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SIMPLE HOMOTOPY TYPE OF FINITE 2-COMPLEXES WITH FINITE ABELIAN FUNDAMENTAL GROUP

BY

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ABSTRACT.

THEOREM 1. *Let K be a finite 2-dimensional CW-complex with $\pi_1(K)$ finite and abelian. Then every element of the Whitehead group of K is realizable as the torsion of a self-homotopy equivalence on K.*

THEOREM 2. *Homotopy equivalence and simple homotopy equivalence are the same for finite 2-dimensional CW-complexes with finite abelian fundamental groups.*

0. Introduction. It is known that there exist finite n-complexes for all $n > 2$ which are homotopy equivalent but not simply homotopy equivalent. In this paper, we show that in dimension 2 homotopy type and simple homotopy type are the same when the fundamental group of the finite 2-dimensional complexes is finite and abelian.

The technique used is to show that all the elements of the Whitehead group of a complex K are realizable as torsions of self-equivalences on K.

1. An important example. Dyer and Sieradski **[DS]** showed in 1973 that two 2-dimensional CW-complexes whose fundamental group was Z_n were homotopy equivalent if and only if they were simple homotopy equivalent. The next case to consider would be $Z_n \times Z_m$. We want to show that any 2-dimensional complex with fundamental group $Z_n \times Z_m$ realizes all of its Whitehead group as torsions of self-equivalences. ($\xi(K)$ denotes the group of self-homotopy equivalences of K.)

THEOREM 1.1. *Let K be the standard 2-complex of the presentation $P = \{a, b \mid a^n, b^m, [a, b]\}$. Then every element of* $\mathrm{Wh}(K)$ *is realizable as $\tau(f)$ for some $f \in \xi(K)$.*

To prove the above theorem, we need the following three lemmas.

LEMMA 1.2. *Suppose G is a group with generating set $\{a_i \mid i \in I\}$ for some index set I. Let A: $Z(G) \to Z$ be the augmentation map (i.e., the ring homomorphism taking g to 1 for each $g \in G$). Then each element $\theta \in \ker A$ is of the form*

$$\theta = \sum_i \phi_i(a_i - 1) \quad \textit{for some elements } \phi_i \in Z(G).$$

PROOF. See **[F**, p. 549].

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LEMMA 1.3. *Let K be the standard complex of the presentation $\{a, b \mid a^n, b^m, aba^{-1}b^{-1}\}$ and let $\tilde{K}$ be the universal cover of K with a chosen lift $\tilde{e}_0$ of the single 0-cell in K. Consider $C_i(\tilde{K})$ as a $Z\pi_1$-module generated by the preferred lifts of reach i-cell. Then $H_2(\tilde{K})$ is generated by the set*

$$(a-1)\tilde{R}_a, \quad (b-1)\tilde{R}_b,$$
$$(b-1)\tilde{R}_a + \left(\sum_{i=0}^{n-1} a^i\right)\tilde{R}_{[a,b]}, \quad (a-1)\tilde{R}_b - \left(\sum_{i=0}^{m-1} b\right)\tilde{R}_{[a,b]}.$$

PROOF. Simple calculations will show that the given chains are cycles and thus represent elements of $H_2(\tilde{K}) = Z_2(\tilde{K})$. To show that they generate all of $H_2(\tilde{K})$ (a fact not actually needed in this paper), use a technique similar to Metzler's in [**M2**, p. 330]. □

LEMMA 1.4. *Given a finite 2-dimensional complex K and $Z\pi_1$-module map ϕ: $C_2(\tilde{K}) \to C_2(\tilde{K})$ which commutes with the boundary operator in the sense that $\partial_2\phi = \partial_2$, then there exists a homotopy equivalence f: $K \to K$ such that $\tilde{f}_2 = \phi$ and $\tilde{f}_1$ = identity only if the $Z\pi_1$-module representation of ϕ is invertible.*

PROOF. This is done by modifying the identity map on K using the Puppe action. This technique is patterned after [**DS**, p. 41]. For the original reference, see [**P**].

Suppose f: $K \to K$ is a homotopy equivalence which induces the identity map on $\pi_1(K)$. Let $\tilde{K}$ be the universal cover of K. Since f_*: $\pi_1 \to \pi_1$ is the identity, then f is homotopic to a map which induces the identity on $C_1(\tilde{K})$. Since we are only interested in homotopy classes of maps, we may assume f is that map. Now f induces the following map on the chain complex of the universe cover $\tilde{K}$:

$$(1)\qquad \begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ H_2(\tilde{K}) & \xrightarrow{f_2^*} & H_2(\tilde{K}) \\ \partial\downarrow & & \partial\downarrow \\ C_2(\tilde{K}) & \xrightarrow{\tilde{f}_2} & C_2(\tilde{K}) \\ \partial\downarrow & & \partial\downarrow \\ C_1(\tilde{K}) & \xrightarrow{\tilde{f}_1=\mathrm{id}} & C_1(\tilde{K}) \\ \partial\downarrow & & \partial\downarrow \\ C_0(\tilde{K}) & \xrightarrow{\tilde{f}_0=\mathrm{id}} & C_0(\tilde{K}) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

Consider the map id: $K \to K$. The induced maps on the chain complex of $\tilde{K}$ will all be the identity. Replace $\tilde{f}_2$ by ϕ in diagram (1). Now we have two chain maps that agree except on $C_2(\tilde{K})$. That means that ϕ-id_2 will commute in the following diagram:

$$
(2)\qquad \begin{array}{ccc} C_2(\tilde{K}) & \overset{\varphi\text{-id}}{\rightarrow} & C_2(\tilde{K}) \\ \partial\downarrow & & \partial\downarrow \\ C_1(\tilde{K}) & \overset{0}{\rightarrow} & C_1(\tilde{K}) \end{array}
$$

Consequently, the image of ϕ-id will be in $H_2(\tilde{K}) \equiv \pi_2(K)$. Let R be a 2-cell in K (from the CW-decomposition). Then $[\phi\text{-id}](\tilde{R}) \in H_2(\tilde{K}) \equiv \pi_2(K)$. Let p be the representative of the image of $\tilde{R}$ in $\pi_2(K)$. Also let p represent the actual map p: $S^2 \to K$. Now define g by the composition of the following maps:

$$
K \overset{h}{\rightarrow} K \vee S^2 \overset{\mathrm{id}\vee p}{\rightarrow} K,
$$

where h is the identity on $K \setminus R$ and maps R onto $R \vee S^2$ by mapping some $(S^1, e) \subset (R, e_0)$ to e_0 and the interior disk to S^2. The rest of R gets "stretched" to cover R. Now $\tilde{g}_2(\tilde{R}) = \phi(\tilde{R})$.

We can modify g using the above technique on the other 2-cells in the decomposition of K to obtain our hypothesized f. Using the Five Lemma on diagram (1), f will be a homotopy equivalence if and only if ϕ is an isomorphism.

That $\tau(f) = [M]$ comes from direct computation, using Cohen's §15 and (22.8) in [C]. □

Proof of Theorem 1.1. Given an element $\phi \in \mathrm{Wh}(K)$, we want to construct a homotopy equivalence f: $K \to K$, so that $\tau(f) = \phi$. By Lemma 1.4 we merely need to produce an invertible $Z\pi_1$-matrix M such that M commutes with the boundary operator and $[M] = \phi \in \mathrm{Wh}(K)$. Since in this case K has three 2-cells, we need $M \in \mathrm{GL}_3(Z\pi_1)$.

Let $\bar{a} = a - 1$ and $\bar{b} = b - 1$. Then by Lemma 1.3, any matrix of the following form (where the *rows* represent the images of $\tilde{R}_a$, $\tilde{R}_b$, $\tilde{R}_{[a,b]}$) will commute with the boundary operator:

$$
\begin{bmatrix} 1 + \phi_{11}\bar{a} + \psi_{11}\bar{b}, & \phi_{12}\bar{a} + \psi_{12}\bar{b}, & \psi_{11}\sum a^i - \phi_{12}\sum b^i \\ \phi_{21}\bar{a} + \psi_{21}\bar{b}, & 1 + \phi_{22}\bar{a} + \psi_{22}\bar{b}, & \psi_{21}\sum a^i - \phi_{12}\sum b^i \\ \phi_{31}\bar{a} + \psi_{31}\bar{b}, & \phi_{32}\bar{a} + \psi_{32}\bar{b}, & 1 + \psi_{31}\sum a^i - \phi_{32}\sum b^i \end{bmatrix}
$$

where ϕ_{ij}, ψ_{ij} are appropriate elements of $Z[\pi_1(K)] = Z[Z_n \times Z_m]$.

Now any element of $\mathrm{Wh}(Z(Z_n \times Z_m))$ may be represented by a 2×2-matrix, see Bass [**B**, p. 183] and Lam [**L**, p. 143]. Let $\left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right)$ be that matrix, where r, s, t, $u \in Z(Z_n \times Z_m)$. Consider the augmentation map $Z(Z_n \times Z_m) \to Z$ which maps the elements of $Z_n \times Z_m$ to 1. Call the images of r, s, t and u respectively r', s', t' and $u' \in Z$. Then we may transform the matrix $\left(\begin{smallmatrix} r' & s' \\ t' & u' \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} \pm 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ via row and column operators.

We can choose our representative $\left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right)$ without changing the element in $\mathrm{Wh}(Z(Z_n \times Z_m))$ in order that $\left(\begin{smallmatrix} r' & s' \\ t' & u' \end{smallmatrix}\right)$ transforms to $\left(\begin{smallmatrix} +1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Now use the same row and column operations on $\left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right)$ that we did on $\left(\begin{smallmatrix} t' & s' \\ t' & u' \end{smallmatrix}\right)$. We get a matrix which maps to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ when $a, b \mapsto 1$. Using Lemma 1.2, this new matrix must be of the form

$$\begin{bmatrix} 1 + \alpha_1 \bar{a} + \beta_1 \bar{b}, & \phi_1 \bar{a} + \psi_1 \bar{b} \\ \psi_2 \bar{a} + \beta_2 \bar{b}, & 1 + \phi_2 \bar{a} + \psi_2 \bar{b} \end{bmatrix} \quad \text{where } \alpha_i, \beta_1, \phi_1, \psi_i \in Z(Z_n \times Z_m)$$

Therefore, we may represent any element of $\mathrm{Wh}(Z(Z_n \times Z_m))$ in the form (2). If we choose the constants correctly for our matrix (1), and manipulate that matrix to get the form (2), then we will be done. First, letting $\phi_{31} = \psi_{31} = \phi_{32} = \psi_{32} = 0$, we get

$$\begin{bmatrix} 1 + \phi_{11}\bar{a} + \psi_{11}\bar{b}, & \phi_{12}\bar{b} + \psi_{12}(-\bar{a}), & \psi_{11}\sum a + \phi_{12}\sum b^i \\ \phi_{21}\bar{a} + \psi_{21}\bar{b}, & 1 + \phi_{22}\bar{b} + \psi_{22}(-\bar{a}), & \psi_{21}\sum a + \phi_{22}\sum b^i \\ 0 & 0 & 1 \end{bmatrix}.$$

We then use row 3 to clear column 3 to the equivalent matrix

$$\begin{bmatrix} 1 + \phi_{11}\bar{a} + \psi_{11}\bar{b}, & \phi_{12}\bar{b} + \psi_{12}(-\bar{a}) \\ \phi_{21}\bar{a} + \psi_{21}\bar{b}, & 1 + \phi_{22}\bar{b} + \phi_{22}(-\bar{a}) \end{bmatrix}$$

which, with appropriate choice of constants, is equal to the matrix (2). Consequently, we may produce a matrix M which represents any given element of $\mathrm{Wh}(K)$, by Lemma 1.4, such that M represents $\tilde{f}_2$ for some map $f\colon K \to K$, with $\tilde{f}_1 =$ identity. M is invertible since it represents an element of $\mathrm{Wh}(K)$. Therefore, f is a homotopy equivalence. Since $\tilde{f}_1$ is the identity map, then $\tau(f) = M$, and therefore all the elements of $\mathrm{Wh}(K)$ are representable. □

COROLLARY 1.5. *Let L be a finite 2-dimensional complex with $\pi_1(L) = Z_n \times Z_m$ and $\chi(L) = k$. Then L is simple homotopy equivalent to $K \vee (k-2)(S^2)$, where K is the complex of Theorem* 1.1.

PROOF. Since the complex K of Theorem 1.1 realizes all of its Whitehead group as the torsions of self-equivalence on K, then by Cohen [**C**, Theorem 24.4] any complex homotopy equivalent to K is simple homotopy equivalent to K. By Dyer [**D3**], homotopy type of any finite 2-complex with fundamental group $Z_n \times Z_m$ is determined by the Euler characteristic. Since our example K has minimal Euler characteristic (see Swan [**Sw**, Proposition 2.1]), the other Euler characteristics may be obtained by wedging K with the appropriate number of copies of S^2. If K realizes all of its torsion by self-equivalences then so does $K \vee S^2$. For if $f\colon K \to K$ is a homotpy equivalence, let $f \vee \mathrm{id}\colon K \vee S^2 \to K \vee S^2$. Then $\tau(f) = \tau(f \vee \mathrm{id})$.

Consequently, given a 2-complex L with $\pi_1(L) = Z_m \times Z_n$ and $\chi(L) = k$, then by Dyer [**D3**] L is homotopy equivalent to $K \vee (k-2)S^2$. But since $K \vee (k-2)S^2$ realizes all of its torsion by self-equivalence we know by Cohen [**C**] that L is simple homotopy equivalent to $K \vee (k-2)S^2$. □

COROLLARY 1.6. *Let K, L be finite 2-dimensional CW-complexes with fundamental groups $Z_n \times Z_m$. Then the following are equivalent.*

(a) $\chi(K) = \chi(L)$,

(b) *K is homotopy equivalent to L.*

(c) *K is simple homotopy equivalent to L.*

2. The general abelian case. We want to prove that simple homotopy type and homotopy type agree for finite 2-complexes with finite abelian fundamental group. This general situation is unlike the $Z_n \times Z_m$ case in that homotopy type does not depend only on Euler characteristic. When the Euler characteristic is minimal, homotopy type depends also on bias (see Metzler [**M1**], Sieradski [**S**] and Browning [**Br**]).

According to Browning [**Br**, Theorem 1.7] and Sieradski [**S**, Theorem 2], any finite 2-complex of minimal Euler characteristic with finite abelian fundamental group is homotopy equivalent to the standard complex of some twisted (or "untwisted") presentation of the form

$$\{a_i | a_i^{n_i}, [a_1^r, a_2], [a_i, a_j], i < j, j \neq 2\},$$

where $r < n_1$ and $(r, n_1) = 1$.

So if we are to use our previous techniques for showing that homotopy type and simple type agree, we not only have to show that the standard complex of the presentation

$$\{a_i | a_i^{n_i}; [a_i, a_j]; i < j; j = 1, \dots, N\}$$

realizes all of its Whitehead torsion by self-equivalences, we also have to show that the standard complex of any twisted presentation realizes all of its torsion by self-equivalences.

LEMMA 2.1. *Let G be a finite abelian group, and let $P = \{a_i | a_i^{n_i}, [a_1^r, a_2], [a_i, a_j]; i < j; j \neq 2; i, j = 1, \dots, N\}$, where $r < n_1$ and $(r, n_1) = 1$, be a twisted presentation of G. Let $K(P) = K$ be the standard complex of P. Then all of the torsion of* $\mathrm{Wh}(K)$ *is realizable as $\tau(f)$ for some $f \in \xi(K)$.*

PROOF. The proof proceeds as that of Theorem 1.1. Let $\tilde{K}$ be the universal cover of K. Then the following are elements of $H_2(\tilde{K})$:

$$\begin{aligned} &(a_i - 1)\tilde{R}_i, \quad (a_1^r - 1)\tilde{R}_2 - \left(\textstyle\sum a_2^k\right)\tilde{R}_{12}, \\ &(a_j - 1)\tilde{R}_i + \left(\textstyle\sum a_i^k\right)\tilde{R}_{ij}, \qquad ij \neq 12,\ i < j, \\ &(a_i - 1)\tilde{R}_j - \left(\textstyle\sum a_j^k\right)\tilde{R}_{ij}, \qquad ij \neq 12,\ i < j. \end{aligned}$$

The above restrictions on ij tell us that if we try to use these elements of $H_2(\tilde{K})$ to modify the first two rows of the identity matrix, the first column will have no $(a_1 - 1) = \bar{a}_1$ component. So what we will do is ignore the first row and attempt to get our result using the second and third rows.

Consider the map ϕ: $C_2(\tilde{K}) \to C_2(\tilde{K})$ which is the identity except:

$$\begin{aligned}
\tilde{R}_2 \mapsto 1 \cdot \tilde{R}_2 &+ C_{11}\left[(a_1^r - 1)\tilde{R}_2 - \left(\sum a_2^k\right)\tilde{R}_{12}\right] + C_{12}(a_2 - 1)\tilde{R}_2 \\
&+ \sum_{h=3}^{N} C_{1h}\left[(a_h - 1)\tilde{R}_2 - \left(\sum a_2^k\right)\tilde{R}_{2h}\right] \\
&+ C_{21}\left[(a_1 - 1)\tilde{R}_3 - \left(\sum a_3^k\right)\tilde{R}_{13}\right] \\
&+ C_{22}\left[(a_2 - 1)\tilde{R}_3 - \left(\sum a_3^k\right)\tilde{R}_{23}\right] + C_{23}(a_3 - 1)\tilde{R}_3 \\
&+ \sum_{h=4}^{N} C_{2h}\left[(a_b - 1)\tilde{R}_3 + \left(\sum a_3^k\right)\tilde{R}_{3h}\right], \\
\tilde{R}_3 \mapsto 1 \cdot \tilde{R}_3 &+ C_{31}\left[(a_1^r - 1)\tilde{R}_2 - \left(\sum a_2^k\right)\tilde{R}_{12}\right] + C_{32}(a_2 - 1)\tilde{R}_2 \\
&+ \sum_{h=3}^{N} C_{3h}\left[(a_h - 1)\tilde{R}_2 - \left(\sum a_2^k\right)\tilde{R}_{2h}\right] \\
&+ C_{41}\left[(a_1 - 1)\tilde{R}_3 - \left(\sum a_3^k\right)\tilde{R}_{13}\right] \\
&+ C_{42}\left[(a_2 - 1)\tilde{R}_3 - \left(\sum a_3^k\right)\tilde{R}_{23}\right] + C_{43}(a_3 - 1)\tilde{R}_3 \\
&+ \sum_{h=4}^{N} C_{4h}\left[(a_h - 1)\tilde{R}_3 + \left(\sum a_3^k\right)\tilde{R}_{3h}\right],
\end{aligned}$$

where the C's are arbitrary elements of $Z[\pi_1 K]$.

If the matrix M representing ϕ is invertible, ϕ will (by Lemma 1.4) represent the induced map on $C_2(\tilde{K})$ of a homotopy equivalence which induces the identity on $C_1(\tilde{K})$, since $\partial_2 \circ M = \partial_2$.

M will have the following form:

$$\begin{array}{c|ccccc|}
 & \tilde{R}_1 & \tilde{R}_2 & \tilde{R}_3 & \cdots, & R_{ij} \\
\tilde{R}_1 & 1, & 0, & 0, & \cdots, & 0 \\
\tilde{R}_2 & 0, & 1 + C_{11}\bar{a}_1^r + \sum_{h=2}^{N} C_{1h}\bar{a}_h, & \sum_{h=1}^{N} C_{2h}\bar{a}_h, & 0,\ldots,0, & \phi_{ij_2} \\
\tilde{R}_3 & 0, & C_{31}\bar{a}_1^r + \sum_{h=2}^{N} c_{2h}\bar{a}_h, & 1 + \sum_{h=2}^{N} c_{2h}\bar{a}_h, & 0,\ldots,0, & \phi_{ij_3} \\
\vdots & & & \bigcirc & & \bigcirc \\
\hline
\tilde{R}_{ij} & & & \bigcirc & & I_{\binom{n}{2}}
\end{array}$$

where ϕ_{ij_g} $(g = 1, 2)$ is the appropriate row vector, i.e.,

$$\phi_{12_1} = C_{11}\left(\sum a_2^k\right),$$
$$\phi_{2h_1} = -C_{1h}\left(\sum a_3^k\right), \qquad 4 \leqslant h \leqslant N,$$
$$\phi_{13_1} = -C_{21}\left[\sum a_3^k\right],$$
$$\phi_{23_1} = -C_{22}\left(\sum a_3^k\right) - C_{13}\left(\sum a_2^k\right),$$
$$\phi_{3h_1} = C_{2h}\left(\sum a_3^k\right), \qquad 4 \leqslant h \leqslant N,$$
$$\phi_{12_2} = -C_{31}\left(\sum a_2^k\right),$$
$$\phi_{2h_2} = -C_{3h}\left(\sum a_2^k\right), \qquad 4 \leqslant h \leqslant N,$$
$$\phi_{13_2} = -C_{41}\left(\sum a_3^k\right),$$
$$\phi_{3h} = C_{4h}\left(\sum a_3^k\right) \quad \text{and}$$
$$\phi_{23_2} = -C_{42}\left(\sum a_2^k\right) - C_{33}\left(\sum a_2^k\right),$$
$$\phi_{ij_g} = 0 \quad \text{otherwise.}$$

As before, this matrix is Whitehead equivalent to

$$\begin{bmatrix} 1 + C_{11}(a_1^r - 1) + \sum_{h=2}^{N} C_{1h}(a_h - 1), & \sum_{h=1}^{N} C_{2h}(a - 1) \\ C_{31}(a_1^r - 1) + \sum_{h=2}^{N} C_{3h}(a_h - 1), & 1 + \sum_{h=1}^{N} C_{4h}(a_h - 1) \end{bmatrix}.$$

To show that all elements of $\mathrm{Wh}(Z(G))$ can be represented in the above fashion, we invoke Lemma 1.2 using $\{a_1^r, a_2, \ldots, a_N\}$ as the generators of G. □

Since we only need to consider the above twisted presentations, from Sieradski's Theorem 2 in [**2**, Lemma 2.1], they give us the following two theorems.

THEOREM 2.2. *Let K be a finite 2-dimensional CW-complex with finite abelian fundamental group. Then all of the elements of* $\mathrm{Wh}(K)$ *are realizable as the torsions of self-equivalences on K.*

THEOREM 2.3. *Homotopy equivalence and simple homotopy equivalence are the same for finite 2-dimensional CW-complexes with finite abelian fundamental group.*

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