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H^p-CLASSES ON RANK ONE SYMMETRIC SPACES OF NONCOMPACT TYPE. I. NONTANGENTIAL AND PROBABILISTIC MAXIMAL FUNCTIONS

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ABSTRACT. Two kinds of H^p-classes of harmonic functions are defined on a general rank one symmetric space of noncompact type. The first one is introduced by using a nontangential maximal function. The second is related to the diffusion generated by the Laplace-Beltrami operator. The equivalence of the two classes is proven for $0 < p < \infty$.

0. Introduction. The aim of the present work is to prove an analogue, for a rank one symmetric space of noncompact type, of a result due to Burkholder, Gundy and Silverstein [**2**] for the Euclidean space $\mathbf{R}^2_+$ and to Burkholder and Gundy [**1**] for $\mathbf{R}^{n+1}_+$. Our results generalize and extend those of Debiard [**5**] who did his work on the generalized half plane equivalent to the hyperbolic ball of $\mathbf{C}^n$.

In §1 we start with some general background about symmetric spaces of noncompact type with special emphasis on those of rank one. The reader is referred to Helgason [**8** or **9**] and Korányi [**11**] for more details. We give then some basic definitions and illustrate them with an example. We end this section by stating the main result of this work (Theorem I) that consists in the equivalence of two H^p-classes of harmonic functions (H^p_* and H^p_N) previously defined. §2 contains several facts about Brownian motion in a Riemannian manifold that are used later on in the paper. A general reference for these is McKean [**16**]. §§3 and 4 are devoted to the proof of each of the two parts of Theorem I. In §3 we prove the inclusion $H^p_N \subset H^p_*$ following the techniques of [**1** and **5**] adapted to the symmetric space. It is remarkable that for this part of the proof we do not need to require harmonicity. In §4 the opposite inclusion is proven. Although we also follow here the path marked by [**1** and **5**] the problems that one has to face to extend the results are more serious. In particular, the techniques to prove Proposition 8 and Proposition 12 are new. Also for the part of Lemma 7 that deals with $0 < p < 1$ we had to adapt to a symmetric space a technique due to De Giorgi [**7**] for the Euclidean space. The author is indebted to Dr. Choi of the University of Chicago who showed him De Giorgi's method.

The results proven here are, in part, contained in the author's Ph.D. Dissertation presented at Washington University in St. Louis in May 1983. The author wishes to thank Adam Korányi for his help and encouragement.

1. Symmetric spaces. Definitions. A symmetric space of noncompact type is a Riemannian manifold of negative curvature that verifies the following property:

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for every point $x \in \mathbf{X}$ there exists an involutive isometry whose only fixed point is x. The rank of the space is one if any maximal totally geodesic, flat submanifold has dimension one. Let $\mathbf{X}$ be a rank one symmetric space of noncompact type. If G is the connected group of isometries of $\mathbf{X}$ and K is the isotropy subgroup of a fixed point $\mathbf{o} \in \mathbf{X}$ then G is a semisimple Lie group with finite center and K is a maximal compact subgroup of G. Denote by $\mathbf{g}$ and $\mathbf{k}$ the Lie algebras of G and K respectively and choose a vector subspace $\mathbf{p}$ of $\mathbf{g}$ such that $\mathbf{g} = \mathbf{k} + \mathbf{p}$ (direct sum). This type of decomposition of $\mathbf{g}$ is called a Cartan decomposition. The endomorphism θ with eigenvalue 1 on $\mathbf{k}$ and eigenvalue -1 on $\mathbf{p}$ is called the Cartan involution of the given decomposition.

We fix a maximal abelian subspace of $\mathbf{p}$ that we denote as $\mathbf{a}$. In our case (rank one) this subspace, and its corresponding Lie group, that we denote as A, have dimension one, and can be identified with $\mathbf{R}$. Let $\mathbf{a}^*$ be the dual of $\mathbf{a}$; if $\lambda \in \mathbf{a}^*$ we write

$$\mathbf{g}_\lambda = \{X \in \mathbf{g}: [H, X] = \lambda(H)X,\ \ H \in \mathbf{a}\}.$$

The nonzero elements λ for which $\mathbf{g}_\lambda \neq 0$ are called the restricted roots. In the rank one case we can have two or four roots that we write as $\{\pm\lambda\}$ or $\{\pm\lambda, \pm 2\lambda\}$. We will do all our calculations for the four roots case; the two roots case is always simpler. From now on we fix an element $H \in \mathbf{a}$ such that $\lambda(H) = 1$, then $\mathbf{a} = \{sH: s \in \mathbf{R}\}$.

The space $\mathbf{n} = \mathbf{g}_\lambda + \mathbf{g}_{2\lambda}$ is a nilpotent Lie algebra. Its image under the Cartan involution is $\overline{\mathbf{n}} = \mathbf{g}_{-\lambda} + \mathbf{g}_{-2\lambda}$. We denote by N and $\overline{N}$ the analytic subgroups of G with Lie algebras $\mathbf{n}$ and $\overline{\mathbf{n}}$ respectively. The group G can be decomposed as a product $G = \overline{N}AK$ (Iwasawa decomposition) and every element $g \in G$ can be written uniquely $g = nak$. Thus, every element $x \in \mathbf{X}$ can be written uniquely as $na \cdot \mathbf{o}$, $(n \in \overline{N},\ a \in A)$ where $n = \exp(X + Y)$, $(X \in \mathbf{g}_{-\lambda},\ Y \in \mathbf{g}_{-2\lambda})$, and $a = \exp(sH)$, $(s \in \mathbf{R})$. We will write $x = (n, s)$.

The group A acts on $\overline{N}$ by conjugation: $n^s = \exp sH \cdot n \cdot \exp(-sH)$. We can define on $\overline{N}$ a smooth gauge, homogeneous relative to the cojugation by A, as follows,

$$|n| = |\exp(X + Y)| = (|X|^4 + 16|Y|^2)^{1/4}$$

where $|X|$ and $|Y|$ denote the canonical lengths of $X \in \mathbf{g}_\lambda$ and $Y \in \mathbf{g}_{2\lambda}$ in the Lie algebra $\mathbf{g}$ (cf. Helgason [**9**, p. 414]). This gauge is a C^∞-function on $\overline{N} - \{e\}$ verifying the properties $|n^{-1}| = |n|$, $|nn'| \leq |n| + |n'|$ and $|n^s| = \mathbf{e}^{-s}|n|$ and therefore defines a distance in $\overline{N}$ (cf. Cygan [**4**]). Haar measure on $\overline{N}$ is the measure inherited through the exponential mapping from the Lebesgue measure of $\overline{\mathbf{n}}$. It will be denoted as $|\cdot|$ or dn. This Haar measure is unimodular. The corresponding L^p-norms in $\overline{N}$ will be written as $\|\cdot\|_p$. If $b(r) = \{n \in \overline{N}: |n| < r\}$ is a gauge ball, its Haar measure is proportional to r^m where m is the homogeneous dimension of $\overline{N}$ defined as follows: let ρ be the half sum of the positive roots with multiplicities counted; then $m = 2\rho(H)$. We have,

$$|b(r)| = \int_{(|X|^4 + 16|Y|^2)^{1/4} < r} dX\, dY = cr^m.$$

In the symmetric space X we denote the volume element as dx and the geodesic ball of center x and radius r as $B_r(x)$. The invariant gradient and the Laplace-Beltrami operator of $\mathbf{X}$ are written as ∇ and Δ respectively.

A function F is *harmonic* in X if it is locally integrable and it verifies the mean value property

$$F(g \cdot \mathbf{o}) = \int_K F(gk \cdot x)\, dk$$

where dk is the normalized Haar measure of K, $g \in G$, and $x \in \mathbf{X}$. A theorem of Godement (cf. Korányi [**11**, Theorem 1.1 and Remark 3 on p. 383]) proves that F is harmonic if and only if $\Delta F = 0$. Harmonic functions verifying some positive boundedness condition can be reproduced by Poisson integrals of their boundary values. Knapp and Williamson have proved [**10**, Theorem 3.2] that if F is harmonic in $\mathbf{X}$ and

$$\sup_{s \in \mathbf{R}} \int_{\overline{N}} |F(n,s)|\, dn < \infty,$$

then

$$F(n,s) = \int_{\overline{N}} P_s(u^{-1}n)\, d\nu(n)$$

where ν is a finite measure on $\overline{N}$. Therefore $\overline{N}$ plays the role of boundary of $\mathbf{X}$. A point (n,s) is approaching the boundary when $s \to \infty$. The Poisson kernel $P(n,s) = P_s(n) = \mathbf{e}^{-2\rho(H(\exp(-sH)n))}$ verifies the homogeneity property $P_{s+s'}(n^{s'}) = P_s(n)\mathbf{e}^{ms'}$ and the normalization $\int_{\overline{N}} P_s(n)\, dn = 1$. We also can observe that $P(e,s) = P_s(e) = \mathbf{e}^{-2\rho(H(\exp(-sH)))} = \mathbf{e}^{ms}$ and this function is invariant under the action of N, the dual of $\overline{N}$ under the Cartan involution.

We define next *admissible domains* in the space $\mathbf{X}$. These domains will play the same role the nontangential domains play in the Euclidean space. The admissible domain Γ_α is defined as

$$\Gamma_\alpha = \{x = (n,s) \colon |n| < \alpha \mathbf{e}^{-s},\ s \in \mathbf{R}\}.$$

We also define truncated versions of Γ_α:

$$\Gamma_\alpha^{(\sigma)} = \Gamma_\alpha \cap \{(n,s) \colon -\sigma < s < \sigma\}, \qquad (\sigma > 0).$$

The region Γ_α can be though of as a cone in $\mathbf{X}$ having "vertex" at the boundary point $e \in \overline{N}$. We can translate this region, acting with $n \in \overline{N}$, and obtain the corresponding cone, $n \cdot \Gamma_\alpha$, with "vertex" at n.

REMARKS. (1) The domains Γ_α are invariant under the action of the group A: let $x = n \exp sH \cdot \mathbf{o} \in \Gamma_\alpha$. Then

$$\exp s'Hn \exp sH \cdot \mathbf{o} = n^{s'} \exp(s'+s)H \cdot \mathbf{o} \quad \text{and} \quad |n^{s'}| = \mathbf{e}^{-s'}|n| < \alpha \mathbf{e}^{-(s'+s)}.$$

(2) Given $\alpha > 0$ there exists $r > 0$ such that $\Gamma_\alpha \subset \bigcup_{s \in \mathbf{R}} \exp sH \cdot B_r(\mathbf{o})$; just choose r larger than the distance from $\mathbf{o}$ to the "walls" of Γ_α.

(3) For $\alpha' > \alpha > 0$, there exists $r > 0$ such that for any $x \in \Gamma_\alpha$ the geodesic ball $B_r(x)$ is contained in $\Gamma_{\alpha'}$. To see this, it is enough to establish the property at a fixed level, say $s = 0$, using a compactness argument and then act with the group A.

To illustrate some of the concepts that we have just introduced, we consider the following example. Let $D = \{(z_0, z_1) \in \mathbf{C}^2 \colon \operatorname{Im} z_0 - |z_1| > 0\}$ and attach to it the Bergmann metric (i.e., the Hermitian metric that is invariant under all the biholomorphisms of D). To obtain a transitive group of isometries of D we consider homogeneous coordinates (z_0, z_1, z_2). Then D is the set of equivalence classes $[\lambda(z_0, z_1, z_2)]$ such that $\frac{i}{2}(z_0\bar{z}_2 - \bar{z}_0 z_2) + |z_1|^2 < 0$. This space is invariant under the group G of matrices $g \in SL(3, \mathbf{C})$ that keep invariant the quadratic form that appears in the last inequality. Equivalently, if

$$J = \begin{pmatrix} 0 & 0 & -\frac{i}{2} \\ 0 & 1 & 0 \\ -\frac{i}{2} & 0 & 0 \end{pmatrix},$$

then $G = \{g \in SL(3, \mathbf{C}): {}^t\bar{g}Jg = J\}$. If $z = (z_0, z_1) = (t + i(y + |z_1|^2), z_1) \in D$ and $y = \mathbf{e}^{-s}$, $s \in \mathbf{R}$, then

$$z = \begin{pmatrix} z_0 \\ z_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2i\bar{z}_1 & t + i|z_1|^2 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}^{-s/2} & & \\ & 1 & \\ & & \mathbf{e}^{s/2} \end{pmatrix} \cdot \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}.$$

We write this expression as $z = [t, z_1]a_s \cdot \mathbf{o}$. The subgroup $\overline{N} = \{[t, z_1]: t \in \mathbf{R}, z_1 \in \mathbf{C}\}$ is nilpotent and acts transitively on the boundary of D, $\partial D = \{(z_0, z_1) \in \mathbf{C}: \operatorname{Im} z_0 = |z_1|^2\}$, as follows,

$$[t, z_1] \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} t + i|z_1|^2 \\ z_1 \\ 1 \end{pmatrix}.$$

We can identify $\overline{N}$ and ∂D in this way.

The subgroup $A = \{a_s: s \in \mathbf{R}\}$ is abelian and normalizes $\overline{N}$. It acts as a group of dilations on $\overline{N}$: $a_s \cdot [t, z_1] = a_s[t, z_1]a_{-s} = [t\mathbf{e}^{-2s}, z_1\mathbf{e}^{-s}]$. The dilation a_s of magnitude $\mathbf{e}^{-s}$ is homogeneous of degree 2 on t and of degree 1 on z_1. The homogeneous gauge $\|[t, z_1]\| = (16t^2 + |z_1|^4)^{1/4}$ verifies the required properties and defines a metric in $\overline{N}$. For this metric $|b(r)| = \pi r^4$, therefore the homogeneous dimension is in this case 4.

The admissible domains for this space are

$$\Gamma_\alpha = \{([t, z_1], s): (16t^2 + |z_1|^4)^{1/4} < \alpha\mathbf{e}^{-s}\},$$

and the Poisson kernel is

$$P_0([t, z_1]) = \mathbf{e}^{-2\rho(H([t, z_1]))} = \left(\frac{1}{(1 + |z_1|^2)^2 + t^2}\right)^2$$

(cf. Helgason [**9**, p. 415]).

We define now the two maximal functions that are the object of our study. From now on F will always represent a function on $\mathbf{X}$.

Given F, its *nontangential maximal function* is defined as

$$N_\alpha F(n) = \sup\{|F(x)|: x \in n \cdot \Gamma_\alpha\}$$

for every $n \in \overline{N}$. We will also use a truncated version, $N_\alpha^{(\sigma)}F$, defined replacing $\Gamma_\alpha^{(\sigma)}$ for Γ_α.

Consider the diffusion of paths $\mathbf{x}_t(\omega)$ generated by the Laplace-Beltrami operator of $\mathbf{X}$ (cf. §2). The *Brownian maximal function* of F is defined as

$$F^*(\omega) = \sup\{|F(\mathbf{x}_t(\omega))|: t \geq 0\}.$$

The function F^* estimates F along the Brownian path.

Using these two different maximal functions we define the corresponding H^p-classes of harmonic functions. The *maximal H^p-space* is defined, for p in the range $0 < p < \infty$, as

$$H^p_N = \{F: \Delta F = 0,\ N_\alpha F \in L^p(\overline{N})\}.$$

If $F \in H^p_N$ we define $\|F\|_{H^p_N} = \|N_\alpha F\|_p$. A priori the space H^p_N may depend on α. We will see soon that this is not the case and that all the "norms" $\|F\|_{H^p_N}$ are equivalent.

The *probabilistic H^p-space* is defined as

$$H^p_* = \left\{F: \Delta F = 0, \sup_{s\in\mathbf{R}} \int_{\overline{N}} \mathbf{E}_{(n,s)}[(F^*)^p]\,dn < \infty\right\}$$

where $\mathbf{E}_{(n,s)}$ represents the mathematical expectation corresponding to the probability distribution $\mathbf{P}_{(n,s)}$ (cf. §2). For $F \in H^p_*$ we write

$$\|F\|^p_{H^p_*} = \sup_{s\in\mathbf{R}} \int_{\overline{N}} \mathbf{E}_{(n,s)}[(F^*)^p]\,dn.$$

The aim of the present work is to prove the following result that asserts the equivalence of the two H^p-spaces defined.

THEOREM I. *Let F be a harmonic function in $\mathbf{X}$ and $0 < p < \infty$.*

(1) *There exists a constant C, independent of p, such that*

$$\|F\|^p_{H^p_*} \le C\|F\|^p_{H^p_N}.$$

(2) *There exists a constant C_p such that*

$$\|F\|^p_{H^p_N} \le C_p\|F\|^p_{H^p_*}.$$

2. Brownian motion in X. The Brownian motion of the symmetric space is the diffusion generated in $\mathbf{X}$ by the Laplace-Beltrami operator. It can be considered as a family of probability distributions $\mathbf{P}_x$ defined on the set Ω of all continuous paths in $\mathbf{X}$. For each x, $\mathbf{P}_x$ is supported on the set of paths in Ω that start at x. The density function for this probability, $p(t,x,y)$, is given by the fundamental solution of the heat equation, $du/dt = \frac{1}{2}\Delta u$, with pole at x (cf. McKean [**16**, pp. 90–91]). This diffusion is then invariant under the action of the group G.

A positive (possibly infinite) real function τ on Ω is called a *stopping time* if the event $\{\tau < t\}$ is measurable with respect to the family $\{\mathbf{x}_s : s \le t\}$ for every $t > 0$. An example of a stopping time that we will use below is the first hitting time of the closure of a set B, $\tau = \inf\{t : \mathbf{x}_t \in B\}$. The diffusion in $\mathbf{X}$ verifies the strong Markov property: if τ is a stopping time of $\mathbf{x}_t$ then, conditional on $\tau < \infty$ and $\mathbf{x}_\tau = x$, the future, $\mathbf{x}_{\tau+t}$, is independent of the past, $\mathbf{x}_s$, $s < \tau$, and is identical in law to the motion starting at x (cf. McKean [**16**, p. 90]).

Debiard, Gaveau and Mazet [**6**] have proved that the lifetime of this diffusion is almost surely infinite. Malliavin and Malliavin [**15**, pp. 196–197] have obtained limit laws and in particular proved that if we write $\mathbf{x}_t = (n_t, s_t)$, then, as $t \to \infty$, $s_t \to \infty$ and n_t converges to a limit n_∞ that we will also denote as $\mathbf{x}_\infty$. The following lemma relates the limit distribution of the Brownian motion and the Poisson kernel.

LEMMA 1. *The limit distribution of* $\mathbf{x}_\infty$, $\mathbf{P}_{(n,s)}(\mathbf{x}_\infty \in U)$, $U \subset \overline{N}$, *defines a measure on* $\overline{N}$ *(harmonic measure) whose density, relative to the Haar measure, is the Poisson kernel.*

PROOF. We will see first that $\mathbf{P}_{(n,s)}(\mathbf{x}_\infty \in U)$ is a harmonic function in $(n,s) \in \mathbf{X}$. Let $(n,s) = g \cdot \mathbf{o}$, and $\partial B_x = \{b = gk \cdot x \in \mathbf{X}: k \in K\}$ for $g \in G$ and $x \in \mathbf{X}$ fixed. Let τ be the first hitting time of ∂B_x. Then $\tau < \infty$, a.s. and by the strong Markov property we have

$$\mathbf{P}_{g\cdot\mathbf{o}}(\mathbf{x}_\infty \in U) = \int_{\partial B_x} \mathbf{P}_{g\cdot\mathbf{o}}(\mathbf{x}_\tau \in db)\mathbf{P}_b(\mathbf{x}_\infty \in U),$$

where db represents the measure on ∂B_x induced by dk. Invariance under rotations (i.e. under action of the group K) gives $\mathbf{P}_{g\cdot\mathbf{o}}(\mathbf{x}_\tau \in db) = db$ and therefore

$$\mathbf{P}_{g\cdot\mathbf{o}}(\mathbf{x}_\infty \in U) = \int_K \mathbf{P}_{gk\cdot x}(\mathbf{x}_\infty \in U)\, dk,$$

that is, the mean value property. To finish the proof we show that for every open set $U \subset \overline{N}$,

$$\lim_{s\to\infty} \mathbf{P}_{(n,s)}(\mathbf{x}_\infty \in U) = \begin{cases} 1 & \text{if } n \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Without loss of generality take $U = b(r)$ and $n = e$. For $s' > 0$

$$\mathbf{P}_{(e,s)}(\mathbf{x}_\infty \in b(r)) = \mathbf{P}_{(e,s+s')}(\mathbf{x}_\infty \in b(r\mathbf{e}^{-s'})) \leq \mathbf{P}_{(e,s+s')}(\mathbf{x}_\infty \in b(r)).$$

If $\lim_{s\to\infty} \mathbf{P}_{(e,s)}(\mathbf{x}_\infty \in b(r)) = \varepsilon < 1$ then, for $s > 1$,

$$1 > \varepsilon \geq \mathbf{P}_{(e,s)}(\mathbf{x}_\infty \in b(r)) = \mathbf{P}_{(e,0)}(\mathbf{x}_\infty \in b(r\mathbf{e}^s)).$$

But, because $(n_t, s_t) \to \mathbf{x}_\infty \in \overline{N}$ a.s., the last term converges to 1 as $s \to \infty$.

After this result we can write

$$\mathbf{P}_{(n,s)}(\mathbf{x}_\infty \in du) = P_s(u^{-1}n)\, du.$$

Given $u \in \overline{N}$ we can define the conditional probability $\mathbf{P}^u_{(n,s)}(U)$ as the Radon-Nikodym derivative

$$\frac{\mathbf{P}_{(n,s)}(U, \mathbf{x}_\infty \in du)}{\mathbf{P}_{(n,s)}(\mathbf{x}_\infty \in du)}.$$

It gives a measure on the paths in Ω that start at (n,s) and end at u and is invariant under the action of the subgroup of G that fixes the point u, namely uAu^{-1}.

The Green function $g(x,y)$ for the operator $\frac{1}{2}\Delta$ can be written, in terms of the fundamental solution of the corresponding heat equation, as

$$g(x,y) = \int_0^\infty p(t,x,y)\, dt.$$

It can be interpreted in probabilistic terms, as the average time spent at the point y by the motion starting at x.

3. $H^p_N \subset \dot{H}^p_*$. Our first lemma gives the independence of α in the definition of H^p_N. It is a generalization of Lemma 2.1, p. 330 in Debiard [**5**].

LEMMA 2. *Given $\beta \geq \alpha \geq 0$ there exists a constant c depending only on α, β and m, such that for every measurable function F the inequality*

$$|\{N_\beta F > \lambda\}| \leq c|\{N_\alpha F > \lambda\}|$$

holds for every $\lambda > 0$.

PROOF. Fix $\lambda > 0$ and let ϕ be the characteristic function of the set $\{N_\alpha F > \lambda\}$. Let ϕ_* be its Hardy-Littlewood maximal function defined as

$$\phi_*(u) = \sup_{r>0} \left\{ \frac{1}{|b(r)|} \int_{u \cdot b(r)} |\phi(n)|\, dn \right\}.$$

We claim that if $\mu = (\alpha/(\alpha+\beta))^m$, then $\{N_\beta F > \lambda\} \subset \{\phi_* > \mu\}$. Let $n \in \{N_\beta F > \lambda\}$. Then there exists $(n', s') \in n \cdot \Gamma_\beta$ such that $|F(n', s')| > \lambda$. Observe that $n' \cdot b(\alpha \mathbf{e}^{-s'}) \subset n \cdot b((\alpha+\beta)\mathbf{e}^{-s})$ and that $n' \cdot b(\alpha \mathbf{e}^{-s'}) \subset \{N_\alpha F > \lambda\}$. Therefore we can write

$$\phi_*(n') \geq \frac{1}{|b(\alpha+\beta)\mathbf{e}^{-s'})|} \int_{n' \cdot b(\alpha \mathbf{e}^{-s'})} \phi(u)\, du = \frac{|b(\alpha \mathbf{e}^{-s'})|}{|b((\alpha+\beta)\mathbf{e}^{-s'})|} = \left(\frac{\alpha}{\alpha+\beta}\right)^m$$

as claimed. The (1,1) weak type inequality for ϕ_* (cf. Korányi-Vági [**14**, p. 580]) gives now

$$|\{N_\beta F > \lambda\}| \leq |\{\phi_* \geq \mu\}| \leq \frac{C}{\mu}\|\phi\|_1 = \frac{C}{\mu}|\{N_\alpha F > \lambda\}|$$

as desired.

In order to prove the first part of Theorem 1 we only need to assume that the function F is measurable. Theorem 1, part (1) is an easy consequence of the following proposition.

PROPOSITION 3. *There exists a constant c, depending only on α, such that for any measurable function F, the inequality*

$$\sup_{s \in \mathbf{R}} \int_{\overline{N}} \mathbf{P}_{(n,s)}(F^* > \lambda)\, dn \leq c|\{N_\alpha F > \lambda\}|$$

holds for every $\lambda > 0$.

Let us see that integrating both sides of this inequality against an appropriate measure we obtain part (1) of Theorem I.

$$\begin{aligned}
\|F\|^p_{H^p_N} &= \int_{\overline{N}} (N_\alpha F(n))^p\, dn \\
&= \int_0^\infty p\lambda^{p-1}|\{N_\alpha F > \lambda\}|\, d\lambda \\
&\geq \frac{1}{c}\int_0^\infty p\lambda^{p-1} \sup_{s\in\mathbf{R}} \int_{\overline{N}} \mathbf{P}_{(n,s)}(F^* > \lambda)\, dn\, d\lambda \\
&\geq \frac{1}{c}\int_{\overline{N}}\int_0^\infty p\lambda^{p-1}\mathbf{P}_{(n,s)}(F^* > \lambda)\, d\lambda\, dn \\
&\geq \frac{1}{c}\int_{\overline{N}} \mathbf{E}_{(n,s)}[(F^*)^p]\, dn.
\end{aligned}$$

Therefore,

$$\|F\|^p_{H^p_N} \geq \frac{1}{c} \sup_{s\in\mathbf{R}} \int_{\overline{N}} \mathbf{E}_{(n,s)}[(F^*)^p] = \frac{1}{c}\|F\|^p_{H^p_*}.$$

To prove Proposition 3 we need a sequence of lemmas for which the following notation is given. Let F be a measurable function. For $\alpha > 0$ and $\lambda > 0$ we write $A_\lambda = \{N_\alpha F > \lambda\}$, $B_\lambda = \bigcup_{n\in A^c_\lambda} n \cdot \Gamma_\alpha$, where A^c_λ is the complement of A_λ in $\overline{N}$. The set ∂B_λ will be the topological boundary of B_λ in X, and B^c_λ will be $X - B_\lambda$. The characteristic function of a set U is denoted as χ_U. For the remainder of this section $\alpha > 0$ and $\lambda > 0$ are fixed.

LEMMA 4. *If $|A_\lambda| < \infty$, then*

$$\sigma = \inf\{s \in \mathbf{R}\colon \exists n \in \overline{N},\ (n,s) \in B^c_\lambda\} > -\infty.$$

PROOF. If $(n,s) \in B^c_\lambda$, then the inclusion $\{n' \in \overline{N}\colon |n^{-1}n'| < \alpha\mathbf{e}^{-s}\} \subset A_\lambda$ holds. Taking measures on both sides we obtain $\infty > |A_\lambda| > c\alpha^m\mathbf{e}^{-ms}$, and therefore

$$s > \frac{1}{m}\log\left(\frac{c\alpha^m}{|A_\lambda|}\right).$$

LEMMA 5. (1) *Let $\tau'_s = \inf\{t\colon s_t = s'\}$ and n_{τ_s} the $\overline{N}$-coordinate of $\mathbf{x}_{\tau_s}$. Then, for $s < s'$, the measure $\nu(du) = \int_{\overline{N}} \mathbf{P}_{(n,s)}(n_{\tau_{s'}} \in du)\,dn$ is Haar measure over $\overline{N}$.*

(2) *For any $s \in \mathbf{R}$ the measure $\nu(du) = \int_{\overline{N}} \mathbf{P}_{(n,s)}(\mathbf{x}_\infty \in du)\,dn$ is Haar measure over $\overline{N}$.*

PROOF. (1) Let U be a finite measurable subset of $\overline{N}$ with finite measure. Then

$$\nu(U) = \int_{\overline{N}}\int_{\overline{N}} \mathbf{P}_{(n,s)}(n_{\tau_{s'}} \in du)\chi_U(u)\,dn.$$

The invariance of the diffusion under the action of $\overline{N}$ gives

$$\begin{aligned}\nu(U) &= \int_{\overline{N}}\int_{\overline{N}} \mathbf{P}_{(e,s)}(n_{\tau_{s'}} \in du)\chi_U(nu)\,dn \\ &= \int_{\overline{N}}\int_{\overline{N}} \mathbf{P}_{(e,s)}(n_{\tau_{s'}} \in du)\chi_{Uu^{-1}}(n)\,dn \\ &= |U|\int_{\overline{N}} \mathbf{P}_{(e,s)}(n_{\tau_{s'}} \in du) = |U|.\end{aligned}$$

The proof of (2) follows exactly the same pattern. We only need to replace $n_{\tau_{s'}}$ for $\mathbf{x}_\infty$ all along the proof.

Our last lemma of this series gives a lower bound for the probability of the diffusion that starts on ∂B_λ and hits the boundary at the set A_λ. This lower bound allows us to estimate, in the proof of Proposition 3, the probability of hitting A_λ by estimating the probability of hitting ∂B_λ.

LEMMA 6. *There exists a strictly positive constant $C(\alpha)$ such that*

$$\inf_{x\in\partial B_\lambda} \mathbf{P}_x(\mathbf{x}_\infty \in A_\lambda) \geq C(\alpha).$$

PROOF. Let $x = (n,s) \in \partial B_\lambda$. Then $n \cdot b(\alpha \mathbf{e}^{-s}) \subset A_\lambda$ and by the invariance of the diffusion under the action of the group G,

$$\begin{aligned}\mathbf{P}_{(n,s)}(\mathbf{x}_\infty \in A_\lambda) &\geq \mathbf{P}_{(n,s)}(\mathbf{x}_\infty \in n \cdot b(\alpha \mathbf{e}^{-s})) \\ &= \mathbf{P}_{(e,s)}(\mathbf{x}_\infty \in b(\alpha \mathbf{e}^{-s})) \\ &= \mathbf{P}_{(e,0)}(\mathbf{x}_\infty \in b(\alpha)) = C(\alpha) > 0.\end{aligned}$$

Observe that $C(\alpha)$ is strictly positive: as a function of x, $\mathbf{P}_x(\mathbf{x}_\infty \in b(\alpha))$ is harmonic and bounded by 0 and 1, and therefore strictly positive in $\mathbf{X}$.

We have now the necessary elements to prove Proposition 3.

PROOF OF PROPOSITION 3. By the definition of B_λ, if $(n,s) \in B_\lambda$ then $|F(n,s)| \leq \lambda$. Thus, if $F^*(\omega) > \lambda$, the path ω hits (the closure of) B_λ^c and therefore hits its boundary ∂B_λ. More formally, if $\tau(\omega) = \inf\{t : \mathbf{x}_t \in \partial B_\lambda\}$ and $(n,s) \in B_\lambda$, then

$$\{\mathbf{x}_0 = (n,s),\ F^* > \lambda\} \subset \{\mathbf{x}_0 = (n,s),\ \tau < \infty\}.$$

This implies $\mathbf{P}_{(n,s)}(F^* > \lambda) \leq \mathbf{P}_{(n,s)}(\tau < \infty)$ for $(n,s) \in B_\lambda$. If $|A_\lambda| < \infty$, let σ be as in Lemma 4. Then for $s < \sigma$ and $n \in \overline{N}$ the point (n,s) is in B_λ, and therefore Lemma 5(2), the strong Markov property, Lemma 6 and the preceding argument give

$$\begin{aligned}|A_\lambda| &= \int_{\overline{N}} \mathbf{P}_{(n,s)}(\mathbf{x}_\infty \in A_\lambda)\,dn = \int_{\overline{N}} \mathbf{E}_{(n,s)}[\chi_{\{\tau<\infty\}} \mathbf{P}_{\mathbf{x}_\tau}(\mathbf{x}_\infty \in A_\lambda)]\,dn \\ &\geq C(\alpha) \int_{\overline{N}} \mathbf{P}_{(n,s)}(\tau < \infty)\,dn \geq C(\alpha) \int_{\overline{N}} \mathbf{P}_{(n,s)}(F^* > \lambda)\,dn\end{aligned}$$

which is the desired result for $s < \sigma$. To obtain the proof for every s it will be enough to show that the function $\int_{\overline{N}} \mathbf{P}_{(n,s)}(F^* > \lambda)\,dn$ is decreasing in s. Let $\tau_r = \inf\{t : s_t = r\}$ and for $\tau_r < \infty$ define

$$(F^{\tau_r})^*(\omega) = \sup_{t > \tau_r} |F(\mathbf{x}_t(\omega))|.$$

Suppose $s < r$. Then

$$\mathbf{P}_{(n,s)}(F^* > \lambda) \geq \mathbf{P}_{(n,s)}((F^{\tau_r})^* > \lambda)$$

for every $n \in \overline{N}$ (observe that when $s < r$, $\mathbf{P}_{(n,s)}(\tau_r < \infty) = 1$). This inequality, the strong Markov property and Lemma 5(1) give

$$\begin{aligned}\int_{\overline{N}} \mathbf{P}_{(n,s)}(F^* > \lambda)\,dn &\geq \int_{\overline{N}} \mathbf{P}_{(n,s)}((F^{\tau_r})^* > \lambda)\,dn \\ &\geq \int_{\overline{N}} \int_{\overline{N}} \mathbf{P}_{(n,s)}(n_{\tau_r} \in du) \mathbf{P}_{(u,r)}((F^{\tau_r})^* > \lambda)\,dn \\ &= \int_{\overline{N}} \int_{\overline{N}} \mathbf{P}_{(n,s)}(n_{\tau_r} \in du) \mathbf{P}_{(u,r)}(F^* > \lambda)\,dn \\ &= \int_{\overline{N}} \mathbf{P}_{(u,r)}(F^* > \lambda)\,du\end{aligned}$$

for every $s < r$, as desired.

4. $H^p_* \subset H^p_N$. In this section we will prove the second part of Theorem I. First we need to prove several results that will be needed in the main proof.

LEMMA 7. *For every p, $0 < p < \infty$, there exists a constant C such that for every harmonic function F and any $x \in X$ the inequality*

$$\sup_{y \in B_\sigma(x)} |F(y)|^p \leq C \int_{B_{\sigma+1}(x)} |F(y)|^p \, dy$$

holds for every $\sigma > 0$.

PROOF. If $1 \leq p < \infty$ then the maximum principle for harmonic functions and Jensen's Theorem for the function $|\cdot|^p$ give

$$\begin{aligned} \sup_{y \in B_r(x)} |F(y)|^p &\leq \sup_{y \in \partial B_r(x)} |F(y)|^p \\ &\leq \sup_{x' \in \partial B_r(x)} \frac{1}{|B_{R-r}|} \int_{B_{R-r}(x')} |F(y)|^p \, dy \\ &\leq \frac{1}{|B_{R-r}|} \int_{B_R(x)} |F(y)|^p \, dy. \end{aligned} \tag{4.1}$$

For $r = \sigma$ and $R = \sigma + 1$ we obtain the desired result. To extend it to $0 < p < 1$ we use the following technique due to De Giorgi [**7**]: Let $\lambda = 1 - p/2$. From inequality (4.1) with $p = 2$ we can obtain

$$\sup_{y \in B_r(x)} |F(y)| \leq |B_{R-r}|^{-1/2} \sup_{y \in B_R(x)} |F(y)|^\lambda \left(\int_{B_R(x)} |F(y)|^p \, dy \right)^{1/2}. \tag{4.2}$$

Let $\sigma_{i+1} = \sigma_i + (\frac{1}{2})^{i+1}$ and $\sigma_0 = \sigma$. For $r = \sigma_i$ and $R = \sigma_{i+1}$ inequality (4.2) takes the form

$$\sup_{y \in B_{\sigma_i}(x)} |F(y)| \leq |B_{(1/2)^{i+1}}|^{-1/2} \left(\sup_{y \in B_{\sigma_{i+1}}(x)} |F(y)|^\lambda \right) \left(\int_{B_{\sigma_{i+1}}(x)} |F(y)|^p \, dy \right)^{1/2}.$$

Let $I = (\int_{B_{\sigma_{i+1}}(x)} |F(y)|^p \, dy)^{1/p}$ and $\overline{F}(x) = |F(x)|/I$. Then

$$\sup_{y \in B_{\sigma_i}(x)} \overline{F}(y) \leq |B_{(1/2)^{i+1}}|^{-1/2} \left(\sup_{y \in B_{\sigma_{i+1}}(x)} \overline{F}(y) \right)^\lambda.$$

Iterating we obtain

$$\sup_{y \in B_\sigma(x)} \overline{F}(y) \leq |B_{1/2}|^{-1/2} |B_{(1/2)^2}|^{-\lambda/2} \cdots |B_{(1/2)^{i+1}}|^{-\lambda^i/2} \left(\sup_{y \in B_{\sigma_{i+1}}(x)} \overline{F}(y) \right)^{\lambda^{i+1}}.$$

The volume of the geodesic ball of radius R is given by

$$|B_R| = C_0 (\sinh R)^{m_1} (\sinh 2R)^{m_2} \geq C_1 R^{m_1 + m_2}$$

(cf. Korányi-Taylor [**13**, p. 172]). Then

$$\begin{aligned} &|B_{1/2}|^{-1/2} |B_{(1/2)^2}|^{-\lambda/2} \cdots |B_{(1/2)^{i+1}}|^{-\lambda^i/2} \\ &\quad \leq C_1^{-\frac{1}{2} \sum_{k=0}^i \lambda^k} \left(2^{\frac{1}{2} \sum_{k=0}^i (k+1) \lambda^k} \right)^{m_1 + m_2}, \end{aligned}$$

and this last term approaches $C_1^{-1/2(1-\lambda)}2^{(m_1+m_2)/2(1-\lambda)^2}$ when $i \to \infty$. The fact that $\overline{F}$ is bounded in $B_{\sigma+1}(x)$ implies that $\lim_{i\to\infty}(\sup_{y\in B_{\sigma_{i+1}}(x)} \overline{F}(y))^{\lambda^{i+1}} = 1$. Then we obtain $\sup_{y\in B_\sigma(x)} \overline{F}(y) \le C$ and therefore the claimed result follows.

Our next result asserts that under certain conditions the maximum of a harmonic function in a truncated domain does not grow very fast relative to the size of the truncation.

PROPOSITION 8. *Let F be a harmonic function satisfying*

$$\sup_{s\in\mathbf{R}} \int_{\overline{N}} |F(n,s)|^p \, dn < \infty.$$

Then, for any p, $0 < p < \infty$, any $\alpha > 0$ and any $\sigma > 0$,

$$\|N_\alpha^{(\sigma)} F\|_p < \infty.$$

Moreover, there exists $\sigma_0 > 0$ (that may depend on F) such that for any $\sigma > \sigma_0$ the inequality

$$\|N_{2\alpha}^{(\sigma+1)} F\|_p \le C \|N_\alpha^{(\sigma)} F\|_p$$

holds for a constant C depending only on α and p.

PROOF. If $\lim_{\sigma\to\infty} \|N_\alpha^{(\sigma)} F\|_p < \infty$ for some α, then Lemma 2 implies that the same holds for every α. Suppose that for every $\alpha > 0$, $\lim_{\sigma\to\infty} \|N_\alpha^{(\sigma)} F\|_p = \infty$. Fix $\alpha = 1$. Let $r > 0$ be such that $\{(n,0) \in \Gamma_1\} \subset B_r(\mathbf{o})$. If $\overline{\sigma} > \sigma + r$, then $\Gamma_1^{(\sigma)} \subset B_{\overline{\sigma}}(\mathbf{o}) \subset B_{\overline{\sigma}+1}(\mathbf{o})$. By Lemma 7,

$$|N_1^{(\sigma)} F(u)|^p \le C \int_{u\cdot B_{\overline{\sigma}+1}} |F(x)|^p \, dx.$$

Integrating over $u \in \overline{N}$ and using Fubini's Theorem we obtain

$$\begin{aligned} \int_{\overline{N}} |N_1^{(\sigma)} F(u)|^p \, du &\le C \int_{B_{\overline{\sigma}+1}} \int_{\overline{N}} |F(u\cdot x)|^p \, du \, dx \\ &\le C |B_{\overline{\sigma}+1}| \sup_{s\in\mathbf{R}} \int_{\overline{N}} |F(u,s)|^p \, du \end{aligned}$$

and the first part of the proposition follows.

To prove the second part suppose, as before, that $\lim_{\sigma\to\infty} \|N_\alpha^{(\sigma)} F\|_p = \infty$ for any $\alpha > 0$. Then there exists $\sigma_0 > 0$ (that may depend on F) such that for any $\sigma > \sigma_0$

$$\|N_\alpha^{(\sigma)} F\|_p^p \ge \sup_{s\in\mathbf{R}} \int_{\overline{N}} |F(n,s)|^p \, dn.$$

Let $M_\alpha^{(\sigma)} F(u) = \sup\{|F(x)| : x \in u \cdot (\Gamma_\alpha^{(\sigma+1)} - \Gamma_\alpha^{(\sigma)})\}$. Observe that there exists $r > 0$, depending only on α, such that $\Gamma_\alpha^{(\sigma+1)} - \Gamma_\alpha^{(\sigma)} \subset B_r((e,\sigma)) \cup B_r((e,-\sigma))$, and therefore

$$M_\alpha^{(\sigma)} F(u) \le \sup\{|F(x)| : x \in u \cdot (B_r((e,\sigma)) \cup B_r((e,-\sigma)))\}.$$

Lemma 7 and Fubini's Theorem now give

$$\begin{aligned}\int_{\overline{N}} |M_\alpha^{(\sigma)}F(u)|^p\,du &\le C\int_{B_{r+1}((e,\sigma))}\int_{\overline{N}} |F(u\cdot x)|^p\,du\,dx\\ &\quad + C\int_{B_{r+1}((e,-\sigma))}\int_{\overline{N}} |F(u\cdot x)|^p\,du\,dx\\ &\le 2C|B_{r+1}|\sup_{s\in\mathbf{R}}\int_{\overline{N}} |F(n,s)|^p\,dn.\end{aligned}$$

Considering that $|N_\alpha^{(\sigma+1)}F(u)|^p \le |N_\alpha^{(\sigma)}F(u)|^p + |M_\alpha^{(\sigma)}F(u)|^p$ we obtain, for $\sigma > \sigma_0$,

$$\|N_\alpha^{(\sigma+1)}F\|_p^p \le \|N_\alpha^{(\sigma)}F\|_p^p + C'\sup_{s\in\mathbf{R}}\int_{\overline{N}} |F(n,s)|^p\,dn \le (C'+1)\|N_\alpha^{(\sigma)}F\|_p^p$$

where $C' = 2C|B_{r+1}|$ is independent of σ.

Our next aim is to prove a Harnack-type inequality (Proposition 11). Working in that direction we prove the following two lemmas.

LEMMA 9. *Given $\alpha > \alpha' > 0$ there exists a constant C such that for every harmonic function F and every $n \in \overline{N}$,*

$$\sup_{x\in n\cdot\Gamma_\alpha} \|\nabla F(x)\| \le C \sup_{x\in n\cdot\Gamma_\alpha} |F(x)|.$$

PROOF. Without loss of generality take $n = e$. Choose $r > 0$ such that if $g\cdot\mathbf{o} \in \Gamma_{\alpha'}$ then $g\cdot B_r \subset \Gamma_\alpha$. Schauder interior estimates (cf. Courant-Hilbert [**3**, p. 335]) provide a constant C, depending only on r, such that

$$\|\nabla F(\mathbf{o})\| < C\sup_{x\in B_r} |F(x)|.$$

Let $x \in \Gamma_{\alpha'}$. There exists an isometry $g \in G$ such that $g\cdot\mathbf{o} = x$. The function F_g defined as $F_g(y) = F(g\cdot y)$ is also harmonic; therefore,

$$\|\nabla F(x)\| = \|\nabla F_g(\mathbf{o})\| \le C\sup_{y\in B_r}|F_g(y)| \le C\sup_{y\in\Gamma_\alpha}|F(y)|.$$

LEMMA 10. *Given r_1 and r_2 $(0 < r_1 < r_2)$, there exists a constant C such that for any two concentric balls B_1 and B_2 in $\mathbf{X}$ with radii r_1 and r_2 respectively, any harmonic function F and any two points $x, x' \in B_1$ we have*

$$|F(x) - F(x')| \le C\,\mathrm{dist}(x,x')\sup_{y\in B_2}|F(y)|,$$

where $\mathrm{dist}(x,x')$ *stands for the geodesic distance from x to x'.*

PROOF. Let $\gamma(s)$ be a geodesic in $\mathbf{X}$ joining x to x'. The Schwarz inequality and the estimates for the gradient given in Lemma 9 give

$$\begin{aligned}|F(x) - F(x')| &= \left|\int_0^{\mathrm{dist}(x,x')} \langle\gamma'(s)|\nabla F(\gamma'(s))\rangle\,ds\right|\\ &\le C\sup_{y\in B_2}|F(y)|\int_0^{\mathrm{dist}(x,x')}|\gamma'(s)|\,ds\\ &\le C\sup_{y\in B_2}|F(y)|\,\mathrm{dist}(x,x').\end{aligned}$$

PROPOSITION 11. *Fix $\alpha > 0$. Given c there exists $r_0 > 0$ such that for any harmonic function F, any $\sigma > 0$ and any $x_0 \in u \cdot \Gamma_\alpha^{(\sigma)}$,*

$$|F(x) - F(x_0)| \leq cN_{2\alpha}^{(\sigma+1)}F(u) \qquad (x \in B_{r_0}(x_0)).$$

PROOF. There exists $r > 0$ (depending only on α) such that for any $x_0 \in u\cdot\Gamma_\alpha^{(\sigma)}$, the ball $B_r(x_0)$ is contained in $u\cdot\Gamma_{2\alpha}^{(\sigma+1)}$. Using Lemma 10 with $r_1 = \frac{1}{2}r$ and $r_2 = r$ we obtain

$$\begin{aligned}|F(x) - F(x_0)| &\leq C \operatorname{dist}(x, x_0) \sup_{y\in B_r(x_0)} |F(y)| \\ &\leq C \operatorname{dist}(x, x_0) N_{2\alpha}^{(\sigma+1)} F(u) \qquad (x \in B_{r_1}(x_0)).\end{aligned}$$

Take $r_0 = \inf\{c/C, r_1\}$ to obtain the desired result.

Our next result gives a lower bound for the probability (relative to the diffusion conditioned to finish at a point on $\overline{N}$) of hitting a set of a given size located inside an admissible region. Such a lower bound will be independent of the ending point, depending only on the sizes of the admissible region and the set we try to hit.

PROPOSITION 12. *Given $\alpha > 0$ and $r > 0$ there exist $\sigma_1 > 0$ and $C_1 > 0$ such that for any $u \in \overline{N}$, any $s_1 \in \mathbf{R}$, any $x = (n, s) \in u \cdot \Gamma_\alpha$ with $s < -(\sigma_1 - s_1)$ and any $x_0 = (n_0, s_0) \in u \cdot \Gamma_\alpha$ with $s_0 > s_1$ we have that*

$$\mathbf{P}^u_{(n,s)}(\mathbf{x}_t \text{ hitting } B_r(x_0)) \geq C_1.$$

PROOF. It is enough to prove the result for $s_1 = 0$, because the invariance under the action of the subgroup uAu^{-1} will give afterwards the general result. Let $u \in \overline{N}$ and take r small enough such that if $x_0 \in u \cdot \Gamma_\alpha$ then $B = B_r(x_0) \subset u \cdot \Gamma_{2\alpha}$. This r is independent of u and α, depending only on the coordinate s_0 of the point x_0. Denote by τ_B the first hitting time of B. We want to obtain a lower bound for

$$\mathbf{P}^u_{(n,s)}(\tau_B < \infty) = \frac{\mathbf{P}_{(n,s)}(\tau_B < \infty,\ \mathbf{x}_\infty \in du)}{\mathbf{P}_{(n,s)}(\mathbf{x}_\infty \in du)}.$$

To evaluate the numerator we use the strong Markov property,

$$\mathbf{P}_{(n,s)}(\tau_B < \infty,\ \mathbf{x}_\infty \in du) = \int_{\partial B} \mathbf{P}_{(n,s)}(\mathbf{x}_{\tau_B} \in db,\ \tau_B < \infty)\mathbf{P}_b(\mathbf{x}_\infty \in du),$$

and then obtain a lower bound for $\mathbf{P}_b(\mathbf{x}_\infty \in du)$ when $b \in \partial B$. To evaluate the denominator we will obtain an upper bound for $\mathbf{P}_{(n,s)}(\mathbf{x}_\infty \in du)$ when $(n, s) \in u\cdot\Gamma_\alpha$ and $s < -\sigma_1$. To obtain these estimates we may assume $u = e$. Let $\overline{B}$ be a closed ball with center $\mathbf{o}$ containing $\{(n, 0) : |n| < 2\alpha\}$. Then it will also contain $\exp(-s_0 H) \cdot B$. The Poisson kernel $P(n, s)$ has lower and upper bounds in the compact $\overline{B}$: $0 < l \leq P(n, s) \leq L < \infty$ for $(n, s) \in \overline{B}$. Therefore, if $x_1 = (n_1, s_1) \in B$ the point $(n, s) = \exp(s_0 H)\cdot x_1$ is in $\overline{B}$. We obtain $P(n, s) = P_s(n) = P_{s_0+s_1}(n_1^{s_0}) = P_{s_1}(n_1)\mathbf{e}^{ms_0}$, and then $l\mathbf{e}^{-ms_0} \leq P(x_1) \leq L\mathbf{e}^{-ms_0}$. These estimates provide the following inequalities:

$$\left.\frac{\mathbf{P}_b(\mathbf{x}_\infty \in du)}{du}\right|_{u=e} \geq l\mathbf{e}^{ms_0}$$

and

$$\left.\frac{\mathbf{P}_x(\mathbf{x}_\infty \in du)}{du}\right|_{u=e} \leq L\mathbf{e}^{ms}$$

for $x = (n, s) \in \Gamma_\alpha$. Both together give

$$\mathbf{P}_x^u(\tau_B < \infty) \geq \frac{l}{L}\mathbf{e}^{m(s_0-s)}\mathbf{P}_x(\tau_B < \infty).$$

To evaluate $\mathbf{P}_x(\tau_B < \infty)$ we use the Green function of the symmetric space

$$g(x, y) = \int_0^\infty p(t, x, y)\, dt.$$

Let $g(x, B) = \int_B g(x, y)\, dy$. Then

$$g(x, B) = \int_B \int_0^\infty p(t, x, y)\, dt\, dy = \int_0^\infty p_t(x, B)\, dt,$$

where $p_t(x, B) = \int_B p(t, x, y)\, dy$. The strong Markov property gives

$$p_t(x, B) = \mathbf{P}_x(\mathbf{x}_t \in B, \tau_B < \infty) = \mathbf{E}_x\left|\int_B p_{t-\tau_B}(\mathbf{x}_{\tau_B}, y)\, dy, \tau_B < t\right|.$$

Then

$$\begin{aligned}
\int_0^\infty p_t(x, B)\, dt &= \mathbf{E}_x\left|\int_0^\infty \chi_{\{\tau_B < t\}} \int_B p_{t-\tau_B}(\mathbf{x}_{\tau_B}, y)\, dy\, dt\right| \\
&= \mathbf{E}_x\left|\int_B \int_0^\infty \chi_{\{\tau_B < t\}} p_{t-\tau_B}(\mathbf{x}_{\tau_B}, y)\, dt\, dy\right| \\
&= \mathbf{E}_x\left|\int_B \chi_{\{\tau_B < \infty\}} \int_{\tau_B}^\infty p_{t-\tau_B}(\mathbf{x}_{\tau_B}, y)\, dt\, dy\right| \\
&= \mathbf{E}_x\left|\int_B g(\mathbf{x}_{\tau_B}, y)\, dy,\ \tau_B < \infty\right| \\
&= \mathbf{E}_x\left|g(\mathbf{x}_{\tau_B}, B),\ \tau_B < \infty\right|.
\end{aligned}$$

By rotation invariance the function $g(b, B)$ is constant for $b \in \partial B$, depending only on r; therefore $g(x, B) = c\mathbf{P}_x(\tau_B < \infty)$. The asymptotic behavior of $g(x, y)$ (cf. Korányi-Taylor [**13**, Proposition 1.2]) implies that $g(x, B) \approx c\mathbf{e}^{-md}$ as $d \to \infty$, where d is the geodesic distance from x to the center of B. When $s_0 - s$ is large and the points x and x_0 are inside $u \cdot \Gamma_\alpha$ this distance is of the order of $s_0 - s$, and then

$$\mathbf{P}_x^u(\tau_B < \infty) \geq c\frac{l}{L}\mathbf{e}^{m(s_0-s)}\mathbf{e}^{m(s-s_0)} = C_1.$$

PROOF OF THEOREM I(2). Let $F \in H_*^p$. Observe that

$$\|F\|_{H_*^p}^p = \sup_{s\in\mathbf{R}} \int_{\overline{N}} \mathbf{E}_{(n,s)}[(F^*)^p]\, dn < \infty$$

implies

$$\sup_{s\in\mathbf{R}} \int_{\overline{N}} |F(n, s)|^p\, dn < \infty.$$

Then Proposition 8 implies that, for any $\sigma > 0$, $\|N_\alpha^{(\sigma)}F\|_p < \infty$, and that, for $\sigma > \sigma_0$ (where σ_0 may depend on F), $\|N_{2\alpha}^{(\sigma+1)}F\|_p \leq C_0\|N_\alpha^{(\sigma)}F\|_p$. The constant C_0, which we will use below, depends only on p and m. We will prove that

$$\|N_\alpha^{(\sigma)}F\|_p^p = \int_{\overline{N}} (N_\alpha^{(\sigma)}F(n))^p\, dn \leq C_3 \sup_{s\in\mathbf{R}} \int_{\overline{N}} \mathbf{E}_{(n,s)}[(F^*)^p]\, dn$$

with the constant C_3 independent of σ or F. The result we want to prove will follow from here by Lebesgue's monotone convergence theorem. Let

$$U = \left\{u \in \overline{N}: N_\alpha^{(\sigma)}F(u) \geq (1/2C_0)^{1/p} N_{2\alpha}^{(\sigma+1)}F(u)\right\}.$$

A simple calculation (cf. Burkholder-Gundy [**1**, Lemma 5]) gives

$$\int_{\overline{N}} (N_\alpha^{(\sigma)}F(u))^p\,du \leq 2\int_U (N_\alpha^{(\sigma)}F(u))^p\,du,$$

and then we can find $\varepsilon > 0$ such that

$$\int_{\overline{N}} (N_\alpha^{(\sigma)}F(u))^p\,du \leq 2C_0 \int_{U_\varepsilon} (N_\alpha^{(\sigma)}F(u))^p\,du \tag{4.3}$$

where $U_\varepsilon = U \cap b(\varepsilon)$. For every $u \in \overline{N}$ take an $x_0 = (n_0, s_0) \in u\cdot\Gamma_\alpha^{(\sigma)}$ such that $|F(x_0)| > \frac{1}{2}N_\alpha^{(\sigma)}F(u)$. Let $c = \frac{1}{4}(1/2C_0)^{1/p}$ in Proposition 11. Then there exists $r_0 > 0$ such that if $x \in B_{r_0}(x_0) = B$, then $|F(x) - F(x_0)| \leq cN_{2\alpha}^{(\sigma+1)}F(u)$. Thus, if $u \in U_\varepsilon$ we obtain that for $x \in B$, $|F(x)| > \frac{1}{4}N_\alpha^{(\sigma)}F(u)$, and therefore if $y = (n,s) \in u\cdot\Gamma_\alpha$ with $s < -(\sigma_1 + \sigma)$, we have

$$\mathbf{P}_y^u\left(F^* \geq \tfrac{1}{4}N_\alpha^{(\sigma)}F(u)\right) \geq \mathbf{P}_y^u(\tau_B < \infty) \geq C_1 \tag{4.4}$$

where σ_1 and C_1 are those of Proposition 12. Observe that given $\varepsilon > 0$ there exists $\sigma_\varepsilon > 0$ such that for any $s < -\sigma_\varepsilon$ and any $u \in \overline{N}$ with $|u| < \varepsilon$ the set $\{(n,s): |n| \leq \frac{1}{2}\alpha\mathbf{e}^{-s}\}$ is contained in $u\cdot\Gamma_\alpha$ ($|u^{-1}n| < |u| + |n| < \varepsilon + \frac{1}{2}\alpha\mathbf{e}^{-s}$; take $\sigma_\varepsilon = \log(2\varepsilon/2\alpha)$). If $y = (n,s) \in \Gamma_{\alpha/2}$ with $s < \min\{-\sigma_\varepsilon, -(\sigma_1+\sigma)\}$, then $y \in u\cdot\Gamma_\alpha$ for every $u \in U_\varepsilon$. Estimate (4.4) then gives

(4.5)

$$\begin{aligned}
\mathbf{E}_{(n,s)}[(F^*)^p] &= \int_{\overline{N}} \mathbf{E}_{(n,s)}^u[(F^*)^p]P_s(u^{-1}n)\,du \\
&\geq \int_{U_\varepsilon} \mathbf{E}_{(n,s)}^u[(F^*)^p]P_s(u^{-1}n)\,du \\
&\geq \left(\frac{1}{4}\right)^p \int_{U_\varepsilon} (N_\alpha^{(\sigma)}F(u))^p \mathbf{P}_{(n,s)}^u\left(F^* > \frac{1}{4}N_\alpha^{(\sigma)}F(u)\right) P_s(u^{-1}n)\,du \\
&\geq \left(\frac{1}{4}\right)^p C_1 \int_{U_\varepsilon} (N_\alpha^{(\sigma)}F(u))^p P_s(u^{-1}n)\,du.
\end{aligned}$$

Integrating over $|n| < \frac{1}{2}\alpha\mathbf{e}^{-s}$ for $s < \min\{-\sigma_\varepsilon, -(\sigma_0 + \sigma_1)\}$, we obtain

$$\begin{aligned}
&\int_{|n|<\alpha\mathbf{e}^{-s}/2} \mathbf{E}_{(n,s)}[(F^*)^p]\,dn \\
&\qquad \geq \left(\frac{1}{4}\right)^p C_1 \int_{|n|<\alpha\mathbf{e}^{-s}/2} \int_{\overline{N}} (N_\alpha^{(\sigma)}F(u))^p P_s(u^{-1}n)\chi_{U_\varepsilon}(u)\,du\,dn \\
&\qquad \geq \left(\frac{1}{4}\right)^p C_1 \int_{\overline{N}} (N_\alpha^{(\sigma)}F(u))^p \chi_{U_\varepsilon}(u) \int_{|n|<\alpha\mathbf{e}^{-s}/2} P_s(u^{-1}n)\,dn\,du.
\end{aligned}$$

We need to estimate $\int_{|n|<\alpha\mathbf{e}^{-s}/2} P_s(u^{-1}n)\,dn$ uniformly on $u \in U_\varepsilon$. Take $s < \min\{-\overline{\sigma}_\varepsilon, -(\sigma_1+\sigma)\}$ where $\overline{\sigma}_\varepsilon = \log(4\varepsilon/\alpha) > \sigma_\varepsilon$. Then $\varepsilon < \frac{1}{4}\alpha\mathbf{e}^{-s}$, and therefore if $|u| < \varepsilon$, we have

$$\{u^{-1}n\colon |n| < \tfrac{1}{2}\alpha\mathbf{e}^{-s}\} \supset \{\nu\colon |\nu| < \tfrac{1}{4}\alpha\mathbf{e}^{-s}\}$$

and we obtain

$$\begin{aligned}\int_{|n|<\alpha\mathbf{e}^{-s}/2} P_s(u^{-1}n)\,dn &\geq \int_{|n|<\alpha\mathbf{e}^{-s}/4} P_s(n)\,dn \\ &= \int_{|n|<\alpha/4} P_s(n^s)\mathbf{e}^{-ms}\,dn \\ &= \int_{|n|<\alpha/4} P_0(n)\mathbf{e}^{ms}\mathbf{e}^{-ms}\,dn = C_4 > 0.\end{aligned}$$

From (4.5) and (4.3) we obtain finally

$$\begin{aligned}\|F\|^p_{H^p_*} &\geq \int_{|n|<\alpha\mathbf{e}^{-s}/2} \mathbf{E}_{(n,s)}[(F^*)^p]\,dn \\ &\geq \left(\frac{1}{4}\right)^p C_1C_4 \int_{U_\varepsilon} (N^{(\sigma)}_\alpha F(u))^p\,du \\ &\geq \left(\frac{1}{4}\right)^p \frac{C_1C_2}{C_0}\frac{1}{2}\int_{\overline{N}} (N^{(\sigma)}_\alpha F(u))^p\,du\end{aligned}$$

as desired.

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