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A RIGIDITY PROPERTY FOR THE SET OF ALL CHARACTERS INDUCED BY VALUATIONS

BY

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ABSTRACT. If K is a field and G a finitely generated multiplicative subgroup of K then every real valuation on K induces a character $G \to \mathbf{R}$. It is known that the set $\Delta(G) \subseteq \mathbf{R}^n$ of all characters induced by valuations is polyhedral. We prove that $\Delta(G)$ satisfies a certain rigidity property and apply this to give a new and conceptual proof of the Brewster-Roseblade result [4] on the group of automorphisms of K stabilizing G.

1. Introduction.

1.1. Let K be a field and $k \subseteq K$ a subfield. By a real valuation on K over k we mean a homomorphism w: $K^\times \to \mathbf{R}$ of the multiplicative group of K into the additive group of the reals which is trivial on $k^\times$ and satisfies $w(a+b) \geqslant \min\{w(a), w(b)\}$ for all $a, b \in K^\times$ with $a + b \neq 0$. Throughout the paper G shall denote a finitely generated multiplicative subgroup of K. Then w induces a character $\chi = w|_G$: $G \to \mathbf{R}$ on G and so, following G. M. Bergman [1], we consider the set $\Delta(G)$ of all characters of G induced by real valuations on K over k. Thus $\Delta(G)$ is a subset of the real vector space $G^* = \operatorname{Hom}(G, \mathbf{R}) \cong \mathbf{R}^n$, where $n = \operatorname{rk} G$ is the torsion free rank of G. Since valuations on the subfield $k(G)$ can always be extended to the field K, $\Delta(G)$ depends only on $k(G)$ and not on K.

In [2] we have shown that the subset $\Delta(G) \subseteq \mathbf{R}^n$ has some special geometric features. In particular, we established Bergman's conjecture that $\Delta(G)$ is a rational spherical polyhedron, i.e., a finite union of finite intersections of closed half spaces given by inequalities with integer coefficients. Here we continue the investigation of the geometry of $\Delta(G)$ by showing that it decomposes into a cartesian product $\Delta(G) = \mathbf{R}^m \times \Delta_1$ of an affine space $\mathbf{R}^m$ and a polyhedron $\Delta_1 \subseteq \mathbf{R}^{n-m}$ satisfying a certain rigidity property.

1.2. We sketch our main result. In the situation above the group G contains two interesting subgroups $A \leqslant B \leqslant G$, which are in some sense dual to one another. On the one hand we have the unique maximal subgroup A subject to the condition that $k(A)$ is algebraic over k (thus A only consists of all elements of G which are algebraic over k); and on the other hand G contains a unique minimal subgroup B subject to the condition that $k(G)$ is purely transcendental over $k(B)$ (equivalently, the number of generators of G/B coincides with $\operatorname{tr}\deg_{k(B)} k(G)$, the transcendence degree of $k(G)$ over $k(B)$). Note that the factors G/B and B/A are torsion free.

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Any subgroup $H \leqslant G$ gives rise to a short exact sequence of $\mathbf{R}$-vector spaces $(G/H)^* \rightarrowtail G^* \twoheadrightarrow H^*$, so that we can identify $(G/H)^*$ with the kernel of the restriction map res: $G^* \twoheadrightarrow H^*$. Moreover, we have $\operatorname{res}\Delta(G) = \Delta(H)$. Our first result, then, exhibits the subspaces $(G/A)^*$ and $(G/B)^*$ in terms of the polyhedron $\Delta(G)$. The following terminology will be convenient: We say that the vector subspace $V \leqslant \mathbf{R}^n$ is an affine cartesian factor of the polyhedron $\Delta \subseteq \mathbf{R}^n$ if $\Delta = \pi^{-1}(\pi(\Delta))$, where π is the canonical projection $\mathbf{R}^n \twoheadrightarrow \mathbf{R}^n/V$ (so that $\Delta \cong V \times \pi(\Delta)$). Every polyhedron has a unique maximal affine cartesian factor.

THEOREM A. *If $A \leqslant B \leqslant G$ as above, then $(G/B)^* \subseteq \Delta(G) \subseteq (G/A)^*$. In fact we have*

(a) *$(G/A)^*$ is the subspace spanned by $\Delta(G)$,*

(b) *$(G/B)^*$ is the maximal affine cartesian factor of $\Delta(G)$.*

By the *Bergman carrier* $\mathfrak{C}(\Delta(G))$ of $\Delta(G)$ we mean the uniquely determined (finite) set of subspaces $X \leqslant G^*$ such that the union $\bigcup\{X \mid X \in \mathfrak{C}(\Delta(G))\}$ contains $\Delta(G)$ and is minimal with respect to that property. By [**2**, Theorem 1.1], the dimension of each $X \in \mathfrak{C}(\Delta(G))$ is equal to the transcendence degree m of $k(G)$ over k. We stress that nontriviality of A and G/B above should be considered as a degenerate situation. In the nondegenerate, "generic", case the polyhedron $\Delta(G)$ has a certain rigidity property which is the main result of this paper.

THEOREM B. *If A and G/B are finite, then G^* is spanned by the one-dimensional intersections $X_1 \cap X_2 \cap \cdots \cap X_m$ of spaces X_i in the Bergman carrier $\mathfrak{C}(\Delta(G))$.*

In the general case, Theorem B applies, of course, to the polyhedron $\Delta(B) = \Delta(B/A)$ in $(B/A)^*$.

1.3. As an immediate consequence we obtain a new and more conceptual proof of Roseblade's key result [**4**, Theorem D]. This application is in the spirit of, but much stronger than, Bergman's proof of Zalesskii's conjecture [**1**].

Let Γ be the group of all automorphisms of $k(G)$ over k stabilizing G. Then both A and B are Γ invariant so that Γ acts on A, B/A and G/B. The action on A and G/B is easy to understand: the automorphism group induced on A is a subgroup of the Galois group of $k(A)$ over k and hence is finite; and the automorphism group induced on G/B is the full group $\mathrm{GL}_r(\mathbf{Z})$, $r = \operatorname{rk}(G/B)$. Thus the interesting part is the action on B/A.

The action of Γ on B induces an action on its character group B^*. Since Γ permutes the valuations on K over k, the subset $\Delta(B) \subseteq B^*$ is invariant under this action. Hence Γ permutes the subspaces of the Bergman carrier $\mathfrak{C}(\Delta(B))$. Bergman knew that if $A \neq B$, then $\mathfrak{C}(\Delta(B))$ is nonempty and consists of proper subspaces of G^*, whence his conclusion that Γ contains a subgroup of finite index stabilizing a proper subspace of G^* and therefore a subgroup of infinite index in G. But now, with Theorem B available, we know that Γ permutes a set of one-dimensional subspaces of B^* spanning $(B/A)^*$. Hence Γ contains a subgroup of finite index fixing $(B/A)^*$, and therefore B/A, pointwise. In other words we have

COROLLARY. *The automorphism group induced by Γ on B/A is finite.*

This is the core of the Brewster-Roseblade result [**4**, Theorem D]. Their full result, asserting that, in fact, the automorphism group induced by Γ on B is finite, can be deduced from the Corollary by an elementary argument which we sketch in the Appendix.

We observe that in the definition of $\Delta(G)$, G could be a torsion free subgroup in $K^{\times}$ of finite rank, in which case $\Delta(G)$ coincides with $\Delta(G_1)$ for any finitely generated subgroup G_1 of maximum rank in G. The Corollary holds as before.

1.4. The proof of Theorem A is given in §2. As to Theorem B, the only ingredients in its proof, apart from Theorem A, are two geometric properties of $\Delta(G)$ which we established in [**2**]: homogeneity and concavity. §§3, 4 and 5 contain a purely elementary-geometric analysis of these two conditions leading eventually to the rigidity result.

Finally, although we have followed Bergman in assuming that k is a field, similar results hold in the more general case of k being a Dedekind domain; we briefly sketch the necessary changes in §6.

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2. Proof of Theorem A.

2.1. Let us first substantiate the remark we made immediately after stating Theorem A: that nontriviality of G/B is to be considered a degenerate situation. As above let K be a field, $k \subseteq K$ a subfield and $G \leqslant K$ a finitely generated multiplicative subgroup of K. Let kG denote the group algebra of G, $k[G]$ the subalgebra of K generated by G over k, and I the kernel of the canonical projection $kG \twoheadrightarrow k[G]$.

PROPOSITION 2.1. *If $H \leqslant G$ is a subgroup with torsion free factor G/H, then the following statements are equivalent*:

(i) *I is controlled by H, that is, $I = (I \cap kH)kG$.*

(ii) *$k[G]$ is induced from $k[H]$, that is, the canonical map $kH \twoheadrightarrow k[H]$ induces an isomorphism $k[G] \cong k[H] \otimes_{kH} kG$.*

(iii) $\operatorname{tr}\deg_{k(H)} k(G) = \operatorname{rk}(G/H)$.

(iv) $\Delta(G) = \operatorname{res}^{-1}\Delta(H)$.

PROOF. (i) $\Leftrightarrow$ (ii) Tensoring the short exact sequence $I \cap kH \rightarrowtail kH \twoheadrightarrow k[H]$ with kG over kH yields the short exact sequence

$$(I \cap kH)kG \rightarrowtail kG \twoheadrightarrow k[H] \otimes_{kH} kG,$$

whence the assertion.

(ii) $\Leftrightarrow$ (iii) Let X be a complement of H in G. Then $kG \cong kH \otimes_k kX$, whence

$$k[H] \otimes_{kH} kG \cong k[H] \otimes_k kX.$$

This is just the Laurent polynomial ring of transcendence degree, over $k[H]$, equal to rk X, and so, in particular, a domain. Hence the canonical homomorphism $k[H] \otimes_{kH} kG \rightarrowtail k[G]$ is an isomorphism if and only if the transcendence degrees (over $k[H]$, say) of the two domains are the same.

(iii) $\Rightarrow$ (iv) Let X be a complement of H in G and $\chi = (\chi_H, \chi_X) \in G^*$ with $\chi_H \in \Delta(H)$ and $\chi_X \in X^*$. Let v: $k(H)^\times \to \mathbf{R}$ be a valuation on $k(H)$ inducing χ_H. Any basis of X generates $k(G)$ over $k(H)$ and so must be algebraically independent. Hence $k(G)$ is just the field of fractions of the group ring $k(H)X$ and v can be extended to a valuation w: $k(G)^\times \to \mathbf{R}$ inducing χ_X. This shows that $\chi = (\chi_H, \chi_X) \in \Delta(G)$, as asserted.

(iv) $\Rightarrow$ (iii) By [**2**, Theorem 1.1] the dimension of $\Delta(G)$ coincides with the transcendence degree of $k(G)$ over k and similarly for $\Delta(H)$. Hence $\operatorname{rk}(G/H) = \dim \Delta(G) - \dim \Delta(H) = \operatorname{tr}\deg_{k(H)} k(G)$, as asserted.

2.2. The proof of Theorem A is now easily completed. The elements $x \in G$ which are algebraic over k are characterized by the property that $v(x) = 0$ for all valuations on K over k—in other words, by the property that $\Delta(G) \subseteq (G/\operatorname{gp}(x))^*$, where $\operatorname{gp}(x)$ stands for the subgroup generated by x. This shows that $(G/A)^*$ is the intersection of all rational hyperspaces of $\mathbf{R}^n$ containing $\Delta(G)$—a subspace of $\mathbf{R}^n$ is said to be rational if it is induced by a subspace of Q^n. But by [**2**, Theorem 1.1] we know that $\Delta(G)$ is a rational polyhedron, and so the intersection of all rational subspaces containing $\Delta(G)$ is, in fact, the subspace spanned by $\Delta(G)$. This proves assertion (a). As to (b), note that if H is a subgroup of G, then $\Delta(G)$ is the cartesian product of $(G/H)^*$ and $\Delta(H)$ if and only if $\Delta(G) = \operatorname{res}^{-1} \Delta(H)$, and so, in view of Proposition 2.1, $(G/B)^*$ is the maximal *rational* subspace of G^* with $\Delta(G) = \operatorname{res}^{-1} \Delta(H)$. Since $\Delta(G)$ is a rational polyhedron the assertion follows.

3. Polyhedrons.

3.1. We have to recall some affine geometric notation from [**2**]. A subset S of the affine space $\mathbf{R}^n$ is said to be a *convex polyhedron* if it can be written as the intersection of finitely many closed affine half spaces in $\mathbf{R}^n$. The *dimension* of the convex polyhedron S is defined to be the dimension of the affine subspace of $\mathbf{R}^n$ spanned by S and denoted $\dim S$. A subset $\Delta \subseteq \mathbf{R}^n$ is said to be a *polyhedron* if Δ can be written as the union

$$\Delta = C_1 \cup C_2 \cup \cdots \cup C_s \tag{3.1}$$

of a finite number of convex polyhedrons C_i. The *dimension* of Δ, denoted $\dim \Delta$, is the maximum number $\dim C_i$ as $1 \leqslant i \leqslant s$. The polyhedron Δ is said to be *homogenous* if the decomposition (3.1) can be chosen such that $\dim \Delta = \dim C_i$ for all $1 \leqslant i \leqslant s$.

In order to describe the local behaviour of a polyhedron Δ at a point $x \in \Delta$ it is convenient to introduce the *local cone of* Δ *at* x, denoted $\operatorname{LC}_x(\Delta)$. It consists of all points $y \in \mathbf{R}^n$ with the property that the segment $\{x + \rho(y - x) \mid 0 \leqslant \rho \leqslant \varepsilon\}$ joining x and $x + \varepsilon(y - x)$ is contained in Δ for some $\varepsilon > 0$. Note that $\operatorname{LC}_x(\Delta)$ is a polyhedron which contains and is homeomorphic to an open neighbourhood of x in Δ. A point $x \in \Delta$ is said to be *regular* if its local cone $\operatorname{LC}_x(\Delta)$ is an affine subspace of dimension $= \dim \Delta = m$. A nonregular point is said to be *singular*. By $\operatorname{reg} \Delta$ and $\operatorname{sing} \Delta$ we denote the set of all regular and singular points of Δ, respectively. By the *essential part* of Δ, denoted $\operatorname{ess} \Delta$, we mean the closure of $\operatorname{reg} \Delta$ in $\mathbf{R}^n$. $\operatorname{ess} \Delta$ could also be described as the union of all m-dimensional C_i's occurring in (3.1). Although this second description does not make it apparent that ess Δ is independent of the decomposition (3.1), it does show that this is a homogenous polyhedron of dimension m. A similar argument shows that the set $\mathfrak{C}(\Delta)$ of all affine subspaces of $\mathbf{R}^n$ which are the local cone of some regular point of Δ is finite. This is the *carrier* of Δ. The union of all affine subspaces in $\mathfrak{C}(\Delta)$ contains Δ and is minimal with respect to this property.

3.2. Let us consider the special case of an n-dimensional polyhedron in $\mathbf{R}^n$ in somewhat more detail. We prove

LEMMA 3.1. *If Δ is an n-dimensional homogenous polyhedron in $\mathbf{R}^n$, but not equal to $\mathbf{R}^n$, then* sing Δ *is an $(n-1)$-dimensional homogenous polyhedron.*

PROOF. As Δ is a finite union of finite intersections of closed half spaces, the set theoretic complement Δ^c is a finite union of finite intersections of *open* half spaces. But the intersection of finitely many open half spaces in $\mathbf{R}^n$ is either empty or n-dimensional, whence the closure $\overline{\Delta^c}$ is an n-dimensional homogenous polyhedron. A point $x \in \Delta$ is singular if and only if $\mathrm{LC}_x(\Delta) \neq \mathbf{R}^n$ which means that every neighbourhood of x in $\mathbf{R}^n$ contains points not in Δ. In other words we have $\operatorname{sing}\Delta = \Delta \cap \overline{\Delta^c}$, and thus, in particular, sing Δ is a polyhedron. Clearly, $\dim(\operatorname{sing}\Delta) \leqslant n-1$. On the other hand $\mathbf{R}^n$ is the disjoint union of Δ^c, reg Δ and sing Δ, and both reg Δ and Δ^c are nonempty and open in $\mathbf{R}^n$. Hence the complement of sing Δ in $\mathbf{R}^n$ is not connected and so $\dim(\operatorname{sing}\Delta) \geqslant n-1$. Finally, let $x \in \operatorname{sing}\Delta$, and let U be an open ball in $\mathbf{R}^n$ with centre x. As Δ is homogenous, the set reg Δ of regular points is dense in Δ and so $U \cap \operatorname{reg}\Delta \neq \varnothing$. The argument above shows that, in fact, $\dim(U \cap \operatorname{sing}\Delta) = n-1$. Hence the regular points of sing Δ are dense in sing Δ, which amounts to saying that sing Δ is homogenous.

3.3. A polyhedron Δ has, of course, many decompositions of the form (3.1). For later applications, however, it will be crucial that there is a canonical one in favourable cases. Let us say that the union (3.1) is a *convex cell decomposition* of Δ, if the intersection $C_i \cap C_j$ is empty or a common face of both C_i and C_j, for all $i \neq j$. Unfortunately not every polyhedron has a canonical convex cell decomposition (try, e.g., the union of a plane P and a line L intersecting P in a single point). But we can prove

LEMMA 3.2. *Every n-dimensional homogenous polyhedron $\Delta \subseteq \mathbf{R}^n$ has a canonical convex cell decomposition $\Delta = \bigcup_{i=1}^r C_i$, where each C_i is an n-dimensional convex polyhedron and*

$$\mathfrak{C}(\operatorname{sing}\Delta) = \bigcup_{i=1}^{r} \mathfrak{C}(\operatorname{sing} C_i).$$

Note that $\mathfrak{C}(\operatorname{sing} C_i)$ is, of course, just the set of all affine subspaces spanned by the $(n-1)$-dimensional faces of the convex polyhedron C_i. Any face of C_i contained in sing Δ will be called *a face of* Δ. The faces of Δ form a convex cell decomposition of the homogenous polyhedron sing Δ.

PROOF (OF LEMMA 3.2). $\mathfrak{C}(\operatorname{sing}\Delta)$ is a finite set of affine hyperspaces of $\mathbf{R}^n$ and so defines a convex cell decomposition of the affine space $\mathbf{R}^n$,

$$\mathbf{R}^n = \bigcup_{j=1}^{s} D_j,$$

where $D_1, D_2, \ldots, D_s$ are n-dimensional convex polyhedrons. Let $\mathring{D}_j$ denote the interior of the convex cell D_j. Clearly, $\mathring{D}_j$ has empty intersection with $\operatorname{sing}\Delta$, whence

$$\mathring{D}_j = \left(\mathring{D}_j \cap \operatorname{reg}\Delta\right) \cup \left(\mathring{D}_j \cap \Delta^c\right).$$

Thus $\mathring{D}_j$ is the union of two disjoint open sets. As $\mathring{D}_j$ is connected, one of these must be empty, that is, either $\mathring{D}_j \subseteq \Delta$ or $\mathring{D}_j \subseteq \Delta^c$. It follows that Δ is the union of all convex cells D_j contained in Δ, whence the lemma.

COROLLARY 3.3. *Let $\Delta \subseteq \mathbf{R}^n$ be an n-dimensional homogenous polyhedron and $L \subseteq \mathbf{R}^n$ a line which intersects but is not contained in Δ. Then the carrier $\mathfrak{C}(\operatorname{sing}\Delta)$ of* sing Δ *contains a hyperspace X which intersects L in a single point.*

PROOF. By Lemma 3.2 we may assume that Δ is convex. Then we proceed by induction on n. As $\Delta \cap L$ is neither empty nor all of L, there is a point $x \in L \cap \operatorname{sing}\Delta$. Hence there is an $(n-1)$-dimensional face F of Δ with $x \in L \cap F$. Let Y be the affine subspace spanned by F. Then $Y \in \mathfrak{C}(\operatorname{sing}\Delta)$, and if $L \cap Y$ is the singleton set $\{x\}$, we put $X = Y$ and are done. Otherwise, $L \subseteq Y$ and, by induction, there is an $(n-2)$-dimensional face F_1 of F such that the intersection of L with the subspace Z_1 spanned by F_1 is a singleton. But F_1 is the face of exactly two $(n-1)$-dimensional faces, one of which is F (see e.g. [**3**, Satz 5.2]). The subspace X spanned by the other one cannot contain L, whence the corollary.

4. The effect of total concavity.

4.1. Let $\Delta \subseteq \mathbf{R}^n$ be a polyhedron. Recall from [**2**] that Δ is said to be *concave at a point* $x \in \Delta$ if the convex hull of the local cone $\mathrm{LC}_x(\Delta)$ is an affine subspace of $\mathbf{R}^n$. Δ is *totally concave* if it is concave at all points $x \in \Delta$.

LEMMA 4.1. *If Δ is a homogenous n-dimensional totally concave polyhedron in $\mathbf{R}^n$, then $\Delta = \mathbf{R}^n$.*

PROOF. According to the proof of Lemma 3.2 the affine space $\mathbf{R}^n$ has a (finite) convex cell decomposition $\mathbf{R}^n = C_1 \cup C_2 \cup \cdots \cup C_r$ with each C_i n-dimensional, such that Δ is the union of a subset of the C_i's. If $\Delta \neq \mathbf{R}^n$, then there is a pair of indices i, j such that C_i and C_j have a common $(n-1)$-dimensional face F, $C_i \subseteq \Delta$ and $C_j \not\subseteq \Delta$. Then Δ is not concave at the regular points of F.

4.2. Let $\Delta \subseteq \mathbf{R}^n$ be an arbitrary homogenous m-dimensional polyhedron. Then the set $\operatorname{reg}\Delta$ of all regular points of Δ is dense in Δ, hence every point $x \in \Delta$ is contained in the essential part of $\Delta \cap X$ for some $X \in \mathfrak{C}(\Delta)$; that is, one has the canonical decomposition

$$\Delta = \bigcup_{X \in \mathfrak{C}(\Delta)} \operatorname{ess}(\Delta \cap X). \tag{4.1}$$

Note that each of the polyhedrons $\text{ess}(\Delta \cap X)$ is m-dimensional, homogenous and contained in some $X = \mathbf{R}^m$, whence has a canonical convex cell decomposition by Lemma 3.2.

PROPOSITION 4.2. *Let $\Delta \subseteq \mathbf{R}^n$ be a homogenous polyhedron and $X \in \mathfrak{C}(\Delta)$. If Δ is totally concave, then every space Z in the carrier $\mathfrak{C}(\text{sing}(\text{ess}(\Delta \cap X)))$ is the intersection $Z = X \cap Y$ of X with some other $Y \in \mathfrak{C}(\Delta)$.*

PROOF. Z is the local cone of some regular point z of the polyhedron $\text{sing}(\text{ess}(\Delta \cap X))$. If $z \notin \text{ess}(\Delta \cap Y)$ for all $Y \in \mathfrak{C}(\Delta)$ other than X, then there is an open neighbourhood U of z in $\mathbf{R}^n$ with

$$\text{ess}(\Delta \cap Y) \cap U = \varnothing, \quad \text{all } X \neq Y \in \mathfrak{C}(\Delta).$$

Hence $\Delta \cap U = \text{ess}(\Delta \cap X) \cap U$ by (4.1). This shows that the behaviour of Δ in a neighbourhood of z is given by $\text{ess}(\Delta \cap X)$. From that we infer that the local cones of Δ and $\text{ess}(\Delta \cap X)$ at z coincide and are totally concave. But since $\text{LC}_x(\text{ess}(\Delta \cap X))$ is homogenous and of dimension $= \dim X$, Lemma 4.1 applies and $\text{LC}_z(\text{ess}(\Delta \cap X)) = X$, contradicting the assumption that z be a singular point of $\text{ess}(\Delta \cap X)$. This shows that there is some $Y \in \mathfrak{C}(\Delta)$, $Y \neq X$, with $z \in X \cap Y$. The same arguments apply for all points in a neighbourhood of z in $\text{sing}(\text{ess}(\Delta \cap X))$; and so, as $\mathfrak{C}(\Delta)$ is finite, one of the subspaces $Y \in \mathfrak{C}(\Delta)$, $Y \neq X$, must contain such a neighbourhood and hence Z. Since $\dim Z = \dim(\text{sing}(\text{ess}(\Delta \cap X))) = \dim \Delta - 1 \geqslant \dim(X \cap Y)$, we have $Z = X \cap Y$, as asserted.

4.3. As a consequence of Proposition 4.2 we obtain that homogenous totally concave polyhedrons behave somewhat similarly to the set of all singular points of an n-dimensional convex polyhedron in $\mathbf{R}^n$ (compare Corollary 3.3).

PROPOSITION 4.3. *Let Δ be a homogenous and totally concave polyhedron in $\mathbf{R}^n$, and $L \subseteq \mathbf{R}^n$ a line which intersects but is not contained in Δ. Then there is some $X \in \mathfrak{C}(\Delta)$ which intersects L in a single point.*

PROOF. As Δ is homogenous, there is some $X \in \mathfrak{C}(\Delta)$ such that L intersects but is not contained in $\text{ess}(\Delta \cap X)$. If $L \subseteq X$, then we are done; otherwise Corollary 3.3 applies for $\text{ess}(\Delta \cap X)$ and yields $Z \in \mathfrak{C}(\text{sing}(\text{ess}(\Delta \cap X)))$ such that $Z \cap L$ is a singleton. By Proposition 4.2 there is $Y \in \mathfrak{C}(\Delta)$ with $X \cap Y = Z$ and Y cannot contain L since $X \neq Y$. Hence $Y \cap L$ consists of a single point.

4.4. Define the vector carrier $\mathfrak{V}\mathfrak{C}(\Delta)$ to be the set of vector subspaces $V \leqslant \mathbf{R}^n$ such that some affine translate of V lies in $\mathfrak{C}(\Delta)$. This, of course, is equal to $\mathfrak{C}(\Delta)$ in case Δ is a cone.

COROLLARY 4.4. *Let $\Delta \leqslant \mathbf{R}^n$ be a polyhedron which is homogeneous and totally concave. Then the intersection V of all spaces in $\mathfrak{V}\mathfrak{C}(\Delta)$ is an affine cartesian factor of Δ.*

PROOF. Let π be the projection $\pi\colon \mathbf{R}^n \to \mathbf{R}^n/V$. We must show that $\pi^{-1}(\pi(\Delta)) = \Delta$ or, equivalently, that $\Delta + V = \Delta$. Now $\Delta + V = \Delta$ if and only if $z + \mathbf{R}v \subseteq \Delta$ for all $z \in \Delta$, $v \in V$; that is, if and only if every line parallel to V which meets Δ lies wholly in Δ. But a line is parallel to V if and only if it is parallel to each $X \in \mathfrak{B}\mathfrak{C}(\Delta)$ or, equivalently, each $X \in \mathfrak{C}(\Delta)$. Thus Proposition 4.3 implies that V is an affine cartesian factor of Δ.

Proof of Theorem B.

5.1. Let K be a field, $k \subseteq K$ a subfield and $G \leqslant K^{\times}$ a finitely generated multiplicative subgroup of K. Since positive multiples of valuations over k are again valuations over k, the polyhedron $\Delta(G) \subseteq G^*$ is a cone. Moreover, we have shown in [**2**] that $\Delta(G)$ is homogenous and totally concave.

As to homogeneity, we can do better than this. If $H \leqslant G$ is a subgroup, then the restriction map res: $G^* \twoheadrightarrow H^*$ is $\mathbf{R}$-linear and maps $\Delta(G)$ onto $\Delta(H)$. This shows that not only $\Delta(G)$ is homogenous but also every image of $\Delta(G)$ under linear maps induced by embeddings of subgroups. We leave it to the reader to extend this observation to arbitrary linear maps, that is, to prove

THEOREM 5.1. *The polyhedron $\Delta(G) \subseteq G^* = \mathbf{R}^n$ has the property that all its images under $\mathbf{R}$-linear maps π: $\mathbf{R}^n \to \mathbf{R}^n$ are homogenous.*

We shall really need the result only for the case when π is induced by the embedding of a subgroup of corank 1.

5.2. It is now clear that Theorem B is an immediate consequence of Theorem A and the following purely geometric result. (Observing that, as Δ is a cone, the carrier and vector carrier of Δ coincide.)

PROPOSITION 5.2. *Let $\Delta \subseteq \mathbf{R}^n$ be a polyhedron with the following properties.*

(i) *Δ is totally concave.*

(ii) *All images of Δ under linear maps $\mathbf{R}^n \to \mathbf{R}^m$ are homogeneous.*

(iii) *Δ has no nontrivial affine cartesian factor.*

Then the subspace $\mathbf{R}^n$ spanned by $\mathfrak{B}\mathfrak{C}(\Delta)$ is spanned by one-dimensional subspaces of the form $X_1 \cap X_2 \cap \cdots \cap X_r$, $X_i \in \mathfrak{B}\mathfrak{C}(\Delta)$.

5.3. The proof of Proposition 5.2 relies on

LEMMA 5.3. *Let V be an n-dimensional vector space over any field K and $\mathfrak{C}$ a finite family of $(n-1)$-dimensional subspaces. If $\mathfrak{C}$ has the property that it contains complements to every one-dimensional subspace $L \leqslant V$ in V, then V is spanned by the one-dimensional subspaces of the form $X_1 \cap X_2 \cap \cdots \cap X_{n-1}$, $X_i \in \mathfrak{C}$.*

PROOF (BY INDUCTION ON n). If $n = 2$, the spaces in $\mathfrak{C}$ are one-dimensional themselves and there are at least two different ones. So let $n > 2$, pick $X \in \mathfrak{C}$, and consider

$$\mathfrak{C}_X = \{ X \cap Y | X \neq Y \in \mathfrak{C} \}$$

which is a finite family of $(n-2)$-dimensional subspaces of X. For every one-dimensional subspace $L \leqslant X$ there is $Y \in \mathfrak{C}$ with $V = L \oplus Y$. Note $L \nleqslant Y$ so that $Y \neq X$, whence $X = L \oplus (X \cap Y)$. By induction X is spanned by one-dimensional subspaces of the form $X \cap Y_i \cap \cdots \cap Y_{n-2}$, $Y_i \in \mathfrak{C}$. This holds for all $X \in \mathfrak{C}$ and $\mathfrak{C}$ contains more than one space.

5.4. PROOF OF PROPOSITION 5.2. Let $X \in \mathfrak{V}\mathfrak{C}(\Delta)$, $L \leqslant X$ a one-dimensional subspace and π: $\mathbf{R}^n \twoheadrightarrow \mathbf{R}^{n-1}$ a linear map with kernel L. Let $m = \dim \Delta$. If $\dim \pi(\Delta) = m - 1$, then $L \subseteq Y$ for every Y in the carrier $\mathfrak{V}\mathfrak{C}(\Delta)$. But that would mean, by Corollary 4.4, that L is a cartesian factor of Δ. Therefore $\dim \pi(\Delta) = m$. But certainly the dimension of $\pi(X)$ is equal to $m - 1$, and so, as $\pi(\Delta)$ is homogenous, $\pi(X)$ must be contained in $\pi(Y)$ for some $Y \in \mathfrak{V}\mathfrak{C}(\Delta)$, $X \neq Y$. In other words $X \subseteq Y + L$ and hence $X = (Y \cap X) \oplus L$. This shows that the family

$$\mathfrak{C}_X = \{ X \cap Y | X \neq Y \in \mathfrak{V}\mathfrak{C}(\Delta) \}$$

of subspaces of X satisfies the assumption of Lemma 5.3. Hence X is spanned by one-dimensional intersections of subspaces in the carrier of Δ. This holds for all $X \in \mathfrak{V}\mathfrak{C}(\Delta)$, whence the assertion.

This completes the proof of Proposition 5.2 and hence that of Theorem B.

6. Note on a generalization. The assumption that the valuations considered on the field K are trivial when restricted to the base field k is unnecessarily strong. One can take an arbitrary discrete valuation v: $k^{\times} \to \mathbf{Z}$ and consider the set $\Delta^v(G)$ of all characters of G induced by a real valuation on K extending v. Then $\Delta^v(G)$, although not necessarily a cone, is homogeneous and totally concave and does have the property in Proposition 5.1 (see [**2**]).

The proof of all the intermediate results in §§2–5 now carry through for $\Delta^v(G)$ (as well as for the global sets $\Delta^{k/D}(G)$ for $D \subseteq k$ a Dedekind domain; see [**2**]) yielding appropriate versions of Theorems A and B. More precisely, in Theorem A, we must replace "the subspace spanned by $\Delta(G)$" by "the subspace spanned by the vector carrier of $\Delta^v(G)$" and in Theorem B we must replace "$\mathfrak{C}(\Delta(G))$" by "$\mathfrak{V}\mathfrak{C}(\Delta^v(G))$".

However, it should be noticed that by [**2**, Theorem C1], $\Delta(G) = \Delta^0(G)$ is the local cone of $\Delta^v(G)$ at infinity, and so the vector carrier of $\Delta^v(G)$ *contains* the Bergman carrier of $\Delta^0(G)$. Moreover, it can be shown (by using Proposition 2.1 or a geometric argument involving total concavity) that the maximal affine cartesian factors of $\Delta^0(G)$ and $\Delta^v(G)$ coincide. Hence the rigidity result for $\Delta^v(G)$ is no stronger than the rigidity result for $\Delta^0(G)$—and this is the reason why we restricted attention to the field case.

Appendix. We sketch the proof that our Corollary in the introduction implies the full result of Theorem D in [**4**]. We show that, in fact, the automorphism group induced by Γ on B is finite.

We have shown that the automorphism groups induced by Γ on both B/A and A are finite. Thus it suffices to show that any automorphism ρ which is trivial on both B/A and A has finite order.

Let $F \leqslant k(B)$ be the fixed field of ρ and let $\bar{F}$ be its algebraic closure in $k(B)$. Then $|\bar{F}:F|$ is finite and so, after passing to a suitable power of ρ, we may assume that $\bar{F} = F$. We claim now that $k(B) = F$; we suppose not and derive a contradiction.

Let $k(B) = F(x_1, \dots, x_s)$ with $\{x_i\}_{i=1}^{s} \subset B$ multiplicatively independent. By the definition of B the x_i cannot be algebraically independent over F; choose $x_1, \dots, x_t$

so that $x_1, \dots, x_{t-1}$ are algebraically independent over F and x_t is algebraic over $F(x_1, \dots, x_{t-1})$. Let $P(X_1, \dots, X_t) \in F[X_1, \dots, X_t]$ be the minimal polynomial of x_t over $F[x_1, \dots, x_{t-1}]$.

Since ρ fixes both B/A and A, we have $\rho(x_i) = x_i a_i$ with $a_i \in A \subseteq F$. Also, as F is the fixed field of ρ and is algebraically closed in $k(B)$, the a_i are multiplicatively independent. Now, $P(x_1 a_1, \dots, x_t a_t) = 0$ and so $P(X_1 a_1, \dots, X_t a_t)$ is a multiple, clearly a F-scalar multiple, of $P(X_1, \dots, X_t)$. Comparing coefficients of monomials and recalling that the a_i are multiplicatively independent, we observe that P is a monomial in the X_i—an evident contradiction.

Thus $F = k(B)$ and so ρ (or, in the original statement, some power of ρ) is trivial.

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