TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 293, Number 2, February 1986

THE GAUSS MAP FOR SURFACES: PART 2. THE EUCLIDEAN CASE

BY

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ABSTRACT. We study smooth maps $t\colon M \to G_2^c$ of a Riemann surface M into the Grassmannian G_2^c of oriented 2-planes in $\mathbf{E}^{2+c}$ and determine necessary and sufficient conditons on t in order that it be the Gauss map of a conformal immersion $X\colon M \to \mathbf{E}^{2+c}$. We sometimes view t as an oriented riemannian vector bundle; it is a subbundle of $\mathbf{E}_M^{2+c}$, the trivial bundle over M with fibre $\mathbf{E}^{2+c}$. The necessary and sufficient conditions obtained for simply connected M involve the curvatures of t and $t^\perp$, the orthogonal complement of t in $\mathbf{E}_M^{2+c}$, as well as certain components of the tension of t viewed as a map $t\colon M \to S^C(1)$, where $S^C(1)$ is a unit sphere of dimension C that contains G_2^c as a submanifold in a natural fashion. If t satisfies a particular necessary condition, then the results take two different forms depending on whether or not t is the Gauss map of a conformal *minimal* immersion. The case $t\colon M \to G_2^2$ is also studied in some additional detail.

In [**5, 6**], Hoffman and Osserman study the following question: Let M be a Riemann surface and $t\colon M \to G_2^c$ be a smooth map into the Grassmannian of 2-planes in $(2 + c)$-space; when is t the Gauss map of a conformal immersion $X\colon M \to \mathbf{E}^{2+c}$? In [**5, 6**] necessary and sufficient conditions are established when M is simply connected, in order for t to be a Gauss map.

The purpose of this paper is primarily to redo the work of Hoffman and Osserman from the point of view established in [**12, 13**]. One reason for doing this is to free their results of its dependence on the use of complex variables in order to allow and suggest generalizations from the case of surfaces to higher dimensional manifolds.

Also, to some extent, the necessary and sufficient conditions established in [**5, 6**] in order for t to be a Gauss map are somewhat mysterious (to me at least) and could use some illumination. In particular, we obtain corresponding conditions which are stated directly in terms of traditional geometric invariants. We will briefly describe these invariants and also describe how they appear in the theorems of this paper.

Let $\bar{g}_0$ be the standard metric on G_2^c. We say $t\colon M \to G_2^c$ is conformal if $g_0 = t^*\bar{g}_0$ induces the given conformal structure on M, where t_* does not vanish. One may view t not only as a map but quite naturally as an oriented riemannian rank 2 vector bundle over M, a subbundle of the trivial bundle of $\mathbf{E}_M^{2+c}$ over M with fibre $\mathbf{E}^{2+c}$. As such we can define $k\colon M \to \mathbf{R}$ to be the curvature of t with respect to the

Received by the editors December 5, 1984.

1980 *Mathematics Subject Classification*. Primary 53A05; Secondary 53C42.

Key words and phrases. Riemann surface, Gauss map of a conformal immersion, Grassmannian, normal bundle, tension.

element of area associated to g_0 where t_* is regular, i.e., has maximal rank, and to be zero where t_* is not regular—indeed, t is flat as a bundle where t: $M \to G_2^c$ is not regular. Certainly k is a very natural measure of the curvature of the bundle t. Also $-1 \leqslant k \leqslant 1$. Theorem 2 states that if M is a simply connected noncompact Riemann surface and t is conformal with $k = -1$ where $t_* \neq 0$, then t is the Gauss map of a conformal minimal immersion. The corollary to Proposition 2 says the conditions on t are also necessary.

Let $t^\perp$ be the orthogonal complement of the bundle t in $\mathbf{E}_M^{2+c}$; we call $t^\perp$ the normal bundle of t. It is also the case that $t^\perp$ is an oriented riemannian vector bundle over M. Let g be a riemannian metric on M. We define a section N_g in $\Lambda^2 t^\perp$, where $\Lambda^2 t^\perp$ is the second exterior power of $t^\perp$. The projection of $N_g(p)$ for $p \in M$ onto the unit decomposable 2-vector $\lambda \in \Lambda^2 t^\perp(p)$ is the curvature of the "2-plane λ" in $t^\perp$ with respect to the area element associated to g, the normal curvature of λ. The Grassmannian G_2^c is viewed as the space of unit decomposable 2-vectors in $\Lambda^2 \mathbf{E}^{2+c}$; hence $G_2^c \subset S^C(1)$, the sphere of unit 2-vectors in $\Lambda^2 \mathbf{E}^{2+c}$. Let $\bar{q}$ denote the second fundamental form of G_2^c in $S^C(1)$. Set $q = t^*\bar{q}$; if t is an immersion, then q is the component of the second fundamental form of t: $M \to S^C(1)$ orthogonal to G_2^c. In Theorem 1 we show that a necessary condition on t in order for it to be a Gauss map is that

$$\tfrac{1}{2}\operatorname{tr}_g(q) = N_g, \tag{0}$$

where g is any riemannian metric that induces the conformal structure on M. Note that we may view $\frac{1}{2}\operatorname{tr}_g(q)$ as the component of the tension of t: $M \to S^C(1)$ orthogonal to G_2^c. Equation (0) is one of our necessary and sufficient conditions that correspond to one of those of Hoffman and Osserman.

As is pointed out in the remark following Proposition 1, the map t, if it possibly is a Gauss map, already determines whether any possible conformal immersion with it as a Gauss map would be minimal or not, and if the immersion cannot be minimal then t already determines the direction of its mean curvature. Assuming (0) holds and $k \neq -1$ where t is conformal and regular, we define in §3 a riemannian metric $\tilde{g}$ which induces the conformal structure on M. Let τ denote the tension of t: $M \to G_2^c$ with respect to the metrics $\tilde{g}$ on M and g_0 on G_2^c. Associated to τ is a differential 1-form on M, $\tilde{\tau}$, which involves only the component of τ in the direction of the mean curvature just mentioned (see §3 for a complete explanation of what this means). Theorem 3 states that if M is simply connected, (0) holds, $k \neq -1$ where t is conformal and regular, and $d\tilde{\tau} = 0$, then t is a Gauss map. Theorem 3 also points out that the scalar mean curvature H of any conformal immersion with Gauss map t is given by

$$H = H_0 \exp\left(\frac{1}{2}\int \tilde{\tau}\right),$$

where $H_0 \in \mathbf{R}$, and the theorem also presents a representation for any conformal immersion with Gauss map t using H. The equation $d\tilde{\tau} = 0$ is the other one of our necessary and sufficient conditions that corresponds to one of those of Hoffman and Osserman.

In the last section of this paper, we focus on t: $M \to G_2^2$. In this setting equation (0) takes on an especially interesting form. Since $G_2^2 = S^2(1/\sqrt{2}) \times S^2(1/\sqrt{2})$, we let t_i: $M \to S^2(1/\sqrt{2})$ be the ith component of t for $i \in \{1,2\}$. If g is a riemannian metric on M, for $i \in \{1,2\}$, let $\varepsilon_i(g)$ and $\rho_i(g)$ be the energy density and Jacobian of t_i: $M \to S^2(1/\sqrt{2})$, respectively, with respect to the metric g on M and the standard metric on $S^2(1/\sqrt{2})$. Then, according to Proposition 5, equation (0) becomes

$$\varepsilon_1(g) + \rho_1(g) = \varepsilon_2(g) + \rho_2(g),$$

where g is a riemannian metric that induces the conformal structure on M. In fact, we point out in Corollary 3 of Proposition 6 that

$$\varepsilon_i(g) + \rho_i(g) = \|\mathbf{H}\|^2$$

for $i \in \{1,2\}$, if t: $M \to G_2^2$ is the Gauss map of an immersion with induced metric g and mean curvature $\mathbf{H}$.

Finally, Theorem 5 presents a characterization of the Gauss map of a conformal minimal immersion into S^3, the 3-sphere in $\mathbf{E}^4$. The theorem states that under the assumption that M is simply connected, t is the Gauss map of such an immersion if and only if t: $M \to G_2^2$ is a conformal minimal immersion with flat normal bundle $t^\perp$ and $k > -1$.

This paper draws heavily on **[13]**, which we refer to as Part 1 throughout this paper. The notation of Part 1 is used in this part and the ideas introduced in Part 1 are employed extensively here. In particular, the space of admissible maps, α, plays an integral role in Part 2.

All maps that appear in this paper are C^∞ with the exception of k which need not be continuous when t is not an immersion.

1. Preliminaries. Let M be a connected Riemann surface, i.e., M possesses a conformal structure. The conformal structure orients M. Let G_2^2 be the Grassmannian of oriented 2-planes in $\mathbf{E}^{2+c}$ and let $\bar{g}_0$ denote the riemannian metric on G_2^c. Suppose t: $M \to G_2^c$ is a smooth map; as in Part 1 we often view t as an oriented riemannian 2-plane bundle over M, a vector subbundle of $\mathbf{E}_M^{2+c}$, the trivial bundle over M with fibre $\mathbf{E}^{2+c}$. We let $t^\perp$ denote the orthogonal complement of t in $\mathbf{E}_M^{2+c}$ and call $t^\perp$ the normal bundle. The splitting $\mathbf{E}_M^{2+c} = t \oplus t^\perp$ induces orthogonal projections $(\cdots)^\top : \mathbf{E}_M^{2+c} \to t$ and $(\cdots)^\perp : \mathbf{E}_M^{2+c} \to t^\perp$.

If t: $M \to G_2^c$ is given, we show the following in Part 1: There exists over M the rank 4 vector bundle β defined by

$$\beta_p = \left\{ \Phi\colon T_pM \to \mathbf{E}^{2+c} \text{ linear map} \,\middle|\, \Phi(T_pM) \subset t(p) \right\}$$

for all $p \in M$. If $\Phi \in \beta_p$, we define its second fundamental form h_Φ by

$$h_\Phi(u,v) = t_*(u)(\Phi(v)) \tag{1}$$

for all $(u,v) \in T_pM \times T_pM$. We view $t_*(u)$ as an element of $\mathrm{GL}(t(p), t^\perp(p)) \cong T_{t(p)}G_2^c$ (see §2 of Part 1 for details). Contained in β is the *space of admissible maps*, α, which consists of all $\Phi \in \beta$ for which h_Φ is symmetric. For each $p \in M$, α_p is a subspace of β_p of dimension $a(p)$; if $a\colon M \to \{0,1,2,3,4\}$ is constant, then α is a subbundle of β. Contained in α_p for each $p \in M$ is α_p^+, the space of admissible maps at p with positive determinant. A map $\Phi \in \alpha_p$ has positive determinant if and only if Φ maps a positively oriented frame of T_pM into a positively oriented frame of $t(p)$. It may happen that $\alpha_p^+ = \varnothing$ for some $p \in M$. Also we let $\alpha^+ = \bigcup_{p\in M}\alpha_p^+$.

We will often think of a conformal structure on a 2-dimensional vector space V as a linear map $J\colon V \to V$ such that $J^2 = -1$, for J determines an orientation—that determined by (v, Jv) for $0 \neq v \in V$—and a collection of inner products which are multiples of one another—those for which J is an isometry. Each plane $\pi \in G_2^c$ has a conformal structure induced from its orientation and the inner product $\bar{g}$ on $\mathbf{E}^{2+c}$; we denote this conformal structure by J_π. For each $p \in M$, let J_p be the conformal structure on T_pM.

Let $X\colon M \to \mathbf{E}^{2+c}$ be an immersion with Gauss map t that induces the given conformal structure on M, i.e., X is a *conformal immersion*. Since t is a Gauss map it follows that condition [C] of Part 1 holds, i.e., if $\Phi = dX$ then Φ is a section in α with values in α^+ and $(d\Phi)^\top = 0$. But, in addition, for each $p \in M$, Φ_p must "pull back" the conformal structure on $t(p)$, $J_{t(p)}$, to that given on T_pM, J_p, in order that X induce the given conformal structure on M. This suggests that we introduce the following rank 2 vector subbundle κ of β where, for each $p \in M$,

$$\kappa_p = \left\{ \Phi \in \beta_p \middle| J_{t(p)} \circ \Phi = \Phi \circ J_p \right\}.$$

Note that any nonzero member of κ_p has positive determinant and any two members of κ_p differ by a composition with a similarlity transformation of $t(p)$, i.e., a scalar multiple of rotation of $t(p)$.

Hence, according to condition [C] of Part 1 and the definition of κ, in order for a section Φ in β defined over a simply connected open subset U of M to be the differential of a conformal immersion $X\colon U \to E^{2+c}$ with Gauss map t, it is necessary and sufficient that

$$[\mathrm{K}] \qquad \Phi_p \in (\alpha_p \cap \kappa_p) - \{0\} \quad \text{and} \quad (d_p\Phi)^\top = 0 \quad \text{for all } p \in U.$$

Here $d_p\Phi$ denotes the exterior differential of Φ at p. When condition [K] holds for some section Φ in $\beta|U$, we say *t is a Gauss map on U.*

As we just observed, in order for $t\colon M \to G_2^c$ to be a Gauss map it is necessary that $\alpha_p^+ \cap \kappa_p = (\alpha_p \cap \kappa_p) - \{0\} \neq \varnothing$ for all $p \in M$. We now turn to a characterization of $\dim(\alpha_p \cap \kappa_p)$. We define the *mean curvature* $\mathbf{H}(\Phi)$ for $\Phi \in \kappa_p - \{0\}$, p arbitrary, by

$$(2) \qquad \mathbf{H}(\Phi) = \frac{1}{2}\sum_{i=1}^{2} h_\Phi\left(\Phi^{-1}(e_i), \Phi^{-1}(e_i)\right) = \frac{1}{2}\sum_{i=1}^{2} t_*\left(\Phi^{-1}(e_i)\right)(e_i),$$

where e_1, e_2 is an orthonormal frame of $t(p)$. The second equality in (2) follows from (1). We leave it to the reader to check that $\mathbf{H}(\Phi)$ is well defined. Note that $\mathbf{H}(\Phi) \in t^\perp(p)$ for all $\Phi \in \kappa_p - \{0\}$. Of course, if $\Phi = d_pX$ for an immersion X with Gauss map t, then $\mathbf{H}(\Phi)$ is the mean of curvature of X at p.

Note that J_π is a rotation by $+90°$ for all $\pi \in G_2^c$. Thus $J_{t(p)} \circ \Phi \in \kappa_p$ if $\Phi \in \kappa_p$. We will delete the subscript $t(p)$ in $J_{t(p)}$ in what follows. Also the reader is reminded that α is the set of all $\Phi \in \beta$ such that h_Φ is symmetric.

LEMMA 1. *Let $\Phi \in \kappa_p - \{0\}$ for some $p \in M$. Then h_Φ is symmetric if and only if* $\mathbf{H}(J \circ \Phi) = 0$.

PROOF. Let e_1, e_2 be a positively oriented orthonormal frame of $t(p)$ and define v_1, $v_2 \in T_pM$ by $\Phi(v_i) = e_i$ for $i \in \{1,2\}$. Then note that $J \circ \Phi(v_1) = e_2$ and $J \circ \Phi(v_2) = -e_1$. Hence by (2) and (1)

$$\begin{aligned}\mathbf{H}(J \circ \Phi) &= \tfrac{1}{2}\left[t_*(-v_2)(e_1) + t_*(v_1)(e_2)\right]\\ &= \tfrac{1}{2}\left[h_\Phi(v_1, v_2) - h_\Phi(v_2, v_1)\right].\end{aligned}$$

PROPOSITION 1. *For any $p \in M$ the following hold:*
(i) $\dim(\alpha_p \cap \kappa_p) = 2$ *if and only if* $\mathbf{H}(\Phi) = 0$ *for all* $\Phi \in \kappa_p - \{0\}$.
(ii) $\dim(\alpha_p \cap \kappa_p) = 0$ *if and only if* $\mathbf{H}(\Phi) \neq 0$ *for all* $\Phi \in \kappa_p - \{0\}$.
(iii) *If* $\dim(\alpha_p \cap \kappa_p) = 1$, *then* $\mathbf{H}(\Phi) \neq 0$ *for all* $\Phi \in \alpha_p^+ \cap \kappa_p$.

PROOF. First note that $\kappa_p = \{J \circ \Phi \mid \Phi \in \kappa_p\}$. Hence $\mathbf{H}(\Phi) = 0$ for all $\Phi \in \kappa_p - \{0\}$, if and only if $\mathbf{H}(J \circ \Phi) = 0$ for all $\Phi \in \kappa_p - \{0\}$, if and only if h_Φ is symmetric for all $\Phi \in \kappa_p$ (by Lemma 1), if and only if $\kappa_p \subset \alpha_p$. This proves (i); a similar argument proves (ii). For (iii), just note that if $\mathbf{H}(\Phi) = \mathbf{H}(J \circ \Phi) = 0$ for some $\Phi \in \kappa_p - \{0\}$, then $\mathbf{H}(\Psi) = 0$ for all $\Psi \in \kappa_p - \{0\}$ by (2) since $\Psi^{-1} = r\Phi^{-1} + s(J \circ \Phi)^{-1}$ for appropriate reals r and s.

REMARK. Choose v_1, $v_2 \in T_pM$ so that $v_2 = Jv_1$. Then define

$$\mathcal{H}_p = \operatorname{span}\left\{ \frac{1}{2} \sum_{i=1}^{2} t_*(v_i)(e_i) \,\middle|\, e_1, e_2 \text{ is a positively oriented orthonormal frame of } t(p) \right\}.$$

Now let $X\colon M \to \mathbf{E}^{2+c}$ be a conformal immersion with Gauss map t. According to [K] and Proposition 1, X is minimal at p if and only if $\dim(\mathcal{H}_p) = 0$; also if X is not minimal then $\mathbf{H}(p)$, the mean curvature of X at p, lies in $\mathcal{H}_p$ which is 1-dimensional. Thus the Gauss map of X already determines whether or not X is minimal at p, and if X is not minimal at p, then the Gauss map determines the direction of the mean curvature at p. Hoffman and Osserman already pointed this out in [**5**].

2. The dimension of $\alpha_p \cap \kappa_p$. We now study $\dim(\alpha_p \cap \kappa_p)$ more directly in terms of the geometry of $t\colon M \to G_2^c$ and the curvatures of the vector bundles t and $t^\perp$.

We remind the reader that we defined the Weingarten map $A\colon t^\perp \to \beta$ as the adjoint of t_* (see §2 of Part 1 for details). The curvature k of the oriented riemannian vector bundle t at p, with respect to μ, a fixed area element on M, is given by

$$k(p) = \sum_{\alpha=3}^{2+c} \det\big(A^{e_\alpha}\colon T_pM \to t(p)\big),$$

where A^e denotes the value of A at e, and $e_3, \ldots, e_{2+c}$ is an orthonormal frame of $t^\perp(p)$. The determinant of A^{e_α} may be defined as follows: Let v_1, $v_2 \in T_pM$ such that $\mu(v_1 \wedge v_2) = 1$ and let e_1, e_2 be a positively oriented orthonormal frame of $t(p)$; then define reals l_{ij}^α for $i, j \in \{1,2\}$ and $\alpha \in \{3, \ldots, 2+c\}$, by

$$l_{ij}^\alpha = \big(t_*(v_j)(e_i), e_\alpha\big). \tag{3}$$

Since A and t_* are adjoints, we see that

$$\big(A^{e_\alpha}(v_j), e_i\big) = l_{ij}^\alpha$$

for $i, j \in \{1,2\}$ and $\alpha \in \{3, \ldots, 2+c\}$. Then $\det(A^{e_\alpha}) = |l_{ij}^\alpha|$, where $|l_{ij}^\alpha|$ denotes the determinant of the 2×2-matrix $(l_{ij}^\alpha)_{1 \leqslant i,j \leqslant 2}$. In particular, we also have

$$k(p) = \sum_\alpha |l_{ij}^\alpha|. \tag{4}$$

If t is regular at p, i.e., $\operatorname{rank}(t_{*|p}) = 2$, so that $g_0 = t^*\bar{g}_0$ is an inner product at p we may suppose $\mu_p = \mu_{0|p}$, where μ_0 is the differential 2-form defined as follows: If u_1, u_2 is a positively oriented frame of T_pM, then

$$\mu_0(u_1, u_2) = g_0(u_1, u_1)g_0(u_2, u_2) - g_0(u_1, u_2)^2.$$

Hence, when t is regular at p so that $\mu_p = \mu_{0|p}$, we may suppose v_1, v_2 given above in the definition of l_{ij}^α, form a positively oriented g_0-orthonormal frame of T_pM. Unless stated otherwise $k(p)$ at a point p where t is regular will be the curvature of t at p with respect to $\mu_{0|p}$. It is easy to check that $k(p) = 0$ when t is not regular at p. We call k the *pre-Gaussian curvature* of t (see §2 of Part 1 for motivation of this definition).

If $\pi \in G_2^c$, then a positively oriented orthonormal frame $e_1, \ldots, e_{2+c}$ of $\mathbf{E}^{2+c}$ is said to be "π-adapted" if e_1, e_2 is a positively oriented orthonormal frame of π. If $l \in T_\pi G_2^c \cong \mathrm{GL}(\pi, \pi^\perp)$, then $\bar{g}_0(l, l) = \|l\|^2$, i.e., if $e_1, \ldots, e_{2+c}$ is a π-adapted frame of $\mathbf{E}^{2+c}$, and reals l_i^α for $i \in \{1,2\}$, $\alpha \in \{3, \ldots, 2+c\}$ are defined by $l_i^\alpha = (l(e_i), e_\alpha)$, then

$$\bar{g}_0(l, l) = \sum_{i,\alpha} (l_i^\alpha)^2. \tag{5}$$

Assume t is regular at p and reals l_{ij}^α are defined as in (3) with v_1, v_2 a positively oriented g_0-orthonormal frame of T_pM. Then, by (3) and (5),

$$\delta_{rs} = g_0(v_r, v_s) = \sum_{i,\alpha} l_{ir}^\alpha l_{is}^\alpha \tag{6}$$

for $r, s \in \{1,2\}$.

LEMMA 2. *Suppose t is regular at p. Then $|k(p)| \leqslant 1$ and $k(p) = \pm 1$ if and only if*

$$l_{12}^{\alpha} = \mp l_{21}^{\alpha} \quad \textit{and} \quad l_{22}^{\alpha} = \pm l_{11}^{\alpha}$$

for $\alpha \in \{3, \ldots, 2 + c\}$.

PROOF. Note, by (4), $k(p)$ may be viewed as the dot product of the $(2c)$-tuples $(l_{11}^3, l_{21}^3, l_{11}^4, l_{21}^4, \ldots, l_{11}^{2+c}, l_{21}^{2+c})$ and $(l_{22}^3, -l_{12}^3, l_{22}^4, -l_{12}^4, \ldots, l_{22}^{2+c}, -l_{12}^{2+c})$. These $(2c)$-tuples are unit vectors by (6). Hence, the lemma follows directly from the Cauchy-Schwarz inequality.

PROPOSITION 2. $\dim(\alpha_p \cap \kappa_p) = 2$ *if and only if either*
(i) $t_{*|p} = 0$, *or*
(ii) *t is conformal (and regular) at p and $k(p) = -1$.*

PROOF. Throughout this proof let v_1, v_2 be a positively oriented frame of T_pM and $e_1, \ldots, e_{2+c}$ a $t(p)$-adapted frame of $\mathbf{E}^{2+c}$. Then define reals l_{ij}^{α} as in (3). The vectors v_1, v_2 and hence the reals l_{ij}^{α} will have various additional properties as needed in the course of the proof.

Suppose $\dim(\alpha_p \cap \kappa_p) = 2$ and $t_{*|p} \neq 0$. We will show t is conformal at p and $k(p) = -1$. Choose $\Phi \in \kappa_p - \{0\}$ and suppose $\Phi(v_i) = e_i$ for $i \in \{1, 2\}$. Then, by (1),

$$l_{ij}^{\alpha} = \big(h_{\Phi}(v_j, v_i), e_{\alpha}\big)$$

for $i, j \in \{1, 2\}$ and $\alpha \in \{3, \ldots, 2 + c\}$. By (i) of Proposition 1 and Lemma 1, both h_{Φ} and $h_{J \circ \Phi}$ are symmetric. The symmetry of h_{Φ} and $h_{J \circ \Phi}$ implies in turn that

$$l_{12}^{\alpha} = l_{21}^{\alpha} \quad \text{and} \quad l_{22}^{\alpha} = -l_{11}^{\alpha} \tag{7}$$

for $\alpha \in \{3, \ldots, 2 + c\}$. By (5), $(l_{1j}^3, l_{2j}^3, \ldots, l_{1j}^{2+c}, l_{2j}^{2+c})$ are the components of $t_*(v_j)$ with respect to an orthonormal frame of $T_{t(p)}G_2^c$. Hence, by (7), $t_*(v_1)$ and $t_*(v_2)$ are orthogonal to one another and have the same (nonzero) length. Hence t is conformal (and regular) at p. Thus, by multiplying v_1 and v_2 by an appropriate scalar if necessary, we may suppose v_1, v_2 is a g_0-orthonormal frame, too. Then (7) and Lemma 2 imply $k(p) = -1$.

If $t_{*|p} = 0$, then $\dim(\alpha_p) = 4$, so clearly $\dim(\alpha_p \cap \kappa_p) = 2$. Therefore suppose t is conformal and regular at p and $k(p) = -1$. Assume $\Phi \in \kappa_p - \{0\}$. Since t is conformal at p, g_0 induces the conformal structure on M at p. Since Φ: $T_pM \to t(p)$ is also conformal, an appropriate multiple of Φ, $m\Phi$, is an isometry from T_pM, with the inner product g_0, onto $t(p)$. Now choose v_1, v_2 so that $(m\Phi)(v_i) = e_i$; hence v_1, v_2 is a g_0-orthonormal frame, and $l_{ij}^{\alpha} = (h_{m\Phi}(v_j, v_i), e_{\alpha})$ by (1). But, by Lemma 2, $k(p) = -1$ implies (7) holds. In particular, $h_{m\Phi}$ is symmetric. Thus $m\Phi$ and hence $\Phi \in \alpha_p$.

COROLLARY. *If X: $M \to \mathbf{E}^{2+c}$ is a conformal minimal immersion with Gauss map t, then t is conformal with respect to the induced conformal structure on M and $K = -\rho$, where K and ρ are the Gaussian curvature of X and the Jacobian of the Gauss map t, respectively.*

PROOF. Since $\Phi = dX$ is a section in $\alpha \cap \kappa$ and $\mathbf{H}(\Phi) = 0$ on M, Proposition 1 implies $\alpha \cap \kappa$ is a rank 2 vector bundle. Hence Proposition 2 implies that t is conformal and $k = -1$ where $t_* \neq 0$. But for the Gaussian curvature of X, K, it is the case that $K = \rho k$, since ρ is the ratio of the area elements induced on M by t and X. So where $t_* \neq 0$, clearly $K = -\rho$. Where $t_* = 0$, both K and ρ are zero; so here too $K = -\rho$.

REMARK. It is well known that the Gauss map of a minimal immersion is conformal. That $K = -\rho$ follows immediately from [**4**, Theorem 1].

If M is simply connected and t: $M \to G_2^c$ is the Gauss map of a conformal minimal immersion, it is known that t is the Gauss map of infinitely many essential different conformal (minimal) immersions (see Hoffman and Osserman [**5**] or Remark 2 after Theorem 2 of this paper). But are there other immersions with Gauss map t which induce conformal structures other than the given one on M? If $a = 3$ on M, i.e., α is a rank 3 vector bundle, then one may show there exists a 3-plane $\mathbf{E}^3 \subset \mathbf{E}^{2+c}$ such that t: $M \to G_2(\mathbf{E}^3) \subset G_2^c$ (see §3 of Part 1). Hence, in this case, in general there exist nonconformal immersions of the Riemann surface M with Gauss map t since the first and third fundamental forms of an immersion X: $M \to \mathbf{E}^3$ need not be conformally equivalent. But if $a = 2$ on a dense (and necessarily open) subset of M, then the given conformal structure is the only one induced by any immersion of M with Gauss map t. This is a consequence of

PROPOSITION 3. *Suppose t is a Gauss map of a conformal immersion which is minimal at $p \in M$. If $a(p) = 2$, then the given conformal structure at p, is the only conformal structure induced by an immersion with Gauss map t.*

PROOF. By Proposition 1, $\dim(\alpha_p \cap \kappa_p) = 2$. Since $a(p) = 2$, it must be that $\kappa_p = \alpha_p$. Let J_p^* be another conformal structure at p and let $\kappa_p^* = \{\Phi \in \beta_p \mid J_{t(p)} \circ \Phi = \Phi \circ J_p^*\}$. Clearly $\alpha_p \cap \kappa_p^* = \kappa_p \cap \kappa_p^* \neq \{0\}$ if there exists an immersion with Gauss map t which induces the conformal structure J_p^*. But $\kappa_p \cap \kappa_p^* \neq \{0\}$ obviously implies $J_p^* = J_p$, the given conformal structure.

REMARK. If an immersion X: $M \to \mathbf{E}^{2+c}$ is minimal at $p \in M$, then necessarily $a(p) \geqslant 2$ by Proposition 1. It turns out that $a(p) = 2$ if and only if the normal bundle of X at p is not flat.

We now turn our attention to determining when $\alpha_p \cap \kappa_p \neq \{0\}$. In §2 of Part 1 we introduced *the span of* $t_{*|p}$, S_p, where

$$S_p = \operatorname{span}\left\{ t_*(v)(e) \middle| v \in T_pM,\ e \in t(p)\right\}.$$

Clearly $S_p \subset t^{\perp}(p)$; also we showed in Proposition 1 of Part 1 that $\dim(S_p) = 4 - a(p)$.

The Grassmannian G_2^c is regarded as the set of unit decomposable 2-vectors in $\Lambda^2\mathbf{E}^{2+c}$. Hence, in a natural way, G_2^c is a submanifold of $S^C(1)$, the set of unit 2-vectors in $\Lambda^2\mathbf{E}^{2+c}$; of course, $C = \binom{c+2}{2} - 1$. Let $\bar{q}$ denote the *second fundamental form of* G_2^c as a submanifold of $S^C(1)$. We showed in §4 of Part 1 that for all $l \in T_\pi G_2^c \cong \mathrm{GL}(\pi, \pi^{\perp})$

$$\tfrac{1}{2}\bar{q}(l, l) = l \wedge l(\pi), \tag{8}$$

which makes sense since π is regarded as the positive unit 2-vector in the second exterior power of the 2-plane π, $\Lambda^2\pi$. Now let $q = t^*\bar{q}$. It is easy to see that q_p takes values in $\Lambda^2 S_p$ for all $p \in M$. Hence, if $a(p) \geqslant 2$, so that $\dim(S_p) < 2$, it follows that $q_p = 0$. Assuming $a(p) \leqslant 2$, if $\lambda \in G_2(S_p)$ let $r^\lambda \colon t^\perp(p) \to \lambda$ be the orthogonal projection onto λ.

LEMMA 3. *Suppose $a(p) \leqslant 2$. If $v \in T_pM$ and $\lambda \in G_2(S_p) \subset \Lambda^2 S_p$, then*

$$\tfrac{1}{2}(q(v,v),\lambda) = \det(r^\lambda \circ t_*(v)).$$

PROOF. Let $e_1, e_2, \ldots, e_{2+c}$ be a $t(p)$-adapted frame such that $\lambda = \operatorname{span}\{e_3, e_4\}$. Then, since $t(p) = e_1 \wedge e_2$ and $\lambda = e_3 \wedge e_4$, (8) implies

$$\begin{aligned}\tfrac{1}{2}(q(v,v),\lambda) &= (t_*(v) \wedge t_*(v)(e_1 \wedge e_2), e_3 \wedge e_r)\\ &= \det\big[(t_*(v)(e_i), e_\alpha)\big]_{\substack{1 \leqslant i \leqslant 2\\ 3 \leqslant \alpha \leqslant 4}} = \det(r^\lambda \circ t_*(v)).\end{aligned}$$

If g is a riemannian metric on M, we define $\operatorname{tr}_g(q)$ by

$$\operatorname{tr}_g(q) = \sum_{i=1}^{2} q(v_i, v_i),$$

where v_1, v_2 is a g-orthonormal frame field defined on M.

We discuss for a moment the curvatures of $t^\perp$, the normal curvatures of t. Let g be a riemannian metric on M. If λ is an oriented 2-plane in $t^\perp(p)$, we may define the normal curvature of λ with respect to g as follows: Let $e_1, \ldots, e_{2+c}$ be a t-adapted frame field of $\mathbf{E}^{2+c}$ defined near p such that $e_3(p)$, $e_4(p)$ is a positively oriented frame of λ. Then define $(A^\lambda)^{e_i}\colon T_pM \to \lambda$ for $i \in \{1, 2\}$, by letting

$$(A^\lambda)^{e_i}(u) = -\sum_{\alpha=3}^{4} (de_i(u), e_\alpha)e_\alpha \tag{9}$$

for all $u \in T_pM$. By $\det((A^\lambda)^{e_i})$ we mean the determinant of a matrix representing $(A^\lambda)^{e_i}$ with respect to orthonormal frames of T_pM and λ. Then, the *normal curvature of* λ with respect to g, N_g^λ, is defined by

$$N_g^\lambda = \sum_{i=1}^{2} \det((A^\lambda)^{e_i}). \tag{10}$$

Of course, one may check that N_g^λ is independent of the choice of e_1, e_2 in $t(p)$ and e_3, e_4 in λ as well as the fact that N_g^λ actually depends on the area element on M associated to g rather than g. Perhaps the easiest way to see this is to introduce 1-forms $\omega_i^\alpha = (de_i, e_\alpha)$ for $i \in \{1,2\}$ and $\alpha \in \{3, \ldots, 2+c\}$, and show that, at p,

$$N_g^\lambda \mu_g = \sum_{i=1}^{2} \omega_i^3 \wedge \omega_i^4,$$

where μ_g is the area element on M associated to g. Also one may show that if X is an immersion with Gauss map t which induces the metric g on M, then the normal curvature of $\lambda \in G_2(t^\perp(p))$ for X, at p, is given by N_g^λ.

If $e_1, e_2, \ldots, e_{2+c}$ is a $t(p)$-adapted frame of $\mathbf{E}^{2+c}$ and g is a riemannian metric on M, set $N_g^{\alpha\beta} = N_g^\lambda$ if λ is the 2-plane having e_α, e_β (in that order) for a positively oriented frame, for distinct $\alpha, \beta \in \{3, \ldots, 2+c\}$. Then define $N_g\colon M \to t^\perp$ at p by

$$N_g(p) = \sum_{\alpha<\beta} N_g^{\alpha\beta} e_\alpha \wedge e_\beta.$$

It is straightforward to check that $N_g(p)$ is independent of the choice of the $t(p)$-adapted frame $e_1, e_2, \ldots, e_{2+c}$. Also since $N_g(p)$ actually only depends on the associated area element μ_g of g, it follows that $N_g(p)$ is determined up to a scalar multiple, in fact, up to a positive scalar multiple by the orientation of M. We call $N_g(p)$ the *primary normal curvature* 2*-vector* at p (with respect to g). Clearly $N_g(p) \in \Lambda^2 t^\perp(p)$. In fact, the following lemma holds.

LEMMA 4. $N_g(p) \in \Lambda^2 S_p$.

PROOF. It follows by letting $e_i = \Phi(v)$ in (1) of Part 1 and (9) that, for all $v \in T_pM$,

$$(11) \qquad (A^\lambda)^{e_i}(v) = -\sum_{\alpha=3}^{4} (t_*(v)(e_i), e_\alpha) e_\alpha,$$

where $e_1, \ldots, e_{2+c}$ is a $t(p)$-adapted frame of $\mathbf{E}^{2+c}$, and e_3, e_4 is a positively oriented frame of λ. If λ is orthogonal to S_p, i.e., λ contains a vector orthogonal to S_p, then clearly $\operatorname{rank}((A^\lambda)^{e_i}) < 2$ for $i \in \{1,2\}$; consequently $\det((A^\lambda)^{e_i}) = 0$. Therefore $N_g^\lambda = 0$ by (10) if λ is orthogonal to S_p, and thus $N_g(p) \in \Lambda^2 S_p$.

PROPOSITION 4. *Let g be a riemannian metric which induces the given conformal structure on M, i.e., $J^*g = g$. Then $\alpha_p \cap \kappa_p \neq \{0\}$ if and only if $\frac{1}{2}\operatorname{tr}_g(q) = N_g$ at p.*

PROOF. If $a(p) > 2$, then $\dim(S_p) < 2$. Hence $q_p = 0$, and thus $\operatorname{tr}_g(q_p) = 0$. But Lemma 4 also implies $N_g = 0$ since $\Lambda^2 S_p = 0$. However when $a(p) > 2$, clearly $\alpha_p \cap \kappa_p \neq \{0\}$ because of dimensional reasons. Therefore, for the remainder of this proof we assume $a(p) \leqslant 2$, i.e., $\dim(S_p) \geqslant 2$.

First observe that $\frac{1}{2}\operatorname{tr}_g(q) = N_g$ if and only if

$$(12) \qquad \tfrac{1}{2}\operatorname{tr}_g(q, \lambda) = N_g^\lambda$$

for all $\lambda \in G_2(S_p)$. This is so since both $\operatorname{tr}_g(q)$ and N_g lie in $\Lambda^2 S_p$, which is spanned by $G_2(S_p)$, and since $(\operatorname{tr}_g(q), \lambda) = \operatorname{tr}_g(q, \lambda)$ and $(N_g, \lambda) = N_g^\lambda$ for all $\lambda \in G_2(S_p)$.

Let $P\colon \beta \oplus \beta \to \mathbf{R}$ be the nondegenerate skew-symmetric tensor field introduced in §2 of Part 1. It is immediate from the definition of P that for any $\Phi, \Psi \in \beta_p$

$$(13) \qquad \nu P(\Phi, \Psi) = (\Phi(v_1), \Psi(v_2)) - (\Phi(v_2), \Psi(v_1)),$$

where $\nu = \mu_0(v_1 \wedge v_2) > 0$. For each $\lambda \in G_2(S_p)$, define $\alpha^{(\lambda)}$ to be the P-orthogonal complement of $A^{(\lambda)} = \{A^z \mid z \in \lambda\}$ in β_p. Clearly, by Lemma 2 of Part 1, the collection $\{\alpha^{(\lambda)} \mid \lambda \in G_2(S_p)\}$ is the same as the collection of all 2-planes in β_p containing α_p. This leads to our second observation: $\alpha_p \cap \kappa_p \neq \{0\}$ if and only if $\alpha^{(\lambda)} \cap \kappa_p \neq \{0\}$ for all $\lambda \in G_2(S_p)$.

Using these two observations the proof of the proposition reduces to showing the following: *For all* $\lambda \in G_2(S_p)$, $\alpha^{(\lambda)} \cap \kappa_p \neq \{0\}$ *if and only if* (12) *holds.*

Pick $\lambda \in G_2(S_p)$. Let v_1, v_2 be a positively oriented g-orthonormal frame of T_pM. Also let $e_1, e_2, \ldots, e_{2+c}$ be a $t(p)$-adapted frame of $\mathbf{E}^{2+c}$ with e_3, e_4 a positively oriented frame of λ. Now define $\chi_1, \chi_2 \in \kappa_p$ by $\chi_1(v_i) = e_i$ for $i \in \{1,2\}$, and $\chi_2 = J \circ \chi_1$. Clearly χ_1, χ_2 is a basis of κ_p. Define $P^\alpha\colon \beta_p \to \mathbf{R}$ for $\alpha \in \{3,4\}$ by

$$P^\alpha(\Phi) = P(A^{e_\alpha}, \Phi)$$

for all $\Phi \in \beta_p$. By definition, $\alpha^{(\lambda)} = \ker(P^3) \cap \ker(P^4)$. Hence $\alpha^{(\lambda)} \cap \kappa_p \neq \{0\}$ if and only if $P^3|\kappa_p$, $P^4|\kappa_p$ are linearly dependent. The linear dependence may be expressed using the basis χ_1, χ_2 of κ_p. Hence $\alpha^{(\lambda)} \cap \kappa_p \neq \{0\}$ if and only if

$$\begin{vmatrix} P^3(\chi_1) & P^3(\chi_2) \\ P^4(\chi_1) & P^4(\chi_2) \end{vmatrix} = 0. \tag{14}$$

Define reals l^α_{ij} as in (3). Since by (13),

$$\begin{aligned} \nu P^\alpha(\chi_i) &= \nu P(\chi_i, A^{e_\alpha}) \\ &= (\chi_i(v_1), A^{e_\alpha}(v_2)) - (\chi_i(v_2), A^{e_\alpha}(v_1)) \\ &= (t_*(v_2)(\chi_i(v_1)) - t_*(v_1)(\chi_i(v_2)), e_\alpha) \end{aligned}$$

for $i \in \{1,2\}$, $\alpha \in \{3,4\}$, (14) becomes

$$\begin{vmatrix} l^3_{12} - l^3_{21} & l^3_{11} + l^3_{22} \\ l^4_{12} - l^4_{21} & l^4_{11} + l^4_{22} \end{vmatrix} = 0,$$

i.e.,

$$\begin{vmatrix} l^3_{11} & l^3_{21} \\ l^4_{11} & l^4_{21} \end{vmatrix} + \begin{vmatrix} l^3_{12} & l^3_{22} \\ l^4_{12} & l^4_{22} \end{vmatrix} = \begin{vmatrix} l^3_{11} & l^3_{12} \\ l^4_{11} & l^4_{12} \end{vmatrix} + \begin{vmatrix} l^3_{21} & l^3_{22} \\ l^4_{21} & l^4_{12} \end{vmatrix}.$$

Using (11), this last equation states that

$$\det(r^\lambda \circ t_*(v_2)) + \det(r^\lambda \circ t_*(v_1)) = \det((A^\lambda)^{e_1}) + \det((A^\lambda)^{e_2}), \tag{15}$$

where $r^\lambda\colon t^\perp(p) \to \lambda$ is the orthogonal projection onto λ. Finally, applying Lemma 3 and (10), we obtain

$$\tfrac{1}{2}\operatorname{tr}_g(q, \lambda) = N_g^\lambda.$$

An immediate consequence of condition [K] and Proposition 4 is our first theorem.

Theorem 1. *If $t\colon M \to G_2^c$ is a smooth map for which $\frac{1}{2}\operatorname{tr}_g(q) \neq N_g$ for some (and hence any) metric g which induces the conformal structure on M, then t is not a Gauss map.*

Remark 1. The equation $\frac{1}{2}\operatorname{tr}_g(q) = N_g$ corresponds to equation (2.20) of [5]. Both these equations represent the algebraic aspect to the necessary and sufficient conditions on t in order for it to be locally a Gauss map.

REMARK 2. If we view t: $M \to S^C(1)$, then we may view $\frac{1}{2}\mathrm{tr}_g(q)$ as the component of the tension $\tau = \mathrm{tr}_g \nabla(t_*)$ (with respect to the metric g on M and the standard metric on $S^C(1)$) which is orthogonal to G_2^c; cf. [**4**, §3].

3. The p.d.e. in [K]. We now turn to study the p.d.e. $(d\Phi)^\top = 0$, where Φ is a section in $\alpha \cap \kappa$, under the regularity assumption that $\alpha \cap \kappa$ is a vector bundle necessarily of rank 1 or 2.

Let us consider first the situation in which $\alpha \cap \kappa = \kappa$, i.e., $\mathrm{rank}(\alpha \cap \kappa) = 2$, for the given t: $M \to G_2^c$. In order to carry out our investigations in this situation we make the further assumptions that M is not compact and is simply connected. Since both t (viewed as a vector bundle) and $\alpha \cap \kappa$ are of rank 2 and M is not compact, there exists a globally defined positively oriented orthonormal frame field e_1, e_2 in t and a globally defined nowhere vanishing section χ in $\alpha \cap \kappa$. Set $E = e_1 - ie_2$; it is a globally defined $\mathbf{E}^{2+c} \otimes \mathbf{C}$-valued vector field on M. Hence there exists a globally defined differential form ξ of type $(1,0)$ such that $\chi = \mathrm{Re}[E\xi]$. Also $(d\chi)^\top = \mathrm{Re}[E\xi \wedge \eta]$ for some differential form η of type $(0,1)$. Any section Φ in $\alpha \cap \kappa$ may be written $\Phi = \mathrm{Re}[Eu\xi]$, where u: $M \to \mathbf{C}$ is smooth, and

$$(d\Phi)^\top = \mathrm{Re}[E(du - u\eta) \wedge \xi]. \tag{16}$$

Hence, by [K] and (16), for t to be the Gauss map of a conformal immersion it is sufficient to find a nowhere vanishing u: $M \to \mathbf{C}$ satisfying $\bar\partial u - u\eta = 0$, where $\bar\partial u$ is the $(0,1)$-part of du. Thus it is sufficient to solve

$$\bar\partial w = \eta \tag{17}$$

and set $u = e^w$. Since M is not compact and is simply connected, and hence is conformally equivalent to either the unit disk in $\mathbf{C}$ or $\mathbf{C}$ itself, it follows from Theorem 4 of [**3**] that (17) has global solutions. In fact, the general solution of (17) is of the form $w_0 + f$, where w_0 is a particular solution of (17) and f: $M \to \mathbf{C}$ is holomorphic. Hence, if M is not compact, is simply connected, and $\mathrm{rank}(\alpha \cap \kappa) = 2$, then t is a Gauss map. Also, by Proposition 1(i), any conformal immersion with Gauss map t is necessarily minimal. We summarize these results in

THEOREM 2. *Let M be a simply connected noncompact Riemann surface and suppose t: $M \to G_2^c$ is a smooth conformal map. If $k = -1$, where $t_* \neq 0$, then t is the Gauss map of a conformal minimal immersion X: $M \to \mathbf{E}^{2+c}$.*

PROOF. Proposition 2 implies $\alpha \cap \kappa$ is a rank 2 vector bundle so indeed the foregoing remarks apply.

REMARK 1. Theorem 2 is a partial converse to the corollary to Proposition 2. Also Theorem 2 corresponds to Case 1 of Theorem 2.6 of Hoffman and Osserman [**5**], since $V \equiv 0$ (see [**5**]) corresponds to $\alpha \cap \kappa$ being a rank 2 vector bundle.

It is clear that if X is a conformal immersion with Gauss map t, then so is the immersion $sX + X_0$, where $0 \neq s \in \mathbf{R}$ and $X_0 \in \mathbf{E}^{2+c}$. If Y is a conformal immersion with Gauss map t and not one of the immersions $sX + X_0$, we will say that Y is *essentially different* from X.

REMARK 2. Assume now that M is just noncompact. Let X be a conformal minimal immersion with Gauss map t; necessarily $\alpha \cap \kappa$ is a rank 2 vector bundle. We may use $\mathrm{Re}[\partial X]$ for χ above, where $\partial X = (\partial X/\partial z)\,dz$ in a local complex coordinate z. Any section Φ in $\alpha \cap \kappa$ may be written as $\Phi = \mathrm{Re}[u\partial X]$, and condition [K] implies Φ is the differential of a conformal immersion Y with Gauss map t if and only if u is a nonvanishing globally defined holomorphic function. The immersion Y is essentially different from X if and only if u is not a (constant) real-valued function. If we let u be a nonreal constant function, then we obtain a conformal minimal immersion Y which is a multiple of what is called an associate minimal surface to X (see [**9**, p. 117]). But any of these associate minimal surfaces Y induces a riemannian metric on M which is a constant multiple of the one induced on M by X. If u is a nonconstant holomorphic function, then the corresponding conformal minimal immersion Y induces a metric on M which is not a constant multiple of the one induced on M by X.

We now assume that $\alpha \cap \kappa$ is a line bundle over M, i.e., $\mathrm{rank}(\alpha \cap \kappa) = 1$, for the given map t. Under this assumption we assume M is simply connected. Hence $\alpha \cap \kappa$ has a globally defined nowhere vanishing section which we denote by χ. By Proposition 1(iii) we may and, in fact, do suppose that $\mathbf{H}(\chi)$ is a unit vector field. This determines χ up to sign. Let $\tilde{g}$ be the metric induced on M by χ, i.e., $\tilde{g}(u, v) = (\chi(u), \chi(v))$ for all $(u, v) \in TM \oplus TM$. Note that $\tilde{g}$ is independent of the sign of χ.

Any nowhere vanishing section Φ in $\alpha \cap \kappa$ may be written as

$$\Phi = H^{-1}\chi, \tag{18}$$

where necessarily $\mathbf{H}(\Phi) = H \cdot \mathbf{H}(\chi)$; thus Φ's "scalar mean curvature" is H.

Let $e_1, e_2, \ldots, e_{2+c}$ be a t-adapted frame field of $\mathbf{E}^{2+c}$ defined on some open subset of M such that $e_3 = \mathbf{H}(\chi)$ (we may have to change the sign of χ if $c = 1$ in order to accomplish this). Then $\chi = e_1\psi^1 + e_2\psi^2$, where ψ^1, ψ^2 are 1-forms such that $\tilde{g} = (\psi^1)^2 + (\psi^2)^2$. For any section $\Phi = H^{-1}\chi$ in $\alpha \cap \kappa$,

$$(d\Phi)^{\top} = d(H^{-1}) \wedge \left(e_1\psi^1 + e_2\psi^2\right) + H^{-1}\left(\psi_1^2 - \omega_1^2\right) \wedge \left(e_1\psi^2 - e_2\psi^1\right),$$

where the connection form $\psi_1^2 = -\psi_2^1$ is defined by $d\psi^1 = -\psi_2^1 \wedge \psi^2$ and $d\psi^2 = -\psi_1^2 \wedge \psi^1$, and $\omega_1^2 = (de_1, e_2)$. Writing the e_1 and e_2 components of the p.d.e. $(d\Phi)^{\top} = 0$ of $[K]$ for $\Phi = H^{-1}\chi$ leads to

$$\begin{cases} d(H^{-1}) \wedge \psi^1 = H^{-1}\left(\omega_1^2 - \psi_1^2\right) \wedge \psi^2 = -H^{-1} * \left(\omega_1^2 - \psi_1^2\right) \wedge \psi^1, \\ d(H^{-1}) \wedge \psi^2 = -H^{-1}\left(\omega_1^2 - \psi_1^2\right) \wedge \psi^1 = -H^{-1} * \left(\omega_1^2 - \psi_1^2\right) \wedge \psi^2, \end{cases} \tag{19}$$

where $*$ is the Hodge star operator associated with $\tilde{g}$. However, (19) may be written more succinctly as

$$d\left(\log|H|\right) = *\left(\omega_1^2 - \psi_1^2\right). \tag{20}$$

The integrability condition for (20), i.e., $(d\Phi)^{\top} = 0$ for $\Phi = H^{-1}\chi$, is

$$d * \left(\omega_1^2 - \psi_1^2\right) = 0. \tag{21}$$

We now wish to find an interpretation for $*(\omega_1^2 - \psi_1^2)$. It turns out to be intimately related to the tension of t with respect to the metric $\tilde{g}$ on M and the standard metric $\bar{g}_0$ on G_2^c. From the work of Hoffman and Osserman [5], it is no surprise that the tension is involved.

We again return to the geometry of G_2^c. For $i \in \{1,2\}$ and $\alpha \in \{3,\ldots,2+c\}$, let E_α^i be the TG_2^c-valued vector field along t defined by $E_\alpha^i(e_j) = \delta_j^i e_\alpha$, where δ_j^i is Kronecker's delta, and let ω_A^B be 1-forms defined by $\omega_A^B = (de_A, e_B)$ for $A, B \in \{1,2,\ldots,2+c\}$. Then E_α^i's and ω_i^α's are dual to one another in the sense that $t_* = \sum_{i,\alpha} E_\alpha^i \omega_i^\alpha$. The Cartan structural equations imply

$$dE_\alpha^i = -\sum_j E_\alpha^j \omega_j^i + \sum_\beta E_\beta^i \omega_\alpha^\beta.$$

Let v_1, v_2 be vector fields in TM dual to ψ^1, ψ^2, i.e., $\chi(v_i) = e_i$ for $i \in \{1,2\}$. Then define l_{ij}^α as in (3); one easily sees that

$$t_* = \sum_{i,j,\alpha} E_\alpha^i l_{ij}^\alpha \psi^j, \tag{22}$$

i.e., $\omega_i^\alpha = \sum_j l_{ij}^\alpha \psi^j$. Note that $l_{ij}^\alpha = l_{ji}^\alpha$ by (1) since χ is a section of α. For $\beta \in \{3,\ldots,2+c\}$, define $m^\beta = \frac{1}{2}(l_{11}^\beta + l_{22}^\beta)$; then by (2) $\mathbf{H}(\chi) = \sum_\beta m^\beta e_\beta = e_3$ so that $m^3 = 1$ and $m^\beta = 0$ for $\beta > 3$. Now

$$\begin{aligned} 0 = dt_* = &\sum_{i,j,k,\alpha} \left(-E_\alpha^i \omega_i^j \wedge l_{jk}^\alpha \psi^k + \delta_i^j E_\alpha^i dl_{jk}^\alpha \wedge \psi^k - E_\alpha^i l_{ij}^\alpha \psi_k^j \wedge \psi^k\right) \\ &+ \sum_{i,k,\alpha,\beta} E_\alpha^i \omega_\beta^\alpha \wedge l_{ik}^\beta \psi^k, \end{aligned}$$

where i, j, k range over 1,2 and α, β range over $3,4,\ldots,2+c$. Setting the coefficients of E_α^i in the preceding equation equal to zero we conclude that

$$\begin{aligned} dl_{12}^\alpha(v_1) = {}& dl_{11}^\alpha(v_2) - l_{11}^\alpha \psi_1^2(v_1) + l_{22}^\alpha \omega_1^2(v_1) - l_{12}^\alpha\left(\omega_1^2(v_2) + \psi_1^2(v_2)\right) \\ &- \sum_\beta \left(l_{11}^\beta \omega_\alpha^\beta(v_2) - l_{12}^\beta \omega_\alpha^\beta(v_1)\right), \end{aligned}$$

and

$$\begin{aligned} dl_{21}^\alpha(v_2) = {}& dl_{22}^\alpha(v_1) + l_{22}^\alpha \psi_1^2(v_2) - l_{11}^\alpha \omega_1^2(v_2) + l_{12}^\alpha\left(\omega_1^2(v_1) + \psi_1^2(v_1)\right) \\ &- \sum_\beta \left(l_{22}^\beta \omega_\alpha^\beta(v_1) - l_{21}^\beta \omega_\alpha^\beta(v_2)\right). \end{aligned}$$

By (22), $*t_* = \sum_{i,j,\alpha} E_\alpha^i l_{ij}^\alpha * \psi^j$. After computing $*d*t_*$, replace $dl_{12}^\alpha(v_1)$ and $dl_{21}^\alpha(v_2)$ by the right-hand sides of the preceding equations. This gives the formula we need for the tension $\tau = *d*t_*$ of t with respect to the metrics $\tilde{g}$ and $\bar{g}_0$ on M and G_2^c, respectively. The formula is

$$\begin{aligned} \tfrac{1}{2}\tau = {}& \sum_\alpha E_\alpha^1 \left\{ dm^\alpha(v_1) - m^\alpha\left(\omega_1^2 - \psi_1^2\right)(v_2) - \sum_\beta m^\beta \omega_\alpha^\beta(v_1) \right\} \\ &+ \sum_\alpha E_\alpha^2 \left\{ dm^\alpha(v_2) + m^\alpha\left(\omega_1^2 - \psi_1^2\right)(v_1) - \sum_\beta m^\beta \omega_\alpha^\beta(v_2) \right\}, \end{aligned}$$

or

$$\tfrac{1}{2}\tau = \sum_{i,\alpha} E^i_\alpha \Big\{ dm^\alpha + m^\alpha * (\omega^2_1 - \psi^2_1) - \sum_\beta m^\beta \omega^\beta_\alpha \Big\} (v_i).$$

Since $m^3 = 1$, and $m^\beta = 0$ for $\beta > 3$, this last equation becomes

$$\tfrac{1}{2}\tau = \sum_i E^i_3 * (\omega^2_1 - \psi^2_1)(v_i) + \sum_{i,\alpha} E^i_\alpha \omega^\alpha_3 (v_i).$$

We can easily solve for $*(\omega^2_1 - \psi^2_1)$; in fact, we obtain

$$*(\omega^2_1 - \psi^2_1) = \tfrac{1}{2}\sum_i \bar{g}_0(\tau, E^i_3)\psi^i = \tfrac{1}{2}\sum_i (\tau(e_i), e_3)\psi^i.$$

If we view (τ, e_3): $t \to \mathbf{R}$ by $(\tau, e_3)(e) = (\tau(e), e_3(p))$ for $e \in t(p)$, $p \in M$, then

$$*(\omega^2_1 - \psi^2_1) + \tfrac{1}{2}(\tau, e_3) \circ \chi \tag{23}$$

since χ: $TM \to t$ equals $\sum_i e_i \psi^i$. Set

$$\tilde{\tau} = (\tau, e_3) \circ \chi.$$

Note that $\tilde{\tau}$ is a 1-form which involves only the projection of τ onto e_3, the direction of the mean curvature. Especially noteworthy is the observation that $\tilde{\tau}$ is independent of the sign of χ and hence is well defined even if M is not simply connected. Clearly $\tau = 0$ if and only if $\tilde{\tau} = 0$ and e_3 is parallel in the normal bundle $\tau^\perp$. When $c = 1$, $\tau = 0$ if and only if $\tilde{\tau} = 0$ since e_3 is automatically parallel in the line bundle $t^\perp$.

Hence, by (23), $(d\Phi)^\tau = 0$ for $\Phi = H^{-1}\chi$, or (20) becomes

$$d(\log|H|) = \tfrac{1}{2}\tilde{\tau}, \tag{24}$$

and the integrability condition for (24) is

$$d\tilde{\tau} = 0. \tag{25}$$

Finally note from (18) and (24) that the general solution of $(d\Phi)^\tau = 0$, Φ a nonvanishing section in $\alpha \cap \kappa$, is determined up to an arbitrary nonzero factor. Hence any conformal immersion X with Gauss map t, which is given by $X = \int H^{-1}\chi$, where H satisfies (24), is determined up a homothety and translation of $\mathbf{E}^{2+c}$.

We summarize the foregoing discussion in the statement of the following

Theorem 3. *Let M be a simply connected Riemann surface and t: $M \to G^c_2$ be a smooth map. Also assume $\frac{1}{2}\mathrm{tr}_g(q) = N_g$ on M, where g is a metric that induces the conformal structure on M, and $k \neq -1$ anywhere t_* is conformal and nonvanishing. Then t is a Gauss map if and only if $d\tilde{\tau} = 0$. Any conformal immersion X with Gauss map t is given by $X = \int H^{-1}\chi$, where H is the scalar mean curvature of X. The scalar mean curvature in turn is given by $H = H_0 \exp(\frac{1}{2}\int\tilde{\tau})$, where $H_0 \in \mathbf{R}$.*

Proof. The conditions that $\frac{1}{2}\mathrm{tr}_g(q) = N_g$ on M and $k \neq -1$ anywhere t_* is conformal and nonvanishing imply that $\alpha \cap \kappa$ is a rank 1 vector bundle by Propositions 2 and 4. Hence the discussion preceding the statement of this theorem proves the theorem.

REMARK. Theorem 3 corresponds to Case 2 of Theorem 2.6 and Theorem 2.7 of Hoffman and Osserman [5].

In the following, let $\mathbf{R}^+ = \{r \in \mathbf{R} \mid r > 0\}$.

COROLLARY. *Let M and t: $M \to G_2^c$ satisfy the same hypotheses as those of Theorem 3. In addition, let H: $M \to \mathbf{R}^+$ be a smooth function such that $d(\log H) = \frac{1}{2}\tilde{\tau}$; then there exists a conformal immersion X: $M \to \mathbf{E}^{2+c}$ with Gauss map t and scalar mean curvature H. In particular, if $\tilde{\tau} = 0$, there exist conformal immersions with Gauss map t and constant scalar mean curvature.*

REMARK. This corollary generalizes to the higher codimensions results of Kenmotsu [8].

Now we present the principal uniqueness result discovered by Hoffman and Osserman [5].

THEOREM 4. *Let X: $M \to \mathbf{E}^{2+c}$ be a conformal immersion with Gauss map t and mean curvature $\mathbf{H}$. If $\mathbf{H}(p) \neq 0$ for some $p \in M$, then there exist no conformal immersions essentially different from X with Gauss map t.*

PROOF. Any section Φ in κ may be written as $\Phi = \mathrm{Re}[u\partial X]$, where u: $M \to \mathbf{C}$ is smooth. If Φ is the differential of an immersion, then $(d\Phi)^\top = 0$; this implies $\bar{\partial} u = 0$, i.e., u is holomorphic on M. In a neighborhood of U of p, $\alpha \cap \kappa$ is a rank 1 bundle; hence if Φ is the differential of an immersion, $\Phi \mid U$ is a section in $\alpha \cap \kappa \mid U$, so that u is real-valued on U. Thus $u =$ real constant.

4. A closer look at t: $M \to G_2^2$. It is well known [1], when G_2^2 is regarded as the set of unit decomposable 2-vectors in $\Lambda^2\mathbf{E}^4$ and $\Lambda^2\mathbf{E}^4$ is identified with $\mathbf{E}^6$, that

$$G_2^2 = \left\{ (\xi, \eta) \in \mathbf{E}^3 \times \mathbf{E}^3 = \mathbf{E}^6 \mid \|\xi\|^2 = \|\eta\|^2 = \tfrac{1}{2} \right\},$$

i.e., $G_2^2 = S^2(1/\sqrt{2}) \times S^2(1/\sqrt{2})$ where these spheres lie in 3-planes orthogonal to one another. Let $\bar{g}_i$ be the riemannian metric on the ith factor of G_2^2 for $i \in \{1, 2\}$. Clearly

$$\bar{g}_0 = \bar{g}_1 + \bar{g}_2. \tag{26}$$

In this section we view the second fundamental form $\bar{q}$ of G_2^2 in $S^5(1)$ as real-valued; we can do this, since $\dim(\pi^\perp) = 2$ for each $\pi \in G_2^2$, as follows: For $l \in T_\pi G_2^2$, we redefine $\bar{q}$ by letting

$$\tfrac{1}{2}\bar{q}(l, l)\pi^\perp = l \wedge l(\pi); \tag{27}$$

this makes sense since $l \wedge l(\pi) \in \Lambda^2\pi^\perp$ and we regarded $\pi^\perp$ as the positive unit 2-vector in $\Lambda^2\pi^\perp$. It turns out (cf. proof of Lemma 5 of Part 1) that

$$\tfrac{1}{2}\bar{q}(l, l) = \det(l\colon \pi \to \pi^\perp)$$

for all $l \in T_\pi G_2^2$. We may also write $\bar{q}$ in terms of $\bar{g}_i$, $i \in \{1, 2\}$.

LEMMA 5. $\bar{q} = \bar{g}_2 - \bar{g}_1$.

PROOF. If $\pi = (\xi, \eta) \in S^2(1/\sqrt{2}) \times S^2(1/\sqrt{2}) = G_2^2$, one may show [**1**] that $\pi^\perp = (\xi, -\eta)$. Then

$$\begin{aligned}\bar{q} &= -(d\pi, d\pi^\perp) = -((d\xi, d\eta), (d\xi, -d\eta))\\ &= -[(d\xi, d\xi) - (d\eta, d\eta)] = -\bar{g}_1 + \bar{g}_2.\end{aligned}$$

If t: $M \to G_2^2$ is smooth we let $g_i = t^*\bar{g}_i$ for $i \in \{0,1,2\}$ and $q = t^*\bar{q}$. Of course, (26) and Lemma 5 imply

$$g_0 = g_1 + g_2 \quad \text{and} \quad q = -g_1 + g_2. \tag{28}$$

Let g be a riemannian metric on M. The primary normal curvature, $N_g(p)$, may be redefined to be a scalar, since $\dim(t^\perp(p)) = 2$, by setting

$$N_g(p) = N_g^{t^\perp(p)}. \tag{29}$$

Now let t_i: $M \to S^2(1/\sqrt{2})$ be the ith component of t: $M \to G_2^2 = S^2(1/\sqrt{2}) \times S^2(1/\sqrt{2})$ for $i \in \{1,2\}$. For $i \in \{1,2\}$, let $\varepsilon_i(g)$ be the energy density of t_i with respect to the metrics g and $\bar{g}_i$ (cf. [**2**]). Also, for $i \in \{1,2\}$, define $\rho_i(g)$ by $\mu_{g_i} = \rho_i(g)\mu_g$, where μ_g is the area element associated to g and μ_{g_i} is the "area element" associated to g_i; we call $\rho_i(g)$ the Jacobian of t_i with respect to g and $\bar{g}_i$.

PROPOSITION 5. *Let g be a riemannian metric on M that induces the given conformal structure; then $\alpha_p \cap \kappa_p \neq \{0\}$ if and only if at p*

$$\varepsilon_1(g) + \rho_1(g) = \varepsilon_2(g) + \rho_2(g).$$

PROOF. Proposition 4 states that $\alpha_p \cap \kappa_p \neq \{0\}$ if and only if $\frac{1}{2}\mathrm{tr}_g(q_p) = N_g(p)$; this is clearly also true for q_p and $N_g(p)$ redefined by (27) and (29), respectively. But, by (28),

$$\tfrac{1}{2}\mathrm{tr}_g(q) = \tfrac{1}{2}\mathrm{tr}_g(g_2 - g_1) = \varepsilon_2(g) - \varepsilon_1(g), \tag{30}$$

according to the definition of energy density. On the other hand, arguing as in [**12**, §5], we can show

$$N_g = \rho_1(g) - \rho_2(g). \tag{31}$$

Hence, $\alpha_p \cap \kappa_p \neq \{0\}$ if and only if, at p, $\varepsilon_2(g) - \varepsilon_1(g) = \rho_1(g) - \rho_2(g)$.

COROLLARY. *Suppose t: $M \to G_2^2$ is a Gauss map. Then t_1 is antiholomorphic at p if and only if t_2 is antiholomorphic at p.*

PROOF. The component t_i of t is antiholomorphic at p if and only if $\varepsilon_i(g) + \rho_i(g) = 0$ at p for $i \in \{1,2\}$, where g is a metric on M which induces the given conformal structure.

REMARK. The equation $\varepsilon_1(g) + \rho_1(g) = \varepsilon_2(g) + \rho_2(g)$ corresponds to equation (4.7) of Hoffman and Osserman [**6**].

PROPOSITION 6. *Let $\Phi \in \alpha_p \cap \kappa_p - \{0\}$ and let g be a riemannian metric on M such that g_p is the metric induced on T_pM by Φ. Then, at p, for $i \in \{1,2\}$*

$$\varepsilon_i(g) + \rho_i(g) = \|\mathbf{H}(\Phi)\|^2.$$

PROOF. Let e_1, e_2, e_3, e_4 be a $t(p)$-adapted frame of $\mathbf{E}^4$. Define $v_i \in T_pM$ by $\Phi(v_i) = e_i$ for $i \in \{1,2\}$. Then define scalars l^α_{ij} by (3) for $i, j \in \{1,2\}$ and $\alpha \in \{3,4\}$. Since $\Phi \in \alpha_p$, h_Φ is symmetric; thus $l^\alpha_{ij} = l^\alpha_{ji}$ by (1). Clearly $\mathbf{H}(\Phi) = \frac{1}{2}\sum_{i,\alpha} l^\alpha_{ii} e_\alpha$ by (2); hence

$$\|\mathbf{H}(\Phi)\|^2 = \tfrac{1}{4}\sum_\alpha\Big(\sum_i l^\alpha_{ii}\Big)^2. \tag{32}$$

Since $\alpha_p \cap \kappa_p \neq \{0\}$, Proposition 5 implies $\varepsilon_1(g) + \rho_i(g) = \varepsilon_2(g) + \rho_2(g)$. Hence, it is enough to prove $\sum_{i=1}^2 \varepsilon_i(g) + \rho_i(g) = 2\|\mathbf{H}(\Phi)\|^2$. First, observe by the first equation of (28) that $\varepsilon_1(g) + \varepsilon_2(g) = \varepsilon_0(g)$, the energy density of t with respect to g and $\bar{g}_0$. Next observe that $\rho_1(g) + \rho_2(g) = K_g$, the curvature of t with respect to g; this follows by arguing as in [**12**, §5]. Hence

$$\sum_{i=1}^2 \varepsilon_i(g) + \rho_i(g) = \varepsilon_0(g) + K_g. \tag{33}$$

Since v_1, v_2 is a g-orthonormal frame,

$$\varepsilon_0(g) = \tfrac{1}{2}\sum_{i,j,\alpha}\left(l^\alpha_{ij}\right)^2 \tag{34}$$

and

$$K_g = \sum_\alpha \left|l^\alpha_{ij}\right|. \tag{35}$$

Using (34) and (35) in the right side of (33) and simplifying we obtain

$$\sum_{i=1}^2 \varepsilon_i(g) + \rho_i(g) = \tfrac{1}{2}\sum_\alpha\Big(\sum_i l^\alpha_{ii}\Big)^2.$$

The proposition follows from (32).

COROLLARY 1. *Suppose $t_{*|p} \neq 0$ and g is a riemannian metric on M which induces the given conformal structure. Then $\alpha_p \supset \kappa_p$ if and only if $\varepsilon_i(g) + \rho_i(g) = 0$ for $i \in \{1,2\}$.*

PROOF. The proof is an immediate consequence of Propositions 1 and 6.

Of course, when $\Phi = dX$ for an immersion X: $M \to \mathbf{E}^4$, we obtain information about X.

COROLLARY 2. *Suppose X: $M \to \mathbf{E}^4$ is an immersion with Gauss map t. Let g be the induced metric and let $N = N_g$ be the normal curvature of X. Then $N = \varepsilon_2(g) - \varepsilon_1(g)$.*

PROOF. This follows from Proposition 6 and (31).

COROLLARY 3. *Suppose X: $M \to \mathbf{E}^4$ is an immersion with induced metric g and mean curvature $\mathbf{H}$. Then $\|\mathbf{H}\|^2 = \varepsilon_i(g) + \rho_i(g)$ for $i \in \{1,2\}$.*

COROLLARY 4. *An immersion X: $M \to \mathbf{E}^4$ is minimal if and only if each factor of t is antiholomorphic.*

PROOF. The map t_i is antiholomorphic if and only if $\rho_i(g) = -\varepsilon_i(g)$.

REMARK. Corollaries 2 and 3 of Proposition 6 are new results although the fact that $\|\mathbf{H}\|^2 = \varepsilon_i(g) + \rho_i(g)$ for $i \in \{1,2\}$ is hinted at in the calculations in [7]. Corollary 4 is, of course, well known.

We know of no particularly nice way of writing the integrability condition $d\tilde{\tau} = 0$ for the p.d.e. $(d\Phi)^{\top} = 0$, where Φ is a nowhere vanishing section in $\alpha \cap \kappa$, in terms of the components t_1, t_2 of t when $c = 2$. Thus, since we can do little more than replace the algebraic condition in Theorems 1–3 by ones involving $\varepsilon_i(g) + \rho_i(g)$, $i \in \{1,2\}$, we do not write out the corresponding theorems for the case $c = 2$. We turn instead to a characterization of the Gauss map of a minimal immersion into a 3-sphere S^3 of $\mathbf{E}^4$. But first we need

LEMMA 6. *Suppose t is regular at p. Then $\frac{1}{2}\operatorname{tr}_{g_0}(q) = -kN_{g_0}$ at p.*

PROOF. By (28), $\frac{1}{2}\operatorname{tr}_{g_0}(q) = \det(g_2 - g_1, g_1 + g_2)/\det(g_0)$, where $\det(\ ,\ )$ stands for the polarization of the quadratic form det on 2×2-matrices, and the g_i for $i \in \{0,1,2\}$ in the arguments of these determinants stand for matrices representing the quadratic forms g_i with respect to the same fixed basis. But

$$\det(g_2 - g_1, g_1 + g_2) = \det(g_2) - \det(g_1).$$

Hence $\frac{1}{2}\operatorname{tr}_{g_0}(q) = (\det(g_2) - \det(g_1))/\det(g_0) = \rho_2(g_0)^2 - \rho_1(g_0)^2 = -kN_{g_0}$, since $k = \rho_1(g_0) + \rho_2(g_0)$ and $N_{g_0} = \rho_1(g_0) - \rho_2(g_0)$.

That the normal bundle $t^{\perp}$ is flat is independent of the conformal structure on M since if $N_g = 0$ for *any* riemannian metric on M, then $N_g = 0$ for all riemannian metrics on M.

THEOREM 5. *Let t: $M \to G_2^2$ be a smooth map of a simply connected Riemann surface. Then t is the Gauss map of a conformal minimal immersion X: $M \to S^3$ if and only if t is a conformal minimal immersion with flat normal bundle $t^{\perp}$ and $k > -1$.*

PROOF. Suppose t is a conformal minimal immersion with $N_{g_0} = 0$ and $k > -1$. By [**4**, Proposition 2], t is a conformal harmonic immersion. Since t is harmonic, $\tau = 0$ (cf. [**2**]); thus by Theorem 3 and (23), it is enough to show that $\frac{1}{2}\operatorname{tr}_{g_0}(q) = N_{g_0}$ in order for t to be the Gauss map of a conformal immersion. But we are given that $N_{g_0} = 0$, and by Lemma 6, $\frac{1}{2}\operatorname{tr}_{g_0}(q) = 0$, too. Let X be a conformal immersion with Gauss map t. Since t is conformal, X is pseudoumbilic [**10**], and since t is harmonic, X has parallel mean curvature [**11**]. Also, by Proposition 2, X is nowhere minimal. Thus, by the Lemma on p. 446 of [**4**], X is a conformal minimal immersion into a 3-sphere $S^3 \subset \mathbf{E}^4$. The proof in the other direction is straightforward.

The condition that $N_{g_0} = \rho_i(g_0) - \rho_2(g_0) = 0$ has a nice geometrical interpretation; by the results of [**12**, §5], it says that the tangent planes to $t(M)$ make equal angles with the two $S^2(1/\sqrt{2})$-factors of G_2^2. Also since q is the second fundamental form of t: $M \to S^5(1)$ in the direction normal to G_2^2 in $S^5(1)$, it turns out by Lemma 6 that the map t is a conformal minimal immersion into $S^5(1)$.

COROLLARY. *The Gauss map t: $M \to G_2^2$ of a conformal minimal immersion into $S^3 \subset \mathbf{H}^4$ is a conformal minimal immersion when we view t: $M \to S^5(1)$.*

REMARK. This Corollary is a direct consequence of a result in a paper by H. B. Lawson, Jr., *Complete minimal surfaces in* S^3, Ann. of Math. **92** (1970), 335–374. In that paper Lawson points out that what he calls the bipolar of a minimal surface in S^3 is a conformal minimal immersion into $S^5(1)$. But Lawson's bipolar is just the normal bundle $t^\perp$. Moreover, the map on G_2^2 sending an oriented plane to its oriented normal plane, which thus transforms $t^\perp$ into t, is the restriction to G_2^2 of an isometry of $S^5(1)$.

REMARK. Note, by Lemma 6, that a conformal immersion $t\colon M \to G_2^2$ can be a Gauss map only if $k = -1$ or the normal bundle $t^\perp$ is flat.

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