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ON LINKING DOUBLE LINES

BY
JUAN MIGLIORE

ABSTRACT. A double line is a nonreduced locally Cohen-Macaulay scheme of degree two supported on a line in projective three-space. The heart of this work is to compute the associated Hartshorne-Rao module for such a curve. We can then say exactly when two such curves are in the same liaison class and in fact when they are directly linked. In particular, we find that C is only self-linked in characteristic two.

Introduction. Let k be an algebraically closed field and $S = k[X_0, X_1, X_2, X_3]$. A *double line* $C \subset \mathbf{P}^3_k$ is a nonreduced locally Cohen-Macaulay scheme of degree two supported on a line. The main purpose of this paper is to determine when two such curves can be linked. This is accomplished by a careful study of the Hartshorne-Rao module $M(C) = \bigoplus_{n\in\mathbf{Z}} H^1(\mathbf{P}^3, \mathcal{I}_C(n))$ (cf. [**R1**]) and has a somewhat surprising answer. In addition, we check when C can be self-linked.

In order to state the results, we first recall briefly the description of double lines due to Harris (cf. [**H**, pp. 32–33] for more detail). He gives a geometric description of a double line C in $\mathbf{P}^3$, produces the homogeneous ideal $I(C)$, then verifies that the degree is in fact two and calculates the arithmetic genus.

To specify a double line C in $\mathbf{P}^3$ one has to choose the underlying line λ and then specify for each point $P \in \lambda$ a normal direction to λ. This is done by specifying for each $P \in \lambda$ a plane H_P containing λ. Equivalently, we have a map $\psi\colon \lambda \to \mathbf{P}^3/\lambda$ (where $\mathbf{P}^3/\lambda \cong \mathbf{P}^1$ represents the pencil of planes through λ).

Without loss of generality, let λ be the line given by $X_2 = X_3 = 0$. Then we can write

$$\psi([a_0, a_1, 0, 0]) = F(a_0, a_1)X_2 + G(a_0, a_1)X_3,$$

where F and G are homogeneous polynomials of degree equal to $\deg \psi = d$ (say), with no common zero. We can also think of this as a map of $\mathbf{P}^1$ to $\mathbf{P}^1$:

$$\psi([X_0, X_1]) = [F(X_0, X_1), G(X_0, X_1)].$$

This information is lumped together in the ideal of C, which is

$$I(C) = (X_2^2, X_2X_3, X_3^2, X_2F(X_0, X_1) - X_3G(X_0, X_1)).$$

Note that $X_2F - X_3G$ corresponds to a surface S of degree $d + 1$ which is nonsingular in a neighborhood of λ (since F and G have no common zero). So we can equivalently write

$$I(C) = (I(\lambda)^2, I(S)).$$

Finally, Harris gives an elementary argument to show that the Hilbert polynomial for C is $2x+d+1$. Hence the degree of C is 2 and the arithmetic genus $p_a(C) = -d$.

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We can also compute the Hilbert function for C. For $n \leq 1$ this is trivial. For $n \geq 2$ observe that any element of $I(C)$ of degree n can be uniquely expressed in the form

$$X_2^2 P(X_0, X_1, X_2) + X_2 X_3 Q(X_0, X_1, X_2, X_3) + X_3^2 R(X_0, X_1, X_3) \qquad (n \leq d)$$

or

$$X_2^2 P(X_0, X_1, X_2) + X_2 X_3 Q(X_0, X_1, X_2, X_3) + X_3^2 R(X_0, X_1, X_3) \\ + [X_2 F(X_0, X_1) - X_3 G(X_0, X_1)] S(X_0, X_1) \qquad (n \geq d+1),$$

where P, Q and R are homogeneous of degree $n-2$ and S is homogeneous of degree $n-d-1$. Therefore

$$\dim \left(\frac{S}{I(C)} \right)_n = \begin{cases} \binom{n+3}{3} - \binom{n}{2} - \binom{n+1}{3} - \binom{n}{2} & \text{if } n \leq d, \\ \binom{n+3}{3} - \binom{n}{2} - \binom{n+1}{3} - \binom{n}{2} - (n-d) & \text{if } n \geq d. \end{cases}$$

Turning to liaison, it turns out that the cases $d = 0$ and $d = 1$ are trivial. The main result of the paper is the following:

THEOREM. *Let C and C' be double lines and assume $p_a(C) \leq -2$. Then C is linked to C' if and only if*

(a) *they have the same line λ as support,*

(b) *they have the same arithmetic genus $-d$,*

(c) *the corresponding maps $\psi, \psi' : \lambda \to \mathbf{P}^3/\lambda$ differ by an automorphism of the target $\mathbf{P}^3/\lambda$.*

To achieve this, §1 gives some preliminary results and §2 describes a special type of complete intersection which is used to simplify the computations. The heart of the paper is in §3 where we describe the global sections of the sheaves $\mathcal{O}_C(n)$, compute the Hartshorne-Rao module $M(C)$, and prove the main theorem. §4 gives an interesting interpretation which is purely geometric, from which we establish the fact that double lines of arithmetic genus ≤ -2 are only self-linked in characteristic two. (More generally, we see when C and C' can be directly linked.)

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1. First results. A key strategy for this paper will be to study the sheaves $\mathcal{O}_C(n)$ for all n, which will tell us what we need to known about $M(C)$. In this section we give a first description of $\mathcal{O}_C$, the main point being the exact sequence (3). As a consequence we compute the dimensions of the components of $M(C)$.

To begin, let S be the surface corresponding to $X_2F - X_3G$ as in the Introduction. Then since $I(S) \subset I(C) \subset I(\lambda)$ it follows that $\lambda \subset C \subset S$.

LEMMA 1.1. *On S, $\lambda \cdot \lambda = 1 - d$.*

PROOF. This is an immediate application of the adjunction formula, recalling that $\omega_S = \mathcal{O}_S(d-3)$ (since $\deg S = d+1$) so $K_S = (d-3)H$ (where K_S is the canonical divisor class and H is the class of a hyperplane section), and the geometric genus $\pi(\lambda) = 0$:

$$0 = \pi(\lambda) = \frac{\lambda \cdot \lambda + K_S \cdot \lambda}{2} + 1 = \frac{\lambda \cdot \lambda + (d-3)}{2} + 1. \quad \square$$

Now, we have an exact sequence of sheaves

$$0 \to \mathcal{I}_{C,S} \to \mathcal{I}_{\lambda,S} \to \mathcal{I}_{\lambda,C} \to 0$$

from which we deduce

$$\mathcal{I}_{\lambda,S}/\mathcal{I}_{C,S} \cong \mathcal{I}_{\lambda,C}. \tag{1}$$

Since $\lambda \hookrightarrow C$ we have a natural map $\mathcal{O}_C \to \mathcal{O}_\lambda$, which yields the exact sequence

$$0 \to \mathcal{I}_{\lambda,C} \to \mathcal{O}_C \to \mathcal{O}_\lambda \to 0$$

or, substituting the isomorphism (1) (thinking of this as a sequence of sheaves on S)

$$0 \to \mathcal{I}_{\lambda,S}/\mathcal{I}_{C,S} \to \mathcal{O}_C \to \mathcal{O}_\lambda \to 0. \tag{2}$$

But since $I(C) = (I(\lambda)^2, I(S))$ it follows that $\mathcal{I}_{C,S} = \mathcal{I}^2_{\lambda,S}$. Now, $\mathcal{I}_{\lambda,S}/\mathcal{I}^2_{\lambda,S}$ is the conormal bundle of λ on S (recall S is smooth around λ), which is a line bundle of degree $-(\lambda \cdot \lambda) = d-1$ on $\lambda \cong \mathbf{P}^1$ (cf. [**Ha**, p. 361]). Therefore we can write (2) in the form

$$0 \to \mathcal{O}_\lambda(d-1) \xrightarrow{\alpha} \mathcal{O}_C \xrightarrow{r} \mathcal{O}_\lambda \to 0. \tag{3}$$

The maps α and r will be described following Proposition 3.1.

From this we can quickly compute the dimensions of the components of the Hartshorne-Rao module $M(C)$:

LEMMA 1.2.

$$\dim M_n(C) = \begin{cases} 0, & n \le -d, \\ d+n, & -d \le n \le 0, \\ d-n, & 0 \le n \le d, \\ 0, & n \ge d. \end{cases}$$

PROOF. From the exact sequence of sheaves

$$0 \to \mathcal{I}_C(n) \to \mathcal{O}_{P^3}(n) \to \mathcal{O}_C(n) \to 0$$

we get the associated long exact sequence in cohomology:

$$0 \to H^0(\mathbf{P}^3, \mathcal{I}_C(n)) \to H^0(\mathbf{P}^3, \mathcal{O}(n)) \to H^0(C, \mathcal{O}_C(n)) \to M_n(C) \to 0. \tag{4}$$

Having (3) and twisting, we can compute $h^0(\mathcal{O}_C(n))$ for any n. (Note that

$$h^1(\mathcal{O}_\lambda(n+d-1)) = 0 \quad \text{for } n \ge -d \quad \text{and} \quad h^0(\mathcal{O}_\lambda(n)) = 0 \quad \text{for } n \le -1.)$$

Then recalling the Hilbert function calculation of the Introduction, the result is easily obtained using (4). □

In other words, $\{\dim M_n(C) | n \in \mathbf{Z}\}$ is the sequence

$$\ldots, 0, 0, 1, 2, \ldots, d-1, d, d-1, \ldots, 2, 1, 0, 0, \ldots$$

where the "peak" occurs for $n = 0$. The "diameter" of $M(C)$ is thus $2d-1$, and this gives a necessary condition for liaison:

COROLLARY 1.3. *If two double lines C and C' are linked, then the corresponding maps $\psi: \lambda \to \mathbf{P}^3/\lambda$ and $\psi': \lambda' \to \mathbf{P}^3/\lambda'$ have the same degree d (and, equivalently, C and C' have the same arithmetic genus $-d$).*

REMARK 1.4. It follows that a double line can have a Hartshorne-Rao module of arbitrarily large "diameter", that it can have arbitrarily large components, and that it can have arbitrarily many nonzero components in negative degrees. Furthermore, for $d > 0$, C is "extremal" in its liaison class in the following sense: Since $\dim M_0(C) > \dim M_1(C)$, it follows from Proposition 2.8 of [**M**] that no curve in the liaison class can have a Hartshorne-Rao module which is leftward shift of $M(C)$. $\square$

REMARK 1.5. A double line is also extremal in the sense that (for $d > 0$) there is obviously no curve in the liaison class of strictly smaller degree. This paper describes which other degree two curves are in the liaison class.

For $d = 0$, C is a plane curve and hence a complete intersection. $M(C) = 0$ and C is arithmetically Cohen-Macaulay. C is thus linked to any line and any (plane) conic in $\mathbf{P}^3$, including all other double lines with $d = 0$.

For $d = 1$, $M(C) \cong k$ so C is arithmetically Buchsbaum (cf. [**GMV2**]). C is linked to any pair of skew lines (cf. [**R1**]), as well as to any other double line with $d = 1$.

For $d > 1$, the components of $M(C)$ are as described above, but now there may be some nontrivial module structure. In fact, it was shown in [**GMV2**] that such a C is *never* arithmetically Buchsbaum (i.e. there does exist nontrivial module structure). Because of this added complexity, it is reasonable to expect that not all double lines with the same d will be linked. $\square$

2. Basic links. We are assuming that C is a double line with ideal $(X_2^2, X_2X_3, X_3^2, X_2F(X_0,X_1) - X_3G(X_0,X_1))$, where $\deg F = \deg G = d$. Observe that if T is a reducible quadratic polynomial of the form $(\alpha X_2 + \beta X_3)(\gamma X_2 + \delta X_3)$ (i.e. the corresponding surface is the union of two planes, both containing the line $\lambda: (X_2 = X_3 = 0)$), then $T \in I(C)$.

If T_1 and T_2 are both of this form, with no common factor, then the corresponding complete intersection curve X links C to a curve C' which is again locally Cohen-Macaulay. Since $\deg X = 4$ we have $\deg C' = 2$, and clearly C' is supported on λ. Then by Harris' description,

$$I(C') = (X_2^2, X_2X_3, X_3^2, X_2F'(X_0,X_1) - X_3G'(X_0,X_1)),$$

where F' and G' have no common root. Furthermore, by Corollary 1.3 we have $\deg F' = \deg G' = \deg F = \deg G = d$.

For our purposes it will actually be enough to look at complete intersections of the form (T_1, T_2), where $T_1 = (X_2)(X_2 + \tau X_3)$ and $T_2 = (X_3)(\mu X_2 + X_3)$. Observe that T_1 and T_2 have no common factor (and hence give a link) if and only if the determinant $\left|\begin{smallmatrix} 1 & \mu \\ -\tau & -1 \end{smallmatrix}\right| \neq 0$. We shall call such a link a *basic link*.

Given $I(C)$ and the basic link (T_1, T_2), we can compute the ideal of the residual double line C'. Recall

$$I(C') = (T_1, T_2) : I(C). \tag{5}$$

Since we know $I(C') = (X_2^2, X_2X_3, X_3^2, X_2F'(X_0,X_1) - X_3G'(X_0,X_1))$ with $\deg F' = \deg G' = d$, we have only to identify F' and G' in terms of F, G, τ and μ.

By (5),

$$[X_2F'(X_0,X_1) - X_3G'(X_0,X_1)][X_2F(X_0,X_1) - X_3G(X_0,X_1)] \\ = P(X_0,X_1)(X_2)(X_2+\tau X_3) + Q(X_0,X_1)(X_3)(\mu X_2 + X_3)$$

for some P, Q. Then $FF' = P$ and $GG' = Q$, so one deduces that $F'(-\tau F - G) = G'(\mu G + F)$. But F' and G' have no common root, nor do $(-\tau F - G)$ and $(\mu G + F)$. Therefore, *up to scalar multiplication* we have

$$F' = \mu G + F, \qquad G' = -\tau F - G.$$

Equivalently,

$$\begin{bmatrix} F'(X_0,X_1) \\ G'(X_0,X_1) \end{bmatrix} = \alpha \begin{bmatrix} 1 & \mu \\ -\tau & -1 \end{bmatrix} \begin{bmatrix} F(X_0,X_1) \\ G(X_0,X_1) \end{bmatrix}$$

for some scalar $\alpha \neq 0$. We conclude

PROPOSITION 2.1. *Let C and C' have ideals as above. Then C is (directly) linked to C' by a basic link if and only if* $\left[\begin{smallmatrix} F' \\ G' \end{smallmatrix}\right] = A\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$, *where* $\det A \neq 0$ *and* $\operatorname{trace} A = 0$ (*but the main diagonal is not zero*).

REMARK 2.2. We know that direct linking is a symmetric relation. Hence if we link C to C' as above and then apply the same surfaces to C', we *must* get C back. This is neatly reflected in the fact that

$$\begin{bmatrix} 1 & \mu \\ -\tau & -1 \end{bmatrix} \begin{bmatrix} 1 & \mu \\ -\tau & -1 \end{bmatrix} = \begin{bmatrix} 1-\tau\mu & 0 \\ 0 & 1-\tau\mu \end{bmatrix} = (1-\tau\mu)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad \square$$

REMARK 2.3. Since

$$\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

it follows that C is linked to itself by a sequence of three basic links. (Note that by Hartshorne's theorem (cf. [**R1**]) this implies that $M(C)$ is self-dual.) $\square$

REMARK 2.4. As a corollary to Proposition 2.1, note that C is *self-linked* (cf. [**R2**]) by a basic link if and only if $\operatorname{char} k = 2$ (and we take $\tau = \mu = 0$). We shall say more about self-linkage in §4. $\square$

3. Main results. As indicated earlier, we are interested in studying the sheaf $\mathcal{O}_C(n)$. The key point is the following:

PROPOSITION 3.1. *The global sections of $\mathcal{O}_C(n)$ are of the form*

$$H^0(C, \mathcal{O}_C(n)) = \left\{ A(X_0,X_1) + B(X_0,X_1) \cdot \frac{X_2}{G(X_0,X_1)} \;\middle|\; \begin{array}{l} A \text{ and } B \text{ are polyno-} \\ \text{mials of degree } n \text{ and} \\ d-1+n \text{ respectively,} \\ \text{or zero} \end{array} \right\}.$$

PROOF. We shall prove the inclusion $\supseteq$, and then the result will follow from a check of vector space dimensions, using sequence (3).

Recall that $I(C) = (X_2^2, X_2X_3, X_3^2, X_2F(X_0,X_1) - X_3G(X_0,X_1))$. Let U_0 be the open set $\{X_0 \neq 0, G(X_0,X_1) \neq 0\}$ on $\mathbf{P}^3$, and consider the exact sequence

$$0 \to \mathcal{O}_C(n)(U_0) \to \mathcal{O}_{\mathbf{P}^3}(n)(U_0) \to \mathcal{O}_C(n)(U_0) \to \cdots.$$

For any open set U, $\mathcal{O}_{\mathbf{P}^3}(n)(U) = \{P/Q | \deg P = \deg Q + n,\ Q \text{ never zero on } U\}$. For $\mathcal{I}_C(n)(U) \subset \mathcal{O}_{\mathbf{P}^3}(n)(U)$ we have the further condition $P \in I(C \cap U)$.

Hence it follows that (up to isomorphism) at least the following inclusions must hold:

$$\begin{aligned}\mathcal{O}_C(n)(U_0) &\supseteq \frac{\mathcal{O}_{\mathbf{P}^3}(n)(U_0)}{\mathcal{I}_C(n)(U_0)} \\ &\supseteq \left\{ \frac{A(X_0,X_1)}{X_0^p} + \frac{B(X_0,X_1)\cdot X_2}{X_0^q G(X_0,X_1)} \,\middle|\, \begin{array}{l}\deg A = p+n \\ \deg B = q+d-1+n\end{array} \right\}.\end{aligned}$$

Similarly, on $U_1 = \{X_1 \neq 0, G(X_0,X_1) \neq 0\}$ we have

$$\begin{aligned}\mathcal{O}_C(n)(U_1) &\supseteq \mathcal{O}_{\mathbf{P}^3}(n)(U_1)/\mathcal{I}_C(n)(U_1) \\ &\supseteq \left\{ \frac{D(X_0,X_1)}{X_1^s} + \frac{E(X_0,X_1)\cdot X_2}{X_1^t G(X_0,X_1)} \,\middle|\, \begin{array}{l}\deg D = s+n \\ \deg E = t+d-1+n\end{array} \right\}.\end{aligned}$$

Two such sections patch together to give a section over $U = U_0 \cup U_1$ if and only if $p = q = s = t = 0$, as in the statement of the proposition. But such a section over U in fact extends to a global section of $\mathcal{O}_C(n)$ (i.e. it extends to $\{G(X_0,X_1) = 0\}$) because on C we have $X_2/G = X_3/F$ and F and G have no common root. □

REMARK 3.2. As a result of this proposition, we may view the map α in the exact sequence (3) as multiplication by $X_2/G(X_0,X_1)$ and r as the restriction to the first term. □

From Proposition 3.1 we can immediately describe $M(C)$:

COROLLARY 3.3. *$M(C) \cong [k[X_0,X_1]/(F,G)](d-1)$ as an S-module.*

PROOF. We have the usual exact sequence (4):

$$0 \to I(C)_n \to S_n \to H^0(C, \mathcal{O}_C(n)) \to M_n(C) \to 0.$$

Let $R = S/I(C)$, so $M(C) \cong H^0_*(C, \mathcal{O}_C(n))/R$ (where H^0_* represents the direct sum over all n).

Now given an element $A + B\cdot X_2/G \in H^0(C, \mathcal{O}_C(n))$ as in Proposition 3.1, we see that A becomes zero in $M(C)$, and it is a simple exercise to check that B also becomes zero whenever it is a linear combination $PF + QG$. Also, multiplication by X_2 and X_3 are zero in $M(C)$. □

We can now prove our main result. Note that the cases $d = 0$ and $d = 1$ were taken care of in Remark 1.5.

THEOREM 3.4. *Let C and C' be double lines and assume $p_a(C) \leq -2$. Then C is linked to C' if and only if*

(a) *they have the same line λ as support,*

(b) *they have the same arithmetic genus $-d$,*

(c) *the corresponding maps $\psi, \psi' \colon \lambda \to \mathbf{P}^3/\lambda$ differ by an automorphism of the target $\mathbf{P}^3/\lambda$.*

PROOF. Again, we assume without loss of generality that C is supported on the line $\lambda\colon (X_2 = X_3 = 0)$. Now, assume C is linked to C'. By Remark 2.3 we may assume that C is evenly linked to C', so $M(C) \cong M(C')$ by Hartshorne's theorem (cf. [**R1**]) and Lemma 1.2. (The latter assures that there is no shift.)

Consider the homomorphism $\phi_{1-d,1}\colon S_1 \to \operatorname{Hom}(M_{1-d}(C), M_{2-d}(C))$. By Corollary 3.3, $\ker \phi_{1-d,1} = \langle X_2, X_3\rangle$, so we have (a). We know (b) from Corollary 1.3. Given these, (c) says that $\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right] = A\left[\begin{smallmatrix} F' \\ G' \end{smallmatrix}\right]$ for some invertible 2×2 matrix A. Equivalently, F and G span the same subspace of $k[X_0, X_1]_d$ that F' and G' do.

Consider the homomorphism $\phi_{1-d,d}\colon S_d \to \operatorname{Hom}(M_{1-d}(C), M_1(C))$ and let K be its kernel. Let $I(\lambda)$ be the ideal $(X_2, X_3) \subset S$. From Corollary 3.3 we have that K is generated by $I(\lambda)_d$, $F(X_0, X_1)$ and $G(X_0, X_1)$. We thus have

$$I(\lambda)_d \subset K \hookrightarrow S_d \overset{\phi_{1-d,d}}{\to} \operatorname{Hom}(M_{1-d}(C), M_1(C)).$$

Modding out by $I(\lambda)_d$ we then have

$$\langle F(X_0, X_1), G(X_0, X_1)\rangle = K/I(\lambda)_d \hookrightarrow S_d/I(\lambda)_d \cong k[X_0, X_1]_d.$$

Since both K and $I(\lambda)$ are isomorphism invariants, we get that the subspace $\langle F, G\rangle$ of $k[X_0, X_1]_d$ is also, and so we are done.

Conversely, it suffices to check the following: Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2)$. Then

(1) if $b = c = 0$, certainly $a \neq 0$ and $d \neq 0$ and we have

$$A = a \begin{bmatrix} 1 & 0 \\ \dfrac{d}{a} + 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ \dfrac{d}{a} + 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix},$$

(2) if at least one of b, c is nonzero then there exist scalars $\alpha \neq 0, \tau, \mu, \tau', \mu'$ such that

$$A = \alpha \begin{bmatrix} 1 & \mu \\ -\tau & -1 \end{bmatrix} \begin{bmatrix} 1 & \mu' \\ -\tau' & -1 \end{bmatrix}.$$

(The latter is tedious but straightforward.) Then Proposition 2.1 gives the result. □

4. Projective geometry and self-linkage. In this section we give an interesting geometric interpretation of the preceding sections which was pointed out by A. Landman. It wil then be easy to say when a double line is self-linked, generalizing a result of [**R2**]. For this section, we assume $\operatorname{char} k \neq 2$.

As always, we assume that the double line C is supported on the line $\lambda\colon (X_2 = X_3 = 0)$, and that $d > 1$. The quadric hypersurfaces containing C are then given by homogeneous polynomials of the form $aX_2^2 + bX_2X_3 + cX_3^2$ (which are always reducible). These hypersurfaces thus form a projective space $\mathbf{P}^2$. The ideal generated by two such polynomials is then represented by a line in this $\mathbf{P}^2$, or by a point in the dual space $(\mathbf{P}^2)^*$. (Henceforth, unless stated otherwise, an *ideal* shall be understood to be of this special form.)

For the purposes of liaison we need a complete intersection ideal, that is one whose two generators have no common factor. We shall call these *allowable* ideals. We may thus ask which lines in $\mathbf{P}^2$ (or which points in $(\mathbf{P}^2)^*$) correspond to allowable Ideals. We now find the complement of this set:

LEMMA 4.1. (a) *The "unallowable" ideals form a smooth conic in* $(\mathbf{P}^2)^*$.

(b) *This conic is dual to the "conic of squares" in* $\mathbf{P}^2$.

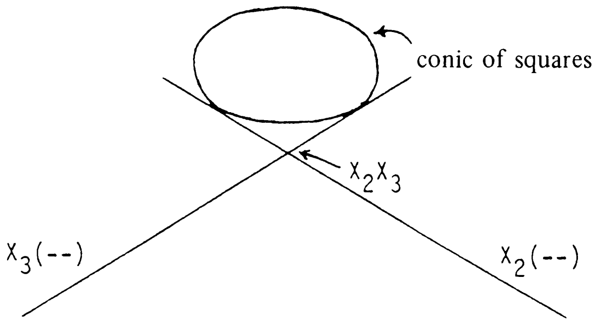

FIGURE 1

PROOF. (a) Let $I = (aX_2^2 + bX_2X_3 + cX_3^2, dX_2^2 + eX_2X_3 + fX_3^2)$. One shows using the resultant that these polynomials have a common root if and only if

$$(af - cd)^2 - (ae - bd)(bf - ce) = 0. \tag{6}$$

Now, the line in $\mathbf{P}^2$ through the points $[a, b, c]$ and $[d, e, f]$ has equation

$$\det \begin{bmatrix} Y_0 & Y_1 & Y_2 \\ a & b & c \\ d & e & f \end{bmatrix}$$

(where Y_0, Y_1, Y_2 are the coordinates for this $\mathbf{P}^2$). Hence the corresponding point in $(\mathbf{P}^2)^*$ is $[ae - bd, -(af - cd), bf - ce]$, and the result follows from (6).

(b) A *line* in $(\mathbf{P}^2)^*$ corresponds to a pencil of lines in $\mathbf{P}^2$, i.e. the collection of those ideals containing a given polynomial L_1L_2. This line in $(\mathbf{P}^2)^*$ will meet the "unallowable" conic twice in general, corresponding to the ideals (L_1L_2, L_1L_3) and (L_1L_2, L_2L_3) (L_3 arbitrary). This line will be tangent to the "unallowable" conic precisely when the corresponding point in $\mathbf{P}^2$ represents a square L_1^2, and so we are done. (Note that the squares really do form a conic in $\mathbf{P}^2$, namely $Y_1^2 - 4Y_0Y_2 = 0$.) □

REMARK 4.2. We get several amusing facts about these ideals for free from this description. For example,

(a) A general ideal (L_1L_2, L_3L_4) contains exactly two squares (up to scalar multiplication).

(b) Two distinct ideals of this form have exactly one polynomial in common (up to scalar multiplication).

(c) Given a *general* polynomial L_1L_2, there are exactly two ideals containing it which are not allowable. □

We now consider our basic links of §2. The set of all polynomials of the form $X_2(X_2 + \tau X_3)$ corresponds to a line in $\mathbf{P}^2$, as does the set of all polynomials of the form $X_3(\mu X_2 + X_3)$. These lines meet at X_2X_3 (which is at infinity) and each is tangent to the conic of squares (by duality, since each represents an unallowable ideal). See Figure 1.

The following is then immediate:

PROPOSITION 4.3. *Given an allowable ideal* (L_1L_2, L_3L_4), *we can find generators which give a basic link if and only if the ideal does not contain the polynomial* X_2X_3.

PROOF. An allowable ideal corresponds to a line is $\mathbf{P}^2$ not tangent to the conic of squares. It then must meet the two distinguished lines either in two distinct points or at the point corresponding to X_2X_3 (see Figure 1). □

We now turn to the question of when a double line in self-linked, i.e. when it can be linked to itself in one step. In the case $d = 0$, C is clearly self-linked since it is a plane curve (and so a complete intersection). For $d = 1$, on the other hand, it is shown in [**R2**] that C is self-linked if and only if $\operatorname{char} k = 2$. This was generalized in [**GMV1**], where it is shown that if C_n is the curve defined by the ideal $((I(\lambda))^n, X_0X_3 - X_1X_2)$, then C is never self-linked for $n > 2$. ($n = 1$ is Rao's result.) We now give a different generalization:

THEOREM 4.4. *A double line* C *of arithmetic genus* < -1 (*i.e.* $d \geq 2$) *is self-linked if and only if* $\operatorname{char} k = 2$.

PROOF. In view of Remark 2.4, the only thing left to prove is that for $\operatorname{char} k \neq 2$, C cannot be self-linked by a nonbasic link.

By Proposition 4.3, "almost all" links are basic. Hence we have only to consider allowable ideals of the form (X_2X_3, L_1L_2). Clearly we may write this as $(X_2X_3, aX_2^2 + bX_3^2)$, where $a \neq 0$ and $b \neq 0$. Then writing $I(C)$ and the linked ideal $I(C')$ as in §2, a similar calculation shows

$$\begin{bmatrix} F' \\ G' \end{bmatrix} = \alpha \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix}.$$

Therefore C is not self-linked. □

REMARK 4.5. We can now generalize Proposition 2.1 to say that C is directly linked to C' if and only if

$$\begin{bmatrix} F' \\ G' \end{bmatrix} = A \begin{bmatrix} F \\ G \end{bmatrix},$$

where $\det A \neq 0$ and $\operatorname{trace} A = 0$. □

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DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60201

Current address: Department of Mathematics, Drew University, Madison, New Jersey 07940