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HYPERREFLEXIVITY AND A DUAL PRODUCT CONSTRUCTION[1]

BY

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ABSTRACT. We show that an example of a nonhyperreflexive CSL algebra recently constructed by Davidson and Power is a special case of a general and natural reflexive subspace construction. Completely different techniques of proof are needed because of absence of symmetry. It is proven that if $\mathscr{S}$ and $\mathscr{T}$ are reflexive proper linear subspaces of operators acting on a separable Hilbert space, then the hyperreflexivity constant of $(\mathscr{S}_\perp \otimes \mathscr{T}_\perp)^\perp$ is at least as great as the product of the constants of $\mathscr{S}$ and $\mathscr{T}$.

This paper was inspired by the interesting "key example" in the recent paper [**2**] by Davidson and Power in which a nonhyperreflexive CSL algebra was constructed. In an attempt to completely understand this result we obtained a distance constant inequality of a more general nature, which we present here.

Let H, K be separable Hilbert spaces—finite or infinite dimensional—and let $\mathscr{S}$, $\mathscr{T}$ be linear subspaces of $L(H)$, $L(K)$, which are reflexive in the Loginov-Shulman sense. ($\mathscr{S}$ is *reflexive* iff whenever $T \in L(H)$ is such that $Tx \in [Sx]$, $x \in H$, then $T \in \mathscr{S}$, where $[\cdot]$ means closure.) Let $\mathscr{K}(\mathscr{S})$, $\mathscr{K}(\mathscr{T})$ be the constants of hyperreflexivity of $\mathscr{S}$ and $\mathscr{T}$ as defined in [**4**]. We recall that a subspace $\mathscr{S}$ of $L(H)$ is *hyperreflexive* if there is a constant C such that for operators T in $L(H)$,

$$d(T, \mathscr{S}) \leqslant C \sup\{\|P^\perp TQ\|: P, Q \text{ are projections with } P^\perp \mathscr{S} Q = 0\},$$

and the optimal constant is denoted $\mathscr{K}(\mathscr{S})$. If $\mathscr{S}$ is reflexive but not hyperreflexive, then we define $\mathscr{K}(\mathscr{S}) = +\infty$. We make use of preannihilator techniques, and refer the reader to [**1, 4, 5, 7**] for details. As shown in [**4**], the reflexive subspaces of $L(H)$ are precisely those for which the preannihilator in $\mathscr{L}_* \equiv C_1$ is the $\|\cdot\|_1$-closed linear span of rank $\leqslant 1$ operators, where $\|\cdot\|_1$ denotes trace-class norm. Since $\mathscr{S}_\perp$, $\mathscr{T}_\perp$ are generated by rank $\leqslant 1$ operators, so is the tensor product of preannihilators $\mathscr{S}_\perp \otimes \mathscr{T}_\perp$. By this we mean the $\|\cdot\|_1$-closed linear subspace of the ideal of trace-class operators on $L(H \otimes K)$ generated by the elementary tensors $\{f \otimes g\colon f \in \mathscr{S}_\perp, g \in \mathscr{T}_\perp\}$, where $H \otimes K$ denotes the usual tensor product Hilbert space. (When we write $\mathscr{S} \otimes \mathscr{T}$, we will mean the σ-weakly closed linear subspace of $L(H \otimes K)$ generated by $\{S \otimes T\colon S \in \mathscr{S}, T \in \mathscr{T}\}$.) Thus the annihilator

$$(\mathscr{S}_\perp \otimes \mathscr{T}_\perp)^\perp = \{A \in L(H \otimes K)\colon \operatorname{Tr}(Ah) = 0,\ h \in \mathscr{S}_\perp \otimes \mathscr{T}_\perp\}$$

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is a reflexive subspace of $L(H \otimes K)$. We will call this the *dual* product of $\mathscr{S}$ and $\mathscr{T}$ since it is in a sense dual to the usual tensor product, and will adopt the notation $\mathscr{S} * \mathscr{T} = (\mathscr{S}_\perp \otimes \mathscr{T}_\perp)^\perp$. We extend this term, and notation, to arbitrary σ-weakly closed subspaces.

For reflexive $\mathscr{S}$ and $\mathscr{T}$, Theorem 8 states that $\mathscr{S} * \mathscr{T}$ is the smallest reflexive subspace containing $\mathscr{S} \otimes L(K) + L(H) \otimes \mathscr{T}$. In special cases (and perhaps in general) this coincides with the σ-weak closure of $\mathscr{S} \otimes L(K) + L(H) \otimes \mathscr{T}$. This is the case in finite dimensions (Proposition 1).

It is clear that for subspaces $\mathscr{S}_i$ we have $(\mathscr{S}_1 * \mathscr{S}_2) * \mathscr{S}_3 = \mathscr{S}_1 * (\mathscr{S}_2 * \mathscr{S}_3)$ (or rather, equivalence), so we may drop parentheses with no ambiguity. An n-fold dual product $\mathscr{S}_1 * \cdots * \mathscr{S}_n$ will have the n-fold tensor product $(\mathscr{S}_1)_\perp \otimes \cdots \otimes (\mathscr{S}_n)_\perp$ as preannihilator.

The main result of this paper, Theorem 9, states that the inequality $\mathscr{K}(\mathscr{S} * \mathscr{T}) \geqslant \mathscr{K}(\mathscr{S}) \cdot \mathscr{K}(\mathscr{T})$ always holds for reflexive *proper* subspaces $\mathscr{S}$, $\mathscr{T}$. (If either $\mathscr{S} = L(H)$ or $\mathscr{T} = L(K)$, then $\mathscr{S} * \mathscr{T} = L(H \otimes K)$, so the inequality need not hold. These are the only exceptions.)

A special case arises when $\mathscr{D}$ is the algebra of 3×3 diagonal operators acting on a 3-dimensional Hilbert space. Then, since it is known (M. D. Choi, unpublished) that $\mathscr{K}(\mathscr{D}) \geqslant \sqrt{9/8}$, the n-fold dual product $\mathscr{D} * \cdots * \mathscr{D}$ has constant $\geqslant (9/8)^{n/2}$. This is seen to be the Davidson-Power example. The subspace $\mathscr{D} * \cdots * \mathscr{D}$ is a bimodule over the n-fold tensor product $\mathscr{D} \otimes \cdots \otimes \mathscr{D}$, a m.a.s.a.; hence

$$\begin{pmatrix} \mathscr{D} \otimes \cdots \otimes \mathscr{D} & \mathscr{D} * \cdots * \mathscr{D} \\ 0 & \mathscr{D} \otimes \cdots \otimes \mathscr{D} \end{pmatrix}$$

is a CSL algebra. It can be shown directly, as in [**2**], or via duality, as in Theorem 12, that the hyperreflexivity constant for this algebra is at least as great as that of $\mathscr{D} * \cdots * \mathscr{D}$. Theorem 9 can be viewed as a generalization of the induction step in the Davidson-Power construction. Since averaging techniques utilizing symmetry do not apply, proofs are necessarily different. Prior to their example, inequalities of this nature were not suspected.

We note that while $\mathscr{D}$ is an algebra, $\mathscr{D} * \mathscr{D}$ is not. Hence, analysis of multi-dual-products such as $\mathscr{D} * \cdots * \mathscr{D}$ requires reflexive subspace theory. Also, we note that Propositions 1 and 3 are not used in the proofs of our main results, but are given for perspective on these.

The next lemma will be used repeatedly.

LEMMA 0. *Let $\mathscr{S} \subseteq L(H)$, $\mathscr{T} \subseteq L(K)$ be linear subspaces. Then $\mathscr{S} * \mathscr{T} \supseteq \mathscr{S} \otimes L(K) + L(H) \otimes \mathscr{T}$.*

PROOF. If $f \in \mathscr{S}_\perp$, $g \in \mathscr{T}_\perp$, then for each $S \in \mathscr{S}$, $T \in \mathscr{T}$, $A \in L(H)$, $B \in L(K)$ we have

$$\begin{aligned} \operatorname{Tr}[(S \otimes B + A \otimes T)(f \otimes g)] &= \operatorname{Tr}[(sf) \otimes (Bg)] + \operatorname{Tr}[(Af) \otimes (Tg)] \\ &= \operatorname{Tr}(Sf) \cdot \operatorname{Tr}(Bg) + \operatorname{Tr}(Af) \cdot \operatorname{Tr}(Tg) = 0 \end{aligned}$$

since $\operatorname{Tr}(Sf) = 0$ and $\operatorname{Tr}(Tg) = 0$. Since the operators $S \otimes B + A \otimes T$ generate $\mathscr{S} \otimes L(K) + L(H) \otimes \mathscr{T}$, and the operators $f \otimes g$ generate $\mathscr{S}_\perp \otimes \mathscr{T}_\perp = (\mathscr{S} * \mathscr{T})_\perp$, we conclude that $(\mathscr{S} * \mathscr{T})_\perp \subseteq (\mathscr{S} \otimes L(K) + L(H) \otimes \mathscr{T})_\perp$, and hence that $\mathscr{S} \otimes \mathscr{T} \supseteq \mathscr{S} \otimes L(K) + L(H) \otimes \mathscr{T}$. □

PROPOSITION 1. *Let H, K be finite dimensional Hilbert spaces, and let $\mathcal{S} \subseteq L(H)$, $\mathcal{T} \subseteq L(K)$ be linear subspaces. Then $\mathcal{S} * \mathcal{T} = \mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$.*

PROOF. Let n_1, n_2 be the dimensions of H, K, respectively, and let m_1, m_2 be the vector space dimensions of $\mathcal{S}$, $\mathcal{T}$, respectively. Then $\dim(L(H)) = n_1^2$, $\dim(L(K)) = n_2^2$, $\dim(\mathcal{S}_\perp) = n_1^2 - m_1$ and $\dim(\mathcal{T}_\perp) = n_2^2 - m_2$. So $\dim(\mathcal{S}_\perp \otimes \mathcal{T}_\perp) = (n_1^2 - m_1)(n_2^2 - m_2)$, and thus

$$\dim(\mathcal{S} \otimes \mathcal{T}) = n_1^2 n_2^2 - \left(n_1^2 - m_1\right)\left(n_2^2 - m_2\right) = n_1^2 m_2 + m_1 n_2^2 - m_1 m_2.$$

If X, Y are finite dimensional vector spaces over $\mathbf{C}$, and if $X_1 \subseteq X$, $Y_1 \subseteq Y$ are linear subspaces, then $(X_1 \otimes Y) \cap (X \otimes Y_1) = X_1 \otimes Y_1$. Thus $(\mathcal{S} \otimes L(K)) \cap (L(H) \otimes \mathcal{T}) = \mathcal{S} \otimes \mathcal{T}$. So

$$\begin{aligned}\dim(\mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T})& \\ &= \dim(\mathcal{S} \otimes L(K)) + \dim(L(H) \otimes \mathcal{T}) - \dim(\mathcal{S} \otimes \mathcal{T}) \\ &= m_1 n_2^2 + n_1^2 m_2 - m_1 m_2 = \dim(\mathcal{S} * \mathcal{T}).\end{aligned}$$

So by Lemma 0 we must have $\mathcal{S} * \mathcal{T} = \mathcal{S} \otimes L(K) + L(H) \otimes \mathcal{T}$. □

If $\mathcal{A} \subseteq L(H)$ is a reflexive algebra and $T \in L(H)$, the Arveson estimate for the distance from T to $\mathcal{A}$ is $\alpha(T, \mathcal{A}) = \sup\{\|P^\perp TP\|: P \in \operatorname{lat} \mathcal{A}\}$. For reflexive subspaces $\mathcal{S} \subseteq L(H)$ the estimate is defined [4] by $\alpha(T, \mathcal{S}) = \sup\{\|P^\perp TQ\|: P, Q$ are projections with $P^\perp \mathcal{S} Q = 0\}$. This agrees with the "$PTP^\perp$" formula when $I \in \mathcal{S}$. There is a "projection free" characterization of this estimate which proves useful. Let $d(\cdot, \cdot)$ denote distance.

LEMMA 2. *Let $\mathcal{S} \subseteq L(H)$ be a reflexive subspace. Then $\alpha(T, \mathcal{S}) = \sup\{d(Tx, \mathcal{S}x): x \in H, \|x\| = 1\}$, $T \in L(H)$.*

PROOF. We have $d(Tx, \mathcal{S}x) = \|P^\perp Tx\|$, where P is the orthogonal projection onto $[\mathcal{S}x]$. Let $\|x\| = 1$ and let Q be the projection onto $\mathbf{C}x$. Then $\|P^\perp Tx\| = \|P^\perp TQ\|$. Since $P^\perp \mathcal{S} Q = 0$, the inequality "$\geqslant$" follows.

Conversely, suppose P, Q are projections with $P^\perp \mathcal{S} Q = 0$. Let $\varepsilon > 0$ be given, and choose $x \in QH$, $\|x\| = 1$, such that $\|P^\perp Tx\| \geqslant \|P^\perp TQ\| - \varepsilon$. Then

$$d(Tx, \mathcal{S}x) \geqslant d(Tx, PH) = \left\|P^\perp Tx\right\| \geqslant \left\|P^\perp TQ\right\| - \varepsilon.$$

Since ε was arbitrary, we have $\|P^\perp TQ\| \leqslant \sup\{d(T, \mathcal{S}x): x \in H, \|x\| = 1\}$. Taking the supremum over all pairs $\{P, Q\}$ with $P^\perp \mathcal{S} Q = 0$ completes the proof. □

Lemma 2 points out that only cyclic projections P need be considered in distance estimate computations. Also, taking this as the definition permits natural extension of the concept to general normed linear spaces.

If $\mathcal{S} \neq L(H)$ is reflexive and $T \notin \mathcal{S}$, we write $\mathcal{K}(T, \mathcal{S}) = d(T, \mathcal{S})/\alpha(T, \mathcal{S})$. So $\mathcal{K}(\mathcal{S}) = \sup\{\mathcal{K}(T, \mathcal{S}): T \in L(H), T \notin \mathcal{S}\}$. By convention $\mathcal{K}(L(H)) = 1$. $\mathcal{S}$ is hyperreflexive if $\mathcal{K}(\mathcal{S}) < \infty$, and is nonhyperreflexive otherwise.

We first give an initial generalization of the Davidson-Power induction step in which use is made of symmetry. The proof is more direct than that of our general result, so is included for perspective.

PROPOSITION 3. *Let $\mathscr{S}$ be a reflexive subspace of $L(H)$, with $\mathscr{S} \neq L(H)$. Let*

$$\tilde{\mathscr{S}} = \begin{pmatrix} * & \mathscr{S} & \mathscr{S} \\ \mathscr{S} & * & \mathscr{S} \\ \mathscr{S} & \mathscr{S} & * \end{pmatrix}$$

be the subspace of all 3×3 operator matrices with diagonal elements arbitrary and off-diagonal elements in $\mathscr{S}$. Then $\mathscr{K}(\tilde{\mathscr{S}}) \geqslant \sqrt{9/8} \cdot \mathscr{K}(\mathscr{S})$.

PROOF. First, suppose $\mathscr{K}(\mathscr{S})$ is finite. Fix $\varepsilon > 0$. Choose $T \in L(H)$ for which $\mathscr{K}(T, \mathscr{S}) \geqslant \mathscr{K}(\mathscr{S}) - \varepsilon$. Let

$$\tilde{T} = \begin{pmatrix} T & T & T \\ T & T & T \\ T & T & T \end{pmatrix}.$$

The averaging technique used in the proof of Theorem 1.1 in [**2**] yields without modification that $d(\tilde{T}, \tilde{\mathscr{S}}) = \frac{3}{2} \cdot d(T, \mathscr{S})$. We need only show that $\alpha(\tilde{T}, \tilde{\mathscr{S}}) \leqslant \sqrt{2}\, \alpha(T, \mathscr{S})$, for then

$$\mathscr{K}(\tilde{\mathscr{S}}) \geqslant \mathscr{K}(\tilde{T}, \tilde{\mathscr{S}}) \geqslant \sqrt{9/8}\, \mathscr{K}(T, \mathscr{S}) > \sqrt{9/8}\, (\mathscr{K}(\mathscr{S}) - \varepsilon),$$

and since ε was arbitrary the desired inequality would follow.

To show that $\alpha(\tilde{T}, \tilde{\mathscr{S}}) \leqslant \sqrt{2}\, \alpha(T, \mathscr{S})$, it is useful to use Lemma 2. Let

$$\tilde{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

be an arbitrary unit vector in $H \otimes H_3$. Then

$$\tilde{T}\tilde{x} = \begin{pmatrix} Tz \\ Tz \\ Tz \end{pmatrix},$$

where $z = x_1 + x_2 + x_3$. Descriptively, we have

$$\tilde{\mathscr{S}}\tilde{x} = \begin{pmatrix} * & \mathscr{S} & \mathscr{S} \\ \mathscr{S} & * & \mathscr{S} \\ \mathscr{S} & \mathscr{S} & * \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} L(H)x_1 + \mathscr{S}x_2 + \mathscr{S}x_3 \\ \mathscr{S}x_1 + L(H)x_2 + \mathscr{S}x_3 \\ \mathscr{S}x_1 + \mathscr{S}x_2 + L(H)x_3 \end{pmatrix}.$$

We consider three cases:

(1) If neither x_1, x_2 nor $x_3 = 0$, then $\tilde{\mathscr{S}}\tilde{x} = H \otimes H_3$, so $d(\tilde{T}\tilde{x}, \tilde{\mathscr{S}}\tilde{x}) = 0$.

(2) If precisely one of x_1, x_2, x_3 is 0, then without loss of generality we may assume $x_1 = 0$ by noting that $\tilde{\mathscr{S}}$ is invariant under the group of unitary transformations corresponding to permutation of basis vectors in H_3. We have $z = x_2 + x_3$, and

$$\tilde{\mathscr{S}}\tilde{x} = \begin{pmatrix} \mathscr{S}x_2 + \mathscr{S}x_3 \\ L(H) \\ L(H) \end{pmatrix},$$

so

$$d(\tilde{T}\tilde{x}, \tilde{\mathscr{S}}\tilde{x}) = d(Tz, \mathscr{S}x_2 + \mathscr{S}x_3) \leqslant d(Tz, \mathscr{S}z).$$

We have $\|z\| \leqslant \sqrt{2}$. If $z = 0$, then $d(\tilde{T}\tilde{x}, \tilde{\mathscr{S}}\tilde{x}) = 0$. If $z \neq 0$, let $w = z/\|z\|$. Then

$$d(\tilde{T}\tilde{x}, \tilde{\mathscr{S}}\tilde{x}) \leqslant \sqrt{2}\, d(Tw, \mathscr{S}w) \leqslant \sqrt{2}\, \alpha(T, S),$$

as desired.

(3) If precisely two of x_1, x_2, x_3 are 0, via permutation as above, we may assume $x_1 = x_2 = 0$. Then $z = x_3$, so $\|z\| = 1$. We have

$$\tilde{\mathscr{S}}\tilde{x} = \begin{pmatrix} \mathscr{S}z \\ \mathscr{S}z \\ L(H) \end{pmatrix},$$

so

$$d(\tilde{T}\tilde{x}, \tilde{\mathscr{S}}\tilde{x}) = \sqrt{2}\, d(Tz, \mathscr{S}z) \leqslant \sqrt{2}\, \alpha(T, \mathscr{S}).$$

Now from cases (1)–(3) we have

$$\begin{aligned} \alpha(\tilde{T}, \tilde{\mathscr{S}}) &= \sup\left\{ d(\tilde{T}\tilde{x}, \tilde{\mathscr{S}}\tilde{x}) \colon \tilde{x} \in H \otimes H_3,\ \|\tilde{x}\| = 1 \right\} \\ &\leqslant \sqrt{2}\, \alpha(T, \mathscr{S}), \end{aligned}$$

as required. For the case $\mathscr{K}(\mathscr{S}) = \infty$, let $n \geqslant 1$ be arbitrary and choose T with $\mathscr{K}(T, \mathscr{S}) \geqslant n$. The same argument as above shows that $\mathscr{K}(\tilde{T}, \tilde{\mathscr{S}}) \geqslant \sqrt{9/8}\, n$. Hence $\mathscr{K}(\tilde{\mathscr{S}}) = +\infty$. □

A simple duality computation shows that the preannihilator of $\tilde{\mathscr{S}}$ in Proposition 3 has the form

$$\tilde{\mathscr{S}}_\perp = \begin{pmatrix} 0 & \mathscr{S}_\perp & \mathscr{S}_\perp \\ \mathscr{S}_\perp & 0 & \mathscr{S}_\perp \\ \mathscr{S}_\perp & \mathscr{S}_\perp & 0 \end{pmatrix},$$

where $\mathscr{S}_\perp$ is the preannihilator of $\mathscr{S}$. The preannihilator of $\mathscr{D}_3$ has the form

$$\begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix},$$

so $\tilde{\mathscr{S}}_\perp = \mathscr{S}_\perp \otimes (\mathscr{D}_3)_\perp$, and hence $\tilde{\mathscr{S}} = (\mathscr{S}_\perp \otimes (\mathscr{D}_3)_\perp)^\perp = \mathscr{S} * \mathscr{D}_3$. This suggests that a generalization is possible. The next proposition is used in place of an averaging technique.

PROPOSITION 4. *Let $\mathscr{S} \subseteq L(H)$, $\mathscr{T} \subseteq L(K)$ be σ-weakly closed subspaces. If $A \in L(H)$ and $B \in L(K)$ are arbitrary, then $d(A \otimes B,\ \mathscr{S} * \mathscr{T}) = d(A, \mathscr{S}) \cdot d(B, \mathscr{T})$.*

PROOF. Let $\mathscr{R} = \mathscr{S} \otimes L(K) + L(H) \otimes \mathscr{T}$. By Lemma 0 we have $\mathscr{R} \subseteq \mathscr{S} * \mathscr{T}$, so for each $S \in \mathscr{S}$ and $T \in \mathscr{T}$ we have $(A - S) \otimes (B - T) - A \otimes B \in \mathscr{R} \subseteq \mathscr{S} * \mathscr{T}$. Thus

$$d(A \otimes B,\ \mathscr{S} * \mathscr{T}) = d((A - S) \otimes (B - T),\ \mathscr{S} * \mathscr{T}) \leqslant \|A - S\| \cdot \|B - T\|.$$

It follows that $d(A \otimes B,\ \mathscr{S} * \mathscr{T}) \leqslant d(A, \mathscr{S}) \cdot d(B, \mathscr{T})$.

For the reverse inequality, let $\varepsilon > 0$ be given and choose $f \in \mathscr{S}_\perp$, $g \in \mathscr{T}_\perp$ with $\|f\|_1 = \|g\|_1 = 1$, such that $\mathrm{Tr}(Af) > d(A, \mathscr{S}) - \varepsilon$, and $\mathrm{Tr}(Bg) > d(B, \mathscr{T}) - \varepsilon$. Let $h = f \otimes g$. We have

$$|\mathrm{Tr}[(A \otimes B)h]| = |\mathrm{Tr}(Af \otimes Bg)| = |\mathrm{Tr}(Af)| \cdot |\mathrm{Tr}(Bg)| \\ > (d(A, \mathscr{S}) - \varepsilon) \cdot (d(B, \mathscr{T}) - \varepsilon).$$

Since h is a norm -1 operator in $\mathscr{S}_\perp \otimes \mathscr{T}_\perp$, and this is the preannihilator of $\mathscr{S} * \mathscr{T}$ by definition, this implies that $d(A \otimes B, \mathscr{S} * \mathscr{T}) > (d(A, \mathscr{S}) - \varepsilon) \cdot (d(B, \mathscr{T}) - \varepsilon)$. Since ε was arbitrary, the proof is complete. □

LEMMA 5. *Let $\mathscr{S} \subseteq L(H)$ be a linear subspace, and let x be a vector in $H \otimes K$. Let F be the smallest projection in $L(H)$ such that $(F \otimes I)x = x$. Let P be the orthogonal projection onto $[\mathscr{S}FH]$. Then $P \otimes I$ is the orthogonal projection onto $[(\mathscr{S} \otimes L(K))x]$.*

PROOF. Let $\{e_1, e_2, \ldots\}$ be any orthonormal basis for K. Then there is a sequence $\{x_i\}$ of vectors in H with $\sum\|x_i\|^2 = \|x\|^2$ such that $x = \sum x_i \otimes e_i$. Let E_i be the projection onto $\mathbf{C}e_i$. If $S \in \mathscr{S}$, $T \in L(K)$, then $(S \otimes TE_i)x = Sx_i \otimes Te_i$. Hence $[(\mathscr{S} \otimes L(K))x]$ contains all vectors of the form $Sx_i \otimes y$ for arbitrary $S \in \mathscr{S}$, $y \in K$, for each i. Let F' be the projection onto the closed span of vectors $\{x_i: i = 1, 2, \ldots\}$. Then $F' \geqslant F$, and we have

$$[(\mathscr{S} \otimes L(K))x] \supseteq [\mathscr{S}F'H] \otimes K \supseteq [\mathscr{S}FH] \otimes K \\ = [(\mathscr{S} \otimes L(K))(FH \otimes K)] \supseteq [(\mathscr{S} \otimes L(K))x],$$

so

$$[(\mathscr{S} \otimes L(K))x] = [\mathscr{S}FH] \otimes K = (P \otimes I)(H \otimes K). \qquad \square$$

LEMMA 6. *Let H be a Hilbert space, let $\mathscr{S} \subseteq L(H)$ be a linear subspace, and let $h \in \mathscr{S}_\perp$ be a rank-1 operator. Then $hP = 0$, where P is the orthogonal projection onto $[\mathscr{S}hH]$.*

PROOF. Write $h = v \otimes u$, where u, v are vectors such that $hw = (w, v)u$, $w \in H$. Then $[\mathscr{S}hH] = [\mathscr{S}u]$. If $S \in \mathscr{S}$ we have $0 = \mathrm{Tr}(Sh) = (Su, v)$, so $[\mathscr{S}u] \perp v$; hence $Pv = 0$. Then $hP = (Pv) \otimes u = 0$. □

LEMMA 7. *Let $\mathscr{S} \subseteq L(H)$, $\mathscr{T} \subseteq L(K)$ be linear subspaces, and let $\mathscr{R} = \mathscr{S} \otimes L(K) + L(H) \otimes \mathscr{T}$. Let $x \in H \otimes K$. Let $F \in L(H)$, $E \in L(K)$ be the smallest projections such that $F \otimes I$ and $I \otimes E$ contain x in their range, and let P be the projection onto $[\mathscr{S}FH]$ and Q the projection onto $[\mathscr{T}EK]$. The projection onto $[\mathscr{R}x]$ is then $(P^\perp \otimes Q^\perp)^\perp$.*

PROOF. We have $[\mathscr{R}x] = [(\mathscr{S} \otimes L(K))x] \vee [(L(H) \otimes \mathscr{T})x]$. By Lemma 5, the projections onto $[(\mathscr{S} \otimes L(K))x]$ and $[(L(H) \otimes \mathscr{T})x]$ are $P \otimes I$ and $I \otimes Q$, respectively. The projection onto $[\mathscr{R}x]$ is then $(P \otimes I) \vee (I \otimes Q)$, and since $P \otimes I$ and $I \otimes Q$ commute this reduces to $P \otimes I + I \otimes Q - P \otimes Q$. The orthogonal complement is then

$$I \otimes I - P \otimes I - I \otimes Q + P \otimes Q = P^\perp \otimes I - P^\perp \otimes Q = P^\perp \otimes Q^\perp,$$

so $\mathrm{proj}[\mathscr{R}x] = (P^\perp \otimes Q^\perp)^\perp$. □

If $\mathscr{S}$ is a linear subspace of $L(H)$, we adopt the notation ref($\mathscr{S}$) to mean the smallest reflexive subspace of $L(H)$ containing $\mathscr{S}$. Thus ref$(\mathscr{S}) = \{T \in L(H): Tx \in [\mathscr{S}x], x \in H\}$.

THEOREM 8. *Let $\mathscr{S} \subseteq L(H)$, $\mathscr{T} \subseteq L(K)$ be reflexive subspaces. Then $\mathscr{S} * \mathscr{T}$ is the smallest reflexive subspace containing $\mathscr{S} \otimes L(K) + L(H) \otimes \mathscr{T}$.*

PROOF. Let $\mathscr{R} = \mathscr{S} \otimes L(K) + L(H) \otimes \mathscr{T}$, and let $\hat{\mathscr{R}} = \text{ref}(\mathscr{R})$. By definition, $\mathscr{R}$ and $\hat{\mathscr{R}}$ have the same closed cyclic subspaces. Since $\mathscr{S} * \mathscr{T}$ is reflexive and contains $\mathscr{R}$ it contains $\hat{\mathscr{R}}$. To show equality, it will suffice to show that every rank-1 operator in $\hat{\mathscr{R}}_\perp$ is in $(\mathscr{S} * \mathscr{T})_\perp = \mathscr{S}_\perp \otimes \mathscr{T}_\perp$. Let h be a rank-1 operator in $\mathscr{R}_\perp$, and let x be a nonzero vector in the range of h. By Lemma 6, $h = hG^\perp$, where G is the projection onto $[\hat{\mathscr{R}}x] = [\mathscr{R}x]$. Let $F \in L(H)$, $E \in L(K)$ be the smallest projections such that $(F \otimes I)x = x = (I \otimes E)x$, and let $P = \text{proj}[\mathscr{S}FH]$, $Q = \text{proj}[\mathscr{T}EK]$. Then by Lemma 7, $G^\perp = P^\perp \otimes Q^\perp$. We have $(F \otimes E)x = x$; hence

$$h = (F \otimes E)h = (F \otimes E)h(P^\perp \otimes Q^\perp) \in (F \otimes E)(\mathscr{L}_*(H \otimes K))(P^\perp \otimes Q^\perp),$$

where $\mathscr{L}_*(H \otimes K)$ denotes the ideal of trace-class operators on $H \otimes K$. Since $\mathscr{L}_*(H \otimes K)$ is the trace-class norm closed span of elementary tensors $\{f \otimes g: f \in \mathscr{L}_*(H), g \in \mathscr{L}_*(K)\}$, the space $(F \otimes E)(\mathscr{L}_*(H \otimes K))(P^\perp \otimes Q^\perp)$ is the closed span of elementary tensors $\{(FfP^\perp) \otimes (EgQ^\perp): f \in \mathscr{L}_*(H), g \in \mathscr{L}_*(K)\}$.

But for f arbitrary and $S \in \mathscr{S}$ we have $\text{Tr}(SFfP^\perp) = \text{Tr}(P^\perp SFf) = 0$ since $P^\perp \mathscr{S} F = 0$. So $FfP^\perp \in S_\perp$. Similarly, $EgQ^\perp \in \mathscr{T}_\perp$ for all $g \in \mathscr{L}_*(K)$. So each $(FfP^\perp) \otimes (EgQ^\perp) \in \mathscr{S}_\perp \otimes \mathscr{T}_\perp$, and hence $h \in \mathscr{S}_\perp \otimes \mathscr{T}_\perp$. □

THEOREM 9. *Let $\mathscr{S} \subseteq L(H)$, $\mathscr{T} \subseteq L(K)$ be reflexive proper subspaces. Then $\mathscr{K}(\mathscr{S} * \mathscr{T}) \geqslant \mathscr{K}(\mathscr{S}) \cdot \mathscr{K}(\mathscr{T})$.*

PROOF. Let $A \in L(H)$, $B \in L(K)$ be arbitrary. By Proposition 4 we have $d(A \otimes B, \mathscr{S} * \mathscr{T}) = d(A, \mathscr{S}) \cdot d(B, \mathscr{T})$. We will show that in general $\alpha(A \otimes B, \mathscr{S} * \mathscr{T}) \leqslant \alpha(A, \mathscr{S}) \cdot \alpha(B, \mathscr{T})$, and hence if $A \notin \mathscr{S}$ and $B \notin \mathscr{T}$ then $\mathscr{K}(A \otimes B, \mathscr{S} * \mathscr{T}) \geqslant \mathscr{K}(A, \mathscr{S}) \cdot \mathscr{K}(B, \mathscr{T})$. Taking of suprema over all such A, B then yields $K(\mathscr{S} * \mathscr{T}) \geqslant \mathscr{K}(\mathscr{S}) \cdot \mathscr{K}(\mathscr{T})$, since by hypothesis $\mathscr{S} \neq L(H)$ and $\mathscr{T} \neq L(K)$.

We utilize Lemma 2. Let x be a unit vector in $H \otimes K$. Let $\mathscr{R} = \mathscr{S} \otimes L(K) + L(H) \otimes \mathscr{T}$. By Theorem 8, $\mathscr{S} * \mathscr{T} = \text{ref}(\mathscr{R})$, so $\mathscr{S} * \mathscr{T}$ and $\mathscr{R}$ have the same cyclic subspaces. As in the proof of Theorem 8, let G be the projection onto $[\mathscr{R}x] = [(\mathscr{S} * \mathscr{T})x]$, and let $F \in L(H)$, $E \in L(K)$ be the smallest projections such that $(F \otimes I)x = (I \otimes E)x$. Let $P = \text{proj}[\mathscr{S}FH]$, $Q = \text{proj}[\mathscr{T}EK]$. By Lemma 7, $G = (P^\perp \otimes Q^\perp)^\perp$. Since $(F \otimes E)x = x$ we have

$$d[(A \otimes B)x, (\mathscr{S} * \mathscr{T})x] = \|G^\perp(A \otimes B)x\| = \|(P^\perp \otimes Q^\perp)(A \otimes B)(F \otimes E)x\|$$

$$= \|((P^\perp AF) \otimes (Q^\perp BE))x\| \leqslant \|P^\perp AF\| \cdot \|Q^\perp BE\| \leqslant \alpha(A, \mathscr{S}) \cdot \alpha(B, \mathscr{T})$$

since $P^\perp \mathscr{S} F = 0$ and $Q^\perp \mathscr{T} E = 0$.

Since x was an arbitrary unit vector, we have

$$\alpha(A \otimes B, \mathscr{S} * \mathscr{T}) \leqslant \alpha(A, \mathscr{S}) \cdot \alpha(B, \mathscr{T}), \text{ as required. } \square$$

The proof of Theorem 9 can be improved slightly to show that for arbitrary $A \in L(H)$, $B \in L(K)$ the inequality $\alpha(A \otimes B, \mathscr{S} * \mathscr{T}) \leqslant \alpha(A, \mathscr{S}) \cdot \alpha(B, \mathscr{T})$ is in fact an equality. We capture this fact.

COROLLARY 10. *Let $\mathscr{S} \subseteq L(H)$ and $\mathscr{T} \subseteq L(K)$ be reflexive subspaces, and let $A \in L(H)$, $B \in L(K)$ be arbitrary. Then $\alpha(A \otimes B, \mathscr{S} * \mathscr{T}) = \alpha(A, \mathscr{S}) \cdot \alpha(B, \mathscr{T})$.*

PROOF. The inequality " $\leqslant$ " is contained in the proof of Theorem 9. For the converse, let F, $P \in L(H)$ and E, $Q \in L(K)$ be arbitrary projections satisfying $P^{\perp} \mathscr{S} F = 0$ and $Q^{\perp} \mathscr{T} E = 0$. Then if $\mathscr{R} = \mathscr{S} \otimes L(K) + L(H) \otimes \mathscr{T}$, we have $(P^{\perp} \otimes Q^{\perp}) \mathscr{R}(F \otimes E) = 0$, so since $\mathscr{S} * \mathscr{T} = \operatorname{ref}(\mathscr{R})$ by Theorem 8, we have $(P^{\perp} \otimes Q^{\perp})(\mathscr{S} * \mathscr{T})(F \otimes E) = 0$. Since $\|(P^{\perp} \otimes Q^{\perp})(A \otimes B)(F \otimes E)\| = \|P^{\perp} AF\| \cdot \|Q^{\perp} BE\|$, we have

$$\begin{aligned}\alpha(A \otimes B, \mathscr{S} * \mathscr{T}) = \sup\{\|L(A \otimes B)M\| &: L, M \text{ are projections} \\ &\text{in } L(H \otimes K) \text{ with } L(\mathscr{S} \otimes \mathscr{T})M = 0\} \\ \geqslant \|P^{\perp} AF\| \cdot \|Q^{\perp} BE\|&.\end{aligned}$$

So since P, F, Q, E were arbitrary, we conclude that $\alpha(A \otimes B, \mathscr{S} * \mathscr{T}) \geqslant \alpha(A, \mathscr{S}) \cdot \alpha(B, \mathscr{T})$. Finally, we note that equality is trivially true if either $\mathscr{S} = L(H)$ or $\mathscr{T} = L(K)$. $\square$

From Corollary 10 and Proposition 4 we conclude that $\mathscr{K}(A \otimes B, \mathscr{S} * \mathscr{T}) = \mathscr{K}(A, \mathscr{S}) \cdot \mathscr{K}(B, \mathscr{T})$ whenever $\mathscr{S}$, $\mathscr{T}$ are reflexive proper subspaces with $A \notin \mathscr{S}$, $B \notin \mathscr{T}$. That is, the basic inequality is an equality when restricted to the class of elementary tensors. It can happen, however, that for some operator $T \in L(H \otimes K)$, which is not an elementary tensor, we have $\mathscr{K}(T, \mathscr{S} * \mathscr{T}) > \mathscr{K}(\mathscr{S}) \cdot \mathscr{K}(\mathscr{T})$, and thus the inequality in Theorem 9 may be strict. The following simple example shows this.

EXAMPLE 11. Let

$$S = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} : \lambda \in \mathbf{C} \right\}$$

be regarded as a subspace of operators acting on 2-dimensional Hilbert space. An elementary computation shows that S is reflexive. An application of [**6**, Lemma 3.3] after interchanging rows, and either [**4**, Proposition 3 or **5**, Theorems 1.1 or 1.2] to the preannihilator

$$S_{\perp} = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix},$$

shows that $\mathscr{K}(S) = 1$. Since

$$\mathscr{S}_{\perp} \otimes \mathscr{S}_{\perp} = \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & * & * \\ 0 & * & 0 & * \\ * & * & * & * \end{pmatrix}$$

we have

$$S * S = \begin{pmatrix} * & * & * & 0 \\ * & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

acting on 4-dimensional Hilbert space. Let

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $P(S * S) \subseteq S * S$, and thus by [**5**, Lemma 1.3] the compression $P(S * S)|_{PH}$ is reflexive with hyperreflexivity constant no greater than that of $S * S$. But this compression has diagram

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and by [**5**, Example 4.7] this has constant $\geqslant \sqrt{9/8}$. Thus $\mathscr{K}(S * S) \geqslant \sqrt{9/8} > 1 = \mathscr{K}(S) \cdot \mathscr{K}(S)$. So in this case the inequality of Theorem 9 is strict. □

THEOREM 12. *Let n be a positive integer, and for $1 \leqslant i \leqslant n$ let $\mathscr{A}_i$ be a reflexive proper subalgebra of $L(H_i)$ for H_i a separable Hilbert space. Suppose the tensor product $\mathscr{A}_1 \otimes \cdots \otimes \mathscr{A}_n$, acting on $H = H_1 \otimes \cdots \otimes H_n$, is reflexive. Let*

$$\mathscr{A} = \begin{pmatrix} \mathscr{A}_1 \otimes \cdots \otimes \mathscr{A}_n & \mathscr{A}_1 * \cdots * \mathscr{A}_n \\ 0 & \mathscr{A}_1 \otimes \cdots \otimes \mathscr{A}_n \end{pmatrix}.$$

Then $\mathscr{A}$ is a reflexive subalgebra of $L(H \otimes H)$, and $\mathscr{K}(\mathscr{A}) \geqslant \mathscr{K}(\mathscr{A}_1) \cdot \mathscr{K}(\mathscr{A}_2) \cdots \mathscr{K}(\mathscr{A}_n)$.

PROOF. $\mathscr{A}$ is an algebra since $\mathscr{A}_1 * \cdots * \mathscr{A}_n$ is a bimodule over $\mathscr{A}_1 \otimes \cdots \otimes \mathscr{A}_n$. A simple calculation shows that

$$\mathscr{A}_\perp = \begin{pmatrix} (\mathscr{A}_1 \otimes \cdots \otimes \mathscr{A}_n)_\perp & L_*(H) \\ (\mathscr{A}_1 * \cdots * \mathscr{A}_n)_\perp & (\mathscr{A}_1 \otimes \cdots \otimes \mathscr{A}_n)_\perp \end{pmatrix}.$$

Since $\mathscr{A}_1 \otimes \cdots \otimes \mathscr{A}_n$ is reflexive, $(\mathscr{A}_1 \otimes \cdots \otimes \mathscr{A}_n)_\perp$ is generated by rank-1 operators. Since $\mathscr{A}_1 \cdots \mathscr{A}_n$ are reflexive, $(\mathscr{A}_1 * \cdots * \mathscr{A}_n)_\perp = (\mathscr{A}_1)_\perp \otimes \cdots \otimes (\mathscr{A}_n)_\perp$ is also generated by rank-1 operators. Hence $\mathscr{A}_\perp$ is generated by rank-1 operators, so $\mathscr{A}$ is reflexive.

To show $\mathscr{K}(\mathscr{A}) \geqslant \mathscr{K}(\mathscr{A}_1) \cdots \mathscr{K}(\mathscr{A}_n)$ we utilize [**4**, Proposition 3]. Let P be the orthogonal projection from $H \oplus H$ onto H. Let $S = \mathscr{A}_1 * \cdots * \mathscr{A}_n$. By Theorem 9, $\mathscr{K}(S) \geqslant \mathscr{K}(\mathscr{A}_1) \cdots \mathscr{K}(\mathscr{A}_n)$. Let $\mathscr{C}_1(\mathscr{A})$, $\mathscr{C}_1(S)$ denote the closed convex hulls of the rank $\leqslant 1$ operators in the unit balls of $\mathscr{A}_\perp$, S, respectively. Then clearly

$$P^\perp \mathscr{C}_1(\mathscr{A}) P = \begin{pmatrix} 0 & 0 \\ \mathscr{C}_1(S) & 0 \end{pmatrix}.$$

Let $R(\mathscr{A})$, $R(S)$ be the largest radii such that $\{f \in \mathscr{A}_\perp : \|f\|_1 \leqslant R(\mathscr{A})\} \subseteq \mathscr{C}_1(\mathscr{A})$ and $\{g \in S_\perp : \|g\|_1 \leqslant R(S)\} \subseteq \mathscr{C}_1(S)$. It follows that $R(\mathscr{A}) \leqslant R(S)$. By [**4**, Proposition 3] we have $\mathscr{K}(\mathscr{A}) = 1/R(\mathscr{A})$ and $\mathscr{K}(S) = 1/R(S)$, so $\mathscr{K}(\mathscr{A}) \geqslant \mathscr{K}(S)$, as required. □

REMARKS. The requirement that $\mathscr{A}_1 \otimes \cdots \otimes \mathscr{A}_n$ be reflexive will be met if each $\mathscr{A}_i$ is finite dimensional, and more generally, if each $\mathscr{A}_i$ has property S_σ (Kraus [**3**]). (It is, of course, an open question whether the tensor product of reflexive algebras is necessarily reflexive.) As in the special case of the "key example" in [**2**], Theorem 12 gives a means of constructing reflexive algebras of arbitrarily large distance constant. If each $\mathscr{A}_i$ is a CSL algebra and so contains a m.a.s.a., then $\mathscr{A}$ will also contain a m.a.s.a., so will be a CSL algebra. A direct sum of such algebras, with increasing constants, will be nonhyperreflexive.

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