1}$). It also follows from (3.4.0.2) and Argument 2.1 that we have a C^ω Nash ε-approximation $g_{ij\alpha}$ of

$f'_{ij\alpha}H_{ij}$ of order r of the form $f'_{ij\alpha}G_{ij}$ for $G_{ij} \in N^{\omega}(\mathbf{R}^n)$. Hence $g_{ij\alpha}$ is a 2ε-approximation of $\tilde{f}_{ij\alpha P_{ij\alpha}}H_{ij}$ of order r. Put

$$g = \sum_{\substack{|\alpha| \leqslant r \\ i,j}} g_{ij\alpha} h_i^{\alpha}.$$

Then by (3.4.0.3) and (3.4.0.4) we see

$$\begin{aligned}
|D^{\beta}(f-g)| &\leqslant \sum_{i,j} \left| D^{\beta}\left\{ fH_{ij} - \sum_{|\alpha| \leqslant r} g_{ij\alpha} h_i^{\alpha} \right\} \right| \\
&\leqslant \sum_{i,j} \left| D^{\beta}\left\{ fH_{ij} - \sum_{|\alpha| \leqslant r} \tilde{f}_{ij\alpha P_{ij\alpha}} h_i^{\alpha} H_{ij} \right\} \right| \\
&\quad + \sum_{\substack{|\alpha| \leqslant r \\ i,j}} \left| D^{\beta}\left\{ \tilde{f}_{ij\alpha P_{ij\alpha}} h_i^{\alpha} H_{ij} - g_{ij\alpha} h_i^{\alpha} \right\} \right| \\
&\leqslant Q\varepsilon
\end{aligned}$$

for all β with $|\beta| \leqslant r$ and some polynomial Q on $\mathbf{R}^n$ which does not depend on ε. Hence, diminishing ε we can approximate f by the C^{ω} Nash function g in the C^r topology. This completes the proof of the first half of Theorem 1.

3.6. Proof of the latter half of Theorem 1. Let $i_0 = \operatorname{codim} M_3$ in $\mathbf{R}^n$. By Corollary 5 in **[9]**, we have $f_0 \in N^{\omega}(\mathbf{R}^n)$ such that $f_0 = f$ on M_3. Hence, considering $f - f_0$ in place of f, we can assume $f \equiv 0$ on M_3. Recall Corollary 6 in **[9]** which states that M_3 is C^{ω} Nash nonsingular, namely there exist $\theta_1, \dots, \theta_t \in N^{\omega}(\mathbf{R}^n)$ such that $\theta_1^{-1}(0) \cap \cdots \cap \theta_t^{-1}(0) = M_3$ and $\operatorname{grad} \theta_1, \dots, \operatorname{grad} \theta_t$ span the normal bundle of M_3 in $\mathbf{R}^n$. Put $\theta = \sum_{i=1}^{t} \theta_i^2$. Then we can add to 3.3 the following conditions: there exist, moreover, Λ a subset of $\{(i, j): 0 \leqslant i \leqslant n, 1 \leqslant j \leqslant m_i\}$, $\bar{f}_{ij\alpha} \in N^{\omega}(U_{ij})$ for all $(i, j) \notin \Lambda$ and $\alpha \in \mathbf{N}^i$ with $|\alpha| \leqslant r$, and minimal polynomials $\bar{P}_{ij\alpha}$ for $\bar{f}_{ij\alpha}$ such that

(3.3.6) $M_3 = \bigcup_{(i,j) \in \Lambda} U_{ij}$;

(3.3.7) if $(i, j) \in \Lambda$ we can replace in (3.3.3) and (3.3.5) i_0 functions of $\{h_{ik}\}_{k=1,\dots,i}$ by some θ_k's and we use the notations $\{\bar{h}_{ijk}\}_{k=1,\dots,i}$ for the new family, here we assume $\bar{h}_{ij1}, \dots, \bar{h}_{iji_0}$ to be the replaced ones;

(3.3.8) $(\bar{f}_{ij\alpha}, \bar{P}_{ij\alpha})$ have Property (A);

(3.3.9) for each $(i, j) \notin \Lambda, f/\theta - \sum_{|\alpha| \leqslant r} \tilde{\bar{f}}_{ij\alpha \bar{P}_{ij}} h_i^{\alpha}$ is r-flat on U_{ij}.

As the proof of the above proceeds in the same way as 3.3 we omit it. We remark only that (3.3.7) implies that if we choose a local coordinate system at each point of $U_{ij} \subset M$ so that $\{\bar{h}_{ijk}\}_{k=1,\dots,i}$ is its part, then M_3 and U_{ij} become linear subspaces.

Recall g in 3.5 which was the required approximation function. We modify g as follows:

$$\bar{g} = \sum_{\substack{(i,j) \notin \Lambda \\ |\alpha| \leqslant r}} \bar{g}_{ij\alpha} h_i^{\alpha} \theta + \sum_{\substack{(i,j) \in \Lambda \\ |\alpha| \leqslant r}} g_{ij\alpha} \bar{h}_{ij}^{\alpha},$$

where $\bar{g}_{ij\alpha}$ are approximations of $\tilde{\bar{f}}_{ij\alpha P_{ij\alpha}} H_{ij}$ defined in the same way as $g_{ij\alpha}$ in 3.5. It is obvious by definition that $\bar{g}$ is a C^{ω} Nash approximation of f and the first part of $\bar{g}$ vanishes on M_3. Hence we need only

(3.6.1) $g_{ij\alpha} \equiv 0$ for $(i, j) \in \Lambda$ and $\alpha_1 = \cdots = \alpha_{i_0} = 0$.

For this it suffices by 3.5 that

(3.6.2) $\overline{P}_{ij\alpha}(z, x) \equiv z$ for the same i, j and α,

which is equivalent to

(3.6.3) $\bar{f}_{ij\alpha} \equiv 0$ on U_{ij} for the same i, j and α.

But this is clearly possible by the above remark and the method of proof of 3.3. Thus we have proved the latter half and, hence, Theorem 1.

REMARK 3.7. In Theorem 1 we do not need the compactness condition on M_3 if any C^ω Nash function on M_3 is extensible to M_1 and M_3 is "C^ω Nash nonsingular" in the sense in 3.6. The reason for this is clear by 3.6.

3.8. PROOF OF COROLLARY 2. Let $0 \leqslant r < \infty$ and U_3 be a semialgebraic set such that $\overline{U}_1 \subset U_3$ and $\overline{U}_3 \subset U_2$. Let $\varphi \in N^r(\mathbf{R}^n)$ such that $\varphi = 1$ on $\overline{U}_1$ and $\varphi = 0$ on $\mathbf{R}^n - U_3$, and apply Theorem 1 to $\varphi f \in N^r(\mathbf{R}^n)$. Then we obtain the required $g \in N^\omega(\mathbf{R}^n)$.

4. The approximation theorem for Nash manifolds. Let $0 \leqslant r \leqslant \omega$. A vector bundle $\xi = (E, p, B)$ is called a *C^r Nash vector bundle* if the total space E and the base space B are C^r Nash manifolds, B is covered by a finite number of semialgebraic coordinate neighborhoods and all the coordinate functions and the projection p are of class C^r Nash. A *C^r Nash bundle map* is naturally defined, and we call a C^r Nash invertible bundle map a *C^r Nash isomorphism* and two C^r Nash vector bundles *C^r Nash isomorphic* if there is a C^r Nash isomorphism between them. Let $1 \leqslant k \leqslant n$ and $G_{n,k}$ denote the Grassmannian of k-linear subspaces in $\mathbf{R}^n$, put

$$E_{n,k} = \{(\lambda, x) \in G_{n,k} \times \mathbf{R}^n | x \in \lambda\}$$

and let p_G: $E_{n,k} \to G_{n,k}$ be the projection. Then the bundle $\xi_G = (E_{n,k}, p_G, G_{n,k})$ naturally has a C^ω Nash vector bundle structure [8].

PROOF OF THEOREM 3. Let k denote the codimension of M in $\mathbf{R}^n$. Let π: $M \to G_{n,k}$ denote the C^{r-1} Nash map defined by $\pi(x) =$ *the normal vector space of M in $\mathbf{R}^n$ at* x, π' a close C^ω Nash approximation of π in the C^0 topology (Theorem 1), and $\pi'^*\xi_G = (\pi'^*E_{n,k}, p_M, M)$, the induced bundle of ξ_G by π'. Here we remark that $\pi'^*\xi_G$ is a Nash vector bundle of class C^r. Let us regard M and $G_{n,k}$ as subsets of $\pi'^*E_{n,k}$ and $E_{n,k}$, respectively, through the zero cross-sections. Define a C^r Nash map φ: $\pi'^*E_{n,k} \to \mathbf{R}^n$ by

$$\varphi(x, y, z) = x + z, \qquad (x, y, z) \in \pi'^*E_{n,k} \subset M \times E_{n,k} \subset M \times G_{n,k} \times \mathbf{R}^n.$$

Then we see easily (cf. the proof of Lemma 7 in [9]) that there exists a C^r Nash tubular neighborhood V of M in $\pi'^*E_{n,k}$ such that $\varphi|_V$ is an imbedding. Put $W = \varphi(V)$ and $\psi = q \circ \varphi^{-1}$: $W \to E_{n,k}$, where q: $\pi'^*E_{n,k} \to E_{n,k}$ is the bundle map. Then $p_M \circ \varphi^{-1}$: $W \to M$ is a C^r Nash tubular neighborhood of M in $\mathbf{R}^n$, ψ is transversal to $G_{n,k}$ and $\psi^{-1}(G_{n,k}) = M$. Apply Theorem 1 to ψ and let ψ': $W \to E_{n,k}$ be a resultant C^ω Nash approximation in the C^r topology such that ψ' is transversal to $G_{n,k}$. It is then easy to see that $M' = \psi'^{-1}(G_{n,k})$ is a C^ω Nash manifold, τ: $M \to M'$ defined by $\tau(x) = (p_M \circ \varphi^{-1}|_W)^{-1}(x) \cap M'$ is a C^r Nash diffeomorphism and, moreover, $\tau \to$ identity as $\psi' \to \psi$ in the C^r topology.

For the second part of the theorem, just use the latter half of Theorem 1.

PROOF OF THE LAST PART OF THEOREM 3. By the first part we can assume M is a C^ω Nash manifold in $\mathbf{R}^n$. Let $p: V \to M$ be a C^ω Nash tubular neighborhood in $\mathbf{R}^n$. Apply the first part to $M_2 \subset \mathbf{R}^n$ Then we have a C^r Nash imbedding $\pi: M_2 \to \mathbf{R}^n$ arbitrarily close to the identity such that $\pi(M_2)$ is of class C^ω Nash. Moreover the above proof says that π can be extended to a C^r Nash diffeomorphism $\tilde{\pi}$ of $\mathbf{R}^n$. Hence choosing π and, hence, $\tilde{\pi}$ close enough to the identity we obtain a required C^r Nash imbedding $p \circ \tilde{\pi}|_M$. Therefore Theorem 3 is proved.

Theorems 1 and 3 tell us the following.

COROLLARY 4.1. *Let $1 \leqslant r < \infty$. Then the C^r Nash diffeomorphism classes of all C^r Nash manifolds is identical with the C^ω Nash diffeomorphism classes of all C^ω Nash manifolds.*

REMARK 4.2. In spite of Corollary 4.1 there is a definite difference between the C^r Nash category for $1 \leqslant r < \infty$ and the C^ω Nash category as shown in [**11**]. An *abstract C^r Nash manifold* means a manifold with a finite system of coordinate neighborhoods of class C^r Nash. To distinguish this we call a usual C^r Nash manifold *affine*. As [**1**, **8** and **11**] pointed out, an abstract C^ω Nash manifold is not necessarily affine. But if $0 \leqslant r < \infty$, then an abstract C^r Nash manifold is always affine [**11**].

PROOF OF COROLLARY 4. We already know the corollary in the cases $r = \omega$ [**9**] and $r = 0$ [**12**]. Hence by Corollary 4.1 it suffices to prove the following. Let $1 \leqslant r < \infty$ and let M be a compact C^r Nash manifold with boundary. Then

(1) M is C^r Nash diffeomorphic to some C^ω Nash manifold with boundary.

For the proof, consider the double DM of M with naturally defined abstract C^r Nash manifold structure. Regard $M \subset DM$. Then, by Remark 4.2, $M \subset DM$ can be contained in some Euclidean space as C^r Nash manifolds. Hence considering the pair $(DM, \partial M)$ we obtain (1) by the last statement in Theorem 3.

THEOREM 4.3. *Let $0 \leqslant r \leqslant \omega$. Let $\zeta = (E, p, M)$ be a C^0 vector bundle over a C^r Nash manifold M. Then ζ has a unique C^r Nash vector bundle structure up to a C^r Nash isomorphism.*

PROOF. As the case of compact M is easy, we assume M is not compact. By Corollary 4 we regard M as the interior of a compact C^r Nash manifold $\tilde{M}$ with boundary. We also regard ζ as the induced bundle $f^*\xi_G$ of ξ_G by some C^0 map f: $M \to G_{n,k}$ for some n, where $k = \dim \zeta$ [**13**]. We know [**13**] that if f is homotopic to a C^0 map g, then $f^*\xi_G$ is equivalent to $g^*\xi_G$. So for the existence of C^r Nash vector bundle structure on ζ we only need to find a C^r Nash map $g: M \to G_{n,k}$ which is homotopic to f. Now f is homotopic to the restriction to M of a C^0 map $\tilde{f}: \tilde{M} \to G_{n,k}$ because $\tilde{M}$ has a collar. But by Theorem 1, $\tilde{f}$ is approximated by a C^r Nash map $\tilde{g}$: $\tilde{M} \to G_{n,k}$. Hence choosing the approximation close enough we obtain the required $g = \tilde{g}|_M$.

PROOF OF THE UNIQUENESS. Assume C^r Nash vector bundles $\zeta_1 = (E_1, p_1, M)$, $\zeta_2 = (E_2, p_2, M)$ and a bundle map $\Psi: \zeta_1 \to \zeta_2$ of C^0 equivalence are given. Then we have to obtain a C^r Nash isomorphism $\Phi: \zeta_1 \to \zeta_2$.

Put $E_3 = \bigcup_{x\in M} L(p_1^{-1}(x), p_2^{-1}(x))$ and $\zeta_3 = (E_3, p_3, M)$, where $L(\ ,\ )$ means the space of linear maps and $p_3\colon E_3 \to M$ is the projection. We want to give naturally ζ_3 a C^r Nash vector bundle structure. Let $\{U_\alpha\}_{\alpha\in A}$ and $\{V_\beta\}_{\beta\in B}$ be finite systems of semialgebraic coordinate neighborhoods of ζ_1 and ζ_2 respectively. Then for each U_α and V_β, $(p_3^{-1}(U_\alpha \cap V_\beta), p_3, U_\alpha \cap V_\beta)$ has the trivial vector bundle structure. Hence the coordinate transformations of ζ_1 and ζ_2 give ζ_3 an abstract C^r Nash vector bundle structure in the sense of Remark 4.2, namely E_3 becomes an abstract C^r Nash manifold. Therefore if $r < \infty$, ζ_3 is a C^r Nash vector bundle by Remark 4.2.

Consider the case $r = \omega$. Assume $E_1, E_2 \subset \mathbf{R}^n$ and regard $M \subset E_1$ and $M \subset E_2$ through the zero cross-sections. For each $x \in M$ let $f_1(x)$ and $f_2(x)$ denote the tangent vector spaces of $p_1^{-1}(x)$ and $p_2^{-1}(x)$ at x respectively. Then we see easily that $f_1, f_2\colon M \to G_{n,k}$ are C^ω Nash maps and that $f_1{}^*\xi_G$ and $f_2{}^*\xi_G$ are C^ω Nash isomorphic to ζ_1 and ζ_2 respectively. Hence we consider $f_1{}^*\xi_G$ and $f_2{}^*\xi_G$ in place of ζ_1 and ζ_2. Put

$$F = \{(\lambda, \mu, T) \in G_{n,k} \times G_{n,k} \times L(\mathbf{R}^n, \mathbf{R}^n)\colon T\lambda \subset \mu, T\lambda^\perp = 0\}$$

and let $q\colon F \to G_{n,k} \times G_{n,k}$ be the projection. Then $\eta = (F, q, G_{n,k} \times G_{n,k})$ is a C^ω Nash vector bundle (see the C^ω Nash manifold structure on $G_{n,k}$ in **[8]**), and $(f_1, f_2)^*\eta$ is C^ω Nash isomorphic to ζ_3. Thus we have given ζ_3 a C^r Nash vector bundle structure in any case.

Let E_4 be the subset of E_3 consisting of linear isomorphisms and p_4 the restriction of p_3 to E_4. Then E_4 is an open semialgebraic subset of E_3 and $\zeta_4 = (E_4, p_4, M)$ is a C^ω Nash fibre bundle with fibre $\mathrm{GL}(k, \mathbf{R})$. Now what we have to prove is that if ζ_4 has a C^0 global cross-section ψ, then it has a C^r Nash one φ.

Let us regard M as contained in $\mathbf{R}^n$ so that $\overline{M}$ is compact and $\overline{M} - M$ consists of one point a (see **[9]**). Then by **[4]** there exists a semialgebraic triangulation of $\overline{M}$ compatible with $\{a\} \cup \{U_\alpha \cap V_\beta\}_{\alpha\in A;\beta\in B}$, i.e. a finite simplicial complex $K \subset \mathbf{R}^n$ and a semialgebraic homeomorphism $\tau\colon |K| \to \overline{M}$ such that a or each $U_\alpha \cap V_\beta$ is the image of a union of some open simplices of K. Put $W_\sigma = \tau(\sigma) - a$ for $\sigma \in K$. Then $\{W_\sigma\}_{\sigma\in K}$ is a finite closed covering of M, moreover refining $\{U_\alpha\}$ and $\{V_\beta\}$ if necessary we can assume $\zeta_4|_{W_\sigma}$ is C^r Nash trivial for each σ.

Put

$$K' = \{\sigma \in K | \tau(\sigma) \ni a\}, \qquad K'' = K - K',$$
$$M' = \bigcup_{\sigma\in K'} W_\sigma \quad \text{and} \quad M'' = \bigcup_{\sigma\in K''} W_\sigma = \overline{M - M'}.$$

For the construction of φ, at first, we will define by induction a C^0 Nash cross-section φ'' of $\zeta_4|_{M''}$. If $\sigma \in K''$ is of dimension 0 we put $\varphi''(W_\sigma) = \psi(W_\sigma)$. So assume we have already defined φ'' on $\bigcup_{\dim\sigma<l} W_\sigma$ so that it is homotopic to the restriction of ψ to $\bigcup_{\dim\sigma<l} W_\alpha$. Then for each $\sigma \in K''$ with $\dim\sigma = l$, the restriction of the homotopy to ∂W_σ can be extended to W_σ so that the extension of φ'' is of class C^0 Nash by the triviality of $\zeta_4|_{W_\sigma}$, the Alexander trick and by Theorem 1 (see **[12]** for the Alexander trick). Hence we have globally φ'' on M''. Next extend φ'' to a C^0 Nash cross-section φ_0 of ζ_4 by induction on $\dim\sigma$ for $\sigma \in K' - \tau^{-1}(a)$ in the same way as above. Hence the case $r = 0$ is proved.

Assume $r > 0$. For each $x \in M$ let ρ_x denote the orthogonal projection of a semialgebraic tubular neighborhood of $p_4^{-1}(x)$ in $\mathbf{R}^n$. Put $P(x, y) = \rho_x(y)$. Then P is a C^{r-1} Nash map from an open semialgebraic set X in $M \times \mathbf{R}^n$ to E_4. Furthermore, approximating ρ_x by a C^r map in the same way as the proof of Theorem 3 we can assume P is of class C^r Nash. Regard φ_0 as a map from M to E_4 and apply Theorem 1 to φ_0. Then we have a C^r Nash map χ: $M \to E_4$ such that the graph of χ is contained in the domain of P, and, hence, $\varphi = P(x, \chi(x))$ is a required cross-section.

Problem. Let M be a C^r Nash manifold. We call a vector bundle over M an *abstract C^r Nash vector bundle* if the total space is an abstract C^r Nash manifold and the same conditions on coordinates as a C^r Nash vector bundle are satisfied. As pointed out in Remark 4.2, an abstract C^r Nash vector bundle over M is always *affine* if $r < \infty$. Is this the case for $r = \omega$?

ADDED IN PROOF. D. Pecker corrected the proofs of [2] in his thesis *Fonction: approximation, extension, factorisation*.

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