

ENTIRE FUNCTIONS WHICH ARE INFINITELY INTEGER VALUED AT A FINITE NUMBER OF POINTS

BY

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ABSTRACT. This paper determines arithmetic limits for the growth rates of entire functions which are infinitely integer valued on a finite set S . The characterization of such functions with growth rate less than the arithmetic limit is complete if there exist exponential polynomials which are infinitely integer valued on S .

1. Introduction. Let K be the rational field \mathbf{Q} or an imaginary quadratic number field. An analytic function $f(z)$ is *infinitely integer valued* at a point z_0 if for all $n \geq 0$, $f^{(n)}(z_0)$ is an integer of K . In this note we continue the study of the family \mathcal{F} of entire functions which are infinitely integer valued at a fixed set of points z_1, z_2, \dots, z_s of the complex plane.

Let $M(R, f)$ be the maximum modulus of $f(z)$ on $|z| = R$. Let $\rho, \sigma_0, \sigma_1, \dots, \sigma_k$ be real numbers with $\rho > 0$, $\sigma_0 > 0$. We say that $f(z)$ has growth rate $(\rho; \sigma_0, \sigma_1, \dots, \sigma_k)$ if for every $\varepsilon > 0$,

$$(i) \quad \log M(R, f) < R^\rho \left(\sigma_0 + \frac{\sigma_1}{\log R} + \dots + \frac{\sigma_k + \varepsilon}{(\log R)^k} \right)$$

for all sufficiently large R , and

$$(ii) \quad \log M(R, f) > R^\rho \left(\sigma_0 + \frac{\sigma_1}{\log R} + \dots + \frac{\sigma_k - \varepsilon}{(\log R)^k} \right)$$

for arbitrarily large R .

In this definition we allow the possibility that $k = -1$; then $f(z)$ has growth rate (ρ) if and only if $f(z)$ has order ρ .

For each fixed k , we order the growth rates $(\rho; \sigma_0, \dots, \sigma_k)$ lexicographically; we do not compare growth rates with different values of k .

We call $(\rho; \sigma_0, \dots, \sigma_k)$ an *arithmetic limit* for \mathcal{F} if

(i) \mathcal{F} contains only countably many functions with growth rate less than $(\rho; \sigma_0, \dots, \sigma_k)$, and

(ii) \mathcal{F} contains uncountably many functions with growth rate greater than $(\rho; \sigma_0, \dots, \sigma_k)$.

The work of E. G. Straus, D. Sato and A. M. Cayford [2,3,7,9] leads to the following.

THEOREM 1.1. *For any $\{z_1, \dots, z_s\}$ and any $\rho \geq s$, there exist 2^{\aleph_0} entire functions of order ρ which are infinitely integer valued at z_1, \dots, z_s . If there exists an entire function $f(z)$ of order $\rho < s$ which is infinitely integer valued at z_1, \dots, z_s ,*

Received by the editors August 9, 1982 and, in revised form, March 25, 1985.

1980 *Mathematics Subject Classification.* Primary 30D15; Secondary 30D99.

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 0002-9947/86 \$1.00 + \$.25 per page

then $f(z)$ satisfies a linear differential equation with integral coefficients. In fact, $\rho = 0$ or 1 and $f(z)$ is either a polynomial or an exponential polynomial, i.e. a Laurent polynomial in an exponential $e^{\lambda z}$.

Hence the arithmetic limit for the orders of functions in \mathcal{F} is equal to s ; in particular, it depends only on the number of points and not on their location.

The existence of a nontrivial function in \mathcal{F} of order $< s$ places a severe restriction on the differences $z_2 - z_1, \dots, z_s - z_1$; In fact, the number of such $(s - 1)$ -tuples is denumerable and readily characterized.

In [7] the arithmetic limit $(\rho; \sigma_0)$ for \mathcal{F} is also determined. The result is as follows.

THEOREM 1.2. *Let*

$$\sigma_0 = \prod_{1 \leq i < j \leq s} |z_i - z_j|^{-2/s},$$

where $\sigma_0 = 1$ if $s = 1$. Then there are 2^{8s} functions in \mathcal{F} with growth rate $> (s; \sigma_0)$, but only countably many functions in \mathcal{F} with growth rate $< (s; \sigma_0)$.

The characterization of functions in \mathcal{F} with growth rate $< (s; \sigma_0)$ is made more difficult by the fact that type limitations are not preserved under multiplication.

In §2 and §3 we determine the arithmetic limits $(\rho; \sigma_0, \dots, \sigma_k)$ for \mathcal{F} .

In §4 and §5 we discuss the case where \mathcal{F} contains nonconstant functions of order $< s$.

2. Generalized Taylor series and growth rates of entire functions.

Given a sequence ζ_1, ζ_2, \dots , with $|\zeta_n| \leq r$ for all n , we can, as in [2], expand any entire function in the generalized Taylor series

$$(2.1) \quad f(z) = \sum_{n=0}^{\infty} a_n (z - \zeta_1)(z - \zeta_2) \cdots (z - \zeta_n),$$

$$a_n = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{(z - \zeta_1) \cdots (z - \zeta_{n+1})} dz, \quad R > r.$$

We define

$$M(R, f) = \max_{|z|=R} |f(z)|,$$

$$\mu(R, f) = \max_n \max_{|z|=R} |a_n| |z - \zeta_1| |z - \zeta_2| \cdots |z - \zeta_n|.$$

If $f(z)$ has finite order, then

$$(2.2) \quad \log \mu(R, f) \leq \log M(R, f) \leq \log \mu(R, f) + O(\log R)$$

(see [4, 1.11]). This enables us to use $\mu(R, f)$ and $M(R, f)$ interchangeably in what follows.

LEMMA 2.1. *Suppose $\phi(x)$ is a twice differentiable function such that $x\phi'(x)$ increases to ∞ for large x . Suppose further that $\lim_{x \rightarrow \infty} (\log \phi(x))/(\log x) = \rho$. Define $p(t)$ by $p(t)\phi'(p(t)) = t$ for all large t . Then if $f(z)$ is entire of order ρ and*

$$(2.3) \quad \log M(R, f) \leq \phi(R),$$

there exists a constant $\delta_1 < 1$ so that

$$(2.4) \quad \log |a_n| \leq \phi(p(n)) - n \log p(n) + O(n^{\delta_1}).$$

PROOF. L'Hospital's Rule implies $\log p(t)/\log t \rightarrow 1/\rho$. From (2.1) we have

$$|a_n| \leq \frac{1}{2\pi} \left\| \int_{|z|=R} \frac{f(z)}{(z - \zeta_1) \cdots (z - \zeta_{n+1})} dz \right\| \leq RM(R, f)(R - r)^{-(n+1)}.$$

Hence

$$(2.5) \quad \log |a_n| \leq \phi(R) - n \log R + O(n/R).$$

Setting $R = p(n)$, we get (2.4).

LEMMA 2.2. Suppose $\psi(x)$ is a twice differentiable function such that $\psi'(x)$ decreases to $-\infty$ for large x . Suppose also that $\lim_{x \rightarrow \infty} (\psi(x))/(x \log x) = -1/\rho$. Let $q(t)$ satisfy $\psi'(q(t)) = -\log t$. Then if $f(z)$ is entire of order ρ and

$$(2.6) \quad \log |a_n| \leq \psi(n),$$

there exists a constant $\delta_2 < \rho$ so that

$$(2.7) \quad \log M(R, f) \leq \psi(q(R)) + q(R) \log R + O(R^{\delta_2}).$$

PROOF. The conditions on ψ imply that $\log q(t)/\log t \rightarrow \rho$. We have

$$\log \mu(R, f) \leq \log \max_n |a_n| (R + r)^n \leq \max_n \left(\psi(n) + n \log R + \frac{nr}{R} \right).$$

A straightforward computation now yields

$$\log M(R, f) \leq \psi(q(R)) + q(R) \log R + O\left(\frac{q(R) \log R}{R}\right)$$

which implies (2.7).

We now combine Lemmas 2.1 and 2.2.

LEMMA 2.3. (i) Let ϕ satisfy the hypotheses of Lemma 2.1. Define $\psi(t) = \phi(p(t)) - t \log p(t)$. Then there exists a constant $\delta_2 < \rho$ so that

$$(2.8) \quad \log M(R, f) \leq \phi(R) + O(R^{\delta_2})$$

if and only if there exists a $\delta_1 < 1$ so that

$$(2.9) \quad \log |a_n| \leq \psi(n) + O(n^{\delta_1}).$$

(ii) On the other hand, let ψ satisfy the hypotheses of Lemma 2.2. Define $\phi(t) = \psi(q(t)) + q(t) \log t$. Then there exists a $\delta_1 < 1$ so that

$$(2.10) \quad \log |a_n| \leq \psi(n) + O(n^{\delta_1})$$

if and only if there exists a $\delta_2 < \rho$ so that

$$(2.11) \quad \log M(R, f) \leq \phi(R) + O(R^{\delta_2}).$$

PROOF. To prove (i), observe that ψ satisfies the conditions in Lemma 2.2 and

$$(2.12) \quad \psi'(t) = (\phi'(p(t)) - t/p(t))p'(t) - \log p(t) = -\log p(t).$$

Thus $p(q(t)) = t$, and $\psi(q(t)) + q(t) \log t = \phi(t)$. Therefore, by Lemmas 2.1 and 2.2, we are done. The proof of (ii) is analogous.

We say that the coefficients a_n in (2.1) have growth rate $[1/\rho; \tau_0, \dots, \tau_k]$ if for every $\varepsilon > 0$, we have

$$(2.13) \quad \log |a_n| < -\frac{1}{\rho} n \log n + n \left(\tau_0 + \frac{\tau_1}{\log n} + \dots + \frac{\tau_k + \varepsilon}{(\log n)^k} \right)$$

for all large n , and

$$(2.14) \quad \log |a_n| > -\frac{1}{\rho} n \log n + n \left(\tau_0 + \frac{\tau_1}{\log n} + \dots + \frac{\tau_k - \varepsilon}{(\log n)^k} \right)$$

for arbitrarily large n .

We order these growth rates lexicographically, and define arithmetic limits in the obvious way.

THEOREM 2.4. *The entire function $f(z)$ given by (2.1) has growth rate $(\rho; \sigma_0, \sigma_1, \dots, \sigma_k)$ if and only if the coefficients a_n have growth rate $[1/\rho; \tau_0, \tau_1, \dots, \tau_k]$, where the relation between $\sigma_0, \sigma_1, \dots, \sigma_k$ and $\tau_0, \tau_1, \dots, \tau_k$ is determined by the relation between the functions $\phi(R)$ and $\psi(t)$ in Lemma 2.3.*

PROOF. We apply the first half of Lemma 2.3 to the function

$$\phi(R) = R^\rho(\sigma_0 + \sigma_1/\log R + \dots + \sigma_k/(\log R)^k).$$

Let $R_t = p(t)$. Using this function in Lemma 2.1, we easily see that $R_t^\rho/(\log R_t)^{k+1} = O(t/(\log t)^{k+1})$. Hence,

$$(2.15) \quad R_t^\rho(\lambda_0 + \lambda_1/\log R_t + \dots + \lambda_k/(\log R_t)^k) = t + O(t/(\log t)^{k+1}),$$

where $\lambda_i = \rho\sigma_i - (i-1)\sigma_{i-1}$; $i = 0, 1, \dots, k$ ($\sigma_{-1} = 0$). Inverting (2.15) we get constants μ_0, \dots, μ_k such that

$$(2.16) \quad R_t^\rho = t(\mu_0 + \mu_1/\log t + \dots + \mu_k/(\log t)^k) + O(t/(\log t)^{k+1}).$$

Taking logarithms in (2.16) we get constants ν_0, \dots, ν_k such that

$$(2.17) \quad \log R_t = \frac{1}{\rho} \log t + \nu_0 + \nu_1/\log t + \dots + \nu_k/(\log t)^k + O(1/(\log t)^{k+1}).$$

Using Lemma 2.3, we get the desired form for $\psi(t)$. The converse computation of $\phi(R)$ from $\psi(t)$ is entirely analogous.

The actual computation of τ_0, \dots, τ_k in terms of $\rho, \sigma_0, \dots, \sigma_k$ is rather cumbersome. We will carry it one step beyond the known fact

$$(2.18) \quad \tau_0 = \frac{1}{\rho} \log(e\rho\sigma_0).$$

Setting $\phi(R) = R^\rho(\sigma_0 + \sigma_1/\log R)$ we get

$$t = R_t \phi'(R_t) = \rho \phi(R_t) + O(R_t^\rho/(\log R_t)^2).$$

Hence

$$\log t = \rho \log R_t + \log(\sigma_0 \rho) + \frac{\sigma_1/\sigma_0}{\log R_t} + O\left(\frac{1}{(\log R_t)^2}\right),$$

$$\log R_t = \frac{1}{\rho} \log t - \frac{1}{\rho} \log(\rho\sigma_0) - \frac{\sigma_1/\sigma_0}{\log t} + O\left(\frac{1}{(\log t)^2}\right),$$

and

$$\begin{aligned}\psi(t) &= \phi(R_t) - t \log R_t \\ &= -\frac{1}{\rho} t \log t + t \left(\frac{1}{\rho} \log(\sigma_0 \rho e) + \frac{\sigma_1/\sigma_0}{\log t} + O\left(\frac{1}{(\log t)^2}\right) \right).\end{aligned}$$

Thus

$$(2.19) \quad \tau_1 = \sigma_1/\sigma_0.$$

3. The arithmetic limits. We expand the function $f \in \mathcal{F}$ in a generalized Taylor series at the points ζ_1, ζ_2, \dots , where all $\zeta_n \in \{z_1, z_2, \dots, z_s\}$. Thus

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_1)^{n_1} \cdots (z - z_s)^{n_s},$$

where $n_i = n_i(n)$ is a nondecreasing function of n and $n_1 + n_2 + \cdots + n_s = n$. We further choose the exponents n_i so that the quantities

$$(3.1) \quad \Psi_i(n) = n_i! \prod_{\substack{j=1 \\ j \neq i}}^s (z_i - z_j)^{n_j}, \quad i = 1, 2, \dots, s,$$

are nearly equal in absolute value.

We define $\psi_i(t)$ by

$$-\psi_i(t) = \log |\Psi_i(t)| = \log \Gamma(t_i + 1) + \sum_{j \neq i} t_j \log |z_i - z_j|,$$

for $i = 1, 2, \dots, s$, where $t = t_1 + \cdots + t_s$. For a given t , we want to choose t_1, \dots, t_s in such a way that $\psi_i(t)$ does not depend on i , that is

$$\psi_1(t) = \cdots = \psi_s(t), \quad t_1 + t_2 + \cdots + t_s = t.$$

Let us call $\psi(t)$ the common value of $\psi_i(t)$ so obtained; thus

$$(3.2) \quad -\psi(t) = \log \Gamma(t_i + 1) + \sum_{j \neq i} t_j \log |z_i - z_j|, \quad i = 1, 2, \dots, s.$$

The solution of (3.2) leads to asymptotic expansions of the form

$$(3.3) \quad t_i = \frac{1}{s} t \left(1 + \frac{\gamma_{i1}}{\log t} + \frac{\gamma_{i2}}{(\log t)^2} + \cdots + \frac{\gamma_{ik}}{(\log t)^k} \right) + O\left(\frac{t}{(\log t)^{k+1}}\right),$$

where $\sum_{i=1}^s \gamma_{ij} = 0$, $j = 1, 2, \dots, k$, and

$$(3.4) \quad \psi(t) = -\frac{1}{s} t \log t + \frac{1}{s} t \left(\gamma_0 + \frac{\gamma_1}{\log t} + \cdots + \frac{\gamma_k}{(\log t)^k} \right) + O\left(\frac{t}{(\log t)^{k+1}}\right).$$

Let $P(z) = (z - z_1) \cdots (z - z_s)$. If we compare terms of order t , then (3.2) yields

$$\begin{aligned}(3.5) \quad \gamma_0 &= 1 + \log s - \gamma_{i1} - \sum_{j \neq i} \log |z_i - z_j| \\ &= 1 + \log s - \gamma_{i1} - \log |P'(z_i)|.\end{aligned}$$

Summing over i , we get

$$(3.6) \quad \gamma_0 = 1 + \log s - 2 \log \prod_{1 \leq i < j \leq s} |z_i - z_j| = 1 + \log s - 2 \log V,$$

where V is the absolute value of the Vandermonde of z_1, \dots, z_s . Thus (3.5) yields

$$(3.7) \quad \gamma_{i1} = \log |V^2/P'(z_i)|.$$

We now compare terms of order $t/\log t$ to get

$$(3.8) \quad \begin{aligned} \gamma_1 &= -\gamma_{i2} + \gamma_{i1} \log s - \sum_{j \neq i} \gamma_{j1} \log |z_i - z_j| \\ &= -\gamma_{i2} + 2 \log V \log s - \log |P'(z_i)| \log s - 2 \log V \log |P'(z_i)| \\ &\quad + \sum_{j \neq i} \log |z_i - z_j| \log |P'(z_i)|. \end{aligned}$$

Summing over i , we obtain

$$(3.9) \quad \begin{aligned} \gamma_1 &= \left(2 - \frac{2}{s}\right) \log V \log s - \frac{4}{s} (\log V)^2 \\ &\quad + \frac{1}{s} \sum_{i=1}^s \sum_{j \neq i} \log |z_i - z_j| \log |P'(z_i)|. \end{aligned}$$

Hence

$$(3.10) \quad \begin{aligned} \gamma_{i2} &= \frac{2}{s} \log V \log s + \frac{4}{s} (\log V)^2 - \frac{1}{s} \sum_{i=1}^s \sum_{j \neq i} \log |z_i - z_j| \log |P'(z_i)| \\ &\quad - \log |P'(z_i)| \log(sV^2) + \sum_{j \neq i} \log |z_i - z_j| \log |P'(z_i)|. \end{aligned}$$

In an analogous manner we can compute the constants γ_{ij} for $i = 1, 2, \dots, s$, $j = 3, \dots, k$.

We now choose the integers n_i to satisfy

$$(3.11) \quad n_i(n) = t_i(n) + O(1).$$

To do this, we let ζ_n be the z_i of lowest index for which $t_i(n) - n_i(n-1)$ is maximal.

THEOREM 3.1. *The arithmetic limit for the generalized Taylor coefficients of functions in \mathcal{F} is $[1/s; \gamma_0/s, \dots, \gamma_k/s]$, and the corresponding arithmetic limit for the growth rates of these functions is $(s; \sigma_0, \sigma_1, \dots, \sigma_k)$, where the relation between the σ_i and the $\tau_i = \gamma_i/s$ is as in Theorem 2.4.*

PROOF. Let $f(z) \in \mathcal{F}$ have generalized Taylor expansion

$$(3.12) \quad f(z) = \sum_{k=0}^{\infty} a_n (z - z_1)^{n_1} \cdots (z - z_s)^{n_s},$$

where the n_i satisfy (3.11). Assume that $\zeta_{n+1} = z_i$, that is $n_i(n+1) = n_i(n) + 1$. Then (3.12) yields

$$(3.13) \quad f^{(n_i)}(z_i) = a_n \Psi_i(n) + \mathcal{R}_n,$$

where the remainder \mathcal{R}_n depends only on the coefficients a_0, \dots, a_{n-1} .

If the growth rate of the a_n is less than $[1/s; \gamma_0/s, \dots, \gamma_k/s]$, then we must have

$$(3.14) \quad \log |f^{(n_i)}(z_i) - \mathcal{R}_n| = \log |a_n| - \psi_i(n) \rightarrow -\infty \quad \text{as } n \rightarrow \infty.$$

Since any two distinct integers of K have difference at least 1 in absolute value, there is at most one integral value $f^{(n_i)}(z_i)$ which makes the left side of (3.14) less than $\log \frac{1}{2}$ for a given \mathcal{R}_n . Thus when n is sufficiently large, the coefficients a_0, a_1, \dots, a_{n-1} determine $f^{(n_i)}(z_i)$, and hence a_n , uniquely. This shows that there can only be a countable number of such $f(z)$.

On the other hand, there is a constant c such that every circle of radius c contains at least two integers of K . Thus if we permit $|f^{(n_i)}(z_i) - \mathcal{R}_n| \geq c$ for all n , we get at least two possible choices for $f^{(n_i)}(z_i)$. Hence we get an uncountable set of functions in \mathcal{F} .

REMARK. If $s = 1$, the remainder \mathcal{R}_n in (3.13) is 0 for all n . Hence if the growth of $f(z)$ is below the arithmetic limit $(1, \sigma_0, \sigma_1, \dots, \sigma_k)$, then $f(z)$ is a polynomial.

4. Functions in \mathcal{F} with growth rate below the arithmetic limit. It is known [3] that a function $f(z) \in \mathcal{F}$ of order $\rho < s$ satisfies a linear differential equation with K -integral coefficients. For more general growth rates, the situation is as follows.

THEOREM 4.1. *Let $F(z) \in \mathcal{F}$ have growth rate less than the arithmetic limit $(s; \sigma_0, \dots, \sigma_k)$. If there exists a nonconstant function $f(z) \in \mathcal{F}$ of order $< s$, then $F(z)$ satisfies a linear differential equation with coefficients in $K[f]$, the ring of polynomials in f with coefficients in K .*

REMARK. If $s = 1$, $F(z)$ is a polynomial by the remark following Theorem 3.1. Hence it satisfies the equation $F^{(n)} \equiv 0$ for some n . Thus we may assume from now on that $s \geq 2$.

For the proof of Theorem 4.1 we need an estimate for the derivatives of entire functions.

LEMMA 4.2. *For any entire function f and any $r > 0$, we have*

$$(4.1) \quad M(R, f^{(n)}) \leq r^{-n} M(R + nr, f).$$

PROOF. By Cauchy's formula we have

$$f'(z) = \frac{1}{2\pi i} \int_{|\zeta - z| = r} \frac{f(\zeta) d\zeta}{(\zeta - z)^2},$$

and hence

$$(4.2) \quad M(R, f') \leq r^{-1} M(R + r, f).$$

Iterating (4.2) n times, we get (4.1).

PROOF OF THEOREM 4.1. Suppose $F(z) \in \mathcal{F}$ and

$$(4.3) \quad M(R, F) < R^s \left(\sigma_0 + \frac{\sigma_1}{\log R} + \dots + \frac{\sigma_k - \delta}{(\log R)^k} \right)$$

for some $\delta > 0$ and R sufficiently large. We construct a function

$$(4.4) \quad \Phi(z) = \sum_{u=0}^U \sum_{v=0}^V \lambda_{uv} f(z)^u F^{(v)}(z),$$

where the λ_{uv} are rational integers, not all zero.

Let m be a positive integer to be chosen sufficiently large later, and set

$$U = [m^{1-1/s}/(\log m)^{k+2}], \quad V = [m^{1/s}(\log m)^{2k+4}].$$

We expand Φ in the generalized Taylor series given by (3.11), (3.12), and choose the λ_{uv} so that

$$(4.5) \quad \Phi^{(\nu_i)}(z_i) = 0, \quad \nu_i = 0, 1, \dots, n_i(m); \quad i = 1, 2, \dots, s.$$

This is a system of $(n_1(m) + 1) + (n_2(m) + 1) + \dots + (n_s(m) + 1) = m + s$ linear equations with K -integral coefficients for the $(U + 1)(V + 1)$ unknowns λ_{uv} . If $K \neq \mathbf{Q}$ we take rational and irrational parts, getting $2m + 2s$ equations with coefficients in $\frac{1}{2}\mathbf{Z}$. These coefficients are rational or irrational parts of

$$\begin{aligned} C_{i,\nu_i,u,v} &= \frac{d^{\nu_i}}{dz^{\nu_i}} (f(z)^u F^{(v)}(z))|_{z=z_i} \\ &= \sum_{k=0}^{\nu_i} \binom{\nu_i}{k} \frac{d^k}{dz^k} (f(z)^u)|_{z=z_i} F^{(v+\nu_i-k)}(z_i). \end{aligned}$$

By Lemma 4.2, with $R = \max_{1 \leq i \leq s} |z_i|$ and $r = 1/m$, we have

$$(4.6) \quad \begin{aligned} |C_{i,\nu_i,u,v}| &\leq 2^{n_i(m)} m^{n_i(m)} M(R + n_i(m)/m, f^U) \\ &\quad \times m^{V+n_i(m)} M(R + (V + n_i(m))/m, F). \end{aligned}$$

From (3.10) and (3.3) we see that $n_i(m) < m(1/s + \varepsilon)$ for large m . Hence for some fixed $C > 1$,

$$m^{n_i(m)} \leq m^{V+n_i(m)} \leq C^m \log m,$$

$$M(R + n_i(m)/m, f^U) < C^m,$$

$$M(R + (V + n_i(m))/m, F) < C.$$

Thus (4.6) yields

$$(4.7) \quad |C_{i,\nu_i,u,v}| < C^m \log m.$$

The number of unknowns is $(U + 1)(V + 1) > m(\log m)^{k+2} > 2m + 2s$. Hence by Siegel's lemma (see [1]), we can find rational integers λ_{uv} , not all zero, satisfying (4.5) and

$$(4.8) \quad |\lambda_{uv}| \leq (C_1^m \log m)^{(2m+2s)/(m(\log m)^{k+2} - (2m+2s))} < C_2^{m/(\log m)^{k+1}}.$$

We now show that if m is sufficiently large, then $\Phi(z)$ vanishes identically. From (4.5) we have $a_0 = a_1 = \dots = a_m = 0$.

Let i be the index with $n_i(m + 2) = n_i(m + 1) + 1$. It suffices to show that $\Phi^{n_i(m+1)}(z_i) = 0$ if m is sufficiently large. For this implies that $a_{m+1} = 0$, and the result follows by induction.

We have

$$\begin{aligned} \Phi^{n_i(m+1)}(z_i) &= a_{m+1} \Psi_i(m + 1) \\ &= \Psi_i(m + 1) \frac{1}{2\pi i} \int_{|z|=R} \frac{\Phi(z) dz}{\prod_{j=1}^s (z - z_j)^{n_j(m+1)} (z - z_i)}. \end{aligned}$$

Thus

$$(4.9) \quad \log |\Phi^{n_i(m+1)}(z_i)| \leq -\psi_i(m+1) + \log M(R, \phi) - (m+1) \log R + (Cm/R).$$

We need only show that for sufficiently large M the right side of (4.9) is negative. Now

$$(4.10) \quad \begin{aligned} \log M(R, \Phi) &\leq \log(U+1) + \log(V+1) + \log \max_{u,v} |\lambda_{uv}| \\ &\quad + U \log M(R, f) + \max_{v \leq V} \log M(R, F^{(v)}) \\ &\leq Cm/(\log m)^{k+1} + CRm^{1-1/s}/(\log m)^{k+2} \\ &\quad - V \log r + \log M(R + Vr, F). \end{aligned}$$

Setting $r = 1/V$ and $L = \log R$, we obtain

$$(4.11) \quad \begin{aligned} \log M(R, \Phi) &\leq Cm(\log m)^{k+1} + CRm^{1-1/s}(\log m)^{k+2} \\ &\quad + R^s \left(\sigma_0 + \frac{\sigma_1}{L} + \cdots + \frac{\sigma_k - \delta}{L^k} \right) + CR^{s-1}. \end{aligned}$$

Substituting (4.11) in (4.9), we get

$$(4.12) \quad \begin{aligned} \log |\Phi^{n_i(m+1)}(z_i)| &\leq \left\{ \frac{1}{s} m \log m - \frac{1}{s} m \left(\gamma_0 + \frac{\gamma_1}{\log m} + \cdots + \frac{\gamma_k}{(\log m)^k} \right) \right. \\ &\quad \left. + R^s \left(\sigma_0 + \frac{\sigma_1}{L} + \cdots + \frac{\sigma_k}{L^k} \right) - m \log R \right\} \\ &\quad + Cm/R - \log R + CR^{s-1} + Cm/(\log m)^{k+1} \\ &\quad + CRm^{1-1/s}/(\log m)^{k+2} - \frac{\delta R^s}{(\log R)^k} + O \left(\frac{m}{(\log m)^{k+1}} \right). \end{aligned}$$

We now choose $R = p(m)$ as in Lemma 2.1. By Theorem 2.4, the quantity in braces is $O(m/(\log m)^{k+1})$. Since $R = O(m^{1/s})$, the term $-\delta R^s/(\log R)^k$ dominates, and the limit is $-\infty$. This completes the proof.

5. Solutions of differential equations with periodic coefficients. We continue to assume in this section that there is a nonconstant function $f \in \mathcal{F}$ of order $< s$. It was shown in [3] and [11] that f is either a polynomial or an exponential polynomial. In the latter case f is clearly periodic.

By Theorem 4.1, every function $F \in \mathcal{F}$ with order $s > 1$ and growth rate below the arithmetic limit $(s, \sigma_0, \sigma_1, \dots, \sigma_k)$ satisfies a linear differential equation with coefficients in $K[f]$.

We now discuss the case where $f(z)$ is periodic, say with period ω . The function $F(z)$ cannot be periodic, since it would then be infinitely integer valued at more than s points—in fact, at infinitely many points $z_i + n\omega$. This would imply that $F(z)$ is of order ≤ 1 , contrary to assumption.

LEMMA 5.1. *Let $F(z)$ be entire and satisfy the differential equation*

$$(5.1) \quad A_0(z)F + A_1(z)F' + \cdots + A_n(z)F^{(n)} = 0,$$

where $A_i(z+\omega) = A_i(z)$, $0 \leq i \leq n$. Then $F(z)$ satisfies a linear difference equation

$$(5.2) \quad C_0F(z) + C_1F(z+\omega) + \cdots + C_pF(z+p\omega) = 0$$

with constant coefficients C_0, C_1, \dots, C_p ; $C_p \neq 0$.

PROOF. The functions $F(z + k\omega)$, $0 \leq k \leq n$, are all solutions of (5.1); hence they are linearly dependent.

Let the zeros of the characteristic polynomial $C_0 + C_1t + \cdots + C_pt^p$ of (5.2) be $e^{\lambda_1}, e^{\lambda_2}, \dots, e^{\lambda_k}$, with multiplicities $M_1 + 1, \dots, M_k + 1$. Then for all integers q , we have

$$(5.3) \quad F(z + q\omega) = \sum_{i=1}^k (a_{i0}(z) + a_{i1}(z)q + \cdots + a_{iM_i}(z)q^{M_i})e^{\lambda_i q}$$

(see, for example, [5, p. 384]). Here the $a_{ij}(z)$ are entire functions which satisfy

$$(5.4) \quad \sum_{j=0}^{M_i} a_{ij}(z + \omega)q^j = e^{\lambda_i} \sum_{j=0}^{M_i} a_{ij}(z)(q + 1)^j.$$

If we set $b_{ij}(z) = a_{ij}(z)e^{-\lambda_i z/\omega}$, then (5.4) becomes

$$(5.5) \quad \sum_{j=0}^{M_i} b_{ij}(z + \omega)q^j = \sum_{j=0}^{M_i} b_{ij}(z)(q + 1)^j.$$

In particular, this shows that $b_{i0}(z) = \sum_{j=0}^{M_i} c_{ij}(z)z^j$, where $c_{ij}(z + \omega) = c_{ij}(z)$. From (5.3) with $q = 0$ we get

$$(5.6) \quad F(z) = \sum_{i=1}^k \sum_{j=0}^{M_i} c_{ij}(z)z^j e^{\lambda_i z/\omega}.$$

LEMMA 5.2. *Let e^{λ_i} , $i = 1, 2, 3, \dots$, be distinct. Then the functions $z^j e^{\lambda_i z/\omega}$, $i = 1, 2, 3, \dots$; $j = 0, 1, 2, \dots$, are linearly independent over the set of functions of period ω .*

PROOF. Assume not. Then there exist functions $F_{ij}(z)$ of period ω , not all zero, with

$$(5.7) \quad \sum_{i=1}^n \sum_{j=0}^m F_{ij}(z)z^j e^{\lambda_i z/\omega} = 0.$$

We may assume that m is minimal and that the number of nonzero $F_{im}(z)$ is minimal. Collecting the nonzero terms of (5.7) with $j = m$, we obtain

$$z^m (F_{n_1 m}(z)e^{\lambda_{n_1} z/\omega} + \cdots + F_{n_k m}(z)e^{\lambda_{n_k} z/\omega}),$$

where the $F_{n_i m}(z)$ are nonzero. Substitution of $z + \omega$ into (5.7) gives an equation with corresponding term

$$z^m (F_{n_1 m}(z)e^{\lambda_{n_1} z/\omega} e^{\lambda_{n_1} \omega/\omega} + \cdots + F_{n_k m}(z)e^{\lambda_{n_k} z/\omega} e^{\lambda_{n_k} \omega/\omega}).$$

Since the $e^{\lambda_{n_i}}$ are distinct and k is minimal, this implies that $k = 1$. Thus we may assume that the highest terms of (5.7) are

$$z^{m-1} (F_{p_1 m-1}(z)e^{\lambda_{p_1} z/\omega} + \cdots + F_{p_r m-1}(z)e^{\lambda_{p_r} z/\omega}) + z^m F(z)e^{\lambda z/\omega},$$

where the $F_{p_k m-1}(z)$ and $F(z)$ are nonzero, and $\lambda = \lambda_i$ for some i .

Substituting $z + \omega$ into (5.7), we get an equation with highest terms

$$z^{m-1}(F_{p_1 m-1}(z)e^{\lambda_{p_1} z/\omega} e^{\lambda_{p_1}} + \cdots + F_{p_r m-1}(z)e^{\lambda_{p_r} z/\omega} e^{\lambda_{p_r}} + m\omega F(z)e^{\lambda z/\omega} e^{\lambda}) + z^m F(z)e^{\lambda z/\omega} e^{\lambda}.$$

Since m is minimal we must have $m\omega F(z)e^{\lambda z/\omega} e^{\lambda} = 0$, which implies $m = 0$. Thus (5.7) reduces to $F(z)e^{\lambda z/\omega} = 0$, which is impossible.

LEMMA 5.3. *Let $F(z)$ be analytic in $0 < |z| < \infty$ with Laurent expansion*

$$(5.8) \quad F(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

Suppose that both $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} a_{-n} z^n$ have order 0. If $F(z)$ satisfies a linear differential equation with rational functions as coefficients, then it is a Laurent polynomial.

PROOF. By clearing denominators, we may assure that the coefficients of the differential equation are polynomials. We wish to show that $a_n = 0$ for all large $|n|$ in (5.8). Since the conditions are symmetric in z and $1/z$, it suffices to prove the result for large positive n . The hypothesis implies that

$$\log |a_n|/n \log n \rightarrow -\infty,$$

where we take $\log 0$ to mean $-\infty$. Thus either $a_n = 0$ for all sufficiently large n , or for every $M > 0$ there is an infinite sequence of indices n_1, n_2, \dots with

$$(5.9) \quad |a_{n_i+p}| \leq |a_{n_i}|/n_i^{pM}, \quad p = 0, 1, 2, \dots$$

Now consider the differential equation satisfied by $F(z)$:

$$(5.10) \quad (b_{00} + b_{01}z + \cdots + b_{0k_0}z^{k_0})F + \cdots + (b_{l0} + b_{l1}z + \cdots + b_{lk_l}z^{k_l})F^{(l)} = 0.$$

We substitute the Laurent series (5.8) into (5.10) and set the coefficient of z^n equal to 0. This gives a system of linear equations E_n for the a_i . If $k = \text{Max}_j(k_j - j)$ and $n = k + \nu$, then $a_0, a_1, \dots, a_{\nu-1}$ do not appear in equation E_n . Consider E_{k+n_i} , where n_i satisfies (5.9). The coefficient of a_{n_i} in E_{k+n_i} is

$$(5.11) \quad B_{n_i} = \sum_j' b_{jk_j} n_i!/(n_i - j)!,$$

where the summation is extended over all j for which $k_j - j = k$. It is clear that B_{n_i} is nonzero for all but a finite number of indices i . Moreover there exists a positive B such that $|B_{n_i}| > B$ for all large i . On the other hand, the coefficients of all terms a_{n_i+p} in E_{k+n_i} are bounded by $C_{n_i}^l$. We choose $M > l$ in (5.9). Then for all large n_i , the contribution of the terms a_{n_i+p} ($p > 0$) is too small to cancel the term $B_{n_i} a_{n_i}$. This contradiction shows that $f(z)$ and $g(z)$ are polynomials, so $F(z)$ is a Laurent polynomial.

We now apply these lemmas to the case where the $A_i(z)$ in (5.1) are exponential polynomials, i.e. Laurent polynomials in a single exponential. Substituting (5.6) into (5.1) and using Lemma 5.2, we get a system of equations

$$(5.12) \quad L_{ij}c_{ij}(z) + L_{i,j+1}c_{i,j+1}(z) + \cdots + L_{iM_i}c_{iM_i}(z) = 0, \\ i = 1, 2, \dots, k; j = 0, 1, \dots, M_i,$$

where the L_{ij} are linear differential operators with exponential polynomial coefficients. Since the $C_{ij}(z)$ are entire of period ω , they can be expressed as Laurent series in $w = e^{2\pi iz/\omega}$ which satisfy the hypotheses of Lemma 5.3. Thus (5.12) with $j = M_i$ implies that the C_{iM_i} satisfy linear differential equations with coefficients which are Laurent polynomials in w . Lemma 5.3 implies that the C_{iM_i} are themselves exponential polynomials. Substituting these into (5.12) and using Lemma 5.3 again, we find that the C_{iM_i-1} are exponential polynomials. Continuing in this way, we find that all the C_{ij} are exponential polynomials. Hence, by (5.6) the order of $F(z)$ is ≤ 1 . Thus we have the following

THEOREM 5.4. *If \mathcal{F} contains nonconstant periodic functions of order $< s$, then all functions in \mathcal{F} with growth rate below the arithmetic limit $(s; \sigma_0, \dots, \sigma_k)$ have order ≤ 1 .*

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