

SOME RESULTS CONCERNING THE BOUNDARY ZERO SETS OF GENERAL ANALYTIC FUNCTIONS

BY

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ABSTRACT. Two results concerning the boundary zero sets of analytic functions on the unit disk Δ are proved. First we consider nonconstant analytic functions f on Δ for which the radial limit function f^* is defined at each point of the unit circumference C . We show that a subset E of C is the zero set of f^* for some such function f if and only if it is a \mathcal{G}_δ that is not metrically dense in any open arc of C . We then give a precise version of an asymptotic radial uniqueness theorem and its converse. The constructions given in the proofs of each of these theorems employ an approximation theorem of Arakeljan.

1. Introduction. Let $\Delta = \{|z| < 1\}$ and $C = \{|z| = 1\}$. We assume throughout that E is a subset of C . For f a continuous function mapping Δ into the extended plane $\hat{C} = C \cup \{\infty\}$, let f^* denote its radial limit function. Thus $f^*(\eta) = \lim_{r \rightarrow 1} f(r\eta)$ for each η in C where the limit exists (finite or infinite).

In this paper we shall apply the following theorem of N. U. Arakeljan [1] to obtain two results concerning the boundary zero sets of general analytic (and meromorphic) functions.

THEOREM A. *Let D be a domain in the complex plane C and let K be a relatively closed proper subset of D . Then every function g continuous on K and analytic in its interior can be uniformly approximated by an analytic function f on D if and only if $D^\infty \setminus K$ is connected and locally connected, where $D^\infty = D \cup \{z_\infty\}$ denotes the Alexandrov (one-point) compactification of D and z_∞ is the added point.*

We first consider functions f for which f^* is defined at each point η of C . When f is a function of this kind, the set $\{\eta \in C: f^*(\eta) = 0\}$ is a \mathcal{G}_δ , that is, a countable intersection of open sets. Concerning bounded analytic functions, J. S. Hwang and P. Lappan [15] proved the following theorem, improving a result given by Cahill [12, Theorem 5]. Both results are based on the work of Z. Zahorski [23].

THEOREM B. *There exists a nonconstant (nonvanishing) bounded analytic function f on Δ having radial limits at each point of C such that $E = \{\eta \in C: f^*(\eta) = 0\}$ if and only if E is a \mathcal{G}_δ of measure 0.*

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In [8], the author gave the corresponding characterization for inner functions, that is, bounded analytic functions f for which $|f^*| = 1$ a.e. It was shown that $E = \{\eta \in C: f^*(\eta) = 0\}$ for an inner function f having radial limits at each point of C if and only if E is a \mathcal{G}_δ contained in an \mathcal{T}_σ (a countable union of closed sets) of measure 0. The main result of §2 is the analogous result for general analytic functions.

THEOREM 1.1. *There exists a nonconstant analytic (or meromorphic) function f on Δ having radial limits at each point of C for which $E = \{\eta \in C: f^*(\eta) = 0\}$ if and only if E is a \mathcal{G}_δ that is not metrically dense in any open arc of C .*

Recall that E is said to metrically dense in a (nonempty) open arc A if $E \cap B$ has positive outer measure for every open subarc B of A . Theorem 1.1 sharpens a result given in [10]. Along with Theorem A, its proof relies on results and methods used in [10] and in recent work of C. L. Belna, P. Colwell and G. Piranian [5] in which the boundary zero sets of the class of all inner functions are characterized.

Our second application concerns radial approach to zero at a specified asymptotic rate. K. F. Barth and W. J. Schneider [4], and K. F. Tse [22] proved an asymptotic radial uniqueness theorem for arbitrary analytic functions on Δ . Along with the converse given by P. Gauthier (see [22, Theorem B]), this result may be stated as follows.

THEOREM C. *Let μ be a positive decreasing function on $[0, 1)$ such that $\lim_{r \rightarrow 1} \mu(r) = 0$. Then there exists a nonconstant analytic (or meromorphic) function f on Δ such that $f(r\eta) = O[\mu(r)]$ for each $\eta \in E$ if and only if E is of first category.*

In §3 we prove a more precise version of Theorem C.

THEOREM 1.2. *Let μ be a positive decreasing function on $[0, 1)$ such that $\lim_{r \rightarrow 1} \mu(r) = 0$. Then there exists a nonconstant analytic (or meromorphic) function f on Δ such that $E = \{\eta \in C: f(r\eta) = O[\mu(r)]\}$ if and only if E is first-category \mathcal{T}_σ set.*

Necessity in Theorem 1.2 follows from an elementary proof of the Barth-Schneider-Tse theorem given in [6]. The proof of sufficiency again uses Theorem A and the methods of §2.

In §4, analogues for continuous subharmonic functions are given. These are easily derived from previous results for subharmonic functions (see, for example, [9]), and the theorems for analytic functions stated in this section. In the final section, §5, we state some open problems.

2. Proof of Theorem 1.1. We start by establishing notation and proving some elementary results. For f a meromorphic function on Δ , let $C_\rho(f, \eta)$ denote the radial cluster set of f at $\eta \in C$. Thus, by definition,

$$C_\rho(f, \eta) = \bigcap_1^\infty \overline{\{f(r\eta): (n-1)/n \leq r < 1\}}, \quad \eta \in C.$$

Here and in the sequel, \overline{W} (respectively $\text{int } W$) denotes the closure (respectively interior) of a subset W of C . Let $Z_\rho(f) = \{\eta \in C: 0 \in C_\rho(f, \eta)\}$.

PROPOSITION 2.1. *The set $Z_\rho(f)$ is a \mathcal{G}_δ .*

PROOF. For each positive integer n , let

$$F_n = \{ \eta \in C : |f(r\eta)| \geq 1/n \text{ for } (n-1)/n \leq r < 1 \}.$$

By the continuity of f , each F_n is closed. Furthermore,

$$C \setminus Z_\rho(f) = \bigcup_{n=1}^{\infty} F_n.$$

Thus $Z_\rho(f)$ is the complement of an \mathcal{F}_σ , that is, a \mathcal{G}_δ . This completes the proof.

If $\eta \in C$ and $C_\rho(f, \eta) \neq \hat{C}$, then $C_\rho(f, \eta)$ is said to be *subtotal*.

DEFINITION 2.1. *Let \mathcal{M}_S be the class of all meromorphic functions f on Δ for which $C_\rho(f, \eta)$ is subtotal for a residual set of η in C (that is, the complement of a first-category set).*

Observe that if $\eta \in C$ and $f^*(\eta)$ is defined, then $C_\rho(f, \eta) = \{f^*(\eta)\}$ so that $C_\rho(f, \eta)$ is subtotal. This leads to the next result.

PROPOSITION 2.2. *If f is a meromorphic function on Δ such that $f^*(\eta)$ is defined for each η in C , then $f \in \mathcal{M}_S$ and $Z_\rho(f) = \{ \eta \in C : f^*(\eta) = 0 \}$ is a \mathcal{G}_δ .*

The next theorem is a generalization of the Lusin-Privalov radial uniqueness theorem (see [14, Theorem 8.3(i)]) proved by E. F. Collingwood and, independently, by F. Bagemihl and W. Seidel.

THEOREM D. *Let f be a meromorphic function on Δ and let A be an open arc of C . If $C_\rho(f, \eta)$ is subtotal for a set of η of second category in A , and $b \in C_\rho(f, \eta)$ for a set of η which is metrically dense in A , then $f \equiv b$.*

The following is a corollary of Theorem D and Proposition 2.2 that completes the proof of necessity in Theorem 1.1.

COROLLARY 2.1. *If f is nonconstant meromorphic function on Δ such that $f^*(\eta)$ is defined for each η in C , then $Z_\rho(f) = \{ \eta \in C : f^*(\eta) = 0 \}$ is a \mathcal{G}_δ that is not metrically dense in any arc of C .*

In [10], it is shown that if E is a subset of C that is not metrically dense in any open arc of C , then there exists a nonconstant analytic function f on Δ whose modulus has radial limits at each point of C such that $f^*(\eta) = 0$ for $\eta \in E$. We shall now refine this result to prove sufficiency in Theorem 1.1. Our procedure will be similar in its use of the following lemma proved in [7].

LEMMA 2.1. *If for every open arc A of C , the set E is not both metrically dense and of second category in A , then there exists a closed subset F of C such that $E \setminus F$ has zero measure and $E \cap F$ is of first category.*

Here, the set E is a \mathcal{G}_δ that is not metrically dense in any open arc of C , and the lemma insures the existence of a closed set F such that $E \setminus F$ is a G_δ of measure 0, and $E \cap F$ is a first-category \mathcal{G}_δ (in fact, a nowhere-dense \mathcal{G}_δ). The set $E \setminus F$ is handled using Theorem B. For $E \cap F$, we shall use the following proposition.

PROPOSITION 2.3. *If E is a first-category \mathcal{G}_δ , then there exists an analytic function f on Δ such that f^* is defined and finite at each point of C and $E = \{\eta \in C: f^*(\eta) = 0\}$.*

In order to give the proof, we shall need some additional notation and a lemma. For each $\eta \in C$ and $r \in (0, 1)$, let

$$R(\eta) = \{z \in \mathbf{C} \setminus \{0\}: z/|z| = \eta\} \cup \{0\},$$

and let $d(\eta, r)$ be the open disk in Δ of radius r whose circumference is tangent to C at η . In addition, for any subset E of C , let $R(E) = \bigcup_{\eta \in E} R(\eta)$ and $d(E, r) = \bigcup_{\eta \in E} d(\eta, r)$. If A is any proper open arc of C , let S_A be the open sector of Δ defined by

$$S_A = \{z \in \Delta \setminus \{0\}: z/|z| \in A\}.$$

Let $\arg \eta$, $\eta \in A$, be a continuous argument of the identity function on A and let \mathbf{Z} denote the integers.

Suppose now that V is a closed nowhere-dense subset of C and A a proper open arc of C . Select $(\eta_j)_{j \in \mathbf{Z}} \subset A \setminus V$ such that $\arg \eta_j$ is an increasing function of j in \mathbf{Z} and $\lim_{j \rightarrow \pm \infty} \eta_j$ are the endpoints of A . For each $j \in \mathbf{Z}$, let F_j be the closed nowhere-dense subset of C consisting of the intersection of V with the open arc A_j having endpoints η_j and η_{j+1} . Given $\varepsilon \in (0, 1)$, choose $(r_j)_{j \in \mathbf{Z}}$ so that $\lim_{j \rightarrow \pm \infty} r_j = 0$, and for each j , $r_j \in (0, \varepsilon)$ and $d(F_j, r_j) \subset S_A$.

LEMMA 2.2. *Let $\varepsilon \in (0, 1)$. There exists an analytic function f on Δ such that*

- (i) f^* exists at each point of C ,
- (ii) $V \cap A = \{\eta \in C: f^*(\eta) = 0\}$,
- (iii) $|f(z) - 1| < \varepsilon$, $z \in \Delta \setminus \bigcup_{j \in \mathbf{Z}} d(F_j, r_j)$, and
- (iv) $|f(z)| \leq 1 + \varepsilon$, $z \in \Delta \cap R(A \cap V)$.

PROOF. For each $j \in \mathbf{Z}$, let

$$D_j = [C \setminus R(F_j)] \cup \Delta, \quad K_j = [D_j \setminus d(F_j, r_j)] \cup [R(F_j) \cap d(F_j, r_j)],$$

and define g_j by

$$g_j(z) = \begin{cases} 0, & z \in D_j \setminus d(F_j, r_j), \\ \ln \frac{2r_j}{1 - |z|}, & z \in R(F_j) \cap d(F_j, r_j). \end{cases}$$

Let $t_0 > 0$ such that $\exp(t) - 1 \leq 2t$ for all $t \in [0, t_0]$ and assume without loss of generality that $\varepsilon < t_0$. Note that for each j , the set K_j is a relatively closed subset of the domain D_j such that $D_j^\infty \setminus K_j$ is connected and locally connected, and g_j is continuous on K_j and analytic in its interior. Therefore, by Theorem A there exists for every $j \in \mathbf{Z}$ an analytic function f_j on D_j such that

$$(2.1) \quad |f_j(z) - g_j(z)| < \varepsilon/2^{(|j|+2)}, \quad z \in K_j.$$

From the definition of g_j and the domain D_j , it follows that

$$(2.2) \quad |f_j(z)| < \varepsilon/2^{(|j|+2)}, \quad z \in D_j \setminus d(F_j, r_j),$$

and $f_j|_\Delta$ has radial limits at each point of C for every $j \in \mathbf{Z}$.

Define $h(z) = \sum_{j \in \mathbf{Z}} f_j(z)$ for $z \in \Delta$. By (2.2) and the choice of $(r_j)_{j \in \mathbf{Z}}$ so that $d(F_j, r_j) \subset S_{A_j}$ for each j and $\lim_{j \rightarrow \pm\infty} r_j = 0$, we see that the sum is uniformly convergent on compact subsets and each radius of Δ ,

$$(2.3) \quad |h(z)| \leq \sum_{j \in \mathbf{Z}} |f_j(z)| \leq \frac{\varepsilon}{2}, \quad z \in \Delta \setminus \bigcup_1^\infty d(F_j, r_j),$$

(2.4)

$$|h(z) - f_k(z)| \leq \sum_{j \neq k} |f_j(z)| \leq \varepsilon \left(\frac{1}{2} - \frac{1}{2^{|k|+2}} \right), \quad z \in \Delta \cap R(A_k), k \in \mathbf{Z},$$

and h^* is defined at each point of C with

$$(2.5) \quad |h^*(\eta)| \leq \varepsilon/2, \quad \eta \in C \setminus (A \cap V),$$

and

$$(2.6) \quad \lim_{r \rightarrow 1} \operatorname{Re} h(r\eta) = +\infty, \quad \eta \in A \cap V.$$

It also follows from (2.1) and (2.4) that

$$(2.7) \quad \operatorname{Re} h(z) \geq -\varepsilon/2, \quad z \in \Delta \cap R(A \cap V).$$

Let $f = \exp(-h)$. Since h^* is defined at each point of C and (2.5) and (2.6) hold, it follows that f^* is defined at each point of C with $V \cap A = \{\eta \in C: f^*(\eta) = 0\}$. Furthermore, (iii) and (iv) follow from (2.3) and (2.7) together with the inequalities

$$|\exp(z) - 1| \leq \exp(|z|) - 1 \leq 2|z|, \quad |z| \leq t_0.$$

This completes the proof of the lemma.

We turn now to the proof of Proposition 2.3.

PROOF. We put aside the trivial case and assume that $E \neq \emptyset$. Since E is a \mathcal{G}_δ of first category, it is contained in a closed nowhere-dense set V . (Recall that a \mathcal{G}_δ that is dense in an arc is residual in it.) Also, there exists a descending sequence $(U_n)_1^\infty$ of open subsets of C such that $E = \bigcap_1^\infty U_n$. For each n , we have $U_n = \bigcup_k A_{nk}$, where (A_{nk}) is a finite or infinite sequence of mutually disjoint open arcs each having nonempty intersection with E . Furthermore, for any pair of allowed positive integers n and k , we have $A_{n+1,k} \subseteq A_{nj}$ for some positive integer j .

We proceed by induction to define $(A_{nkj})_{j \in \mathbf{Z}}, (F_{nkj})_{j \in \mathbf{Z}}$ and $(r_{nkj})_{j \in \mathbf{Z}}$ relative to the arc A_{nk} for all such arcs. Here and in the remainder of the proof, we use subscripted notation corresponding to that preceding Lemma 2.2.

$n = 1$. Choose $(A_{1kj})_{j \in \mathbf{Z}}, (F_{1kj})_{j \in \mathbf{Z}}$ and $(r_{1kj})_{j \in \mathbf{Z}}$ relative to A_{1k} as preceding Lemma 2.2 with $\varepsilon = 1/2^{(1+k)}$ for each k .

Inductive hypothesis. Assume that $n \geq 1$ and $(A_{nkj})_{j \in \mathbf{Z}}, (F_{nkj})_{j \in \mathbf{Z}}$ and $(r_{nkj})_{j \in \mathbf{Z}}$ have been selected relative to A_{nk} as preceding Lemma 2.2 with $\varepsilon = 1/2^{(n+k)}$ for each k . Assume also that if $1 \leq n' \leq n$, $A_{nk} \subseteq A_{n'k'}$ and $\eta \in A_{nk} \cap V$, then the sets F_{nkj} and $F_{n'k'j'}$ that contain η along with the associated r_{nkj} and $r_{n'k'j'}$ satisfy $F_{nkj} \subseteq F_{n'k'j'}$ and $r_{nkj} \leq r_{n'k'j'}$.

$n + 1$. For each arc $A_{n+1,k}$, there exists a unique positive integer k' such that $A_{n+1,k} \subseteq A_{nk'}$. Noting the simple structure of the sets $F_{nk'j'} \cap A_{n+1,k}$ for $j' \in \mathbf{Z}$, the sets $(A_{n+1,kj})_{j \in \mathbf{Z}}$ and $(F_{n+1,kj})_{j \in \mathbf{Z}}$ can be constructed as preceding Lemma 2.2 so that each $F_{n+1,kj}$ is contained in some $F_{nk'j'}$. Then choose $(r_{n+1,kj})_{j \in \mathbf{Z}}$ in $(0, 1/2^{(n+k+1)})$ with $r_{n+1,kj} \leq r_{nk'j'}$ and $\lim_{j \rightarrow \pm \infty} r_{n+1,kj} = 0$.

It is verified without difficulty that the inductive hypothesis holds with $n + 1$ replacing n , and the induction is completed. Therefore, there exist $(A_{nkj})_{j \in \mathbf{Z}}$, $(F_{nkj})_{j \in \mathbf{Z}}$ and $(r_{nkj})_{j \in \mathbf{Z}}$ such that the inductive hypothesis holds for all allowed positive integers n and k . The following observations will be used.

(I) $[\Delta \setminus \bigcup_k \bigcup_{j \in \mathbf{Z}} d(F_{nkj}, r_{nkj})]_{n=1}^\infty$ forms a monotone nondescending sequence of sets such that for every compact subset K of Δ , only finitely many have nonempty intersection with K .

(II) If n is a positive integer and $\eta \notin U_n$, then $\Delta \cap R(\eta)$ does not intersect any of the sets $\bigcup_k \bigcup_{j \in \mathbf{Z}} d(F_{mkj}, r_{mkj})$ for $m \geq n$ since $(U_l)_1^\infty$ is a descending sequence and each set $\bigcup_{j \in \mathbf{Z}} d(F_{mkj}, r_{mkj})$ is contained in the sector $S_{A_{mk}}$.

(III) If n is a positive integer and $\eta \in U_n$, then for $1 \leq m \leq n$, either $\eta \in V \cap U_m$ in which case $\eta \in F_{mkj}$ for some k and j , or else $\eta \notin V \cap U_m$ and $r\eta \in \Delta \setminus \bigcup_k \bigcup_{j \in \mathbf{Z}} d(F_{mkj}, r_{mkj})$ for $r \in [0, 1)$ sufficiently close to 1.

For each pair of positive integers n and k , let f_{nk} be as in Lemma 2.2 relative to $\epsilon = 1/2^{(n+k)}$ with F_j and r_j replaced by F_{nkj} and r_{nkj} respectively. Let $(h_m)_1^\infty$ be an enumeration of the functions $f_{nk} + 1/2^{(n+k)}$. Define $f = \prod_1^\infty h_m$. From (I), we see that f is uniformly product convergent on compact subsets of Δ and therefore analytic. If $\eta \in C \setminus E$, then η is contained in at most finitely many U_n , and it follows from (II), (III) and (iii) of Lemma 2.2 that f is uniformly product convergent on the radius $\Delta \cap R(\eta)$. From (ii) of the lemma and the definition of the h_m , we have h_m^* is defined and nonvanishing at each point of C , so we conclude that f^* is defined and not zero at each point of $C \setminus E$. Finally, since $\lim_{m \rightarrow \infty} h_m^*(\eta) = 0$ for each $\eta \in E$, it follows from (iv) of the lemma that $f^*(\eta) = 0$ for all $\eta \in E$. The proof of Proposition 2.3 is thereby completed.

Proof of Sufficiency in Theorem 1.1. Putting aside the trivial case when $E = \emptyset$, let E be a nonempty \mathcal{G}_δ that is not metrically dense in any open arc of C . By Lemma 2.1 there exists a closed set F such that $E \setminus F$ is a \mathcal{G}_δ of measure 0 and $F \cap E$ is a first-category \mathcal{G}_δ . By Theorem B, there is a bounded analytic function g on Δ having radial limits at each point of C and for which $E \setminus F = \{\eta \in C: g^*(\eta) = 0\}$. From Proposition 2.3, there exists an analytic function h on Δ such that h^* is defined and finite at each point of C with $E \cap F = \{\eta \in C: h^*(\eta) = 0\}$. Then $f = gh$ is an analytic function on Δ having radial limits at each point of C for which $E = \{\eta \in C: f^*(\eta) = 0\}$. Since $\emptyset \neq E \neq C$, the function f is nonconstant. The proof is thereby completed.

3. Proof of Theorem 1.2. For necessity, we restate the proof given in [6]. Let f be a nonconstant meromorphic function on Δ . Suppose that $E = \{\eta \in C: f(r\eta) = O[\mu(r)]\}$. Then $E = \bigcup_1^\infty E_n$, where

$$E_n = \{\eta \in C: |f(r\eta)| \leq n\mu(r) \text{ for } 1 - 1/n \leq r < 1\}$$

for each n . By the continuity of f , each E_n is closed. If $\text{int } E_n \neq \emptyset$ for some n , say A is an open arc contained in $\text{int } E_n$, then by the definition of E_n the function f is continuously 0 at each point of A . Then the Schwarz reflection principle and the identity theorem imply that $f \equiv 0$, a contradiction. We conclude that each E_n is nowhere dense, and necessity follows.

Sufficiency. If $E = \emptyset$, then take $f(z) = \exp(z)$. Suppose now that E is a nonempty first-category \mathcal{F}_σ . Then $E = \bigcup_1^\infty E_n$, where $(E_n)_1^\infty$ is a sequence of mutually disjoint closed nowhere-dense sets (see, for example, [8, Theorem 3.2]) and there exists a decreasing sequence $(r_n)_1^\infty$ in $(0, 1/2)$ converging to 0 such that $\{d(E_n, r_n)\}_{n=1}^\infty$ is a sequence of mutually disjoint compact sets. Define

$$K = \left[\Delta \setminus \bigcup_{n=1}^\infty d(E_n, r_n) \right] \cup [R(E) \cap \Delta].$$

The following properties can be verified.

- (i) The set K is a relatively closed proper subset of Δ such that $\Delta^\infty \setminus K$ is connected and locally connected.
- (ii) For every $\eta \in E$, there exists a positive integer n such that $d(E_n, r_n) \cap R(\eta) \subset K$.
- (iii) If $\eta \notin E$, then there exists a sequence $(z_n)_1^\infty$ in $R(\eta) \cap \Delta$ converging to η such that $z_n \in \text{int } K$ for all positive integers n .

Assume without loss of generality that μ is continuous and let

$$(3.1) \quad g(z) = \begin{cases} 0, & z \in \Delta \setminus \bigcup_{n=1}^\infty d(E_n, r_n), \\ \ln \frac{1}{\mu(|z|)} - \ln \frac{1}{\mu(1 - 2r_n)}, & z \in d(E_n, r_n) \cap R(E_n), n = 1, 2, \dots \end{cases}$$

Then G is continuous on K and analytic (identically 0) in $\text{int } K$. By property (i) and Theorem A, there exists a function h analytic on Δ such that

$$|h(z) - g(z)| < 1, \quad z \in K.$$

Using properties (ii) and (iii), there exists $\delta_n \in (0, \infty)$ such that for all $r \in [0, 1)$ we have

$$(3.2) \quad \text{Re } h(r\eta) \geq -\delta_n + \ln \frac{1}{\mu(r)}, \quad \eta \in E_n,$$

for $n = 1, 2, \dots$, and

$$(3.3) \quad \liminf_{r \rightarrow 1} \text{Re } h(r\eta) < +\infty, \quad \eta \in C \setminus E.$$

Let $f = \exp(-h)$. Then f is a nonvanishing analytic function on Δ , and (3.2) and (3.3) insure that f has the required properties. This completes the proof.

4. Subharmonic analogues. For u ($\neq -\infty$) subharmonic on Δ , the radial limit function (taking values in $\hat{\mathbf{R}} = \mathbf{R} \cup \{\pm \infty\}$) is denoted by u^* . Since every analytic function $f \neq 0$ corresponds to a continuous subharmonic function $u = \log|f|$, the

natural analogue of the zeros of f^* are the $-\infty$ -points of u^* . It turns out that many boundary zero set results for analytic functions have subharmonic analogues. M. G. Arsove [2] and P. J. Rippon [20] were the first to give analogues of the Lusin-Privalov and Barth-Schneider-Tse theorems. These along with other results can be found in [9]. Putting together the results of the previous sections with ones given in [9], we arrive at the following theorems.

THEOREM 4.1. *There exists a continuous subharmonic function $u \leq 0$ on Δ such that u^* exists at each point of C and $E = \{\eta \in C: u^*(\eta) = -\infty\}$, if and only if E is a \mathcal{G}_δ of measure 0.*

THEOREM 4.2. *There exists a continuous subharmonic function $u \leq 0$ such that u^* is defined at each point of C , $u^* = 0$ a.e. and $E = \{\eta \in C: u^*(\eta) = -\infty\}$, if and only if E is a \mathcal{G}_δ contained in an \mathcal{T}_σ of measure 0.*

THEOREM 4.3. *There exists a continuous subharmonic function u such that u^* is defined at each point of C and $E = \{\eta \in C: u^*(\eta) = -\infty\}$, if and only if E is a \mathcal{G}_δ that is not metrically dense in any open arc of C .*

THEOREM 4.4. *Let μ be a decreasing function on $[0, 1)$ such that $\lim_{r \rightarrow 1} \mu(r) = -\infty$. Then there exists a continuous subharmonic function u on Δ such that*

$$E = \left\{ \eta \in C: \limsup_{r \rightarrow 1} [u(r\eta) - \mu(r\eta)] < \infty \right\}$$

if and only if E is a first-category \mathcal{T}_σ .

We note that since the functions constructed for Theorem B and in §3 for Theorem 1.2 are nonvanishing, sufficiency in Theorems 4.1 and 4.4 can be proved using harmonic functions.

5. Concluding remarks. There are a variety of open questions in this subject. There are many important classes of functions for which a radial uniqueness theorem and its converse, or a complete characterization of the boundary zero sets has not been given.

(1) *Bounded analytic functions.* The classical Riesz radial uniqueness theorem [19] along with the converse of Privalov [18, p. 214] can be stated as follows.

THEOREM E. *There exists a nonconstant bounded analytic function f on Δ such that $f^*(\eta) = 0$ for each $\eta \in E$ if and only if E has zero measure.*

Based on this theorem and elementary topological considerations, it is natural to conjecture that the exact characterization of the boundary zero sets (for nonconstant functions in this class) is the class of $\mathcal{T}_{\sigma\delta}$ sets of measure 0.

Å. Samuelsson [21] considered asymptotic radial descent to 0 for bounded analytic functions. He derived a Hausdorff measure condition for sets where this occurs depending on the descent rate to 0. It is not known if this condition taken together with the topological necessary condition of Theorem 1.2 gives a theorem that is sharp.

(2) *General analytic functions.* The Lusin-Privalov radial uniqueness theorem [17, pp. 187–189] and its converse [7] can be given in the following way.

THEOREM F. *There exists a nonconstant analytic (or meromorphic) function f on Δ such that $f^*(\eta) = 0$ for $\eta \in E$ if and only if for every open arc A of C , the set E is not both metrically dense and of second category in A .*

In this case, the natural conjecture is that the class of boundary zero sets consists of the $\mathcal{T}_{\sigma\delta}$ sets of the type given in the theorem.

(3) *Functions having prescribed growth.* It is not known whether the converse to the Lusin-Privalov theorem can be proved using functions f such that $|f(z)| \leq \nu(|z|)$, where ν is a prescribed positive continuous function on $[0, 1)$ satisfying $\lim_{r \rightarrow 1} \nu(r) = \infty$. Barth and Schneider [3] have shown that there exists a function f with $|f(z)| \leq \nu(|z|)$, $z \in \Delta$, such that $f^*(\eta) = 0$ a.e. in C . If Barth and Schneider's result could be modified to insure that for a prescribed first-category set E , the function f satisfies $f^*(\eta) = 0$ for each $\eta \in E$, and $\limsup_{r \rightarrow 1} |f(r\eta)| < \infty$ when $\eta \in C \setminus \bar{E}$, then we would have the converse using essentially the same proof as given in [7].

There is a similar question whether the converse in Theorem C can be proved using functions of prescribed growth.

(4) *Univalent functions.* Beurling [11] (see also [14, pp. 56–57]) proved that the boundary zero set of a univalent function cannot have positive logarithmic capacity. The converse to this result has not been proved.

(5) *Inner functions.* Cargo [13] has shown that for a nonconstant inner function f , the set E of $\eta \in C$ where $f^*(\eta) = 1$ is of first category (and measure 0). It is not known if there is a converse. If so, there would be different characterizations for the zero sets (see [5]) and the 1-points.

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