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ON K_3 OF TRUNCATED POLYNOMIAL RINGS

BY

JANET AISBETT

ABSTRACT. Group homology spectral sequences are used to investigate K_3 of truncated polynomial rings. If F is a finite field of odd characteristic, we show that relative K_2 of the pair $(F[t]/(t^q), (t^k))$, which has been identified by van der Kallen and Stienstra, is isomorphic to $K_3(F[t]/(t^k),(t))$ when q is sufficiently large. We also show that $H_3(\mathrm{SL}\,\mathbf{Z}[t]/(t^k); \mathbf{Z}) = \mathbf{Z}^{k-1} \oplus \mathbf{Z}/24$ and is isomorphic to the associated K_3 group modulo an elementary abelian 2-group.

1. Introduction. There are relatively few rings whose third algebraic K-groups are known fully. Group homology spectral sequence techniques, however, have proved a useful tool, yielding for example K_3 of the quotient rings of the rational integers [**ALSS**]. This paper again applies these methods. Our main results are for $k \geqslant 2$:

1.1 THEOREM. *There are exact sequences*

$$\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}/2 \overset{1\oplus 1\oplus \iota}{\to} K_3\mathbf{Z}[t]/(t^{k+1})$$
$$= K_3\mathbf{Z} \oplus \mathbf{Z}^k \oplus U_{k+1} \to K_3\mathbf{Z}[t]/(t^k) \twoheadrightarrow \mathbf{Z},$$

where U_{k+1} is an elementary abelian 2*-group of rank at most* $[(k+1)/2]$, *and* $\iota = 0$ *if k is even.*

1.2 THEOREM. *If F is a finite field of odd characteristic, there is an isomorphism*

$$\partial^{-1} \circ \Delta_k\colon K_1(F[t]/(t^{2k}),(t))/\{1 - \alpha t^k\colon \alpha \in F\} \to K_3(F[t]/(t^k),(t)).$$

These theorems use, and sharpen, results of Van der Kallen and Stienstra [**S, VKS**]. Stienstra obtains 1.1 modulo information on the torsion group U_k; the theorem with $k = 2$ has essentially been proved by Kassel. Van der Kallen and Stienstra define the isomorphism Δ_k onto the relative K-group $K_2(F[t]/(t^q), (t^k))$, whenever $q \gg k$ and F is a perfect field. Theorem 1.2 with $k = 2$ and 3 has been proved by Snaith, Lluis and Aisbett [**LS, ALS**].

The Hochschild-Serre spectral sequences studied are associated to the reduction $\Pi^k\colon \mathrm{SL}_n R[t]/(t^k) \twoheadrightarrow \mathrm{SL}_n R$, where R is initially any commutative ring with identity for which $SK_1 R = 0$. More general results than those of Theorems 1.1 and 1.2 would need information on certain E^2_{**} terms.

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The remainder of this paper is organized as follows. §2 lists notation, and introduces Hochschild-Serre spectral sequences from our perspective. §3 recursively estimates the second homology groups of the kernel of Π^k, by means of spectral sequences associated to the reduction $\ker \Pi^k \to \ker \Pi^{k-1}$; this also gives information on the $\mathrm{SL}_n R$-coinvariance of the third homology groups. §4 deals with the integral case, to prove 1.1, and §5 deals with the finite fields to prove 1.2. The appendix contains constructive proofs of various module structural details needed elsewhere in the paper.

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2. Notation and conventions. This section introduces notation, describes differentials in Hochschild-Serre spectral sequences, and determines a formula for the d^2-differential which will cover our applications. There are five subsections.

2.1 *General notation and conventions.*

2.1.1 R is a commutative associative ring with identity, such that for some $N > 4$ and all $n \geqslant N$, $SK_1(n, R) = 0$ (i.e. the elementary matrices generate $\mathrm{SL}_n R$) and $K_2(n, R) = K_2 R$. Henceforth assume that $n \geqslant N$.

2.1.2 R_k is the truncated polynomial ring $R[t]/(t^k)$.

2.1.3 Any map π is a homomorphism induced by the reduction $R_k \to R_r$ for some $r < k$ which the context will indicate. π^k is induced by the reduction $R_k \to R_{k-1}$.

2.1.4 All diagrams are commutative exact and all sequences are exact unless otherwise specified.

2.2 *Notation for elements and* $\mathrm{SL}_n R$*-submodules of* $\mathrm{SL}_n R_k$.

2.2.1 $\mathrm{SL}\, R \equiv \mathrm{SL}_\infty R \equiv \varinjlim_n \mathrm{SL}_n R$, where the special linear group $\mathrm{SL}_n R$ includes into $\mathrm{SL}_{n+1} R$ as the upper left corner matrices, say. An elementary matrix is denoted $e_{ij}(\alpha)$, $i \neq j$, $\alpha \in R$. Set $\ddot{e}_{ij}(\alpha) = \mathrm{diag}(1, 1, \ldots, 1 + \alpha, \ldots, (1 + \alpha)^{-1}, \ldots 1)$ when $(1 + \alpha)$ is invertible in R; here, the nontrivial entries are in the ith and jth positions.

2.2.2 $\underline{\alpha}_{ij}$ is the $n \times n$ matrix over R with $\alpha \in R$ in the (i, j)th position and all other entries zero; $\ddot{\underline{\alpha}}_{ij} = \underline{\alpha}_{ii} - \underline{\alpha}_{jj}$. M_n^0 is the $\mathrm{SL}_n R$-module of zero-trace $n \times n$ matrices over R. (Here and elsewhere, subscript ranges are implied to be $\{1, \ldots, n\}$.)

2.2.3 $G_n^k = \ker(\pi_*\colon \mathrm{SL}_n R_k \to \mathrm{SL}_n R)$ so that $\mathrm{SL}_n R_k = \mathrm{SL}_n R \ltimes G_n^k$. Let $i\colon G_n^k \to \mathrm{SL}_n R_k$ be the inclusion. The kernel of $\pi_*\colon G_n^{k+1} \to G_n^k$ is central in G_n^{k+1} and is isomorphic to M_n^0 (identify $e_{ij}(\alpha t^k)$ with $\underline{\alpha}_{ij}$). We denote it $M_n^0(t^k)$, or sometimes just M_n^0. Let $j\colon M_n^0(t^k) \to G_n^{k+1}$ be the inclusion.

2.3 *Commutator relations in* $\mathrm{SL}_n R_k$*: the* $\mathrm{SL}_n R$*-action on* M_n^0. If $x, y \in R_k$ and $x^2 y^2 = 0$,

$$(2.1) \qquad [e_{ij}(x), e_{ab}(y)] = \begin{cases} e_{ib}(xy), & j = a,\ i \neq b, \\ e_{aj}(-xy), & i = b,\ j \neq a, \\ \ddot{e}_{ij}(xy) e_{ij}(-x^2 y) e_{ji}(xy^2), & i = b,\ j = a, \\ 0, & \text{else.} \end{cases}$$

If α, $\beta \in R$, the (left) $\mathrm{SL}_n R$-action on M_n^0 is

$$(2.2)\qquad \left(e_{ij}(\alpha) - 1\right) \cdot \underline{\beta}_{ab} = \begin{cases} \underline{\alpha\beta}_{ib}, & j = a,\ i \neq b, \\ -\underline{\alpha\beta}_{aj}, & a \neq j,\ i = b, \\ \underline{\alpha\ddot{\beta}}_{ij} - \underline{\alpha^2\beta}_{ij}, & a = j,\ i = b, \\ 0, & \text{otherwise.} \end{cases}$$

(The right action differs only by the sign reversal $\alpha \to -\alpha$.)

2.4 *Homology related definitions.*

2.4.1 $H_* X$ denotes the integral homology group $H_*(X; \mathbf{Z})$ of a group X.

2.4.2 $H_1(\mathrm{SL}_n R;\ M_n^0) = HH_1(R, R)$, where HH denotes Hochschild homology [**K4**, 2.16]. Since R is commutative,

$$HH_1(R, R) = R \otimes R / \langle \alpha \otimes \beta\gamma - \alpha\beta \otimes \gamma + \beta \otimes \gamma\alpha \colon \alpha, \beta, \gamma \in R \rangle$$

(e.g. [**I**, p. 108]) and is isomorphic to the R-module of absolute Kähler differentials $\Omega \equiv \Omega^1_{R/\mathbf{Z}}$ (e.g. [**K3**]).

2.4.3 $B_* X$ is the standard normalized bar resolution with $\mathbf{Z}$-basis elements $x_0[x_1| \cdots |x_n]$, $x_i \in X$. The boundary map $\partial_X \colon B_* X \to B_{*-1} X$ is

$$\partial_X x_0\left[x_1| \cdots |x_n\right] = x_0 x_1\left[x_2| \cdots |x_n\right] - \left(\sum_{i=2}^{n} (-1)^i x_0\left[x_1| \cdots |x_{i-1}x_i| \cdots x_n\right]\right) + (-1)^n x_0\left[x_1|x_2 \cdots |x_{n-1}\right].$$

2.4.4 If C is a right $\mathbf{Z}[X]$ coefficient module, and $x \in C \otimes_X B_* X$, we write $\{x\}$ for the class of x with respect to the equivalence relation induced by the boundary map $1 \otimes \partial_X$.

2.4.5 $[x \cap y]$ denotes $[x|y] - [y|x] \in B_2 X$; similarly, $[x_1 \cap x_2 \cap \cdots \cap x_{r-1} \cap x_r]$ is $\Sigma(-1)^\sigma[\sigma x_1|\sigma x_2| \cdots |\sigma x_r]$, where the sum if over the symmetric group Σ_r. Generally, if $z = z_0[z_1| \cdots z_i] \in B_i X$ and $z^1 = z_0^1[z_{i+1}| \cdots z_{i+j}] \in B_j X$, then $[z \cap z^1] \in B_{i+j} X$ is

$$\sum \left\{ (-1)^\sigma z_0 z_0^1\left[z_{\sigma(1)}| \cdots |z_{\sigma(i+j)}\right] \colon \sigma \in \Sigma_{i+j};\ \sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(i) \text{ and } \sigma^{-1}(i+1) < \sigma^{-1}(i+2) < \cdots < \sigma^{-1}(i+j) \right\}.$$

(That is, σ runs over all (i, j)—shuffles—see [**M**, p. 243].) Extend $[-\cap-]$ to a group homomorphism $B_* X \otimes B_* X \to B_* X$.

2.4.6 If X is abelian with operation $+$, $\cap \colon H_i X \otimes H_j X \to H_{i+j} X \times X \xrightarrow{(+)_*} H_{i+j} X$ is the homology (shuffle) product. This operation defines an exterior algebra $\Lambda * X$ which injects into $H_* X$ (see [**B**, p. 123]).

2.4.7 Suppose X_1 and X_2 are subgroups of X with $[X_1, X_2] = 1$. Identify $x \in B_i X_1$ and $x' \in B_j X_2$ with their images in $B_* X$. Then

$$(2.3)\qquad \partial_X[x \cap x'] = \left[\partial_X(x) \cap x'\right] + (-1)^i\left[x \cap \partial_X(x')\right].$$

Note that this is not true for general elements in $B_* X$.

2.5 *Hochschild-Serre spectral sequences.* This subsection reviews aspects of Hochschild-Serre group homology spectral sequences and it introduces more notation [**CE**].

2.5.1 Take an extension $N \overset{i}{\rightarrowtail} Y \overset{f}{\twoheadrightarrow} X$, in a category of S-groups with S-equivariant maps, say. Identify N with $i(N)$. Let C be a right Y-module. Consider the spectral sequence

$$E^2_{**}(C) = H_*(X;\, H_*(N;\, C)) \Rightarrow H_*(Y;\, C) \tag{2.4}$$

induced from the bicomplex $\{C \otimes_Y (B_*Y \otimes B_*X) \equiv (C \otimes_N B_*Y) \otimes_X B_*X;\ \partial = d_{\mathrm{I}} + d_{\mathrm{II}}\}$. B_*X is the standard bar resolution, with $d_{\mathrm{II}} = \partial_Y \otimes 1$ and $d_{\mathrm{I}}(f \otimes g) = (-1)^{|f|} f \otimes \partial_X g$. $B_*Y \otimes B_*X$ is a *left* Y-module via diagonal action, and $C \otimes_N B_*Y$ is a *right* X-module with action $c \otimes y \cdot x = (c \cdot \hat{x}) \otimes (\hat{x}^{-1} y)$ ($\hat{x}$ is any lifting of x to Y). This is well defined, since for all n in N, $c \cdot \hat{x} \otimes \hat{x}^{-1} y = c \cdot \hat{x} n \otimes n^{-1} \hat{x}^{-1} y$.

2.5.2 If $x \in E^c_{a,b}(C)$ and $2 \leqslant c \leqslant a$, then $d^c_{a,b} x$ is calculated by choosing representatives $x_i \in C \otimes_Y (B_{b+i} Y \otimes B_{a-i} X)$ for $0 \leqslant i < c$, such that x_0 represents x and $d_{\mathrm{I}} x_i = -d_{\mathrm{II}} x_{i+1}$; set $d^c_{a,b} x = \{d_{\mathrm{I}} x_{c-1}\}$. Let $d^c_{a,b} x_0$ denote $d_{\mathrm{I}} x_{c-1}$ (of course, this definition is not unique).

2.5.3 Suppose C is a trivial Y-module. Suppose Y_1 and Y_2 are subgroups of Y with $[Y_i, N] = 1$, $i = 1, 2$, and, if $f(Y_i) = X_i$, $[X_1, X_2] = 1$. Let u: $X \to Y$ be any set map section to f with $u(X_i) \subset Y_i$. Suppose

$$x = \sum_{i \in I} (-1)^i \left[x^i_1 | x^i_2 | \cdots | x^i_a\right] \in B_a X_1$$

and

$$v = \sum_{j \in J} (-1)^j \left[v^j_1 | v^j_2 | \cdots | v^j_b\right] \in B_b X_2$$

represent elements in $H_* X$ and $y \in C \otimes_N B_s N$ is a cycle. Then in the spectral sequence (2.4),

(2.5)

$$\begin{aligned} d^2_{a+b,s}\{y \otimes [x \cap v]\} = {} & \left\{\left[d^2_{a,s}(y \otimes x) \cap v\right]\right\} + (-1)^{ab}\left\{\left[d^2_{b,s}(y \otimes v) \cap x\right]\right\} \\ & + \left\{ \sum_{i \in I} \sum_{j \in J} (-1)^{a+i+j} \left[\left[\left[u(x^i_1)^{-1}, u(v^j_1)^{-1}\right]\right] \cap y\right] \right. \\ & \left. \otimes \left[\left[x^i_2 | \cdots | x^i_a\right] \cap \left[v^j_2 | \cdots | v^j_b\right]\right] \right\}. \end{aligned}$$

Note. (1) In the process of calculating the differential we prove that $y \otimes [x \cap v]$ represents an element in $H_{a+b}(X;\, H_s(N;\, C))$.

(2) The restricted definitions of x, v and y ensure that one can choose differential representatives on the right-hand side of (2.5) such that the shuffle products are cycles—again, this comes from the calculation. Indeed, as pointed out by the referee, if N is central in Y, then the coefficients in (2.4) are trivial and $d^2_{a,s}\{[(y \otimes x) \cap v]\}$ may be defined using a homology shuffle product

$$\cap\colon H_{a-2}(X_1;\, H_{s+1}(N;\, C)) \otimes H_b(X_2) \to H_{a+b-2}(X;\, H_{s+1}(N;\, C)).$$

PROOF. Use (2.3) to compute $d_{\mathrm{I}}(y \otimes [x \cap v]) = -d_{\mathrm{II}} z$ for $z \in (C \otimes_N B_{s+1} Y) \otimes_X B_{a+b-1} X$, where

$$-z = \sum_{i \in I} (-1)^{s+i} \Big[\Big[\big(u(x_1^i)\big)^{-1} \Big] \cap y \Big] \otimes \big[[x_2^i | \cdots | x_a^i] \cap v \big]$$
$$+ \sum_{j \in J} (-1)^{a+s+j} \Big[\Big[\big(u(v_1^j)\big)^{-1} \Big] \cap y \Big] \otimes \big[x \cap [v_2^j | \cdots | v_b^j] \big].$$

Then

$$d_{\mathrm{I}} z = \sum_i (-1)^i \Big[\Big[\big(u(x_1^i)\big)^{-1} \Big] \cap y \Big] \otimes \big[\partial_X [x_2^i | \cdots | x_a^i] \cap v \big]$$
$$+ \sum_j (-1)^j \Big[\Big[\big(u(v_1^j)\big)^{-1} \Big] \cap y \Big] \otimes \big[x \cap \partial_X [v_2^j | \cdots | v_b^j] \big]$$
$$+ \sum_i \sum_j (-1)^{a+i+j} \Big[\Big(\big(u(x_1^i)\big)^{-1} - 1 \Big) \Big[\big(u(v_1^j)\big)^{-1} \Big]$$
$$- \Big(u(v_1^j)^{-1} - 1 \Big) \Big[\big(u(x_1^i)\big)^{-1} \Big] \Big) \cap y \Big]$$
$$\otimes \big[[x_2^i | \cdots | x_a^i] \cap [v_2^j | \cdots | v_b^j] \big].$$

Since $(f-1)[g] - (g-1)[f] = \partial_Y([f \cap g] + [fg \mid f^{-1}g^{-1}] - [gf \mid f^{-1}g^{-1}]) + [[f, g]] \sim [[f, g]]$, the expression for $d_{\mathrm{I}} z$ is equivalent to the class in (2.5).

2.5.4 If x and y are as in 2.5.3 with $x = [x_1 \cap x_2 \cap \cdots \cap x_a]$, where $[x_i, x_j] = 1$ $(1 \leqslant i, j \leqslant a)$, then repeated application of (2.5) yields

$$(2.6)\quad d_{a,s}^2 \{ y \otimes x \} = \Big\{ \sum_{\sigma} - \Big[\Big[\Big[\big(u(x_{\sigma(1)})\big)^{-1}, \big(u(x_{\sigma(2)})\big)^{-1} \Big] \Big] \cap y \Big]$$
$$\otimes \big[x_{\sigma(3)} | \cdots | x_{\sigma(a)} \big] \Big\},$$

where σ runs over the alternating group A_a.

3. Low dimensional homology groups of G_n^k. This section inductively looks at low dimensional terms in the spectral sequences

$$(3.1)(k) \qquad H_*\big(G_n^{k-1}; H_* M_n^0(t^{k-1})\big) \Rightarrow H_* G_n^k, \qquad k > 2.$$

The first subsection is concerned with $E_{2,0}^\infty$ and $E_{0,*}^\infty$. Next, the $E_{1,1}^\infty$ term in (3.1) is computed, yielding enough information on $H_2 G_n^k$ to estimate $(E_{2,1}^\infty)_{\mathrm{SL}_n R}$ recursively. This gives us information on the $\mathrm{SL}_n R$-coinvariance of the kernel of $\pi_*^k | H_3 G_n^k$. The final subsection looks at $\ker(: (\pi_*^k H_3 G_n^k)_{\mathrm{SL}_n R} \to (H_3 G_n^{k-1})_{\mathrm{SL}_n R})$.

3.1 LEMMA. (i) π_*: $H_1 G_n^k \to (H_1 G_n^2 \cong M_n^0)$ *is an isomorphism*, $k > 2$.

(ii) *If* $k > r \geqslant 2$, $\operatorname{im}(\pi_*: H_2 G_n^k \to H_2 G_n^r) = \operatorname{im}(\pi_*: H_2 G_n^{r+1} \to H_2 G_n^r)$.

(iii) $j_*(\Lambda^* M_n^0(t^k)) = 0$ *in* $H_* G_n^{k+1}$; *in particular*, $j_*(H_2 M_n^0(t^k)) = 0$. *For torsion-free* R, *in the spectral sequence* (3.1)($k+1$), $E_{0,*}^3 = 0$ *whenever* $* > 0$ *and* $k \geqslant 2$.

PROOF. Consider (3.1)($k+1$), $k \geqslant 2$. For any $\{i, j\} \subset \{1, \ldots, n\}$, fix $m \notin \{i, j\}$. For $\alpha \in R$, define $b_{ij}(\alpha)$ to be $[e_{im}(t) \cap e_{mj}(\alpha t^{k-1})] \in \mathbf{Z} \otimes_{G_n^k} B_2 G_n^k$. Check that this is a cycle. Let u: $G_n^k \to G_n^{k+1}$ be a set map section to the reduction; it can be assumed that the elementary matrix $e_{ab}(\Sigma \alpha_m t^m) \in G_n^k$ is taken to the matrix of the same form in G_n^{k+1}. Apply formula (2.6) of 2.5.4 and the commutator relations 2.3 to compute

$$-d_{2,0}^2\{b_{ij}(\alpha)\} = \{[u(e_{im}(-t)), u(e_{mj}(-\alpha t^{k-1}))]\}$$
$$= \{[e_{im}(-t), e_{mj}(-\alpha t^{k-1})]\}$$
$$= \underline{\alpha}_{ij} \quad \text{if } i \neq j, \text{ else } -\underline{\alpha}_{mm} + \underline{\alpha}_{ii}.$$

Therefore $E_{0,1}^3 = 0$. Since k was arbitrary this proves (i).

(ii) $\{\{b_{ij}(\alpha)\}: 1 \leqslant i, j \leqslant n; \alpha \in R\}$ is a set of generators for $H_2G_n^k/\pi_*H_2G_n^{k+1}$. But also $\{b_{ij}(\alpha)\} \in \ker(\pi_*: H_2G_n^k \to H_2G_n^{k-1})$. Thus in $H_2G_n^{k-1}$, $\pi_*H_2G_n^{k+1} = \pi_*H_2G_n^k$, which implies (ii).

(iii) Consider (3.1)($k+1$), $k \geqslant 2$. By the universal coefficient theorem [**M**, p. 171] there is an inclusion $H_2G_n^k \otimes H_aM_n^0(t^k) \to E_{2,a}^2$. The proof of (i) shows that $d_{2,0}^2(H_2G_n^k) = M_n^0(t^k)$. So, by (2.6), $d_{2,a}^2(E_{2,a}^2)$ contains the homology product $M_n^0(t^k) \cap \Lambda^a M_n^0(t^k)$, which is just $\Lambda^{a+1}M_n^0(t^k)$. $\Lambda^2 M_n^0(t^k) = H_2M_n^0(t^k)$ and, if R is torsion-free, $\Lambda^* M_n^0(t^k) = H_*M_n^0(t^k)$ [**B**, p. 123]. □

3.2 Remarks. Let $J_n = \operatorname{im}(\pi_*: H_2G_n^k \to H_2G_n^2)$, $k > 2$. (This is independent of k, by 3.1(ii).) Then (3.1)(3) contains an exact sequence

$$(3.2) \qquad J_n \overset{s}{\rightarrowtail} \Lambda^2 M_n^0 \overset{d^2}{\twoheadrightarrow} M_n^0.$$

Let L_n be the associated module defined as $\ker([\,,\,]: M_n^0 \otimes M_n^0 \to M_n^0)$, where $[\,,\,]$ is the composite of the homology product with d^2. The identification (2.2.3) of M_n^0 takes the matrix a to $1 + at \in \mathrm{SL}_n R_2$. Thus if $a, b \in M_n^0$, using definition (2.5) $[a, b]$ is the commutator of $1 + at$ and $1 + bt$ evaluated in $\mathrm{SL}_n R_3$. Since $(1+at)^{-1} \equiv 1 - at + a^2t^2 \pmod{t^3}$, this commutator is readily seen to equal $1 + abt^2 - bat^2$. As an element of $M_n^0 \cong \ker(\pi^3: \mathrm{SL}_n R_3 \to \mathrm{SL}_n R_2)$, this is $ab - ba$; thus $[a, b]$ is the usual Lie bracket.

Let $\Omega \rightarrowtail \mathrm{St}(R, R) \overset{\phi}{\twoheadrightarrow} M_n^0$ be the universal $\mathrm{SL}_n R$-central extension of M_n^0 [**K4**, 2.15]. In the Appendix (Lemma A.1(iii)) we show that $(M_n^0 \otimes M_n^0)/I \cong \mathrm{St}(R, R) \oplus R$, where I is the $\mathrm{SL}_n R$-submodule generated by $\underline{1}_{12} \otimes \underline{1}_{23} + \underline{1}_{43} \otimes \underline{1}_{14}$.

We use this to estimate $\ker \pi_*^{k+1} | H_2G_n^{k+1}$, which is the $E_{1,1}^\infty$ term in (3.1) ($k+1$).

3.3 Proposition. (i) *There is a commutative diagram*

$$\begin{array}{ccccc} \Omega \oplus R & \rightarrowtail & \mathrm{St}(R, R) \oplus R & \overset{\phi \oplus 0}{\twoheadrightarrow} & M_n^0 \\ \downarrow & & \downarrow \rho \oplus \psi & & \| \\ \Omega \oplus R/(\operatorname{im} d_{3,0}^2)_{\mathrm{SL}_n R} & \rightarrowtail & \ker \pi_*^{k+1} | H_2G_n^{k+1} & \overset{D}{\twoheadrightarrow} & M_n^0 \end{array} \qquad (k \geqslant 2),$$

(3.3)($k+1$)

where $d_{3,0}^2$ *is the differential in the spectral sequence* (3.1)($k+1$) *and* D *is the restriction of the* $d_{2,0}^2$*-differential in* (3.1)($k+2$).

(ii) $(\ker \pi_*^{k+1} | H_2G_n^{k+1})_{\mathrm{SL}_n R} = \operatorname{im} \psi$.

Proof. Let u be a section to $\pi: G_n^{k+1} \to G_n^k$ with $u(e_{ij}(t^r)) = e_{ij}(t^r)$. Use 3.1 to identify $E_{1,1}^2$ with $M_n^0 \otimes M_n^0$. Use the commutator relations 2.3 to check that

$$x = -[e_{12}(t) \cap e_{24}(t^{k-1}) \cap e_{43}(t)]$$

is a cycle in $\mathbf{Z} \otimes_{G_n^k} B_3G_n^k$. Use (2.6) to compute

$$(3.4)\quad d_{3,0}^2\{x\} = \{[[e_{43}(-t), e_{24}(-t^{k-1})]] \otimes [e_{12}(t)] - [[e_{12}(-t), e_{24}(-t^{k-1})]] \otimes [e_{43}(t)]\}$$
$$= \underline{1}_{12} \otimes \underline{1}_{23} + \underline{1}_{43} \otimes \underline{1}_{14} \in H_1G_n^k \otimes M_n^0(t^k) = M_n^0 \otimes M_n^0.$$

So $I \subset \operatorname{im} d_{3,0}^2$.

As in the proof of 3.1(ii), $H_2G_n^{k+1}/\pi_* H_2G_n^{k+2} \cong M_n^0$ and is represented in $\ker \pi_*^{k+1} = E_{1,1}^\infty$. Moreover by the proof of 3.1(i) and Remarks 3.2, in the spectral sequence (3.1)($k+2$), $d_{2,0}^2 \,|\ker \pi_*^{k+1}$ is induced by the Lie bracket: $(E_{1,1}^2 \cong M_n^0 \otimes M_n^0) \twoheadrightarrow M_n^0$. According to A.1(ii) if $\mathrm{St}(R, R)$ is identified with $(M_n^0 \otimes M_n^0)^0/I$, then also ϕ is induced by the Lie bracket. This gives the right square of the diagram in the proposition statement. Moreover, because $\ker \pi_*^{k+1} = E_{1,1}^\infty = (M_n^0 \otimes M_n^0)/\operatorname{im} d_{3,0}^2$, $\rho \oplus \psi$ has kernel isomorphic to $\operatorname{im} d_{3,0}^2/I$; as a submodule of ($\Omega \oplus R = \ker(\phi \oplus 0)$), $\operatorname{im} d_{3,0}^2/I$ has trivial $\mathrm{SL}_n R$-action. By Lemma A.1(i) $(I)_{\mathrm{SL}_n R} = 0$ so $\operatorname{im} d_{3,0}^2/I \cong (\operatorname{im} d_{3,0}^2)_{\mathrm{SL}_n R}$. This implies the left square of the diagram.

Finally, $(\mathrm{St}(R, R))_{\mathrm{SL}_n R} = 0$ [**K4**, 1.7 and 1.4]; hence part (ii) is implied by the isomorphism $\ker \pi_*^{k+1} \cong (\mathrm{St}(R, R) \oplus R)/(\operatorname{im} d_{3,0}^2)_{\mathrm{SL}_n R}$. □

We next want to estimate $(H_3G_n^k)_{\mathrm{SL}_n R}$. The following lemma will be used in 3.5 to investigate $(E_{2,1}^\infty)_{\mathrm{SL}_n R}$.

3.4 LEMMA. *For $k > 2$, $\mathrm{SL}_n R$ acts trivially on* $\ker(\pi_*^3 \circ \cdots \circ \pi_*^{k-1}\colon \pi_*^k H_2G_n^k \to J_n)$.

PROOF. Denote the kernel of $\pi_*^3 \circ \cdots \circ \pi_*^k$ by U^k when it has domain $H_2G_n^k$ and by $\tilde{U}^k$ when it has domain $\pi_*^{k+1}H_2G_n^{k+1}$. Use (3.3)(k) to fit the sequence $\ker \pi_*^k | H_2G_n^k \rightarrowtail U^k \twoheadrightarrow \tilde{U}^{k-1}$ into the following diagram, the top row of which is therefore exact. (In (3.5), D is the restriction of the $d_{2,0}^2$ differential in the spectral sequence (3.1)($k+1$).)

$$(3.5)\qquad \begin{array}{ccccc}
(\Omega \oplus R)/(\operatorname{im} d_{3,0}^2)_{\mathrm{SL}_n R} & \rightarrowtail & ? & \twoheadrightarrow & \tilde{U}^{k-1} \\
\downarrow & & \downarrow & & \| \\
\ker \pi_*^k & \rightarrowtail & U^k & \twoheadrightarrow & \tilde{U}^{k-1} \\
D \downarrow & & D \downarrow & & \\
M_n^0 & = & M_n^0 & &
\end{array}$$

Inductively assume that both the base and fibre modules in the top row of (3.5) have trivial action ($\tilde{U}^2 = 0$). Therefore, because $H_1\mathrm{SL}_n R = 0$, ? is also a trivial $\mathrm{SL}_n R$-module. Moreover, the middle column of (3.5) fits into the following diagram, i.e., ? $= \tilde{U}^k$. This gives the inductive step.

$$\begin{array}{ccccc}
? & \rightarrowtail & \pi_*^{k+1} H_2G_n^{k+1} & \twoheadrightarrow & J_n \\
\downarrow & & \downarrow & & \| \\
U^k & \overset{m}{\rightarrowtail} & H_2G_n^k & \twoheadrightarrow & J_n \\
D \downarrow & & d_{2,0}^2 \downarrow & & \\
M_n^0 & = & M_n^0 & &
\end{array}$$

□

3.5 PROPOSITION. *In the spectral sequence* (3.1)($k+1$) *for* $k \geqslant 3$,

(i) *there is an epimorphism*: $R \twoheadrightarrow (E^{\infty}_{1,2})_{\mathrm{SL}_n R}$;

(ii) *there is an exact sequence*

$$(3.6) \qquad R \to \left(\ker d^2_{2,1}\right)_{\mathrm{SL}_n R} \twoheadrightarrow \left(\mathrm{Tor}\left(M^0_n, M^0_n\right)\right)_{\mathrm{SL}_n R};$$

hence for some quotient T *of* $(\mathrm{Tor}(M^0_n, M^0_n))_{\mathrm{SL}_n R}$, *there is an exact sequence*

$$(3.7) \qquad R \to \left(\left(\ker d^2_{2,1}/\mathrm{im}\, d^2_{4,0}\right) = E^{\infty}_{2,1}\right)_{\mathrm{SL}_n R} \twoheadrightarrow T.$$

PROOF. (i) $E^{\infty}_{1,2}$ is a quotient of $E^2_{1,2} = H_1G^k_n \otimes H_2M^0_n(t^k)$. By 3.1, $H_1G^k_n \cong M^0_n$, so $E^2_{1,2} \cong M^0_n \otimes \Lambda^2 M^0_n$. Use Lemma A.2(iv) to see that $(E^2_{1,2})_{\mathrm{SL}_n R} \cong R$, where α corresponds to the class of $\underline{\alpha}_{12} \otimes \underline{1}_{23}\underline{1}_{31}$; $(E^{\infty}_{1,2})_{\mathrm{SL}_n R}$ is a quotient of this.

(ii) Using the notation and proof of Lemma 3.4, filter $U^k \otimes M^0_n$ as

$$(3.8) \qquad \tilde{U}^k \otimes M^0_n \to U^k \otimes M^0_n \overset{D\otimes 1}{\twoheadrightarrow} M^0_n \otimes M^0_n,$$

where in the base group $\underline{\alpha}_{rs} \otimes \underline{\beta}_{uv}$ is the image of $\{b_{rs}(\alpha)\} \otimes \underline{\beta}_{uv}$ for $\{b_{rs}\}$ as in 3.1(i)(proof). Hence if $\cap$ is the product $M^0_n \otimes M^0_n \to M^0_n$, the composite $\cap \circ (D \otimes 1)$ induces an epimorphism g: $M^0_n \otimes M^0_n \twoheadrightarrow \Lambda^2 M^0_n = E^2_{0,2}$, equivalent to the product map. This has kernel $\Gamma(M^0_n)$, the Whitehead gamma group which projects onto $M^0_n \otimes \mathbf{Z}/2$ with kernel $S^2M^0_n$, the 2-fold symmetric product of M^0_n (see, for example, [A]). So $(\Gamma(M^0_n))_{\mathrm{SL}_n R}$ is a quotient of $(S^2M^0_n)_{\mathrm{SL}_n R} = R$ (by Lemma A.2(ii)). Moreover, (3.8) restricts to

$$\tilde{U}^k \otimes M^0_n \to \ker \cap \circ (D \otimes 1) \twoheadrightarrow \Gamma\left(M^0_n\right) = \ker g.$$

Since $\tilde{U}^k$ has trivial $\mathrm{SL}_n R$-action, $(\tilde{U}_k \otimes M^0_n)_{\mathrm{SL}_n R} = 0$; so $\ker \cap \circ (D \otimes 1)$ also has $\mathrm{SL}_n R$-coinvariance a quotient of R. U_k is, by definition, $\ker(\pi_*\colon H_2G^k_n \to J_n)$, and D is the restriction of $d^2_{2,0}$. The above shows that the restriction of $d^2_{2,1}$ to the image of $U_k \otimes M^0_n$ in $H_2G^k_n \otimes M^0_n$ is onto $E^2_{0,2}$. Hence we have an exact sequence

$$\ker \cap \circ (D \otimes 1) \to \ker\left(d^2_{2,1} \mid H_2G^k_n \otimes M^0_n\right) \twoheadrightarrow J_n \otimes M^0_n.$$

Apply $H_0(\mathrm{SL}_n R; -)$ to this and insert result $(J_n \otimes M^0_n)_{\mathrm{SL}_n R} = 0$ from Lemma A.2(vi) to see that there is an epimorphism: $R \twoheadrightarrow (\ker d^2_{2,1} \mid H_2G^k_n \otimes M^0_n)_{\mathrm{SL}_n R}$.

Finally, the universal coefficient theorem provides a sequence $H_2G^k_n \otimes M^0_n \rightarrowtail E^2_{2,1} \twoheadrightarrow \mathrm{Tor}(H_1G^k_n, M^0_n)$; because the restriction of $d^2_{2,1}$ to $H_2G^k_n \otimes M^0_n$ is onto $E^2_{0,2}$, there is a sequence

$$\ker d^2_{2,1} \mid H_2G^k_n \otimes M^0_n \to \ker d^2_{2,1} \twoheadrightarrow \mathrm{Tor}\left(H_1G^k_n, M^0_n\right).$$

Apply $H_0(\mathrm{SL}_n R; -)$ to this and identify $H_1G^k_n$ with M^0_n to get (3.6). □

3.6 PROPOSITION. *Whenever* $k > 2$, *there is an exact sequence*

$$(3.9) \qquad H_1(\mathrm{SL}_n R; I) \overset{\partial_k}{\to} \left(\pi^k_* H_3G^k_n\right)_{\mathrm{SL}_n R} \overset{\iota_*}{\to} \left(H_3G^{k-1}_n\right)_{\mathrm{SL}_n R} \twoheadrightarrow \left(\mathrm{im}\, d^2_{3,0}\right)_{\mathrm{SL}_n R},$$

where I *is the* $\mathrm{SL}_n R$*-submodule of* $M^0_n \otimes M^0_n$ *generated by* $\underline{1}_{12} \otimes \underline{1}_{23} + \underline{1}_{43} \otimes \underline{1}_{14}$.

PROOF. Consider the spectral sequence (3.1)(k). There can be no transgressions from $E^3_{3,0}$, as $E^3_{0,2} = 0$ (proof of Proposition 3.1(iii)). Thus there is a sequence $\pi^k_* H_3G^k_n \rightarrowtail H_3G^{k-1}_n \twoheadrightarrow \mathrm{im}\, d^2_{3,0}$. Apply $H_*(\mathrm{SL}_n R; -)$ to this to get (3.9) except that ι_* has kernel $\partial(H_1(\mathrm{SL}_n R; \mathrm{im}\, d^2_{3,0}))$.

Identify $H_1G_n^{k-1} \otimes M_n^0(t^{k-1})$ with $M_n^0 \otimes M_n^0$ via 3.1(i) and identify I with $\{d_{3,0}^2(x)\}$ as in 3.3(proof). As in the last part of the proof of 3.3, $\operatorname{im} d_{3,0}^2/I$ is the trivial $\mathrm{SL}_n R$-module $(\operatorname{im} d_{3,0}^2)_{\mathrm{SL}_n R}$. So $H_1(\mathrm{SL}_n R;\ \operatorname{im} d_{3,0}^2/I) = 0$, implying an epimorphism: $H_1(\mathrm{SL}_n R;\ I) \twoheadrightarrow H_1(\mathrm{SL}_n R;\ \operatorname{im} d_{3,0}^2)$. ∂_k is the composite of this epimorphism and ∂. □

4. K_3 of truncated polynomial rings over the integers. The first three subsections refer to general rings R and concern the $E_{1,2}^*$ terms in the spectral sequences

$$(4.1)(k) \qquad {}^kE_{**}^2 = H_*\left(\mathrm{SL}_n R;\ H_*G_n^k\right) \Rightarrow H_*\mathrm{SL}_n R_k, \qquad k \geqslant 2.$$

Lemmas 4.4 and 4.5 determine $H_1(\mathrm{SL}_n \mathbf{Z};\ \Lambda^2 M_n^0)$ and $H_1(\mathrm{SL}_n \mathbf{Z};\ I)$. This and earlier work yield the main theorem which computes $H_3\mathrm{SL}_n \mathbf{Z}[t]/(t^k)$, $k \geqslant 2$.

4.1 Proposition. *For $k > 2$ there is an exact sequence*

$$(4.2) \qquad (R + N_k)/R \to H_1\left(\mathrm{SL}_n R;\ H_2G_n^k\right) \xrightarrow{\pi_*} H_1\left(\mathrm{SL}_n R;\ J_n \equiv \pi_*^3 H_2G_n^3\right),$$

where N_k is the $\mathrm{SL}_n R$-coinvariance of the image of the $d_{3,0}^2$-differential in the spectral sequence (3.1)(k), *and $R + N_k = R$ if $H_1(\mathrm{SL}_n R;\ M_n^0) = 0$ (see Proposition* 3.3).

Proof. Proposition 3.3 provides a sequence $N_k \rightarrowtail \Omega \oplus R \to \ker \pi_*^k \twoheadrightarrow M_n^0$, in which $(\ker \pi_*^k)_{\mathrm{SL}_n R} \cong \operatorname{im} R = (R + N_k)/N_k$ (implying the splitting $(\Omega \oplus R)/N_k \cong ((R + N_k)/N_k) \oplus (\Omega \oplus R)/(R + N_k)$). Application of $H_*(\mathrm{SL}_n R;\ -)$ to the sequence (3.3)(k) yields

$$(4.3) \quad H_1\left(\mathrm{SL}_n R;\ \ker \pi_*^k\right) \rightarrowtail H_1\left(\mathrm{SL}_n R;\ M_n^0\right)$$
$$= \Omega \to (\Omega \oplus R)/N_k \twoheadrightarrow (\ker \pi_*^k)_{\mathrm{SL}_n R} = (R + N_k)/N_k.$$

Thus $H_1(\mathrm{SL}_n R;\ \ker \pi_*^k) \cong (R + N_k)/R$; its image under the map induced by inclusion is the kernel of

$$\pi_*\colon \left(H_1\left(\mathrm{SL}_n R;\ H_2G_n^k\right) \to H_1\left(\mathrm{SL}_n R;\ \pi_*^k H_2G_n^k\right) \overset{\sigma}{\rightarrowtail} H_1\left(\mathrm{SL}_n R;\ J_n\right)\right),$$

where the injection σ is implied by Lemma 3.4. □

4.2 Proposition. *If $H_2(\mathrm{SL}_n R;\ M_n^0) = 0$ and $s\colon J_n \rightarrowtail \Lambda^2 M_n^0$ is as in* (3.2), *then there is an injection*

$$s_*\colon H_1\left(\mathrm{SL}_n R;\ J_n\right) \to H_1\left(\mathrm{SL}_n R;\ H_2G_n^2 = \Lambda^2 M_n^0\right).$$

If $H_1(\mathrm{SL}_n R;\ M_n^0) = 0$, s_* is an isomorphism.

Proof. Apply $H_*(\mathrm{SL}_n R;\ -)$ to the exact sequence $J_n \overset{s}{\rightarrowtail} \Lambda^2 M_n^0 \twoheadrightarrow M_n^0$ of (3.2) and use the assumption. □

4.3 Proposition. *Suppose $H_i(\mathrm{SL}_n R;\ M_n^0) = 0$ for $i = 1, 2$. Denote E_{**}^* terms in the spectral sequence* (4.1)(k) *by ${}^kE_{**}^*$, $k \geqslant 2$. Then $\pi_*\colon {}^kE_{1,2}^\infty \to {}^2E_{1,2}^\infty$ is an injection.*

Proof. By 4.1 and the assumption, ${}^kE_{1,2}^2$ injects into $H_1(\mathrm{SL}_n R;\ J_n)$ which by 4.2 is isomorphic to ${}^2E_{1,2}^2$. If $j < 2$, $\pi_*\colon H_jG_n^k \to H_jG_n^2$ is an isomorphism (Lemma 3.1) so that $\pi_*\colon {}^kE_{1,2}^r \to {}^2E_{1,2}^r$ is an injection for $r \geqslant 2$. □

The remainder of this section is devoted to computing $K_3\mathbf{Z}[t]/(t^k)$; two preliminary lemmas are required. Let M_n^0R denote the submodule of zero-trace matrices in the $\mathrm{SL}_n R$-module M_nR of $n \times n$ matrices over R.

4.4 LEMMA. *If n is large and p is any prime,*
(i) $H_1(\mathrm{SL}_n \mathbf{Z}/p^2;\ M_n^0\mathbf{Z}/p \otimes M_n^0\mathbf{Z}/p) = 0$; $H_1(\mathrm{SL}_n \mathbf{Z};\ M_n^0\mathbf{Z}/p \otimes M_n^0\mathbf{Z}/p) = 0$; $H_1(\mathrm{SL}_n \mathbf{Z};\ M_n^0\mathbf{Z} \otimes M_n^0\mathbf{Z}) = 0$;
(ii) $H_1(\mathrm{SL}_n \mathbf{Z};\ I) = 0$, *where I is as in* 3.2.

PROOF. (i) Take n large and prime to p and to $p-1$. Consider the spectral sequence

$$(4.4)\qquad E^2_{**}(C) = H_*\big(\mathrm{SL}_n \mathbf{Z}/p;\ H_*\big(M_n^0\mathbf{Z}/p;\ C\big)\big) \Rightarrow H_*\big(\mathrm{SL}_n \mathbf{Z}/p^2;\ C\big),$$

firstly with $C = M_n^0\mathbf{Z}/p$. The proof of [**ALSS**, part 1, VI, 1.1] demonstrates an isomorphism which is dual to $d^2_{2,0}$. Moreover, $M_n^0\mathbf{Z}/p$ may be viewed as a direct summand of $M_n^0\mathbf{Z}/p \otimes M_n^0\mathbf{Z}/p$ using either of the inclusions

$$(4.5)\qquad \Phi\colon \underline{\alpha}_{ij} \mapsto \left[\sum_{k=1}^{n} \underline{\alpha}_{ik} \otimes \underline{1}_{kj}\right] \quad\text{or}\quad \Phi^T\colon \underline{\alpha}_{ij} \mapsto \left[\sum_{k=1}^{n} -\underline{1}_{kj} \otimes \underline{\alpha}_{ik}\right],$$

where if $a \otimes b \in M_n\mathbf{Z}/p \otimes M_n\mathbf{Z}/p$, $[a \otimes b]$ is its image in $M_n^0\mathbf{Z}/p \otimes M_n^0\mathbf{Z}/p$ under the canonical projection $(M_n\mathbf{Z}/p)^{\otimes 2} = (M_n^0\mathbf{Z}/p \oplus \mathbf{Z}/p)^{\otimes 2} \to (M_n^0\mathbf{Z}/p)^{\otimes 2}$. (Since $p \nmid n$, Φ and Φ^T are split by the Lie bracket.)

Now

$$\begin{aligned} E^2_{0,1}\big(M_n^0\mathbf{Z}/p \otimes M_n^0\mathbf{Z}/p\big) &\cong \Big[\big(M_n^0\mathbf{Z}/p\big)^{\otimes 3}\Big]_{\mathrm{SL}_n \mathbf{Z}/p} \\ &= \Phi_*\big(E^2_{0,1}\big(M_n^0\mathbf{Z}/p\big)\big) \oplus \Phi^T_*\big(E^2_{0,1}\big(M_n^0\mathbf{Z}/p\big)\big), \end{aligned}$$

from A.2(i) and (iii). Therefore by naturality,

$$E^2_{0,1}\big(M_n^0\mathbf{Z}/p \otimes M_n^0\mathbf{Z}/p\big) = d^2_{2,0}\big(\Phi_*\big(E^2_{2,0}\big(M_n^0\mathbf{Z}/p\big)\big) + \Phi^T_*\big(E^2_{2,0}\big(M_n^0\mathbf{Z}/p\big)\big)\big).$$

The proof of the first equality in part (i) is completed by recalling that

$$\big(E^2_{1,0}\big(M_n^0\mathbf{Z}/p \otimes M_n^0\mathbf{Z}/p\big) \equiv H_1\big(\mathrm{SL}_n \mathbf{Z}/p;\ M_n^0\mathbf{Z}/p \otimes M_n^0\mathbf{Z}/p\big)\big) = 0,$$

dually to [**ALSS**, part 1, II, 1.4]. Given this, Kassel [**K2**, 3.4] asserts that the second equality holds. This then implies that multiplication by p is an epimorphism on $H_1(\mathrm{SL}_n \mathbf{Z};\ M_n^0\mathbf{Z} \otimes M_n^0\mathbf{Z})$ which, since we are dealing with a finitely generated abelian group (e.g. [**B**, p. 217]) means the group is torsion with trivial p-component. However, for large enough n the inclusion: $M_n^0\mathbf{Z} \to M_{n+m}^0\mathbf{Z}$ induces isomorphisms

$$H_i\big(\mathrm{SL}_n \mathbf{Z};\ M_n^0\mathbf{Z} \otimes M_n^0\mathbf{Z}\big) \to H_i\big(\mathrm{SL}_{n+m}\mathbf{Z};\ M_{n+m}^0\mathbf{Z} \otimes M_{n+m}^0\mathbf{Z}\big), \qquad m \geqslant 1$$

(e.g. [**VK**, §5]). Thus for n sufficiently large, $H_1(\mathrm{SL}_n\mathbf{Z};\ M_n^0\mathbf{Z} \otimes M_n^0\mathbf{Z})$ has no p-component for any prime p, giving (i).

(ii) If $I \subset M_n^0\mathbf{Z} \otimes M_n^0\mathbf{Z}$ is as in 3.2, $I \otimes \mathbf{Z}/p \subset M_n^0\mathbf{Z}/p \otimes M_n^0\mathbf{Z}/p$ is the $\mathrm{SL}_n \mathbf{Z}/p$-module generated by $\underline{1}_{12} \otimes \underline{1}_{23} + \underline{1}_{43} \otimes \underline{1}_{14}$, and equation (A.7) (see A.2(iii) proof) shows that $(I \otimes M_n^0\mathbf{Z}/p)_{\mathrm{SL}_n \mathbf{Z}/p} = \mathbf{Z}/p$, generated by the class of $\underline{1}_{12} \otimes \underline{1}_{23} \otimes \underline{1}_{31} + \underline{1}_{43} \otimes \underline{1}_{14} \otimes \underline{1}_{31}$. Embed $M_n^0\mathbf{Z}/p$ into $I \otimes \mathbf{Z}/p$ with the map $\Phi - \Phi^T$, then argue as for part (i). $\square$

4.5 LEMMA, *If n is large and p is any odd prime,* $H_1(\mathrm{SL}_n\mathbf{Z};\ \Lambda^2 M_n^0\mathbf{Z}/p) = 0$; $H_1(\mathrm{SL}_n\mathbf{Z};\ \Lambda^2 M_n^0\mathbf{Z}/2) = \mathbf{Z}/2$; $H_1(\mathrm{SL}_n\mathbf{Z};\ \Lambda^2 M_n^0\mathbf{Z}) = 0$.

PROOF. If p is odd, $\Lambda^2 M_n^0\mathbf{Z}/p$ is a direct summand of $M_n^0\mathbf{Z}/p \otimes M_n^0\mathbf{Z}/p$, so the first equality follows from 4.4(i).

Define Φ': $M_n^0\mathbf{Z}/2 \to \Lambda^2 M_n^0\mathbf{Z}/2$ to be the composite of Φ (defined in (4.5)) with the homology product on $H_* M_n^0\mathbf{Z}/2$. The map Φ'_*: $E^2_{0,1}(M_n^0\mathbf{Z}/2) \to E^2_{0,1}(\Lambda^2 M_n^0\mathbf{Z}/2)$ between terms in the spectral sequences (4.4) is an isomorphism, by A.2. Arguing as in the proof of 4.4, we conclude that $E^3_{0,1}(\Lambda^2 M_n^0\mathbf{Z}/2) = 0$. Thus the reduction epimorphism: $H_1(\mathrm{SL}_n\mathbf{Z}/4;\ \Lambda^2 M_n^0\mathbf{Z}/2) \to H_1(\mathrm{SL}_n\mathbf{Z}/2;\ \Lambda^2 M_n^0\mathbf{Z}/2)$ is injective; its image is identified as $\mathbf{Z}/2$ in [**ALSS**, part 3, 9.16]. Again apply the Kassel result [**K2**, 3.4] to conclude that $H_1(\mathrm{SL}_n\mathbf{Z};\ \Lambda^2 M_n^0\mathbf{Z}/2) = \mathbf{Z}/2$.

Now $\mathrm{SL}_n\mathbf{Z}$ acts on $\Lambda^2 M_n^0\mathbf{Z}/2$ via reduction to $\mathrm{SL}_n\mathbf{Z}/2$. Thus we can use Lemma A.2 to see that each of the terms in the exact sequence $\Lambda^2 M_n^0\mathbf{Z} \overset{2}{\rightarrowtail} \Lambda^2 M_n^0\mathbf{Z} \twoheadrightarrow \Lambda^2 M_n^0\mathbf{Z}/2$ has $\mathrm{SL}_n\mathbf{Z}$-coinvariance $\mathbf{Z}/2$. In the long exact sequence obtained on application of $H_*(\mathrm{SL}_n\mathbf{Z};\ -)$ to this coefficient sequence, the connecting homomorphism $H_1(\mathrm{SL}_n\mathbf{Z};\ \Lambda^2 M_n^0\mathbf{Z}/2) \to \mathbf{Z}/2$ must therefore be onto, hence injective.

So multiplying the coefficients in $H_1(\mathrm{SL}_n\mathbf{Z};\ \Lambda^2 M_n^0\mathbf{Z})$ by two results in an onto map. Since we are dealing with a finitely generated group, it must be an odd torsion group. The first equality in the statement of the lemma then implies it is trivial. □

4.6 THEOREM. *If $k \geqslant 2$ and n is large there are exact sequences*

$$\mathbf{Z} \oplus \mathbf{Z} \rightarrowtail H_3\mathrm{SL}_n\mathbf{Z}_{k+1} \overset{\pi_*^{k+1}}{\to} H_3\mathrm{SL}_n\mathbf{Z}_k = \mathbf{Z}^{k-1} \oplus \mathbf{Z}/24 \twoheadrightarrow \mathbf{Z},$$

and

$$\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}/2 \overset{1\oplus 1\oplus \iota}{\to} K_3\mathbf{Z}_{k+1} = K_3\mathbf{Z} \oplus \mathbf{Z}^k \oplus U_{k+1} \to K_3\mathbf{Z}_k \twoheadrightarrow \mathbf{Z},$$

where U_k is an elementary 2-group of rank at most $[k/2]$ and $\iota = 0$ if k is even. (Here $\mathbf{Z}_k$ denotes $\mathbf{Z}[t]/(t^k)$.)

PROOF. Take $R = \mathbf{Z}$. If $n > 10$, $H_2(\mathrm{SL}_n\mathbf{Z};\ M_n^0) = 0$ [**K1**], and of course, $\Omega \equiv \Omega^1_{\mathbf{Z}/\mathbf{Z}} = 0$. $H_1(\mathrm{SL}_n\mathbf{Z};\ \Lambda^2 M_n^0) = 0$ by Lemma 4.5, so that by Propositions 3.1 and 4.3, in the spectral sequence (4.1)(k), $k \geqslant 2$, ${}^kE^\infty_{1,2} = 0$ and ${}^kE^2_{2,1} = 0$. Thus there is an epimorphism, induced by the inclusion of G_n^k into $\mathrm{SL}_n R_k$,

$$i_*^k\colon \left(H_3G_n^k\right)_{\mathrm{SL}_n R} \twoheadrightarrow H_3\mathrm{SL}_n R_k / H_3\mathrm{SL}_n R. \tag{4.6}$$

Take $k \geqslant 3$. By (4.6), $\ker \pi_*^k | H_3\mathrm{SL}_n R_k$ is the quotient of

$$V_k = \ker\left(i_*^{k-1} \circ \pi_*^k\colon \left(H_3G_n^k\right)_{\mathrm{SL}_n R} \to H_3\mathrm{SL}_n R_{k-1}\right)$$

by $\ker i_*^k$. Use (3.9) and Lemma 4.4 to identify

$$\ker\left(\ :\left(\pi_*^k H_3G_n^k\right)_{\mathrm{SL}_n R} \to \left(H_3G_n^{k-1}\right)_{\mathrm{SL}_n R}\right) = 0,$$

then use this fact in constructing the exact sequence

$$\left(\ker \pi_*^k | H_3G_n^k\right)_{\mathrm{SL}_n R} \overset{\sigma}{\to} V_k \to \ker\left(i_*^{k-1} | \left(H_3G_n^{k-1}\right)_{\mathrm{SL}_n R}\right). \tag{4.7}$$

By Propositions 3.5 and 3.1, $\operatorname{im}\sigma$ has at most two generators. For the case $k = 2$, we have $H_3\mathrm{SL}_n\mathbf{Z} = \mathbf{Z}/24$ (e.g. [**ALSS**, part 1, VI]) and by A.2(v), $(H_3G_n^2)_{\mathrm{SL}_n R} = \mathbf{Z}$. So $H_3\mathrm{SL}_n R_2 \leqslant \mathbf{Z} \oplus \mathbf{Z}/24$.

Now suppose $n = \infty$. Theorem 2.1 of [**A**] proposes an exact sequence

$$(4.8) \qquad K_2R_r \otimes \mathbf{Z}/2 \to K_3R_r \twoheadrightarrow H_3\mathrm{SL}\,R_r \qquad (r \geqslant 1).$$

There is a canonical epimorphism: $K_3\mathbf{Z}_k \twoheadrightarrow \mathbf{Z}^{k-1}$, with $K_3\mathbf{Z}_{k-1}/\pi_*K_3\mathbf{Z}_k = \mathbf{Z}$; hence $H_3\mathrm{SL}_n R_2 = \mathbf{Z} \oplus \mathbf{Z}/24$ and if $k \geqslant 3$ there is an induced epimorphism: $\ker(K_3\mathbf{Z}_k \to K_3\mathbf{Z}_{k-1}) \twoheadrightarrow \mathbf{Z} \oplus \mathbf{Z}$ (Stienstra [**S**, Theorem 1.13]). As $K_3R_k = H_3\mathrm{SL}\,R_k$ off torsion, an inductive argument from the initial case $k = 3$ shows that in (4.7), $\ker i_*^{k-1} = 0$ and $\operatorname{im}\sigma = \mathbf{Z} \oplus \mathbf{Z} = \ker \pi_*^k \mid H_3\mathrm{SL}_n R_k$. The first of the exact sequences in the theorem statement follows.

The theorem for general n is a consequence of the stability of the generators. The K-theory result is implied by the $n = \infty$ part of the earlier statement, and (4.8), together with the identifications [**Ro**, Theorem 4] $K_2\mathbf{Z}_k = \mathbf{Z}/2 \oplus (\oplus_{i=2}^k \mathbf{Z}/i)$ and $K_3\mathbf{Z} = \mathbf{Z}/48$. □

5. K_3 of truncated polynomial rings over finite fields. Throughout this section, R is a finite field of characteristic p greater than 2, and n is a large integer. We obtain K_3R_k by using a van der Kallen-Stienstra result in conjunction with an estimate of the quotient of group orders $\#K_3R_{k+1}/\#K_3R_k$. The latter is obtained from the spectral sequences and associated maps:

$$\begin{array}{llcl} (5.1)(k+1) & H_*(\mathrm{SL}_n R_k;\ H_*M_n^0(t^k)) & \Rightarrow & H_*\mathrm{SL}_n R_{k+1} \\ & \quad\uparrow (i_*,1) & & \quad\uparrow i_* \qquad (k \geqslant 2). \\ (5.2)(k+1) & H_*(G_n^k;\ H_*M_n^0(t^k)) & \Rightarrow & H_*G_n^{k+1} \end{array}$$

The first subsection reviews various groups $H_i(\mathrm{SL}_n R;\ H_jM_n^0)$ for $j \leqslant 3$. The next four subsections look at E_{**}^* terms of total degree 3 in the spectral sequence (5.1)($k+1$), from which $\ker \pi_*^{k+1} \mid H_3\mathrm{SL}_n R_{k+1}$ is estimated in 5.6. Proposition 5.7 determines $H_3\mathrm{SL}_n R_k/\pi_*H_3\mathrm{SL}_n R_{k+1}$ then obtains the desired quotient of group orders. Finally, the computation of $K_2(R_q, (t^k))$ in [**VKS**] is invoked to give the main theorem, 5.9.

Our constraints on the dimension n are introduced by Proposition 5.1. The results of Lluis [**ALSS**], phrased in terms of the general linear group $\mathrm{GL}_n R$ and the full matrix group M_nR, hold for large n. By restricting to n relatively prime to p and $p-1$, his results are simply expressed in terms of the respective direct summands $\mathrm{SL}_n R$ and M_n^0; however, applying stability theorems such as those in [**VK**], we need only assume n to be "sufficiently large".

5.1 PROPOSITION. (i) $H_i(\mathrm{SL}_n R;\ M_n^0) = 0 = H_i(\mathrm{SL}_n R;\ \Lambda^2 M_n^0)$ *for* $i = 0$ *or* 1.
(ii) $H_0(\mathrm{SL}_n R;\ H_3M_n^0) = (S^2M_n^0)_{\mathrm{SL}_n R} \oplus (\Lambda^3 M_n^0)_{\mathrm{SL}_n R} = R \oplus R$.
(iii) $H_1(\mathrm{SL}_n R;\ M_n^0 \otimes M_n^0) = 0$.
(iv) $H_2(\mathrm{SL}_n R;\ M_n^0) = R$.

PROOF. (i), (ii) and (iv) are derived from Lluis [**ALSS**]. The proof of (iii) is analogous to that used in [**ALSS**] to prove $H_1(\mathrm{SL}_n R;\ \Lambda^2 M_n^0) = 0$. □

5.2 LEMMA. *For $k \geqslant 2$, $R \cong H_1(\mathrm{SL}_n R_k; \wedge^2 M_n^0(t^k))$.*

PROOF. $\mathrm{SL}_n R_k = \mathrm{SL}_n R \ltimes G_n^k$, so that for any coefficient module C,

$$H_1(\mathrm{SL}_n R_k; C) = H_1(\mathrm{SL}_n R; C) \oplus \left(H_1(G^k; C)\right)_{\mathrm{SL}_n R}.$$

Set $C = \wedge^2 M_n^0$ and use 5.1(i), the isomorphism $\pi_*\colon H_1 G_n^k \to M_n^0$ of 3.1(i), and A.2(iv). □

5.3 LEMMA. *For $k \geqslant 2$, $j_* H_3 M_n^0(t^k) = 0$ in $H_* G_n^{k+1}$ and hence $i_* \circ j_* H_3 M_n^0(t^k) = 0$ in $H_* \mathrm{SL}_n R_{k+1}$.*

PROOF. The case $k = 2$ is [**ALS**, 2.7 and 2.8(iii)]. Take $k \geqslant 3$. Then according to Proposition 3.1(iii), $E_{0,1}^3 = 0$ in the spectral sequence (5.2)($k + 1$). Further, it implies $H_2 G_n^k / \pi_* H_2 G_n^{k+1}$ is represented in $E_{1,1}^\infty$.

If β is the homology Bockstein associated to the coefficient sequence $\mathbf{Z} \rightarrowtail \mathbf{Z} \twoheadrightarrow \mathbf{Z}/p$, and $s \geqslant 1$, $\beta H_{s+1}(M_n^0; \mathbf{Z}/p) = H_s M_n^0$ so the lemma is proved by showing that $j_* H_*(M_n^0; \mathbf{Z}/p) = 0$ in $H_*(G_n^k; \mathbf{Z}/p)$, or, dually, that $j^* H^*(G_n^k; \mathbf{Z}/p) = 0$. Consider the cohomology spectral sequence

$$(5.3)(k+1) \quad H^*\left(G_n^k; \mathbf{Z}/p\right) \otimes H^*\left(M_n^0; \mathbf{Z}/p\right) \Rightarrow H^*\left(G_n^{k+1}; \mathbf{Z}/p\right), \qquad k > 2.$$

Denote terms in this sequence by ${}^{k+1}E_*^{**}$.

Dually to the homology results, ${}^{k+1}E_3^{0,1} = 0$ and $\operatorname{im} d_2^{0,1}$ lies in a direct summand of $H^2(G_n^k; \mathbf{Z}/p)$ which is represented in ${}^kE_\infty^{1,1}$. (In (3.3)(k), the map D induced by the homology differential is split since $H_1(\mathrm{SL}_n R; M_n^0) \equiv \Omega \equiv 0$.)

Because $d_2^{0,*} | \wedge^* M_n^{0\#}$ is a derivation with image in the $\mathbf{Z}/p$-vector space $H^2 G_n^k \otimes \wedge^* M_n^{0\#}$, the injectivity of $d_2^{0,1}$ implies that of $d_2^{0,*} | \wedge^* M_n^{0\#}$. Now suppose $\beta^\#$ is the cohomology Bockstein associated to the coefficient sequence $\mathbf{Z}/p \rightarrowtail \mathbf{Z}/p^2 \twoheadrightarrow \mathbf{Z}/p$. Since M_n^0 is a $\mathbf{Z}/p$-vector space, $H^*(M_n^0; \mathbf{Z}/p) = \wedge^* M_n^{0\#} \otimes S^*(\beta^\# M_n^{0\#})$. Thus $(E_3^{0,*} = \ker d_2^{0,*}) = S^*(\beta^\# M_n^{0\#})$. We want to show that $d_3^{0,2} \beta^\# M_n^{0\#} \cong M_n^{0\#}$.

Observe first that the connecting homomorphism $\beta^\#$ commutes with spectral sequence differentials which are transgressive. (This can be seen from the geometric description of the transgression of e.g. [**M**, p. 335].) So $d_3^{0,2} \beta^\# M_n^{0\#} = \beta^\# d_2^{0,1} M_n^{0\#}$ modulo $\operatorname{im} d_2^{1,1}$.

We claim next that $\beta^\# d_2^{0,1} M_n^{0\#}$ is represented in ${}^kE_\infty^{1,2}$ by a submodule isomorphic to $M_n^{0\#}$. To see this, at the cochain level apply the Cartan formula, $\beta^\#(a \otimes b) = (\beta^\# a) \otimes b \pm a \otimes \beta^\# b$, which is a consequence of the definition of the connecting homomorphism in terms of the cochain boundary maps, which are derivations (see e.g. [**M**, pp. 190, 52]). Note that here we are dealing with the cochain bicomplex underlying the spectral sequence (5.3)(k), *not* (5.3)($k + 1$). If $\tilde{x} \neq 0$ in ${}^{k+1}E_2^{0,1}$ and $x \neq 0$ represents $d_2^{0,1} \tilde{x}$ in ${}^kE_2^{1,1}$, then $(1 \otimes \beta^\#)x$ represents $\beta^\# d_2^{0,1} \tilde{x}$ in ${}^kE_2^{1,2}$, and is nonzero because $1 \otimes \beta^\# | {}^kE_2^{*,1}$ is injective. Since there are no higher differentials hitting ${}^kE_r^{1,*}$ terms, $\{(1 \otimes \beta^\#)x\}$ is nonzero in ${}^kE_\infty^{1,2}$. So $\beta^\# d_2^{0,1} M_n^{0\#} \cong M_n^{0\#}$ and is represented in ${}^kE_\infty^{1,2}$.

On the other hand, because the differential is a derivation, $\operatorname{im} d_2^{1,*}$ is represented in

$$({}^kE_\infty^{1,0} \cup {}^kE_\infty^{1,1}) \cup H^*(M_n^0; \mathbf{Z}/p) = {}^kE_\infty^{2,1} \cup H^*(M_n^0; \mathbf{Z}/p)$$

or in a lower filtration. Therefore $\operatorname{im}(\beta^{\#} d_2^{0,1}) \cap \operatorname{im}(d_2^{1,*}) = 0$.

We conclude that $d_3^{0,2} \mid \beta^{\#} M_n^{0\#}$ is injective. Hence so is $d_3^{0,*} \mid S^*(\beta^{\#} M_n^{0\#})$. □

5.4 Lemma. *$H_2(\mathrm{SL}_n R_k;\ M_n^0) \leqslant R \oplus R$ whenever $k \geqslant 2$.*

Proof. In the spectral sequence

$${}^kD_{**}^2 = H_*(\mathrm{SL}_n R;\ H_*(G_n^k;\ M_n^0)) \Rightarrow H_*(\mathrm{SL}_n R_k;\ M_n^0), \quad {}^kD_{1,1}^2 = 0$$

(by 5.1(iii) and 3.1) and ${}^kD_{2,0}^2 = R$ (by 5.1(iv)). Further, there is a sequence

(5.4)(k)

$$(H_2 G_n^k \otimes M_n^0)_{\mathrm{SL}_n R} \to {}^kD_{0,2}^2 = (H_2(G_n^k;\ \mathbf{Z}/p) \otimes M_n^0)_{\mathrm{SL}_n R} \twoheadrightarrow (\mathrm{Tor}(M_n^0, M_n^0))_{\mathrm{SL}_n R},$$

where the final term is isomorphic to $(M_n^0 \otimes M_n^0)_{\mathrm{SL}_n R}$.

Take $k > 2$. Using the notation and proof of 3.4, there are exact sequences

$$U^k \otimes M_n^0 \rightarrowtail H_2 G_n^k \otimes M_n^0 \twoheadrightarrow J_n \otimes M_n^0; \quad \tilde{U}^k \otimes M_n^0 \rightarrowtail U^k \otimes M_n^0 \twoheadrightarrow M_n^0 \otimes M_n^0,$$

where $\mathrm{SL}_n R$ acts trivally on $\tilde{U}_k$. Take $\mathrm{SL}_n R$-coinvariance of these and apply the A.2 results $(J_n \otimes M_n)_{\mathrm{SL}_n R} = 0$ and $(M_n^0 \otimes M_n^0)_{\mathrm{SL}_n R} = R$ to conclude that $(H_2 G_n^k \otimes M_n^0)_{\mathrm{SL}_n R}$ is dominated by R. Thus ${}^kD_{0,2}^2 \leqslant R \otimes R$. When $k = 2$, the same result holds because $H_2 G_n^2 \otimes M_n^0 \cong \Lambda^2 M_n^0 \otimes M_n^0$ and so by A.2(iv) has $\mathrm{SL}_n R$-coinvariance R.

The next lemma shows that there is a surjection $\operatorname{im} d_{2,1}^2 \twoheadrightarrow R$. Since

$$H_2(\mathrm{SL}_n R_k;\ M_n^0) = {}^kD_{2,0}^2 \oplus {}^kD_{1,1}^2 \oplus {}^kD_{0,2}^3,$$

it will prove this lemma.

5.5 Lemma. *In the spectral sequence ${}^kD_{**}^2 \Rightarrow H_*(\mathrm{SL}_n R_k;\ M_n^0)$ of 5.4 (proof), $\operatorname{im} d_{2,1}^2$ maps onto R.*

Proof. The case $k = 2$ is covered by [**ALS**, Theorem 2.2], which shows that $\operatorname{im} d_{2,1}^2$ maps onto $(\mathrm{Tor}(M_n^0, M_n^0))_{\mathrm{SL}_n R}$.

If $k \geqslant 3$, let τ: $M_n^0(t^k) \to M_n^0(t^2)$ be the coefficient isomorphism. The spectral sequence map $\pi_* \tau_*$: ${}^kE_{0,2}^2 \to {}^2E_{0,2}^2$ induces an isomorphism between the terms $(\mathrm{Tor}(M_n^0, M_n^0))_{\mathrm{SL}_n R}$; and $\pi_* \tau_*$ also induces an isomorphism

$$H_2(\mathrm{SL}_n R;\ H_1(G_n^k;\ M_n^0(t^k))) \to H_2(\mathrm{SL}_n R;\ H_1(M_n^0;\ M_n^0(t^2))),$$

because π_*: $H_1(G_n^k;\ \mathbf{Z}/p) \cong H_1(G_n^2;\ \mathbf{Z}/p)$ by 3.1(i). The lemma, and hence Lemma 5.4, follows. □ □

5.6 Proposition. *If $k > 2$, $\#\ker(\pi_*^k$: $H_3 \mathrm{SL}_n R_k \to H_3 \mathrm{SL}_n R_{k-1}) \leqslant \#(R)^3$.*

Proof. The spectral sequence (5.1)(k) has $E_{0,3}^\infty = 0$ (by 5.3), $E_{1,2}^\infty \leqslant R$ (by 5.2) and $E_{2,1}^\infty \leqslant R \oplus R$ (by 5.4). □

We next investigate

$$H_3\mathrm{SL}_n R_{k-1}/\pi_* H_3\mathrm{SL}_n R_k$$

to estimate $\#H_3\mathrm{SL}_n R_k/\#H_3\mathrm{SL}_n R_{k-1}$.

5.7 PROPOSITION. *If $k > 2$, there is an exact sequence*

$$\ker \pi_*^k \rightarrowtail H_3\mathrm{SL}_n R_k \xrightarrow{\pi_*^k} H_3\mathrm{SL}_n R_{k-1} \twoheadrightarrow R.$$

Hence with Proposition 5.6 *we have*

$$(5.6) \qquad \#H_3\mathrm{SL}_n R_k/\#H_3\mathrm{SL}_n R_{k-1} \leqslant (\#R)^2.$$

PROOF. In the spectral sequence (5.1)(k), $E^2_{0,2} = ((H_2 M_n^0(t^k))_{\mathrm{SL}_n R})_{G_n^k} = 0$ by 5.1(i). Therefore $\operatorname{coker} \pi_*^k = \operatorname{im} d^2_{3,0}$, and this is $E^2_{1,1}$ because $H_2\mathrm{SL}_n R_k \equiv K_2(n, R_k) = 0$ [**DS**, 4.4]. Finally, $E^2_{1,1} = H_1(\mathrm{SL}_n R;\ M_n^0(t^k)) \oplus (H_1 G_n^k \otimes M_n^0)_{\mathrm{SL}_n R} \cong R$, following the proof of 5.2 and substituting the isomorphism of 3.1(i) and the results 5.1(i) and A.2(i). □

The main theorem is an easy corollary to 5.7 and the following theorem.

5.8 THEOREM (VAN DER KALLEN AND STIENSTRA). *If $q \gg k$, there is an isomorphism*

$$(5.7) \qquad \Delta_k\colon K_1(R_{2k}, (t))/\{1 - \alpha t^k\colon \alpha \in R\} \to K_2(R_q, (t^k)).$$

PROOF. This is a special case of [**VKS**, 4.3], given that $K_2 R_q = 0$ by the previously quoted result of Dennis and Stein. □

Note that because $K_2 R_k = 0$, $K_3(R_k, (t)) = H_3\mathrm{SL}\, R_k/H_3\mathrm{SL}\, R$.

5.9 THEOREM. *If R is a finite field of odd characteristic p, there is an isomorphism*

$$\partial^{-1} \circ \Delta_k\colon K_1(R_{2k}, (t))/\{1 - \alpha t^k\colon \alpha \in R\} \to K_3(R_k, (t)),$$

where Δ_k is as in 5.8 *and $\partial\colon K_3(R_k, (t)) \to K_2(R_q, (t^k))$ is the connecting homomorphism in the long exact K-sequence ($q \gg k$).*

If n is large then also $H_3\mathrm{SL}_n R_k/H_3\mathrm{SL}_n R \cong K_3(R_k, (t))$ under the map induced by the inclusion $\mathrm{SL}_n R_k \to \mathrm{SL}\, R_k$.

PROOF. If $k = 2$ or 3 the theorem comes from [**ALS**, Theorem 1.1], given the stability of the generators used in that proof. Inductive application of (5.6) yields the group order estimate

$$\#H_3\mathrm{SL}_n R_k/H_3\mathrm{SL}_n R \leqslant (\#R)^{2k-2}.$$

However, the order of the domain of $\partial^{-1} \circ \Delta_k$ is $(\#R)^{2k-2}$; since ∂ is onto, $\#H_3\mathrm{SL}\, R_k/H_3\mathrm{SL}\, R$ must be at least as great. This implies the theorem when $n = \infty$. The general case comes from the stability theorems of, for example, [**VK**]. □

Appendix. This Appendix investigates the $\mathrm{SL}_n R$-structure of various modules related to $M_n^{0\otimes i}$, $i = 2$ or 3. There are 2 subsections.

Notation is as in §2.

A.1 LEMMA. *Let I be the* $\mathrm{SL}_n R$*-submodule of* $M_n^0 \otimes M_n^0$ *generated by* $\underline{1}_{12} \otimes \underline{1}_{23} + \underline{1}_{43} \otimes \underline{1}_{14}$. *Let* $(M_n^0 \otimes M_n^0)^0$ *be the kernel of the quotient map of* $M_n^0 \otimes M_n^0$ *onto its* $\mathrm{SL}_n R$*-coinvariance* R [**K3**, 3.7]. *Let* $\mathrm{St}(R, R)$ *be the additive Steinberg group (defined in* [**K4**, 1.4]) *where by* [**K4**, 2.15] *there is an epimorphism* $\phi\colon \mathrm{St}(R,R) \twoheadrightarrow M_n^0$ *such that* $(\mathrm{St}(R,R), \phi)$ *is the universal* $\mathrm{SL}_n R$*-central extension of* M_n^0, *with kernel* Ω.

Then

(i) $(I)_{\mathrm{SL}_n R} = 0$.

(ii) $(M_n^0 \otimes M_n^0)^0/I \cong \mathrm{St}(R,R)$, *and if* $D^0\colon (M_n^0 \otimes M_n^0)^0/I \to M_n^0$ *is the map induced by the Lie bracket*, $((M_n^0 \otimes M_n^0)^0/I, D^0)$ *is the universal* $\mathrm{SL}_n R$*-central extension of* M_n^0.

(iii) $(M_n^0 \otimes M_n^0)/I \cong \mathrm{St}(R,R) \oplus R$.

PROOF. (i) I is $\mathrm{SL}_n R$-generated by $(e_{15} - 1) \cdot (\underline{1}_{52} \otimes \underline{1}_{23} + \underline{1}_{43} \otimes \underline{1}_{54})$ (use the actions given in 2.3).

(ii) We will use the following definitions:

$$\alpha, \beta, \gamma, \varepsilon \text{ and } \mu \text{ are arbitrary elements in } R;$$

$$D\colon \left(M_n^0 \otimes M_n^0\right)/I \twoheadrightarrow M_n^0 \text{ is the map induced by } [\ ,\ ];$$

$$F_{ij}(\alpha, \beta) = \underline{\alpha}_{ij} \otimes \underline{\beta}_{ji} \in M_n^0 \otimes M_n^0;$$

$$\Psi\colon R \otimes R \to \left(M_n^0 \otimes M_n^0\right)/I \text{ is: } \alpha \otimes \beta \to \left\{F_{12}(\alpha,\beta) + F_{23}(\alpha,\beta) - F_{13}(\alpha,\beta)\right\}.$$

The proof is divided into two parts. The first shows that Ψ is equivariant and that $\mathrm{im}\Psi = \ker D$, so that $((M_n^0 \otimes M_n^0)^0/I,\ D^0)$ is $\mathrm{SL}_n R$-central; the second shows that it is universal by exhibiting a map $\sigma\colon (M_n^0 \otimes M_n^0)^0/I \twoheadrightarrow \mathrm{St}(R,R)$ with $\phi \circ \sigma = D^0$.

A. Define a group homomorphism $g\colon M_n^0 \to ((M_n^0 \otimes M_n^0)/I)/\mathrm{im}\Psi$ by choosing for each pair $i \neq j$, an $m \notin \{i, j\}$ and setting $g(\underline{\alpha}_{ij}) = \{\underline{1}_{im} \otimes \underline{\alpha}_{mj}\}$, and, if $j > 1$, $g(\ddot{\underline{\alpha}}_{1j}) = \{\underline{1}_{j1} \otimes \underline{\alpha}_{1j}\}$. We are going to show that g is an epimorphism. Use 2.3 to check that $\mathrm{im}\Psi$ lies in $\ker D$; hence $[\ ,\ ]$ induces an epimorphism

$$\left(\left(M_n^0 \otimes M_n^0\right)/I\right)/\mathrm{im}\Psi \to M_n^0$$

which is a left inverse to g, so that $\mathrm{im}\Psi = \ker D$ as claimed.

Consider each of the ten elements listed below as representative of the subset of the canonical $\mathbf{Z}$-basis of $M_n^0 \otimes M_n^0$ which can be obtained from (a) the action of the permutation matrices and (b) switching of the modules M_n^0 which form the tensor product. (Recall that $\ddot{\underline{\alpha}}_{ij} = \ddot{\underline{\alpha}}_{im} + \ddot{\underline{\alpha}}_{mj} = -\ddot{\underline{\alpha}}_{ji}$, $m \notin \{i, j\}$.) The subsets are based on subscript configurations, and partition the basis. It thus suffices to show that the projections of these subsets to $((M_n^0 \otimes M_n^0)/I)/\mathrm{im}\Psi$ lie in the image of g.

1.	$\underline{\alpha}_{12} \otimes \underline{\beta}_{34}$	2.	$\ddot{\underline{\alpha}}_{12} \otimes \underline{\beta}_{34}$	3.	$\underline{\alpha}_{12} \otimes \underline{\beta}_{32}$	4.	$\underline{\alpha}_{12} \otimes \underline{\beta}_{13}$
5.	$\underline{\alpha}_{12} \otimes \underline{\beta}_{12}$	6.	$\underline{\alpha}_{12} \otimes \underline{\beta}_{23}$	7.	$\underline{\alpha}_{12} \otimes \ddot{\underline{\beta}}_{13}$	8.	$\underline{\alpha}_{12} \otimes \ddot{\underline{\beta}}_{23}$
9.	$\ddot{\underline{\alpha}} \otimes \ddot{\underline{\beta}}_{13}$	10.	$\underline{\alpha}_{12} \otimes \underline{\beta}_{21}$.				

All elements of type 1–5 *are zero modulo* I. Look at $(e_{m1}(\alpha) - 1) \cdot \underline{1}_{12} \otimes \underline{1}_{34}$, $(e_{m3}(\beta) - 1) \cdot \underline{\alpha}_{12} \otimes \underline{1}_{34}$, $(e_{12} - 1) \cdot \underline{\alpha}_{21} \otimes \underline{\beta}_{34}$, etc., and use the symmetry of the $\mathrm{SL}_n R$-action on the modules M_n^0 forming the tensor product, plus the action of the permutation matrices. I also contains elements of the types $\underline{\gamma\alpha}_{12} \otimes \underline{\beta}_{23} - \underline{\gamma}_{14} \otimes \underline{\alpha\beta}_{43}$ and $\underline{\gamma\alpha}_{23} \otimes \ddot{\underline{\beta}}_{12} - \underline{\alpha}_{43} \otimes \underline{\gamma\beta}_{24}$—consider $(e_{42}(-\alpha) - 1) \cdot \underline{\gamma}_{14} \otimes \underline{\beta}_{23}$, etc. Similarly, it contains elements of type

$$(\text{A.1}) \quad \underline{\gamma\alpha}_{12} \otimes \ddot{\underline{\beta}}_{23} - \underline{\alpha}_{14} \otimes \underline{\gamma\beta}_{42}\big((e_{42}(-\gamma) - 1) \cdot \underline{\alpha}_{14} \otimes \ddot{\underline{\beta}}_{23} + \underline{\gamma\alpha}_{12} \otimes \underline{\gamma\beta}_{42}\big).$$

The class of each element of type 6–8 *is in* im g. As above, $\underline{\varepsilon}_{im} \otimes \underline{\mu}_{mj} \equiv \ddot{\underline{\alpha}}_{im} \otimes \underline{\beta}_{ij} \equiv \underline{\alpha}_{ij} \otimes \ddot{\underline{\beta}}_{jm}$ modulo I, whenever $\alpha\beta = \varepsilon\mu$ and i, j and m are distinct. The same sort of reasoning used above, applied to the generator $\underline{1}_{13} \otimes \underline{1}_{32} + \underline{1}_{42} \otimes \underline{1}_{14}$, shows that modulo I, $-\underline{1}_{im} \otimes \underline{\alpha\beta}_{mj} \equiv \underline{\alpha}_{sj} \otimes \underline{\beta}_{is} \equiv \ddot{\underline{\alpha}}_{js} \otimes \underline{\beta}_{ij} \equiv \underline{\alpha}_{ij} \otimes \ddot{\underline{\beta}}_{is}$ whenever s, m, i and j are mutually distinct.

We can now check that Ψ *is an equivariant map*; e.g. if $m \notin \{1, 2, 3\}$,

$$(e_{im}(-\gamma) - 1) \cdot \Psi(\alpha, \beta) = \underline{\alpha}_{12} \otimes \underline{\gamma\beta}_{2m} - \underline{\alpha}_{13} \otimes \underline{\beta\gamma}_{3m} \equiv 0 \mod I$$

and

$$\begin{aligned}(e_{12}(-\gamma) - 1) \cdot \Psi(\alpha, \beta) &= \underline{\alpha}_{12} \otimes \big(\underline{\beta}\ddot{\underline{\gamma}}_{21} - \underline{\gamma}^2_{12}\big) \\ &\quad - \underline{\alpha\gamma}_{13} \otimes \underline{\beta}_{32} - \underline{\alpha}_{13} \otimes \underline{\beta\gamma}_{32} \equiv 0 \mod I.\end{aligned}$$

The class of each type 10 *element is in* im g. Take $i \neq j$, $\{i, j\} \cap \{a, b\} = \varnothing$ and $\alpha\beta = \varepsilon\mu$. Look at $(e_{21} - 1)(\underline{\alpha\gamma}_{13} \otimes \underline{\beta}_{32} - \underline{\alpha}_{14} \otimes \underline{\gamma\beta}_{42})$ and permutations to see that

$$(\text{A.2}) \qquad -F_{ia}(\alpha\gamma, \beta) + F_{ja}(\alpha\gamma, \beta) + F_{ib}(\alpha, \gamma\beta) - F_{jb}(\alpha, \gamma\beta) \in I.$$

Apply $(e_{ji} - 1)$ to $\underline{\varepsilon}_{ia} \otimes \underline{\mu}_{aj} + \underline{\alpha}_{bj} \otimes \underline{\beta}_{ib}$ to see that

$$(\text{A.3}) \qquad -F_{ja}(\varepsilon, \mu) + F_{ia}(\varepsilon, \mu) + F_{bi}(\alpha, \beta) - F_{bj}(\alpha, \beta) \in I.$$

Hence modulo $I + \operatorname{im}\Psi$,

$$F_{12}(\varepsilon, \mu) + F_{b1}(\alpha, \beta) - F_{b2}(\alpha, \beta) \equiv 0;$$

i.e., $F_{b2}(\alpha, \beta) \equiv F_{12}(\varepsilon, \mu) + F_{b1}(\alpha, \beta) \equiv F_{j2}(\varepsilon, \mu) + F_{bj}(\alpha, \beta)$, by (A.3) so $F_{bj}(\alpha, \beta) \equiv F_{b2}(\alpha, \beta) - F_{j2}(\alpha, \beta) \equiv F_{ba}(1, \alpha\beta) - F_{ja}(1, \alpha\beta)$, by (A.2). Set $a = 1$.

The class of each type 9 *element is in* im g. As shown above $\ddot{\underline{\alpha}}_{12} \otimes \underline{\beta}_{13} - \underline{\alpha}_{14} \otimes \underline{\beta}_{43} \in I$. Apply $(e_{31} - 1)$ to it and use the facts that $F_{31}(\alpha, \beta) + F_{14}(\alpha, \beta) - F_{34}(\alpha, \beta) \in (I + \operatorname{im}\Psi)$, and, modulo I, $-\ddot{\underline{\alpha}}_{12} \otimes \underline{\beta}_{31} \equiv \underline{\alpha}_{34} \otimes \underline{\beta}_{41}$. This gives $\ddot{\underline{\alpha}}_{12} \otimes \ddot{\underline{\beta}}_{13} \equiv -\underline{\alpha}_{31} \otimes \underline{\beta}_{31} \equiv 0$ in $((M_n^0 \otimes M_n^0)/I)/\operatorname{im}\Psi$.

This completes part A.

B. Before going into the main part of the proof, we recall some facts about Hochschild homology [**I**].

If S is any associative ring and N is an S-bimodule, the Hochschild homology groups $HH_*(S, N)$ are defined as the homology of the complex $C_H(S, N) = ((S^{\otimes *}) \otimes N, \delta_H)$, where

$$\begin{aligned}\delta_H(s_1 \otimes \cdots \otimes s_m \otimes x) &= s_1 \otimes \cdots \otimes s_{m-1} \otimes s_m x \\ &\quad + \sum_{i=1}^{m-1} (-1)^i s_1 \otimes \cdots \otimes s_i s_{i+1} \otimes \cdots \otimes x \\ &\quad + (-1)^m s_2 \otimes \cdots \otimes s_n \otimes x s_1.\end{aligned}$$

The Morita equivalence $HH_*(M_n^0, M_n^0) \to HH_*(R, R)$ is induced by the chain equivalence

$$\operatorname{Tr}(X_1 \otimes \cdots \otimes X_m) = \sum_{a,b,\ldots,z=1}^{n} \left(x_{ab}^1 \otimes x_{bc}^2 \otimes \cdots \otimes x_{yz}^{m-1} \otimes x_{za}^m \right)$$

$$\text{where } X_i = \sum_{a,b=1}^{n} x_{ab}^i \, \underline{ab}.$$

Define f: $M_n^0 \otimes M_n^0 \to HH_1(R, R) = \Omega$ by $f(a \otimes b) = \{\operatorname{Tr}(a \otimes b)\}$. In $C_H(M_n^0, M_n^0)$, for any $r \in \mathrm{SL}_n R$ and a, $b \in M_n^0$, compute

(A.4)

$$rar^{-1} \otimes rbr^{-1} = r \otimes [a, b]r^{-1} + a \otimes b + \delta_H(rar^{-1} \otimes r \otimes br^{-1} - r \otimes a \otimes br^{-1}).$$

Thus, since Tr is a chain equivalence,

$$f(rar^{-1} \otimes rbr^{-1}) = f\left(r \otimes [a, b]r^{-1}\right) + f(a \otimes b).$$

As a group, $\mathrm{St}(R, R) \cong \Omega \oplus M_n^0$. Define σ: $(M_n^0 \otimes M_n^0)^0 \twoheadrightarrow \mathrm{St}(R, R)$ by

$$\sigma(a \otimes b) = \left(f(a \otimes b), [a, b]\right).$$

Check that $\sigma(\underline{1}_{12} \otimes \underline{1}_{23} + \underline{1}_{43} \otimes \underline{1}_{14}) = 0$. Hence if we can show that σ is an $\mathrm{SL}_n R$-module map, we will have an induced map

$$\left(M_n^0 \otimes M_n^0\right)^0/I \twoheadrightarrow \mathrm{St}(R, R)$$

which is inverse to the canonical map.

With the identification $\mathrm{St}(R, R) = \Omega \oplus M_n^0$, the $\mathrm{SL}_n R$-action is given by

$$r \cdot (h, u) = \left(f(r \otimes ur^{-1}) + h, rur^{-1}\right); \qquad r \in \mathrm{SL}_n R,\ h \in \Omega,\ u \in M_n^0$$

(to check this, one first checks that the action is well defined, then that it is compatible with the usual description of the action on $\mathrm{St}(R, R)$ defined in terms of the generators $y_{ij}(a)$ [**K4**, 1.4]). Then because $\mathrm{SL}_n R$ acts on M_n^0 by conjugation,

$$\begin{aligned}\sigma(r \cdot (a \otimes b)) = \sigma(rar^{-1} \otimes rbr^{-1}) &= \left(f(rar^{-1} \otimes rbr^{-1}),\ r[a, b]r^{-1}\right) \\ &\overset{(\mathrm{A.4})}{=} \left(f \otimes [a, b]r^{-1} + f(a \otimes b),\ r[a, b]r^{-1}\right) \\ &= r \cdot \left(f(a \otimes b), [a, b]\right) = r \cdot \sigma(a \otimes b).\end{aligned}$$

Thus σ is an $\mathrm{SL}_n R$-module map.

(iii) This is read from the commutative diagram:

$$\begin{array}{ccccc}
\Omega & \overset{\leftarrow}{\rightarrowtail} & \mathrm{im}\Psi & \twoheadrightarrow & ? \\
\downarrow & & \downarrow & & \| \\
\left(M_n^0 \otimes M_n^0\right)^0/I & \rightarrowtail & \left(M_n^0 \otimes M_n^0\right)/I & \twoheadrightarrow & R \\
\downarrow D^0 & & \downarrow D & & \\
M_n^0 & = & M_n^0 & &
\end{array}$$

□

A.2 LEMMA. (i) $(M_n^0 \otimes M_n^0)_{\mathrm{SL}_n R} = R$, *with* $\alpha \leftrightarrow \{\underline{1}_{12} \otimes \underline{\alpha}_{21}\}$,

(ii) $(\Lambda^2 M_n^0)_{\mathrm{SL}_n R} = R/2R$, *and* $(S^2 M_n^0)_{\mathrm{SL}_n R} = R$,

(iii) $((M_n^0)^{\otimes 3})_{\mathrm{SL}_n R} = R \oplus R$ *with* $(\alpha, \beta) \leftrightarrow \{\underline{1}_{12} \otimes \underline{1}_{23} \otimes \underline{\alpha}_{31}\} + \{\underline{1}_{23} \otimes \underline{1}_{12} \otimes \underline{\beta}_{31}\}$,

(iv) $(\Lambda^2 M_n^0 \otimes M_n^0)_{\mathrm{SL}_n R} = R$, *with* $\alpha \leftrightarrow \{\underline{1}_{12}\underline{1}_{23} \otimes \underline{\alpha}_{31}\}$,

(v) $(\Lambda^3 M_n^0)_{\mathrm{SL}_n R} = R$, *with* $\alpha \leftrightarrow \{\underline{1}_{12}\underline{1}_{23}\underline{\alpha}_{31}\}$,

(vi) $(J_n \otimes M_n^0)_{\mathrm{SL}_n R} = 0$.

PROOF. (i) (Kassel [**K3**, 3.7]) The coinvariance is detected by the equivariant map

$$T\colon M_n^0 \otimes M_n^0 \to R, \quad \text{where } T(a \otimes b) = \mathrm{Tr}(ab).$$

(ii) $\Lambda^2 M_n^0$ is $M_n^0 \otimes M_n^0$ quotiented by the group generated by $\{a \otimes b + b \otimes a, a \otimes a;\ a,\ b \in M_n^0\}$, so the map induced by T detects the coinvariance [**K3**, 3.7]. Similarly $S^2 M_n^0$ is the quotient of $M_n^0 \otimes M_n^0$ by the group generated by $\{a \otimes b - b \otimes a;\ a,\ b \in M_n^0\}$; again, the map induced by T detects.

(iii) Let I be the $\mathrm{SL}_n R$-submodule of $M_n^0 \otimes M_n^0$ generated by $\underline{1}_{12} \otimes \underline{1}_{23} + \underline{1}_{42} \otimes \underline{1}_{14}$. Lemma A.1(ii) exhibits a sequence $\mathrm{im}\Psi \rightarrowtail (M_n^0 \otimes M_n^0)/I \overset{D}{\twoheadrightarrow} M_n^0$, where D is induced by the commutator map [,] of 3.2, and $\mathrm{SL}_n R$ acts trivially on $\mathrm{im}\Psi$. Since $(M_n^0)_{\mathrm{SL}_n R} = 0$,

$$D_*\colon \big(\big((M_n^0 \otimes M_n^0)/I\big) \otimes M_n^0\big)_{\mathrm{SL}_n R} \to \big(M_n^0 \otimes M_n^0\big)_{\mathrm{SL}_n R} = R$$

is an isomorphism. We will compute $(I \otimes M_n^0)_{\mathrm{SL}_n R}$ by determining the $\mathrm{SL}_n R$-covariance classes of its generators, $\{\dot{y}_{ab}(\alpha) = (\underline{1}_{12} \otimes \underline{1}_{23} + \underline{1}_{43} \otimes \underline{1}_{14}) \otimes \dot{\underline{\alpha}}_{ab}\colon a \neq b\}$, where $\dot{\underline{\alpha}}_{ij}$ denotes "either $\underline{\alpha}_{ij}$ or $\ddot{\underline{\alpha}}_{ij}$" and $\dot{y}_{ij}$ denotes "either y_{ij} or $\ddot{y}_{ij}$". Congruence in what follows is with respect to coinvariance class.

If $a \notin \{2,3,4\}$, choose $c \notin \{1,2,4,a\}$. Then $y_{ab}(\alpha) = (e_{ac} - 1)\cdot y_{cb}(\alpha)$ (or, if $c = b$, $(e_{ac} - 1)\cdot \ddot{y}_{c2}(\alpha)$, say). Apply this sort of argument again, to conclude

(A.5) $$\text{if } a \notin \{2,3,4\} \text{ or } b \notin \{1,2,4\}, \quad y_{ab}(\alpha) \equiv 0.$$

If $a \notin \{2,3,4\}$ and $b \notin \{1,2,4\}$, $\ddot{y}_{ab}(\alpha) = (e_{ab} - 1)\cdot y_{ba}(\alpha) + y_{ab}(\alpha)$. This together with the identities $\ddot{y}_{ab}(\alpha) = \ddot{y}_{a5}(\alpha) + \ddot{y}_{5b}(\alpha)$ and $\ddot{y}_{a5}(\alpha) = \ddot{y}_{5a}(-\alpha)$, implies

(A.6) $$\ddot{y}_{ab}(\alpha) \equiv 0 \quad \text{unless } \{a, b\} \cap \{2, 4\} \neq \varnothing.$$

Hence we need only consider $\{a, b\} \subset \{1,2,3,4\}$.

$P(i, j)$ is the permutation matrix $e_{ij}e_{ji}(-1)e_{ij}$. Check that if $1 \notin \{a, b\}$, $\dot{y}_{ab} = (e_{15} - 1)\cdot(P(5,1)\cdot \dot{y}_{ab})$, and if $3 \notin \{a, b\}$, $-y_{ab} = (e_{53} - 1)\cdot(P(5,3)\cdot \dot{y}_{ab})$.

This eliminates classes of all the generating set of $I \otimes M_n^0$ other than $\{y_{31}(\alpha)\}$. However, the equivariant function on $M_n^{\otimes 3}$, $:A \otimes B \otimes C \to \mathrm{Tr}(ABC)$, takes $y_{31}(\alpha)$ to α, so there is a copy of R in $(I \otimes M_n^0)_{\mathrm{SL}_n R}$. Thus there is a split sequence

(A.7) $$R = \big(I \otimes M_n^0\big)_{\mathrm{SL}_n R} \overset{\iota}{\to} \big(M_n^{0\otimes 3}\big)_{\mathrm{SL}_n R} \overset{D_*}{\twoheadrightarrow} R,$$

where $\iota(\alpha)$ is the class of $\underline{1}_{12} \otimes \underline{1}_{23} \otimes \underline{\alpha}_{31} + \underline{1}_{43} \otimes \underline{1}_{14} \otimes \underline{\alpha}_{31}$, and $D_*^{-1}(\alpha)$ is the class of $\underline{1}_{12} \otimes \underline{1}_{23} \otimes \underline{\alpha}_{31}$.

(iv) The homology product $M_n^{0\otimes 3} \to \Lambda^2 M_n^0 \otimes M_n$ takes $y_{31}(\alpha)$ to

$$(\cap \otimes 1)\, y_{31}(\alpha) = \underline{1}_{12}\underline{1}_{23} \otimes \underline{\alpha}_{31} - \underline{1}_{14}\underline{1}_{43} \otimes \underline{\alpha}_{31} \equiv -(e_{42} - 1)\cdot(\underline{1}_{14}\underline{1}_{23} \otimes \underline{\alpha}_{31}).$$

$D_*^{-1}(\alpha)$ maps through $\underline{1}_{12}\underline{1}_{23} \otimes \underline{\alpha}_{31} \in \Lambda^2 M_n^0 \otimes M_n^0$ to $\underline{1}_{13} \otimes \underline{\alpha}_{31}$ in $M_n^0 \otimes M_n^0$. Hence $(\Lambda^2 M_n^0 \otimes M_n^0)_{\mathrm{SL}_n R}$ is as described.

(v) By (iv), the homology product induces an epimorphism $R \to (\Lambda^3 M_n^0)_{\mathrm{SL}_n R}$. Observe this is an isomorphism, since there is an $\mathrm{SL}_n R$-invariant function in $(\Lambda^3 M_n^0)^{\#}$, $A \cap B \cap C \to \mathrm{Tr}(ABC - BAC)$, $A, B, C \in M_n^0$, which takes $\underline{1}_{12} \cap \underline{1}_{23} \cap \underline{\alpha}_{31}$ to α.

(vi) J_n is the image under the homology product of L_n (see 3.2) whereas A.1(ii) proves $(L_n/I \equiv \ker D)$ is $\mathrm{SL}_n R$-invariant. Thus, if I' is the image of I under the product, it suffices to show that $(I' \otimes M_n^0)_{\mathrm{SL}_n R} = 0$. But $(I' \otimes M_n^0)_{\mathrm{SL}_n R}$ is generated by the class of $(\cap \otimes 1)y_{31}(\alpha)$, and this is the zero class in $(I' \otimes M_n^0)_{\mathrm{SL}_n R}$ as well as in $(\Lambda^2 M_n^0 \otimes M_n^0)_{\mathrm{SL}_n R}$.

REFERENCES

[**A**] J. Aisbett, *K-groups of rings and the homology of their elementary matrix groups*, J. Austral. Math. Soc. Ser. A **38** (1985), 268–274.

[**ALSS**] J. Aisbett, E. Lluis-Puebla, V. Snaith and C. Soulé, *On* $K_*(\mathbf{Z}/n)$ *and* $K_*(\mathbf{F}_q[t]/(t^2))$, Mem. Amer. Math. Soc., Vol. 57, No. 329, 1985.

[**ALS**] J. Aisbett, E. Lluis-Puebla and V. Snaith, *On* K_3 *of* $\mathbf{F}_q[t]/(t^2)$ *and* $\mathbf{F}_q[t]/(t^3)$, J. Algebra (to appear).

[**B**] K. S. Brown, *Cohomology of groups*, Graduate Texts in Math., no. 87, Springer-Verlag, Berlin and New York, 1982.

[**CE**] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., 1956.

[**DS**] K. Dennis and M. Stein, K_2 *of discrete valuation rings*, Adv. in Math. **18** (1975), 182–238.

[**I**] K. Igusa, *What happens to Hatcher and Wagoner's formula for* $\pi_0 C(M)$ *when the first Postnikov invariant of M is nontrivial*? Lecture Notes in Math., vol. 1046, Springer-Verlag, Berlin and New York, 1984, pp. 104–172.

[**K1**] C. Kassel, *Un calcul d'homologie du groupe linéaire général*, C. R. Acad. Sci. Paris **288** (1979), 481–483.

[**K2**] ______, *Le groupe* $K_3(\mathbf{Z}[\varepsilon])$ *n'a pas de p-torsion pour* $p \neq 2$ *et* 3, Lecture Notes in Math., vol. 966, Springer-Verlag, Berlin and New York, 1982, pp. 114–121.

[**K3**] ______, *K-théorie relative d'un idéal bilatèrie de carré nul*, Lecture Notes in Math., vol. 854, Springer-Verlag, Berlin and New York, 1981, pp. 249–261.

[**K4**] ______, *Calcul algébrique de l'homologie de certains groups de matrices*, J. Algebra **80** (1983), 235–260.

[**LS**] E. Lluis-Puebla and V. Snaith, *Determination of* $K_3(\mathbf{F}_p l[t]/(t^2))$ *for primes* $p \geqslant 5$, Current Trends in Algebraic Topology, CMS Conf. Proc., vol. 2.1, Amer. Math. Soc., Providence, R.I., 1984.

[**L**] J.-L. Loday, *On the boundary map* $K_3(\Lambda/I) \to K_2(\Lambda, I)$, Lecture Notes in Math., vol. 854, Springer-Verlag, Berlin and New York, 1981, pp. 262–268.

[**M**] S. Mac Lane, *Homology*, Springer-Verlag, Berlin, 1963.

[**Ro**] L. Roberts, K_2 *of some truncated polynomial rings*, Lecture Notes in Math., vol. 734, Springer-Verlag, Berlin and New York, 1980, pp. 249–278.

[**S**] J. Stienstra, *On* K_2 *and* K_3 *of truncated polynomial rings*, Lecture Notes in Math., vol. 854, Springer-Verlag, Berlin and New York, 1981, pp. 409–455.

[**VK**] W. van der Kallen, *Homology stability for linear groups*, Invent. Math. **60** (1980), 269–295.

[**VKS**] W. van der Kallen and J. Stienstra, *The relative* K_2 *of truncated polynomial rings*, J. Pure Appl. Algebra **34** (1984), 277–290.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF QUEENSLAND, ST. LUCIA, QUEENSLAND, AUSTRALIA 4067

Current address: Electronics Research Laboratory, Defence Science and Technology Organisation, G.P.O. Box 2151, Adelaide 5001, Australia