SOME GENERALIZED BROWN-GITLER SPECTRA

BY

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ABSTRACT. Brown-Gitler spectra for the homology theories associated with the spectra $K\mathbf{Z}_p^{\wedge}$, bo, and bu are constructed. Complexes adapted to the new Brown-Gitler spectra are produced and a spectral sequence converging to stable maps into these spectra is constructed and examined.

Brown-Gitler spectra have many applications in homotopy theory. At the prime 2 they were constructed in [**4**]. In his thesis, Ralph Cohen constructed Brown-Gitler spectra at odd primes. The purpose of this paper is to produce Brown-Gitler spectra at other homology theories.

Fix a prime p. All homology and cohomology will have $\mathbf{Z}/p$ as coefficients. Let A be the mod p Steenrod algebra and let $\chi: A \to A$ be the canonical antiautomorphism. If $p = 2$, then let $P^k = \mathrm{Sq}^{2k}$ and $\beta = \mathrm{Sq}^1$ be the Bockstein. The theorems of Brown-Gitler and Cohen can be combined to read as follows.

THEOREM 0. *There is a* $\mathbf{Z}/p$-*complete spectrum* $B(pk + 1)$, $k \geqslant 0$, *so that*

(i) $H^*B(pk + 1) \cong A/A\{\chi(\beta^\varepsilon P^i): i > k, \varepsilon = 0 \text{ or } 1\}$,

(ii) *if* $w: B(pk + 1) \to K\mathbf{Z}/p$ *is the generator, then the induced map of reduced homology theories*

$$w_*: B(pk + 1)_m Z \to H_m Z$$

is onto for all CW complexes Z and $m \leqslant 2p(k + 1) - 1$.

We will call $B(pk + 1)$ a Brown-Gitler spectrum over $K\mathbf{Z}/p$. The purpose of this paper is to produce Brown-Gitler spectra over $K\mathbf{Z}_p^{\wedge}$, $Bp\langle 1 \rangle$, and bo. $\mathbf{Z}_p^{\wedge}$ is the integers completed at p.

Our first result concerns Brown-Gitler spectra over $K\mathbf{Z}_p^{\wedge}$. Note that $H^*K\mathbf{Z}_p^{\wedge} = A/A\beta$.

THEOREM 1. *For each $k > 0$ there is a* $\mathbf{Z}/p$-*complete spectrum* $B_1(pk + 1)$ *and a map* $w: B_1(pk + 1) \to K\mathbf{Z}_p^{\wedge}$ *so that*

(i) $w^*: H^*K\mathbf{Z}_p^{\wedge} \to H^*B_1(pk + 1)$ *induces an isomorphism*

$$H^*B_1(pk + 1) \cong A/A\{\beta, \chi P^i, i > k\},$$

(ii) *the induced map of homology theories*

$$B_1(pk + 1)_m Z \to H_m\left(Z; \mathbf{Z}_p^\wedge \right)$$

is onto for all CW complexes Z and $m \leqslant 2p(k + 1) - 1$.

This result is not really new. These spectra have been studied by several authors: Mahowald [10] at $p = 2$ and Kane [9] for odd primes and subsequently by Shimamoto [11] and Goerss [8]. The only point which is new is the proof of (ii).

For our next result let $BP\langle 1 \rangle$ be one of the $p - 1$ factors of mod p connective K-theory completed at p. Then $BP\langle 1 \rangle$ is a ring spectrum and

$$\pi_* BP\langle 1 \rangle = \mathbf{Z}_p^\wedge [v_1],$$

where v_1 is of degree $2(p - 1)$ and

$$H^* BP\langle 1 \rangle = A/A\{ \beta, \beta P^1 \}.$$

Let $q = 2(p - 1)$; then a cohomology calculation shows that there are maps $\chi \overline{P}^i$, $i \geqslant 0$, so that the following diagram commutes:

$$
\begin{array}{ccc}
BP\langle 1 \rangle & \xrightarrow{\chi \overline{P}^i} & \sum^{q^i} K \mathbf{Z}_p^\wedge \\
\downarrow & & \downarrow \\
K \mathbf{Z}/p & \xrightarrow{\chi P^i} & \sum^{q^i} K \mathbf{Z}/p
\end{array}
$$

Here the vertical maps represent generators of $H^* BP\langle 1 \rangle$ and $H^* K \mathbf{Z}_p^\wedge$ as modules over A. Fix a choice for $\chi \overline{P}^i$. There are induced maps of homology theories

$$\chi \overline{P}_*^i : BP\langle 1 \rangle_* \to H_*\left(; \mathbf{Z}_p^\wedge \right).$$

THEOREM 2. *For each* $k \geqslant 0$ *there exists a* $\mathbf{Z}/p$-complete spectrum $BP\langle 1 \rangle^k$ *and a map* w: $BP\langle 1 \rangle^k \to BP\langle 1 \rangle$ *so that*
   (i) $H^* BP\langle 1 \rangle^k \cong A/A\{ \beta, \beta P^1, \chi P^i; i > k \}$,
   (ii) w_*: $BP\langle 1 \rangle_m^k Z \to BP\langle 1 \rangle_m Z$ *is onto* $\bigcap_{i > k} \ker \chi \overline{P}_*^i$ *for all CW complexes Z and* $m \leqslant 2p(k + 1) - 1$.

Now let *bo* be the spectrum of real connective K theory completed at 2. Then

$$H^* bo \cong A/A\{ \mathrm{Sq}^1, \mathrm{Sq}^2 \}.$$

For $i \geqslant 0$ there are maps $\chi \overline{\mathrm{Sq}}^{4i}$ making the following diagram commute:

$$
\begin{array}{ccc}
bo & \xrightarrow{\chi \overline{\mathrm{Sq}}^{4i}} & \sum^{4i} K \mathbf{Z}_2^\wedge \\
\downarrow & & \downarrow m \\
K \mathbf{Z}/2 & \xrightarrow{\chi \mathrm{Sq}^{4i}} & \sum^{4i} K \mathbf{Z}/2
\end{array}
$$

Our result is

THEOREM 3. *For each $k \geqslant 0$ there exists a $\mathbf{Z}/2$-complete spectrum bo^k and a map $w: bo^k \to bo$ so that*

(i) $H^*bo^k \cong A/A\{Sq^1, Sq^2, \chi Sq^{4i}; i > k\}$,

(ii) *if $m \leqslant 8k + 7$, then $w_*: bo_m^k Z \to bo_m Z$ is onto $\bigcap_{i>k} \ker(\chi \bar{S}q^{4i})_*$ for all CW complexes Z.*

These spectra where constructed by Goerss [8] using a different method.

Our method of proof follows that of Brown and Gitler in outline. In §2 we provide acyclic resolutions of the modules we are trying to realize as the cohomology of spectra. In §§3 and 4 we obtain some unstable data necessary for our constructions. In §5 we prove the theorems. §1 is devoted to the detailing of the results of Brown and Gitler and of Cohen. Finally, in §6, we discuss maps into Brown-Gitler type spectra; indeed, we construct a spectral sequence converging to (for example) $[Z, bo^k]^*$ and show it collapses in a range for an important class of spectra Z.

We are working in some good stable category, such as provided by Adams [1] and all our spectra are completed at a prime p. We will make no distinction between a map $\tau: X \to Y$ and its suspension $\tau: \Sigma X \to \Sigma Y$.

1. The results of Brown, Gitler and Cohen. The purpose of this section is to describe, in depth, the results and techniques behind Theorem 0 of the introduction. We need these concepts for our construction. We recall two steps of the work: first we describe a particular acyclic resolution, by A-modules, of $H^*B(pk + 1)$, then we describe a tower of spectra whose homotopy inverse limit is $B(pk + 1)$. First let p be an odd prime. Recall that $q = 2(p - 1)$. Let $\bar{\Lambda}$ be the graded, associative algebra over $\mathbf{Z}/p$ with generators λ_{i-1} of degree $qi - 1$, $i \geqslant 0$, and μ_{i-1} of degree qi, $i \geqslant 0$, subject to the relations

$$\lambda_r \lambda_s = \sum_i a(i, r, s) \lambda_{r+s+1-i} \lambda_{i-1},$$

$$\mu_r \lambda_s = \sum_i b(i, r, s) \mu_{r+s+1-i} \lambda_{i-1},$$

(1.1)

$$\lambda_r \mu_s = \sum_i a(i, r, s) \lambda_{r+s+1-i} \mu_{i-1} + c(i, r, s) \mu_{r+s+1-i} \lambda_{i-1},$$

$$\mu_r \mu_s = \sum_i b(i, r, s) \mu_{r+s+1-i} \mu_{i-1},$$

where

$$a(i, r, s) = (-1)^{i+r} \binom{(p-1)(s+1-i) - 1}{i - p(r+1)},$$

$$b(i, r, s) = (-1)^{i+r+1} \binom{(p-1)(s+1-i) - 1}{i - p(r+1) - 1},$$

$$c(i, r, s) = (-1)^{i+r} \binom{(p-1)(s+1-i)}{i - p(r+1)}.$$

Let Λ be $\bar{\Lambda}$ modulo the left ideal generated by λ_{-1}. Λ is a differential graded algebra with differential given by left multiplication by λ_{-1}.

Now let $p = 2$. Let $\overline{\Lambda}$ be the graded associative $\mathbf{Z}/2$ algebra with generators λ_i of degree i, $i \geqslant -1$, subject to the relations

$$(1.2) \qquad \lambda_r \lambda_s = \sum \binom{i-1}{2i-(s-2r)} \lambda_{r-i} \lambda_{s+i}.$$

Again Λ is $\overline{\Lambda}$ modulo the left ideal generated by λ_{-1}. Left multiplication by λ_{-1} is again a differential. These are the lambda algebras of [2].

If p is odd let $v_i = \lambda_i$ or μ_i; in the case $p = 2$ let $v_i = \lambda_i$. If $(i_1, \ldots, i_q)$ is a q-tuple of integers, $i_j \geqslant -1$, set $v_I = v_{i_1} \cdots v_{i_q}$. This is a monomial of length q. Such a monomial is admissible if $p(i_j + 1) \geqslant i_{j+1} + 2$ when $v_{i_j} = \lambda_{i_j}$ and $\geqslant i_{j+1} + 1$ when $v_{i_j} = \mu_{i_j}$. This is valid at any prime. If p is odd set

$$\Lambda_k = \Lambda\{\mu_{-1}, \lambda_0, \mu_0, \ldots, \lambda_{k-1}, \mu_{k-1}\}$$

and if $p = 2$

$$\Lambda_k = \Lambda\{\lambda_0, \ldots, \lambda_{2k}\}.$$

The notation means that these are left ideals generated by the specified elements.

LEMMA 1.3. (i) *For all primes a basis for* Λ *as a* $\mathbf{Z}/p$ *vector space is given by the set of admissible monomials.*

(ii) Λ_k *is closed under the differential and has a* $\mathbf{Z}/p$ *basis of admissible monomials given by* v_I *with* $i_q \leqslant k - 1$ *if* $p \geqslant 3$ *or* $i_q \leqslant 2k$ *if* $p = 2$.

For the proof see [2, 4 and 6]. This lemma implies that Λ/Λ_k inherits a differential from Λ given by the left action of λ_{-1}; furthermore, Λ/Λ_k has $\mathbf{Z}/p$ basis of monomials v_I with $i_q \geqslant k$ if $p > 2$ or $i_q > 2k$ if $p = 2$.

Let $\Lambda_{q,k}$ be the set of admissible monomials of length q in Λ/Λ_k. If V is a $\mathbf{Z}/p$ vector space let V^* be the dual and $\langle \, , \, \rangle : V^* \otimes V \to \mathbf{Z}/p$ be the canonical pairing. Define

$$\delta_q^* : A \otimes V_{q+1,k}^* \to A \otimes V_{q,k}^*$$

to be the A linear map given by

$$(1.4) \qquad \delta_q^* v_I^* = \sum_J \left\langle v_I^*, \lambda_j v_J \right\rangle \chi P^{j+1} v_J^* + \left\langle v_I^*, \mu_j v_J \right\rangle \chi \beta P^{j+1} v_J^* \quad \text{for } p > 2,$$

$$\delta_q^* \lambda_I^* = \sum_J \left\langle \lambda_I^*, \lambda_j \lambda_J \right\rangle \chi \mathrm{Sq}^{j+1} \lambda_J^* \quad \text{for } p = 2.$$

The sums are over $j \geqslant -1$ and all basis elements v_J of $\Lambda_{q,k}$.

Let

$$(1.5) \qquad M(pk + 1) = A/A\{\chi(\beta^\varepsilon P^i) : i > k, \, \varepsilon = 0 \text{ or } 1\}, \qquad p > 2,$$

$$M(2k + 1) = A/A\{\chi \mathrm{Sq}^i : i > 2k + 1\}, \qquad p = 2.$$

The following is the fundamental result. Let $\varepsilon \colon A \to M(pk + 1)$ be the quotient map.

THEOREM 1.6 [4, 6]. *The following is a long exact sequence*:

$$\cdots \to A \otimes \Lambda_{q+1,k}^* \xrightarrow{\delta_q} A \otimes \Lambda_{q,k}^* \to \cdots \to A \otimes \Lambda_{1,q}^* \xrightarrow{\delta_0} A \xrightarrow{\varepsilon} M(pk + 1) \to 0.$$

Thus by definition, the modules $A \otimes \Lambda^*_{q,k}$ and the maps δ^*_q form an acyclic resolution of $M(pk + 1)$. In the following sections, when k is fixed we will set $C_q = A \otimes \Lambda^*_{q,k}$.

We can now describe the tower whose homotopy inverse limit is $B(pk + 1)$. Fix an integer $k \geqslant 0$. Then let K_q be an Eilenberg-Mac Lane spectrum so that $\pi_* K_q = \Lambda_{q,k}$ and let

$$\delta_q \colon K_q \to \sum K_{q+1}$$

be a map so that in cohomology δ^*_q is as in the theorem above.

THEOREM 1.7. *There exist spectra Y_q and maps $\varepsilon_q \colon Y_q \to \sum K_{q+1}$ so that*
(1) $Y_0 = K_0 = K\mathbf{Z}/p$, $\varepsilon_0 = \delta_0$,
(2) *there are cofibre sequences*

$$K_{q+1} \overset{i_{q+1}}{\to} Y_{q+1} \overset{p_q}{\to} Y_q \overset{\varepsilon_q}{\to} \sum K_{q+1},$$

(3) $\varepsilon_q i_q = \delta_q$,
(4) *the induced map of homology theories*

$$(\varepsilon_q)_* \colon (X_q)_m Z \to (K_{q+1})_{m-1} Z$$

is zero for $m \leqslant 2p(k + 1) - 1$ and all CW complexes Z.

Thus there is a tower

$$\sum K_q$$
$$\uparrow \varepsilon_{q-1}$$
$$(1.8) \quad \cdots \quad \to \quad Y_q \quad \to \quad Y_{q-1} \to \quad \cdots \quad \to \quad Y_1 \quad \to \quad Y_0 = K\mathbf{Z}/2.$$
$$i_q \uparrow$$
$$K_q$$

$B(pk + 1)$ is the homotopy inverse limit of this tower. Theorem 0 of the introduction follows immediately from the above theorem. A bit of unstable information is catalogued in our next lemma. Let Z be a finite complex and let $\hat{\gamma} \in H_m Z$, $m \leqslant 2p(k + 1) - 1$. Then, if DZ is the stable Spanier-Whitehead dual of Z, there is a class dual to $\hat{\gamma}$, $\gamma \in H^0 \sum^m DZ$. This may be realized as a map

$$\gamma \colon \sum^m DZ \to K\mathbf{Z}_p = Y_0.$$

Consider the tower above. The following is an idea of Brown and Peterson [5].

LEMMA 1.9. *Any lifting of γ to Y_q lifts to Y_{q+1}.*

PROOF. The proof follows from Theorem 1.7(4) and a duality argument.

2. Acylic resolutions. Here we present acyclic resolutions of the modules that will be the cohomology of the spectra we will construct. These resolutions will be constructed using ideals in the Λ-algebra and quotient modules of A. In particular, they will not be free over A.

If $p = 2$, let $P^i = \mathrm{Sq}^{2i} \in A$ and $\beta = \mathrm{Sq}^1$ be the Bockstein. Set

$$
\begin{aligned}
(2.1) \qquad M_0(k) &= A/A\{\beta, \chi P^i; \, i > k\}, \\
M_1(k) &= A/A\{\beta, \beta P^1, \chi P^i; \, i > k\}.
\end{aligned}
$$

We will present the resolutions of these modules first. Fix a prime p and an integer $k \geqslant 0$. We define two left ideals in the Λ-algebra:

$$
\begin{aligned}
\Lambda^0 &= \Lambda\{\lambda_i; \, i \geqslant 0\} \quad \text{if } p > 2, \\
(2.2) \qquad \Lambda^0 &= \Lambda\{\lambda_{2i+1}; \, i \geqslant 0\} \quad \text{if } p = 2, \\
\Lambda^1 &= \Lambda\{\lambda_i \lambda_j; \, i, j \geqslant 0\} \quad \text{if } p > 2,
\end{aligned}
$$

and

$$
\Lambda^1 = \Lambda\{\lambda_{2i+1}\lambda_{2j+1}; \, i, j \geqslant 0\} \quad \text{if } p = 2.
$$

Here is the first result.

LEMMA 2.3. (1) Λ^0 *is closed under the differential of* Λ. *If* $p > 2$ *then* Λ^0 *has a* $\mathbf{Z}/p$ *basis of admissible monomials* $v_{i_1} \cdots v_{i_q}$, $q > 0$, *with* $v_{i_q} = \lambda_{i_q}$. *If* $p = 2$, *then* Λ^0 *has a basis of admissible monomials given by* $\lambda_{i_1} \cdots \lambda_{i_q}$, $q > 0$, *with* $i_q \equiv 1 \bmod 2$.

(2) Λ^1 *is closed under the differential of* Λ. *If* $p > 2$, *then* Λ^1 *has a basis of admissible monomials* $v_{i_1} \cdots v_{i_q}$, $q > 1$, *with* $v_{i_q} = \lambda_{i_q}$ *and* $v_{i_{q-1}} = \lambda_{i_{q-1}}$. *If* $p = 2$, *then a basis is given by admissible monomials* $\lambda_{i_1} \cdots \lambda_{i_q}$, $q > 1$, *with* $i_q \equiv 1 \equiv i_{q-1} \bmod 2$.

PROOF. This is a consequence of relations (1.1) and (1.2).

If Λ_k is the ideal defined in §1 (see 1.3), let

$$
\Lambda_k^0 = \Lambda^0/\Lambda^0 \cap \Lambda_k, \qquad \Lambda_k^1 = \Lambda^1/\Lambda^1 \cap \Lambda_k.
$$

Define

$$
\begin{aligned}
\Lambda(q, 0) &= \text{monomials of length } q \text{ in } \Lambda_k^0, \\
\Lambda(q, 1) &= \text{monomials of length } q \text{ in } \Lambda_k^1, \\
\Lambda(1, 1) &= \Lambda(0, 1).
\end{aligned}
$$

We can now define the sequences of A-modules which will be our resolutions.

Let $D_0^0 = A/A\beta$ and $D_q^0 = A \otimes \Lambda^*(q, 0)$ for $q > 0$. Define maps

$$
(2.4) \qquad\qquad\qquad \delta_q^*: D_{q+1}^0 \to D_q^0
$$

by formulae of (1.4). Similarly, let

$$
\begin{aligned}
D_0^1 &= A/A\{\beta, \beta P^1\}, \quad D_1^1 = A/A\beta \otimes \Lambda^*(1, 1), \\
D_q^1 &= A \otimes \Lambda^*(q, 1) \quad \text{for } q > 0.
\end{aligned}
$$

Define maps

$$
(2.5) \qquad\qquad\qquad \delta_q^*: D_{q+1}^1 \to D_q^1
$$

by the formulae (1.4).

THEOREM 2.6. *Let $n = 0$ or 1 and let ε: $D_0^n \to M_n(k)$ be the projection. The following is a long exact sequence of A-modules; that is, an acyclic resolution.*

$$\cdots D_{q+1}^n \overset{\delta_q^*}{\to} D_q^n \to \cdots \to D_1^n \overset{\delta_1^*}{\to} D_0^n \overset{\varepsilon}{\to} M_n(k) \to 0.$$

PROOF. With our conventions for $p = 2$, we see that a $\mathbf{Z}/p$ basis for $A/A\beta$ is given by

$$\chi(P^I) = \chi(P^{i_1}\beta^{\varepsilon_1} \cdots P^{i_q}\beta^{\varepsilon_q})$$

with P^I admissible, and a basis for $A/A\{\beta, \beta P^1\}$ is given by

$$\chi(P^I) = \chi(P^{i_1}P^{i_2}\beta^{\varepsilon_2} \cdots P^{i_q}\beta^{\varepsilon_q})$$

and P^I admissible. With this remark the proofs of the theorems of [**4** and **6**] go through verbatum.

Now let $C_q = A \otimes \Lambda_{q,k}^*$ be as in 1.6. Then there are quotient maps θ_q^*: $C_q \to D_q^n$ so that the following diagram commutes:

$$
\begin{array}{ccc}
C_{q+1} & \overset{\delta_q^*}{\to} & C_q \\
\downarrow \theta_{q+1}^* & & \downarrow \theta_q^* \\
D_{q+1}^n & \overset{\delta_q^*}{\to} & D_q^n.
\end{array}
$$

If $q \geqslant 2$, then θ_q^* is the projection into a direct summand. Let K_q be as in 1.7; then $H^*K_q = C_q$. Set $K_0^0 = K\mathbf{Z}_p^\wedge$, $K_1^0 = BP\langle 1\rangle$ and $K_1^1 = \vee_{i > k}\Sigma^{qi-1}K\mathbf{Z}_p^\wedge$. Furthermore, in the remaining cases, let K_q^n be the Eilenberg-Mac Lane spectrum so that $\pi_*K_q^n = \Lambda(q, n)$. The following proposition is straightforward, but crucial.

PROPOSITION 2.7. *For $q \geqslant 0$ there exist maps θ_q: $K_q^n \to K_q$ and maps δ_q: $K_q^n \to \Sigma K_{q+1}^n$ so that:*
   (i) *θ_q^*: $C_q \to D_q^n$ is the quotient and δ_q^*: $D_{q+1}^n \to D_q^n$ is the differential.*
   (ii) *The following diagram commutes:*

$$
\begin{array}{ccc}
K_q^n & \overset{\delta_q}{\to} & \sum K_{q+1}^n \\
\downarrow \theta_q & & \downarrow \theta_{q+1} \\
K_q & \overset{\delta_q}{\to} & \sum K_{q+1}
\end{array}
$$

(iii) *If $q \geqslant 2$, then there is s_q: $K_q \to K_q^n$ so that $s_q\theta_q$ is the identity.*

PROOF. Only the construction of the map δ_0: $BP\langle 1\rangle \to \vee\Sigma^{qi}K\mathbf{Z}_p^\wedge$ is not obvious. But just set $\delta_0 = \vee_{i > j}\chi\overline{P}^i$ as in Theorem 2 of the introduction.

Now let $p = 2$ and fix $k \geqslant 0$ and set

$$M_2(4k) = A/A\{Sq^1, Sq^2, \chi Sq^{4i}; i > k\}.$$

We wish to find analogues of 2.6 and 2.7 for this module. In the mod 2 Λ-algebra set

$$\Gamma = \Lambda\{\lambda_{2i-1}\lambda_{4j-1}: i, j \geqslant 0\}.$$

LEMMA 2.8. Γ *is closed under the differential of* Λ *and has a basis of admissible monomials* $\lambda_{i_1} \cdots \lambda_{i_q}$ *with* $q \geqslant 2$, $i_{q-1} \equiv 1 \bmod 2$ *and* $i_q \equiv 3 \bmod 4$.

Let

$$\Gamma_k = \Gamma/\Gamma \cap \Lambda_{2k},$$
$$\Gamma_q = \text{all monomials of length } q \text{ in } \Gamma_k, \quad q \geqslant 2,$$
$$\Gamma_1 = \mathbf{Z}/2 \text{ vector space on generators } \lambda_{4i-1}, \quad i > k.$$

Define

$$E_0 = A/A\{Sq^1, Sq^2\}, \quad E_1 = A/ASq^1 \otimes \Gamma_1^*, \quad E_q = A \otimes \Gamma_q^*, \quad q \geqslant 2,$$

and define $\delta_q^*: E_{q+1} \to E_q$ by the formula in (1.4).

THEOREM 2.9. *Let* $\varepsilon: A/A\{Sq^1, Sq^2\} \to M_2(4k)$ *be the projection. The following is a long exact sequence*:

$$\cdots \to E_{q+1} \stackrel{\delta_q^*}{\to} E_q \to \cdots \to E_1 \to E_0 \stackrel{\varepsilon}{\to} M_2(4k) \to 0.$$

PROOF. A vector space basis for $A/A\{Sq^1, Sq^2\}$ is given by

$$\chi Sq^I = \chi Sq^{4i_1} Sq^{2i_2} Sq^{i_3} \cdots Sq^{i_q}$$

with Sq^I admissible. Now proceed as before.

Define

$$K_0^2 = bo, \qquad K_1^2 = \bigvee_{i > k} \Sigma^{4i-1} K\mathbf{Z}_2^\wedge.$$

For $q \geqslant 2$, let K_q^2 be an Eilenberg-Mac Lane spectrum so that $\pi_* K_q^2 \cong \Gamma_q$. Let $C_q = A \otimes \Lambda_{q,2k}^*$ and K_q be so that $H^* K_q = C_q$.

PROPOSITION 2.10. *There exist maps* $\theta_q: K_q^2 \to K_q$ *and* $\delta_q^*: K_q^2 \to \Sigma K_{q+1}^2$ *so that*:
(1) $\theta^*: C_q \to E_q$ *is the quotient and* $\delta_q^*: E_{q+1} \to E_q$ *is the differential.*
(2) *The following diagram commutes*:

$$
\begin{array}{ccc}
K_q^2 & \stackrel{\delta_q}{\to} & \sum K_q^2 \\
\downarrow \theta_q & & \downarrow \theta_{q+1} \\
K_q & \stackrel{\delta_q}{\to} & \sum K_q.
\end{array}
$$

(3) *If* $q \geqslant 2$ *there exists a map* $s_q: K_q \to K_q^2$ *so that* $s_q \theta_q$ *is the identity.*

PROOF. The map δ_0 is just $\bigvee_{i>k} \chi \bar{Sq}^{4i}: bo \to \bigvee \Sigma^{4i} K\mathbf{Z}_2^\wedge$ where $\chi \bar{Sq}^{4i}$ is as in Theorem 3 of the introduction.

3. Adapted complexes. In the study of Brown-Gitler spectra it is important to keep track of unstable homotopy data. This is done with the device of adapted complexes [5 and 6]. In this section we define this idea, then prove that there are complexes adapted to the modules we are trying to realize as the cohomology of spectra.

First the definitions. Suppose we have the following situation: a ring spectrum h and a subalgebra $B \subset A$ so that

$$H^*h \cong A//B = A/A\overline{B},$$

where $\overline{B}$ is the nonunits of B. Furthermore suppose we have modules

$$N(k) \cong A/A\{\overline{B}, Y_k\}$$

where $Y_k \subset \{\chi P^i : i > k\}$. Let $1 \colon h \to K\mathbf{Z}/p$ be the generator of H^*h over A.

DEFINITION 3.1. Let Z be a finite CW complex and

$$\hat{\gamma} \in h_m Z \quad \text{with } m \leqslant 2p(k+1) - 1.$$

Let DZ be the Spanier-Whitehead dual of Z and γ the class dual to $\hat{\gamma}$:

$$\gamma \colon \sum^m DZ \to h.$$

Then $(Z, \hat{\gamma})$ is adapted to $N(k)$ if the following sequence is exact:

$$A\{\overline{B}, Y_k\} \to A \cong H^*K\mathbf{Z}/p \stackrel{(1 \circ \gamma)^*}{\to} H^* \sum^m DZ.$$

Since $\hat{\gamma} \in h_m Z$ this is an adapted complex of degree m.

Let $B_0 \subset A$ be the exterior algebra generated by the Bockstein, $B_1 \subset A$ the exterior algebra generated by β and $P^1\beta - \beta P^1$. If $p = 2$, let A_1 be the algebra generated by Sq^1 and Sq^2. Then

$$H^*K\mathbf{Z}_p^{\wedge} \cong A//B_0, \quad H^*BP\langle 1 \rangle \cong A//B_1, \quad H^*bo \cong A//A_1.$$

THEOREM 3.2. *For each $k \geqslant 0$, there exists a finite CW complex Z_k and $\gamma_k \in H_m(Z_k, \mathbf{Z}_p^{\wedge})$ with $m = 2pk + 1$ so that (Z_k, γ_k) is adapted to $M_0(k)$.*

Let δ_0 be as in 2.7.

THEOREM 3.3. *For each $k \geqslant 0$ there exists a finite CW complex Z_k and $\gamma_k \in BP\langle 1 \rangle_m Z_k$ with $m = 2pk + 2$ so that (Z_k, γ_k) is adapted to $M_1(k)$. Furthermore $(\delta_0)_* \gamma_k = 0$.*

Let δ_0 be as in 2.10.

THEOREM 3.4. *For each $k \geqslant 0$ there exists a finite CW complex Z_k and $\gamma_k \in bo_m Z_k$ with $m = 8k + 4$ so that (Z_k, γ_k) is adapted to $M_2(2k)$. Furthermore $(\delta_0)_* \gamma_k = 0$.*

We prove 3.2 in this section. Because 3.3 and 3.4. require computations in the Adams spectral sequence we postpone these proofs to §4. We begin with a proposition on Spanier-Whitehead duality and then reduce our theorems to two lemmas.

Given an element $a \in A$, we have a right action of a on the homology of any finite complex Z:

$$a^* \colon \hom(H^*Z, \mathbf{Z}/p) \cong H_* Z \to \hom(H^*Z; \mathbf{Z}/p) \cong H_* Z.$$

Since a^* is a natural transformation of homology theories it must be induced by a map of spectra $b \colon K\mathbf{Z}/p \to \sum^s K\mathbf{Z}/p$ where s is the degree of a; $b \in A$.

PROPOSITION 3.5. *The class* b *is* χa. *Thus if* $\phi\colon H_n Z \to H^{-n} DZ$ *is the duality isomorphism the following diagram commutes*:

$$
\begin{array}{ccc}
H_n Z & \overset{a^*}{\to} & H_{n-s} Z \\
\downarrow \phi & & \downarrow \phi \\
H^{-n} DZ & \overset{\chi a}{\to} & H^{-n+s} DZ
\end{array}
$$

PROOF. The first claim follows from Brown-Comenetz duality [3]; the second claim follows from the first.

The following lemma allows us to build adapted complexes. Let k and $N(k)$ be as in 3.1. Let $\varepsilon\colon A \to N(k)$ be the quotient. The following lemma is a result of the definitions.

LEMMA 3.6. *Let* $\mathbf{P} \subset A$ *be a finite set of elements so that* $\mathbf{P}' = \{\varepsilon P\colon P \in \mathbf{P}\}$ *is a basis for* $N(k)$. *Suppose for each* $P \in \mathbf{P}$ *there is a finite CW complex* Z_P *and* $\gamma_P \in H_m X_P$ *for some* m *independent of* P. *Suppose further that*
  (1) $\chi P^* \gamma_P \neq 0$ *and* $m \leqslant 2p(k+1)-1$,
  (2) $\gamma_P = 1_* \hat{\gamma}_P$ *where* 1_* *is the reduction* $h_* X_P \to H_* X_P$.
If $Z = \bigvee_{P \in \mathbf{P}} Z_P$ *and* $\hat{\gamma} = \Sigma \hat{\gamma}_P \in h_m Z$, *then* $(Z, \hat{\gamma})$ *is adapted to* $N(k)$.

PROOF. This follows immediately from 3.5.

The following lemma will allow us to finish the proofs of 3.3 and 3.4.

LEMMA 3.7. (i) *Let* $(Z_k, \hat{\gamma}_k)$ *be adapted to* $M_1(k)$ *and suppose* $H_*(Z_k, \mathbf{Z}_p^{\wedge})$ *is a* $\mathbf{Z}/p$ *vector space. Then* $\delta_{0*} \hat{\gamma}_k = 0$.
  (ii) *Let* $(Z_k, \hat{\gamma}_k)$ *be adapted to* $M_2(k)$ *and suppose* $H_*(Z_k, \mathbf{Z}_2^{\wedge})$ *is a* $\mathbf{Z}/2$ *vector space. Then* $(\delta_0)_* \hat{\gamma}_k = 0$.

PROOF. We prove (i); (ii) is the same. It is sufficient to show that $(\chi \overline{P}^i)_* \hat{\gamma}_k = 0$ for $i > k$. There is a commutative diagram:

$$
\begin{array}{ccc}
BP\langle 1 \rangle_m Z_k & \overset{(\chi \overline{P}^i)_*}{\to} & H_{m-qi}\left(Z_k, \mathbf{Z}_p^{\wedge}\right) \\
\downarrow 1_* & & \downarrow 1_* \\
H_m Z_k & \overset{(P^i)^*}{\to} & H_{m-qi} Z_k
\end{array}
$$

Here we use Proposition 3.5 and the definition of $\chi \overline{P}^i \cdot (P^i)_* 1_* \hat{\gamma}_k = 0$ for degree reasons. Since

$$
1_*\colon H_*\left(Z_k, \mathbf{Z}_p^{\wedge}\right) \to H_* Z_k
$$

is an injection, $(\chi \overline{P}^i)_* \hat{\gamma}_k = 0$.

Now define spaces

$$
C(s,t) = \bigwedge_s \mathbf{C}P^{\infty} \wedge \bigwedge_t B\mathbf{Z}/p.
$$

This is the s-fold smash product of CP^∞ smashed with the t-fold smash product of BZ/p. We assume that $t > 0$. We will werite $C(0, t) = C(t)$. Our complexes Z_k will be wedges of finite subcomplexes of $C(s, t)$.

Now let $B = B_0$ or B_1 or A_1. Then define a vector space map

$$\pi: H^*C(s, t) \to V = H^*C(s, t)/\overline{B}H^*C(s, t).$$

$H^{2s+t}C(s, t)$ is generated by a single element α and $H^nC(s, t) = 0$ for $n < 2s + t$. Suppose $v = \pi(a\alpha) \neq 0$ in V for $a \in A$. Then if $\pi^*: V^* \to H_*C(s, t)$ is the dual, $a^*\pi^*v^* \neq 0$ in $H_{2s+t}C(s, t)$. With this remark, we now supply a lemma on $H^*C(s, t)$.

Recall that if $p = 2$ and $Sq^I = Sq^{i_1} \cdots Sq^{i_q}$ is admissible, the excess is $e(I) = i_1 - i_2 - \cdots - i_q$. If $p > 2$ and $P^I = \beta^{\varepsilon_0}P^{i_1}\beta^{\varepsilon_1} \cdots P^{i_q}\beta^{\varepsilon_q}$, then $e(I) = 2i_1 - \sum_{j=2}^{q} 2(p - 1)i_j - \sum_{i=1}^{q}\varepsilon_i$.

LEMMA 3.8. *Let $P^I \in A$ be admissible so that $\beta P^I \neq 0$. Let $e(I) = e$, $t = \sum_{i=1}^{q}\varepsilon_i$ and $s = (e - t)/2$. If $\alpha \in H^{e+1}C(s, t + 1)$ is the generator, then $\pi P^I\alpha \neq 0$ in $V = H^*C(s, t + 1)/\overline{B}_0 H^*C(s, t + 1)$.*

PROOF. We prove $p > 2$. Let P^J, with $J = \{\delta_0, j_1, \delta_1, \ldots, j_n, \delta_n\}$, be any admissible monomial in A. Let $E = e(J) + \delta_0$, $n = \sum_{i=0}^{k}\delta_i$ and $m = (E - n)/2$. Then, if $\alpha \in H^EC(m, n)$ is the generator, $P^J\alpha \neq 0$. Thus if P^I is as in the hypothesis and $P^I\alpha$ generates a free B_0 submodule of $H^*C(s, t + 1)$. Thus $\pi P^I\alpha \neq 0$ in V.

PROOF OF 3.2. Lemma 3.6 applies. Let $\varepsilon: A \to M_0(k)$ be the projection. Let

$$\mathbf{P} = \left\{ \chi P^I | \beta P^I \neq 0, i_1 \leqslant k \right\}.$$

The classes $\mathbf{P}' = \{\varepsilon\chi P^I | \chi P^I \in \mathbf{P}\}$ is a basis for $M_0(k)$. For each I so that $\chi P^I \in \mathbf{P}$ let s, t and e be as in 3.8 and $X_I \subset C(s, t)$ be a finite subcomplex containing the $2p(k + 2)$ skeleton. Let $n = 2pk - 2pi_1$ and $Z_I = \Sigma^n X_I$. Let $\alpha \in H^{n+e+1}Z_I$. Then $\pi P^I\alpha \neq 0$ in $V = H^*Z_I/\overline{B}_0^*H^*Z_I$ by 3.8. By construction, in degrees $2pk + 1$

$$H_*\left(Z_I; \mathbf{Z}_p^{\wedge}\right) \cong V^* \xrightarrow{\pi^*} H^*Z_I.$$

This inclusion is the reduction map 1_*. Let $\alpha \in H^{n+e+1}Z_I$ be the generator. Set $\hat{\gamma}_I = (\pi P^I\alpha)^*$ and $\gamma_I = \pi^*\hat{\gamma}_I$. The remarks following 3.7 imply that $(P^I)^*\gamma_I \neq 0$.

4. The proof of Theorems 3.3 and 3.4. To construct complexes adapted to $M_1(k)$ and $M_2(k)$ we must do computations in the Adam spectral sequence. Let $h = BP\langle 1\rangle$ or bo. Then we have the spectral sequence

$$E_2^{*,*} \cong \text{Ext}_A^{*,*}\left(H^*h \otimes H^*X; \mathbf{Z}/p\right) \Rightarrow \pi_*h \wedge X = h_*X,$$

where X is a complex of finite type. We will concentrate on the case where $X = C(s, t)$, $t > 0$. Recall that $h \cong A//B$, where $B \subset A$ is either B_1 or A_1. The following lemma eases the computations. It is completely standard.

LEMMA 4.1. *Let h be a ring spectrum so that $H^*h \cong A//B$ for some subalgebra of A. Then there is a canonical isomorphism, natural in X,*

$$\text{Ext}_B\left(H^*X; \mathbf{Z}/p\right) \to \text{Ext}_A\left(H^*h \otimes H^*X; \mathbf{Z}/p\right).$$

Here H^*X is given a left B structure by restriction of scalars. Let

$$\pi\colon H^*X \to V = H^*X/\bar{B}H^*X$$

be as in §3 and let $\varepsilon\colon V \to H^*X$ be any section of the quotient. Then there is an obvious extension

$$B \otimes V \xrightarrow{\delta_0^*} H^*X \to 0.$$

This is the beginning of a minimal B resolution of H^*X. Thus

$$\mathrm{Ext}_A^0 \cong \mathrm{Hom}_B(B \otimes V; \mathbf{Z}/p) \cong V^*.$$

The dual of the quotient gives an inclusion $V^* \subset H_*X$. Now suppose the Adams spectral sequence collapses at E_2. Then there is a projection

$$h_*X \to E_\infty^0 \cong V^*.$$

The following is also standard.

LEMMA 4.2. *If $E_2 = E_\infty$ in the Adams spectral sequence converging to h_*X then the composite*

$$h_*X \to E_\infty^0 \cong V^* \subset H_*X$$

is the reduction induced by $1\colon h \to K\mathbf{Z}/p$.

The major computational result is the following theorem, which was proved by Davis in [7] for $p = 2$. The argument goes through at odd primes with the usual modifications.

THEOREM 4.3. *The Adams spectral sequence converging to $BP\langle 1\rangle_*C(s,t)$ and $bo_*C(t)$ collapses at E_2.*

We now prove 3.3 and 3.4.

LEMMA 4.4. *Let $P^{i_1}P^{i_2}\beta^{\varepsilon_2} \cdots P^{i_q}\beta^{\varepsilon_q} = P^I \in A$ be admissible. Set $e = e(I)$, $t = \sum_{i=2}^q \varepsilon_i$ and $s = (e-t)/2$. If $\alpha \in H^{e+2}C(s,t+2)$ is the generator, then*

$$\pi P^I\alpha \neq 0 \quad in\ V = H^*C(s,t+2)/B_1H^*C(s,t+2).$$

PROOF. B_1 is an exterior algebra on two generators, $Q_0 = \beta$ and $Q_1 = \beta P^1 - P^1\beta$. One calculates that $Q_0Q_1P^I\alpha \neq 0$ in $H^*C(s,t+2)$ and $P^I\alpha$ generates a free B_1 submodule of $H^*C(s,t+2)$. The result follows.

PROOF OF 3.3. We apply Lemmas 3.6 and 3.7. Let $\varepsilon\colon A \to M_1(k)$ be the projection map. Let $\mathbf{P} = \{\chi P^I|Q_0Q_1P^I \neq 0$ and I admissible, $i_1 \leqslant k\}$. Then $\mathbf{P}' = \{\varepsilon\chi P^I|\chi P^I \in \mathbf{P}\}$ forms a basis for $M_1(k)$. For each I so that $\chi P^I \in \mathbf{P}$ let s and t be as in 4.4. Let $X_1 \subset C(s,t+2)$ be a finite subcomplex so that X_I contains the $2p(k+1)$ skeleton and $H^*(X_I; \mathbf{Z}_p^\wedge)$ is a $\mathbf{Z}/p$ vector space. Let $n = 2pk - 2pi_1$ and $Z_I = \Sigma^nX_I$. Let $\alpha \in H^{n+e+2}Z_I$ be the generator. Then

$$\pi P^I\alpha \neq 0 \quad in\ V = H^*Z_I/B_1H^*Z^I$$

by 4.4. So if $\gamma_I = (\pi P^I\alpha)^* \in H_{2pk+2}Z_I$, $(P^I)^*\gamma_I \neq 0$ and Theorem 4.3 guarantees that γ_I is in the image of the reduction

$$BP\langle 1\rangle_*Z_I \to H_*Z_I.$$

Let $\hat{\gamma}_I$ be any pre-image of γ_I. The result now follows by Lemmas 3.6 and 3.7.

In exactly the same manner Theorem 3.4 is a result of the following

LEMMA 4.5. *Let* $Sq^I = Sq^{4i_1}Sq^{2i_2}Sq^{i_3} \cdots Sq^{i_q} \in A$ *be admissible. Let* $t = e(I) + 4$. *If* $\alpha \in H^t C(t)$ *is the generator, then*

$$\pi Sq^I \alpha \neq 0 \quad in \; H^*C(t)/\overline{A}_1 H^*C(t).$$

PROOF. A_1 is generated by Sq^1 and Sq^2. One computes that

$$Sq^1 Sq^2 Sq^1 Sq^2 Sq^I = Sq^{4i_1 + 5}Sq^{2i_2 + 1} \cdots Sq^{i_q}.$$

Thus Sq^I generates a free A_1-submodule of $H^*C(t)$.

5. The construction of $B_0(pk + 1)$, $BP\langle 1 \rangle^k$, **and** bo^k. We are now ready to construct towers of spectra whose homotopy inverse limits are the spectra we are trying to produce. We begin with a general lemma on the homology of the spectra in such towers. Our general set up is this: we have a sequence of spectra

$$K_0 \overset{\delta_0}{\to} \sum K_1 \overset{\delta_1}{\to} \sum^2 K_2 \to \cdots.$$

The cohomology of this sequence is an acyclic resolution of some specified module M over A. We will inductively construct spectra X_q and maps

$$\varepsilon_q \colon X_q \to \sum K_{q+1}$$

so that there are cofiber sequences

$$K_{q+1} \overset{i_{q+1}}{\to} X_{q+1} \overset{p_q}{\to} X_q \overset{\varepsilon_q}{\to} \sum K_{q+1}$$

and the composition $\varepsilon_q i_q = \delta_q$, $X_0 = K_0$, and $\varepsilon_0 = \delta_0$. As the work of the previous sections indicates, we will keep track of further data. There will be a spectrum T and a map $\gamma \colon T \to K_0$ so that the following sequence is exact:

$$H^* \sum K_1 \overset{\delta_0^*}{\to} H^* K_0 \overset{\gamma^*}{\to} H^* T.$$

We will inductively construct liftings γ_q of γ so that the following diagram commutes:

$$
\begin{array}{ccc}
T & \overset{\gamma_{q+1}}{\to} & X_{q+1} \\
\downarrow \mathrm{id} & & \downarrow p_q \\
T & \overset{\gamma_q}{\to} & X_q.
\end{array}
$$

Now $\gamma_0 = \gamma$. Since $\delta_0 p_0 \cong 0$, the composition $p_0 \cdots p_{q+1}$ induces a map $c_q^* \colon M \to H^* X_q$. One sees that the image of the composition $\gamma_q^* c_q^* \colon M \to H^* T$ is the image of γ^*. The following result seems to be well known but is not in the literature (cf. [5, proof of 3.1]).

LEMMA 5.1. *If* $\varepsilon_q \colon X_q \to \sum K_{q+1}$ *exists for* $q \leqslant t$ *and* $\gamma_q \colon T \to X_q$ *exists for* $q \leqslant t + 1$, *then there is an exact sequence*

$$0 \to M \overset{c_q^*}{\to} H^* X_q \overset{i_q^*}{\to} H^* K_q$$

for $q \leqslant t + 1$. *Furthermore, if* $q \leqslant t$, *then* γ_q^* *splits this sequence.*

PROOF. Because the tower of spectra X_q is constructed from an acyclic resolution of M, the sequence is exact. We will know it is split once we know that the image of γ_q^* is exactly the image of γ^*. Consider the following diagram:

$$H^*K_{q+1}$$
$$\downarrow \varepsilon_q^*$$
$$0 \quad \to \quad M \quad \overset{c_q^*}{\to} \quad H^*X_q \quad \overset{i_q^*}{\to} \quad H^*K_q \quad \overset{\delta_{q-1}^*}{\to} \quad H^*K_{q-1}$$
$$\downarrow \gamma_q^*$$
$$H^*T$$

Let $v \in H^*X_q$. Since $i_q \varepsilon_{q-1} = 0$, $\delta_{q-1}^* i_q^* v = 0$. Thus there exists w so that $\delta_q^* w = i_q^* v$. Let $v' = v - \varepsilon_q^* w$. Then $\gamma_q^* v = \gamma_q^* v'$ and there exists μ so that $c_q^* \mu = v'$.

We are now in a position to state our constructive theorem. Let $BP\langle 0 \rangle = K\mathbf{Z}_p^\wedge$. Let $n = 0$ or 1 and let

(5.2) $$BP\langle n \rangle = K_0^n \overset{\delta_0}{\to} \sum K_1^n \overset{\delta_1}{\to} \sum^2 K_2^n \to \cdots$$

be the sequence constructed in 2.7.

THEOREM 5.3. *There exist spectra* X_q, $q \geqslant 0$, *and maps* $\varepsilon_q: X_q \to \sum K_{q+1}^n$ *so that*:
(1) $X_0 = K_0^n = BP\langle n \rangle$ *and* $\varepsilon_0 = \delta_0$.
(2) *There are cofiber sequences*

$$K_{q+1}^n \overset{i_{q+1}}{\to} X_{q+1} \overset{p_q}{\to} X_q \overset{\varepsilon_q}{\to} K_{q+1}^n.$$

(3) $\varepsilon_q i_q = \delta_q$, ε_0 *is the identity.*
(4) *For any CW complex Z the induced map of homology theories*

$$(\varepsilon_q)_*: (X_q)_m Z \to (K_{q+1}^n)_{m-1} Z$$

is zero for $m \leqslant 2p(k+1) - 1$ *for* $q \geqslant 0$ *if* $n = 0$ *and for* $q > 0$ *if* $n = 1$.

The following is an immediate corollary of 5.3 and 2.6. It is Theorem 1 of the introduction.

COROLLARY 5.4. *In Theorem 5.3 set* $n = 0$ *and let* $B_1(pk+1)$ *be the homotopy inverse limit of*

$$\cdots \to X_q \overset{p_q}{\to} X_{q-1} \to \cdots \to K\mathbf{Z}_p^\wedge.$$

Then there is a map

$$\omega: B_1(pk+1) \to X_0 = K\mathbf{Z}_p^\wedge$$

and
(1) $H^*B_1(pk+1) \cong A/A\{\beta, \chi P^i; i > k\}$,
(2) *the induced map for homology theories*

$$\omega_*: B_1(pk+1)_m X \to H_m(X; \mathbf{Z}_p^\wedge)$$

is onto if $m \leqslant 2p(k+1) - 1$.

For the following result recall that if $n = 1$, $\delta_0 \colon X_0 \to \Sigma K_1$ is the map

$$\delta_0 = \bigvee_{i > k} \chi \overline{P}^i \colon BP\langle 1 \rangle \to \bigvee_{i > k} \Sigma^{qi} K \mathbf{Z}_p^{\wedge}, \qquad q = 2(p - 1).$$

The following is also an immediate corollary of 2.6 and 5.3. It is Theorem 2 of the introduction.

COROLLARY 5.5. *In Theorem 5.3 let $n = 1$ and $BP\langle 1 \rangle^k$ be the inverse limit of the X_q. Then there is a map*

$$\omega \colon BP\langle 1 \rangle^k \to BP\langle 1 \rangle.$$

Then
  (1) $H^* BP\langle 1 \rangle^k = A/A\{\beta, P^1\beta, \chi P^i \colon i > k\}$,
  (2) *the induced map of homology theories*

$$\omega_* \colon BP\langle 1 \rangle_m^k X \to BP\langle 1 \rangle_m X$$

is, for $m \leqslant 2p(k + 1) - 1$, onto $\ker(\delta_0^*)_* = \bigcap_{i > k} \ker(\chi \overline{P}^i)_*$.

The proof of 5.3 is by induction on q. We now detail the induction hypothesis. Let

$$(5.6) \qquad \cdots \to Y_q \xrightarrow{p_{q-1}} Y_{q-1} \to \cdots \to Y_1 \to Y_0 = \mathbf{Z}$$

$$i_q \uparrow \qquad\qquad\qquad\qquad \uparrow i_1$$

$$K_q \qquad\qquad\qquad\qquad K_1$$

be the tower whose homotopy inverse limit is $B(pk + 1)$ (see 1.7). Then we have the cofiber sequence

$$K_{q+1} \xrightarrow{i_{q+1}} Y_{q+1} \xrightarrow{p_q} Y_q \xrightarrow{\varepsilon_q} \Sigma K_{q+1}.$$

Let $\theta_q \colon K_q^n \to K_q$ be as in 2.7. Here is the induction hypothesis.

H(t): For $q \leqslant t + 1$ we have spectra X_q and maps ε_q for $q \leqslant t$ so that 5.3(1)–(4) hold and further we have maps

$$\overline{\theta}_q \colon X_q \to Y_q, \qquad q \leqslant t + 1,$$

so that we have a commutative diagram of cofibration sequences for $q \leqslant t$:

$$(5.7) \qquad \begin{array}{ccccccc} K_{q+1}^n & \to & X_{q+1} & \xrightarrow{p_q} & X_q & \xrightarrow{\varepsilon_q} & \Sigma K_{q+1}^n \\ \downarrow \theta_{q+1} & & \downarrow \overline{\theta}_{q+1} & & \downarrow \overline{\theta}_q & & \downarrow \theta_{q+1} \\ K_{q+1} & \to & Y_{q+1} & \xrightarrow{p_q} & Y_q & \xrightarrow{\varepsilon_q} & \Sigma K_{q+1} \end{array}$$

LEMMA 5.8. H(0) *is true.*

PROOF. 5.3(1) and 5.3(2) define X_0 and X_1. 5.3(3) holds by definition of ε_0 and i_0. Proposition 2.7 implies 5.7 holds for $q = 0$. If $n = 1$, then 5.3(4) does not apply. If $n = 0$, then $(\theta_1)_* \colon (K_1^0)_* Z \to (K_1)_* Z$ is injective; thus 5.3(4) follows from 5.7 and 1.7(4).

Now let $(Z_k, \hat{\gamma}_k)$ be the complex adapted to $M_0(k)$ constructed in 3.2 or 3.3. Let $T_k = \Sigma^{2pk+1} DZ_k$ if $n = 0$ or $T_k = \Sigma^{2pk+2} DZ_k$ if $n = 1$. Let $\gamma\colon T_k \to BP\langle n\rangle$ be dual to $\hat{\gamma}_k$ and $1_*\gamma$ be the usual reduction.

LEMMA 5.9. *If* $H(t)$ *holds, then any lifting of* γ *to* X_q *lifts to* X_{q+1} *for* $q < t$.

PROOF. This follows from 5.3(4) and a duality argument if $n = 0$. If $n = 1$, then $(\delta_0)_*\gamma = 0$ by 3.3; therefore γ lifts to X_1. Any lifting to X_q, $q > 0$, lifts to X_{q+1} by 5.3(4).

LEMMA 5.10. $H(t-1)$ *implies* $H(t)$.

PROOF. First, we define $\varepsilon_t\colon X_t \to \Sigma K^n_{t+1}$. Since $t - 1 \geqslant 0$, there exists, by 2.7(3), a map $s_{t+1}\colon K_{t+1} \to K^n_{t+1}$ so that $s_{t+1}\theta_{t+1}$ is the identity.
Define

$$\varepsilon_t = s_{t+1}\varepsilon_t\bar{\theta}_t\colon X_t \to \Sigma K^n_{t+1},$$

and X_{t+1} by 5.3(2). Then

$$\varepsilon_t i_t = \delta_t\colon s_{t+1}\varepsilon_t\bar{\theta}_t i_t = s_{t+1}\delta_t\theta_t = s_{t+1}\theta_{t+1}\delta_t = \delta_t.$$

So 5.3(3) holds.

Second, let $\gamma_t\colon T_k \to X_t$ be any lifting of γ. Then $\bar{\theta}_t\gamma_t$ is a lifting of $1_*\gamma$. So 1.9 implies $\varepsilon_t\bar{\theta}_t\gamma_t = 0$. Thus $\varepsilon_t\gamma_t = 0$ and γ_t lifts to X_{t+1}.

Third, the following diagram commutes:

$$
\begin{array}{ccc}
X_t & \overset{\varepsilon_t}{\to} & \Sigma K^n_{t+1} \\
\downarrow \bar{\theta}_t & & \downarrow \theta_{t+1} \\
Y_t & \overset{\varepsilon_t}{\to} & \Sigma K_{t+1}
\end{array}
$$

Let $\Delta = \varepsilon_t\bar{\theta}_t - \theta_{t+1}\varepsilon_t$. Since K_{t+1} is an Eilenberg-Mac Lane spectrum it is enough to show that Δ is zero in cohomology. By 5.1 it is enough to show that

$$i_t^*\Delta^* = 0 \quad \text{and} \quad \gamma_t^*\Delta^* = 0.$$

But $\Delta i_t = \varepsilon_t\bar{\theta}_t i_t - \theta_{t+1}\varepsilon_t i_t = \theta_t\delta_t - \theta_{t+1}\delta_t \cong 0$ and $\varepsilon_t\gamma_t \cong 0$ and $\varepsilon_t\bar{\theta}_t\gamma_t = 0$ by our above arguments. This shows that $\Delta = 0$. $\bar{\theta}_{t+1}$ is defined because $\theta_{t+1}\varepsilon_t = \varepsilon_t\bar{\theta}_t$. This leaves 5.3(4). But

$$(\theta_{t+1})_*\colon (K^n_{t+1})_*Z \to (K_{t+1})_*Z$$

is injective. We now apply 1.7(4). This proves Lemma 5.10.

Thus we have constructed $B_1(pk+1)$ and $BP\langle n\rangle^k$.

We also have the following theorem. Let $p = 2$ and

$$bo = K^2_0 \overset{\delta_0}{\to} \Sigma K^2_1 \overset{\delta_0}{\to} \Sigma^2 K^2_2 \to \cdots$$

be the sequence constructed in 2.10.

THEOREM 5.11. *There exists spectra* X_q *and* $\varepsilon_q\colon X_q \to \Sigma K^2_{q+1}$, $q \geqslant 0$, *so that:*
(1) $X_0 = K^2_0 = bo$ *and* $\varepsilon_0 = \delta_0$.
(2) *There are cofiber sequences*

$$K^2_{q+1} \overset{i_{q+1}}{\to} X_{q+1} \overset{p_q}{\to} X_q \overset{\varepsilon_q}{\to} \Sigma K^2_{q+1}.$$

(3) $\varepsilon_q i_q = \delta_q$; i_0 is the identity.

(4) For $q > 0$, $(X_q)_m Z \to (K^2_{q+1})_{m-1} Z$ is zero for any CW complex Z and $m \leqslant 8k + 7$.

The proof is the same as in 5.3, except that the tower (5.6) should be replaced by the tower whose inverse limit is $B(4k + 3)$. The following is immediate and is Theorem 3 of the introduction.

COROLLARY 5.12. Let $bo^k = \underleftarrow{\text{holim}}\, X_q$. Then there is a map ω: $bo^k \to bo$ and

(1) $H^* bo^k = A / A\{Sq^1, Sq^2, \chi Sq^{4i}; i > k\}$,

(2) ω_*: $(bo^k)_m Z \to bo_m Z$ is onto $\bigcap_{i>k} \ker(\chi \overline{Sq}^{4i})_*$ for any CW complex Z and $m \leqslant 8k + 7$.

6. Space-like spectra and maps to Brown-Gitler spectra. It is an observation of E. H. Brown that the crucial property of Brown-Gitler spectra is that, with suitable choices, the Adams spectral sequence converging to $\pi_* B(pk + 1) \wedge Z$ collapses at E_1 when Z is a space. In this section, we develop the analogous spectral sequences for the spectra constructed in §5. These spectral sequences are not a priori of Adams type, but they do have remarkable collapsing properties. Furthermore, we show these collapsing theorems for a wider class of spectra, those which we call "space-like".

DEFINITION 6.1. Let Y be a finite CW spectrum. Then Y is space-like of dimension n if there is a spectrum T and a map f: $T \to Y$ so that

(i) $T = \Sigma^n DZ$, where Z is a finite CW complex; and

(ii) f^*: $H^* Y \to H^* T$ is injective.

For example, Theorems 3.4 and 5.11 show that bo^k is space-like of dimension $8k + 4$. Now let $h = K\mathbf{Z}/p$, $K\mathbf{Z}_p^\wedge$, $BP\langle 1 \rangle$, or bo. Then Theorem 1.7, 5.3, or 5.11 gives, for each integer k, a Brown-Gitler spectrum over h. Call it h^k. Then h^k is the inverse limit of a tower of fibrations:

$$h^k = \lim X_q \to \cdots \to X_q \to X_{q-1} \to \cdots \to X_1 \to X_0 = h$$

(6.2)
$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$
$$K_q \qquad\qquad\qquad\qquad K_1$$

In each case, $\pi_n K_q = 0$ for $n < q$. Let Y be a connected CW spectrum. Then, by smashing the tower (6.2) with Y and applying stable homotopy we get a spectral sequence

(6.3) $\{ E^r_{s,t}(Y, h^k) \} \Rightarrow h^k_{t-s} Y$, $E^1_{s,t} = (K_q)_{t-s} Y = \pi_{t-s}(K_q \wedge Y)$.

If $h = K\mathbf{Z}/p$ or $K\mathbf{Z}_p^\wedge$, then 1.7 or 5.3 implies that if Y is the suspension spectrum of a space, then in this spectral sequence

$$E^1_{s,t}(Y, h^k) \cong E^\infty_{s,t}(Y, h^k), \qquad t - s \leqslant 2p(k + 1) - 2.$$

If $h = BP\langle 1 \rangle$ or bo, then 5.3 or 5.11 gives a result only slightly more complicated: Let Y be the suspension spectrum of a space. Then

$$E^2_{s,t}(Y, h^k) \cong E^\infty_{s,t}(Y, h^k)$$

for $t - s \leqslant 2p(k + 1) - 2$ if $h = BP\langle 1 \rangle$ or $t - s \leqslant 8k + 6$ if $h = bo$. In both cases, the only nonzero differential in the prescribed range is

(6.4) d_1: $E^1_{0,t}(Y, h^k) \to E^1_{1,t}(Z, h^k)$.

Using the details provided after 1.3 and before 2.6 and 2.9, these E_1 terms are easily computed. For instance, if $h = bo$, then the differential (6.4) is given by

$$(6.4a) \qquad \sum_{i>k} \chi \mathrm{Sq}_*^{4i} : bo_t Y \to \bigoplus_{i>k} H_{t-4i}(Y; \mathbf{Z}_2^{\wedge}).$$

Dual to this situation is the following. Let $[Y, X]^n = [Y, \Sigma^n X]$. Then if Y is a finite CW spectrum we may apply the functor $[Y, \ ]^*$ to the tower (6.2) to get a spectral sequence

$$(6.5) \qquad \left\{ E_r^{s,t}(Y, h^k) \right\} \Rightarrow [Y, h^k]^{s-t}, \qquad E_1^{s,t} = \left[Y, K_q \right]^{s-t}.$$

The next two propositions comprise the main technical result of this section.

PROPOSITION 6.6. *Let Y be space-like of dimension n. Then, if $h = K\mathbf{Z}/p$ or $K\mathbf{Z}_p^{\wedge}$,*

$$E_1^{s,t}(Y, h^k) \cong E_{\infty}^{s,t}(Y, h^k)$$

for $s - t \geqslant n - 2p(k+1) + 2$.

PROOF. Duality induces an isomorphism of spectral sequences

$$\left\{ E_r^{s,t}(Y, h^k) \right\} \cong \left\{ E_{s,t}^r \left(\sum^n DY, h^k \right) \right\}.$$

Because Y is space-like of dimension n, we have a map

$$Df: \sum^n DY \to Z,$$

where Z is the suspension spectrum of a space. Df_* is injective in mod p homology. Under the hypothesis that $h = K\mathbf{Z}/p$ or $K\mathbf{Z}_p^{\wedge}$, in the tower (6.2), for $s > 0$, K_s is an Eilenberg-MacLane spectrum with $\pi_* K_s$ a $\mathbf{Z}/p$ vector space. Thus, for $s > 0$, there is an injection

$$Df_* : E_1^{s,t+n} \left(\sum^n DY, h^k \right) \to E_1^{s,t+n}(Z, h^k).$$

Since $E_1^{s,t+n}(Z, h^k) \cong E_{\infty}^{s,t+n}(Z, h^k)$ for $t + n - s \leqslant 2p(k+1) - 2$, the result follows.

The next result is proved in an identical manner.

PROPOSITION 6.7. *If $h = BP\langle 1 \rangle$ or $h = bo$ and Y is space-like of dimension n, then*

$$E_2^{s,t}(Y, h^k) \cong E_{\infty}^{s,t}(Y, h^k)$$

for $s - t \geqslant n - 2p(k+1) + 2$ if $h = BP\langle 1 \rangle$ or $s - t \geqslant n - 8k - 6$ if $h = bo$. The only nontrivial differential is

$$d_1 : E_1^{0,t}(Y, h^k) \to E_1^{1,t}(Y, h^k).$$

REMARK. The differential can be described in a manner analogous to (6.4a). 6.6 and 6.7 were also proved in a more conventional manner in [**8**].

We list several immediate consequences of these theorems. We get uniqueness results such as the following.

PROPOSITION 6.8. *Let b^k be any $\mathbf{Z}/2$-complete spectrum with a map $w\colon b^k \to bo$ so that*

(i) *$H^*b^k \cong A/A\{\mathrm{Sq}^1, \mathrm{Sq}^2, \chi\mathrm{Sq}^{4i}\colon i > k\}$ and w^* induces the quotient,*

(ii) *$w_*\colon b_n^k Z \to bo_n Z$ is onto $\bigcap_{i>k} \chi\mathrm{Sq}_*^{4i}$ for $n \leqslant 8k + 7$ and any CW complex Z. Then there is a homotopy equivalence $b^k \cong bo^k$.*

PROOF. Let Z_k be the CW-complex of 3.4 and $T_k \cong \Sigma^{8k+4} DZ_k$. (ii) implies the existence of a map $\phi\colon T_k \to b^k$ so that ϕ^* is injective. Thus b^k is space-like of dimension $8k + 4$. Now apply 6.7.

There are similar results for the other Brown-Gitler type spectra. We also have pairing results. For example, we have the following

PROPOSITION 6.9. *Let $p > 2$. Then for each pair of integers (i, j) there are maps*

$$\mu_{i,j}\colon BP\langle 1\rangle^i \wedge BP\langle 1\rangle^j \to BP\langle 1\rangle^{i+j}$$

so that the following diagram commutes:

$$
\begin{array}{ccc}
BP\langle 1\rangle^i \wedge BP\langle 1\rangle^j & \overset{\mu_{i,j}}{\to} & BP\langle 1\rangle^{i+j} \\
{\scriptstyle w \wedge w}\downarrow & & \downarrow{\scriptstyle w} \\
BP\langle 1\rangle \wedge BP\langle 1\rangle & \to & BP\langle 1\rangle
\end{array}
$$

where the bottom arrow is the ring spectrum multiplication.

PROOF. The existence of the complex in 3.3 and Theorem 5.3 demonstrate that $BP\langle 1\rangle^k$ is space-like of dimension $2pk + 2$. Thus $BP\langle 1\rangle^i \wedge BP\langle 1\rangle^j$ is space-like of dimension $2p(i + j) + 4$. Now apply 6.7.

REMARK. If $p = 2$, then $2p(i + j) + 4 > 2p(i + j + 1) - 1$; thus, the proof of 6.9 fails for this case. Similarly, we fail to get pairings $bo^i \wedge bo^j \to bo^{i+j}$, because $bo^i \wedge bo^j$ is space-like of dimension $8(i + j) + 8$. This is one dimension too high. However, we make the following remark. Let $BP\langle 2\rangle$ be the 2-local spectrum so that $H^*BP\langle 2\rangle \cong A//B_2$ and $\pi_*BP\langle 2\rangle \cong \mathbf{Z}_{(2)}[v_1, v_2]$, where the dimension of v_1 is 2 and the dimension of v_2 is 6. B_2 is the exterior subalgebra of A generated by Sq^1, $Q_1 = \mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^1\mathrm{Sq}^2$, and $Q_2 = \mathrm{Sq}^4Q_1 + Q_1\mathrm{Sq}^4$. One would like to investigate the Adams spectral sequence based on this spectrum; therefore one would like to carry out a detailed analysis of $BP\langle 2\rangle \wedge BP\langle 2\rangle$. For this, one turns to a program such as [**10**], but it is essential to have pairings

$$bo^i \wedge bo^j \to bo^{i+j}.$$

We see no method for obtaining these.

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