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ON THE AMPLENESS OF HOMOGENEOUS VECTOR BUNDLES

BY

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ABSTRACT. A formula is proved which expresses the ampleness of a homogeneous vector bundle over G/P in terms of the distance of the weights of the representation of P to certain dominant weights of G.

Classically, a line bundle L over a compact complex manifold X is said to be *ample* if the map $X \to \mathbf{P}^N$ associated to the sections of some power L^n gives an imbedding of X. This notion has been extended to vector bundles as well [6]. In [9] Sommese generalized the notion of ampleness to the following: Let $\mathbf{E}$ be a holomorphic vector bundle over a compact complex manifold X and let $\xi(\mathbf{E})$ be the tautological line bundle over $\mathbf{P}(\mathbf{E})$. Then $\mathbf{E}$ is called *k-ample* if $\xi(\mathbf{E})^n$ is spanned by global sections for some $n > 0$ and the map $\mathbf{P}(\mathbf{E}) \to \mathbf{P}^N$ associated to the sections of $\xi(\mathbf{E})^n$ has at most k-dimensional fibers. This definition coincides with ampleness in the sense of Grothendieck [6] when $k = 0$.

Sommese [9] proves analogues of Barth-Lefschetz type theorems for k-ample vector bundles, and in [10] he shows that there is a connection between k-ampleness and $(k + 1)$-convexity of holomorphic vector bundles: *If* $\mathbf{E}$ *is k-ample, then* $\mathbf{E}^*$ *is $(k + 1)$-convex* (*in the sense of Andreotti-Grauert*). For $k = 0$ the converse is also true. It should be remarked that the algebro-geometric conditions for k-ampleness are generally easier to verify than the differential geometric conditions for $(k + 1)$-convexity.

In this paper we investigate the ampleness of homogeneous vector bundles $\mathbf{E}$ over simply-connected complex projective algebraic manifolds X. In this case, X is isomorphic to a coset space G/P, where G is a semisimple complex Lie group and P is a parabolic subgroup of G. Any algebraic homogeneous vector bundle over X comes from a rational representation of P on a vector space E. We show in §2 how to calculate the ampleness of $\mathbf{E}$, which we denote by $a(\mathbf{E})$, in terms of the relative positions of the weights in E, $\Lambda(E)$, of a maximal torus $T \subseteq P$ in the lattice of weights Λ of G. One need only compute the distance, measured by the number of Weyl chamber walls crossed in the diagram of G, between the weights $\Lambda(E)$ and certain dominant weights in Λ.

In §3 we review the results of Goldstein [5]. His calculation of the ampleness of the tangent bundle of $X = G/P$ was one of the main inspirations for the generalization to arbitrary homogeneous vector bundles in this paper. In §4 we specialize the

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results of §2 to *irreducible* homogeneous vector bundles. In this case the computation of $a(\mathbf{E})$ is extremely easy to carry out: If μ (respectively λ) is the highest weight in E (respectively in E^*) and λ is dominant, then

$$a(\mathbf{E}) = \dim X - \operatorname{index} \mu,$$

where the index of μ is the number of positive roots $\alpha > 0$ such that $(\alpha, \mu) < 0$. We also show that $\mathbf{E}$ is 0-ample when the weight λ is *regular* and dominant. We conclude the paper with the classical example of Borel-Weil [8], calculating the ampleness of line bundles over $X = G/P$.

1. Definitions and notation.

1.1 *The ampleness map.* Given a finite-dimensional vector space E over $\mathbf{C}$, we define the projectivization of E, denoted $\mathbf{P}(E)$ to be the quotient of $E^* \setminus 0$ by the usual action of $\mathbf{C}^*$. We denote the image of $z \in E^* \setminus 0$ in $\mathbf{P}(E)$ by $[z]$. We use the same notation for the projectivization of a vector bundle $\mathbf{E}$ over a complex manifold X with fiber E. Let π: $\mathbf{P}(\mathbf{E}) \to X$ be the bundle projection (with fiber $\mathbf{P}(E)$) and define $\xi(\mathbf{E})^*$ to be the rank 1 tautological subbundle of $\pi^*\mathbf{E}^*$. Recall that if X is compact, the isomorphism of vector spaces $S^n(E) \cong \Gamma(\mathbf{P}(E), \mathcal{O}(n))$ induces an isomorphism [6]

$$\Gamma(X, S^n(\mathbf{E})) \cong \Gamma(\mathbf{P}(\mathbf{E}), \xi(\mathbf{E})^n), \qquad n \geqslant 1.$$

(Here, $S^n(\cdot)$ denotes the nth symmetric power.) We denote both of these vector spaces by V_n.

DEFINITION [9]. A holomorphic vector bundle $\mathbf{E}$ on a compact complex manifold X is called *k-ample* if:

(1) $\xi(\mathbf{E})^n$ is spanned by global sections for some $n > 0$; i.e., the evaluation map $\mathbf{P}(\mathbf{E}) \times V_n \to \xi(\mathbf{E})^n$ is surjective.

(2) The projectivization α: $\mathbf{P}(\mathbf{E}) \to \mathbf{P}(V_n)$ of the induced map $\xi(\mathbf{E})^{-n} \to V_n^*$ has *at most k-dimensional* fibers.

The number k is independent of the integer n for which $\xi(\mathbf{E})^n$ is spanned [9]. We define the *ampleness of* $\mathbf{E}$, denoted $a(\mathbf{E})$, to be the minimum k for which $\mathbf{E}$ is k-ample, and we call the map α the *ampleness map*. (If $\xi(\mathbf{E})^n$ is not spanned for any $n > 0$, we define $a(\mathbf{E}) = \dim X$.)

It will also be convenient for us to work with the map ν: $\mathbf{E}^* \to S^n(\mathbf{E}^*) \to V_n^*$ (lifted from α) defined as follows: For $z_x \in E_x^*$, $x \in X$, and $s \in V_n$, let $\nu(z_x)(s) := z_x^n(s_x)$. Here, z_x^n stands for the image in $S^n(E^*)_x = S^n(E)_x^*$ of the n-fold tensor $z_x \otimes \cdots \otimes z_x$. From the definitions we easily obtain, for $z_x \in E_x^* \setminus 0$,

$$\alpha[z_x] = [\nu(z_x)] \in \mathbf{P}(V_n), \quad \text{and} \quad \dim \alpha^{-1}\alpha[z_x] = \dim \nu^{-1}\nu(z_x).$$

Thus, $\xi(\mathbf{E})^n$ is spanned if and only if for every $z_x \in E_x^* \setminus 0$ there is a section $s \in V_n$ such that $\nu(z_x)(s) = z_x^n(s_x) \neq 0$. Easy calculations then show that the maps ν and α are *finite* on each fiber E_x^* and $\mathbf{P}(E)_x$, $x \in X$, respectively. Consequently,

$$0 \leqslant a(\mathbf{E}) \leqslant \dim X.$$

1.2 *Homogeneous vector bundles.* As is well known, any locally trivial fiber bundle $\mathbf{E}$ over a space X with fiber E and structure group H is isomorphic (in the appropriate category) to a twisted product $P \times_H E$, where P is the associated principal bundle over X. Here the twisted product $P \times_H E$ is defined to be the quotient of $P \times E$ by the action $h.(p, z) = (p.h^{-1}, h.z)$, $(p, z) \in P \times E$, $h \in H$. We denote the image of a point $(p, z) \in P \times E$ in $P \times_H E$ by $[p, z]$. Thus, $[p.h, z] = [p, h.z]$ for all $h \in H$. Projection onto the first factor $[p, z] \to p.H$ defines the bundle structure of $P \times_H E$ with base $P/H = X$ and fiber E.

Our main interest in this paper is with *homogeneous vector bundles* $\mathbf{E}$ over compact projective algebraic manifolds X. By homogeneous we mean that the group of (algebraic) bundle automorphisms of $\mathbf{E}$ acts transitively on X. In this case, X is isomorphic to a coset space G/H where G and H are algebraic groups and the fiber E is a rational H-module. Thus, $\mathbf{E}$ is isomorphic to $G \times_H E$ as described above. The finite-dimensional vector space of sections $\Gamma(X, \mathbf{E})$ is in a canonical fashion a rational G-module. It is isomorphic to the *induced G-module* $E|^G$ which is defined to be the vector space of all algebraic morphisms s: $G \to E$ which satisfy $s(gh^{-1}) = h.s(g)$ for all $g \in G$, $h \in H$. The action of G on $E|^G$ is by $(g.s)(g_0) := s(g^{-1}g_0)$, for all $g, g_0 \in G$. The relationship between such an s: $G \to E$ and a section $\sigma \in \Gamma(X, \mathbf{E})$ is easily seen by writing a point $x \in X$ as a coset $gH \in G/H$ and expressing $\sigma(x) = \sigma(gH)$ as an element of the twisted product $G \times_H E$: $\sigma(gH) = [g, s(g)]$. We shall not distinguish between these G-modules and write $e|^G$ (or sometimes $E_H|^G$ when we want to emphasize that E is an H-module) for $\Gamma(X, \mathbf{E})$.

Evaluating a section σ at the identity coset $1H$ corresponds to the H-module homomorphism

$$\varepsilon\colon E_H|^G \to E; \quad \varepsilon(s) := s(1).$$

Indeed, we have $\varepsilon(h.s) = (h.s)(1) = s(h^{-1}) = h.s(1) = h.\varepsilon(s)$. A homogeneous vector bundle is then spanned by global sections if and only if ε is surjective. The induced module has the following universal property: Given any G-module M and any H-module homomorphism φ: $M \to E$, there exists a unique G-module homomorphism $\varphi^\wedge$: $M \to E_H|^G$ such that $\varphi = \varepsilon \circ \varphi^\wedge$ (for $m \in M$ define $\varphi^\wedge(m)$: $G \to E$ by $\varphi^\wedge(m)(g) = \varphi(g^{-1}m)$). Also, if $E \to F$ is an H-module homomorphism, then there is an obvious G-module homomorphism of the induced G-modules $E|^G \to F|^G$ which commutes with the evaluation maps (see e.g. [**4**]).

1.3 *Semisimple Lie groups*. From now on, G will denote a *connected*, *simply-connected*, *semisimple complex Lie group*. General references for this section are [**1**, **7**, **3**]. We fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. Let Λ denote the character group of T which we write additively. The elements of Λ are called *weights*. Let $\Phi \subset \Lambda$ denote the *roots* of G relative to T, and Φ^+ (resp. Φ^-) the positive (resp. negative) roots with respect to B. We write $\alpha > 0$ (resp. $\alpha < 0$) if $\alpha \in \Phi^+$ (resp. $\alpha \in \Phi^-$). Further, let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ denote the subset of *simple roots*, $l = \dim T = \operatorname{rank} G$, so that each $\alpha \in \Phi$ can be expressed uniquely as an integral combination $\alpha = \Sigma n_i\alpha_i$ with either all $n_i \geqslant 0$ (i.e., $\alpha > 0$) or all $n_i \leqslant 0$ (i.e., $\alpha < 0$). The $\mathbf{Z}$-module Λ is a free abelian group of rank l with a basis $\{\lambda_1, \ldots, \lambda_l\}$ consisting of *fundamental dominant weights*, where $2(\lambda_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$. The lattice Λ is partially ordered by: $\lambda > \mu$ if $\lambda - \mu$ is a sum of positive roots. A weight $\Sigma n_i\lambda_i$ is said to be *dominant* if all $n_i \geqslant 0$. We denote the dominant weights by Λ^+.

The *Weyl group* $W := N_G(T)/T$ of G acts on Λ and can be realized as the finite linear group generated by the reflections σ_α of $F := \mathbf{R} \otimes_{\mathbf{Z}} \Lambda$ through $\alpha \in \Phi$. For each $\omega \in W$, we denote by $\eta_\omega \in N(T)$ a fixed representative of ω. We let $\sigma_1, \ldots, \sigma_l$ denote the *simple reflections*, i.e., the reflections of F through the simple roots $\alpha_1, \ldots, \alpha_l$, respectively. Every $\omega \in W$ can thus be written as a product of simple reflections $\omega = \sigma_{i(1)} \cdots \sigma_{i(j)}$, and the smallest number j is called the *length* of ω, denoted $l(\omega)$. Let $B = T \ltimes U$ be the semidirect product decomposition of B where U is the unipotent radical of B. (U is just the product of all the *root groups* U_α, $\alpha > 0$.) Then the length function satisfies

$$\dim U\eta_\omega B = \dim B + l(\omega),$$

since $l(\omega)$ is the number of positive roots which are transformed to negative roots by ω [**3**].

The *Weyl chambers* of F are the closures of the connected components of $F \setminus \bigcup H_\alpha$ ($\alpha \in \Phi^+$), where H_α is the hyperplane of F orthogonal to α. The Weyl group W acts simply transitively on the Weyl chambers. There is precisely one chamber, call it F^+, such that $\Lambda \cap F^+ = \Lambda^+$. Thus, the W-orbit of any weight contains exactly one dominant weight. For any weight $\lambda \in \Lambda$ we define the *index* of λ to be the number of positive roots $\alpha > 0$ such that $(\alpha, \lambda) < 0$ [**2**]. Geometrically, the index of λ is just the number of hyperplanes H_α, $\alpha > 0$, *crossed* by a straight line connecting α to a general point of F^+. We also have the following formula:

$$\operatorname{index} \lambda = \min\{ l(\omega) \mid \omega \in W, \omega.\lambda \in \Lambda^+ \}.$$

This follows from the observation that the shortest way for λ to be moved to Λ^+ by W is by reflecting through the s hyperplanes between it and Λ^+, $s = \operatorname{index} \lambda$. Now the product of these s reflections can be rewritten as the product of s simple reflections (see e.g. [**2**, p. 236]).

If V is a rational T-module, we denote by $\Lambda(V)$ the weights of T in V. A *weight space* V_λ, $\lambda \in \Lambda(V)$, is defined to be the subspace of all $v \in V$ such that $t.v = \lambda(t)v$ for all $t \in T$. The *multiplicity* of V_λ is defined to be $\dim V_\lambda$. We often let L_λ denote a T-stable line in V of weight λ. Notice that if V is a G-module, then W permutes the elements of $\Lambda(V)$: $\eta_\omega . V_\lambda = V_{\omega(\lambda)}$ for all $\omega \in W$. A *maximal vector* is a weight vector, $v \in V_\lambda$, which is fixed by U, $v \in V^U$. Let L_λ be the B-stable line spanned by such a v. Then the *highest weight* of the G-submodule V' generated by v is λ, all other weights being $< \lambda$, and L_λ is the unique B-stable line in V'. Moreover, λ is necessarily dominant with multiplicity 1 in V'. The submodule V' is irreducible and isomorphic to the induced G-module $\omega_0\lambda|^G$ where ω_0 is the unique element of W such that $\omega_0(\Phi^+) = \Phi^-$. (We are employing the convention that a weight $\mu \in \Lambda$ will also represent the (abstract) 1-dimensional B-module with weight μ.) Note that $\omega_0\lambda$ is the *lowest* weight of V' in the sense that $\omega_0\lambda \leqslant \mu$ for any $\mu \in \Lambda(V')$.

A *parabolic* subgroup G is a subgroup P which contains a Borel subgroup. A *standard parabolic* subgroup P_I is a parabolic subgroup constructed from a subset $I \subset \Delta$ of simple roots: P_I is the subgroup generated by the fixed Borel subgroup B and all the root groups U_α where α is a root in the $\mathbf{Z}$-linear span of I. Any parabolic subgroup P of G is conjugate to a standard parabolic subgroup P_I. *Throughout this* *paper, when we refer to a parabolic subgroup of G we shall assume that it is a standard parabolic subgroup.* The subgroup of W generated by the simple reflections σ_α, $\alpha \in I$, will be denoted by W_I. Then P is the product of its radical and a semisimple subgroup S, and W_I is isomorphic to the Weyl group of S.

2. The ampleness formula.

2.1 We retain the notation of §1. In particular, G is a connected, simply-connected, complex semisimple Lie group; B a Borel subgroup; U the unipotent radical of B; $P \supset B$ a parabolic subgroup; W the Weyl group of G relative to a fixed maximal torus $T \subset B$; $l(\omega)$ the length of an element $\omega \in W$; Λ (resp. Λ^+) the weights (resp. the dominant weights) of G relative to T; $\Lambda(V)$ the weights of a T-module V.

In this section we derive a formula for the ampleness of a homogeneous vector bundle $\mathbf{E} = G \times_P E$ over G/P based only on the relative positions of the weights of the P-module E in Λ. We let $\mathbf{E}_B := G \times_B E$ denote the pull-back of $\mathbf{E}$ to G/B. Note that the fiber of $\mathbf{E}_B \to \mathbf{E}$ is P/B and that $\Gamma(G/B, \mathbf{E}_B) \cong \Gamma(G/P, \mathbf{E})$.

THEOREM. *Assume $\xi(\mathbf{E})^n$ is spanned by global sections for some $n > 0$ and let $S = (E^*)^U$. Then*

(1) $\Lambda(S) \subset \Lambda^+$;

(2) $a(\mathbf{E}_B) = \max\{l(\omega) | \omega \in W$ *and* $\omega(\mu) \in \Lambda(E^*)$ *for some* $\mu \in \Lambda(S)\}$;

(3) $a(\mathbf{E}) = a(\mathbf{E}_B) - \dim P/B$.

This result can be formulated more cleanly if we use the following notation: for any two subsets A, $B \subset \Lambda$ we will define the *orbital distance* between A and B to be

$$d(A, B) = \min\{l(\omega) \mid \omega \in W, \omega(A) \cap B \neq \varnothing\}.$$

Let $\omega_0 \in W$ be the unique element such that $\omega_0(\Phi^+) = \Phi^-$ and define $\sigma_0 := -\omega_0$, an involution on Λ^+.

COROLLARY. $a(\mathbf{E}) = \dim G/P - d(\sigma_0\Lambda(S), \Lambda(E))$.

PROOF. From the Theorem,

$$\dim G/P - a(\mathbf{E}) = \dim G/B - a(\mathbf{E}_B),$$

where $a(\mathbf{E}) = \max\{l(\omega) | \omega \in W$, and $\omega(\mu) \in \Lambda(E^*)$ for some $\mu \in \Lambda(S)\}$. Now, $l(\omega\omega_0) = \dim G/B - l(\omega)$ [3], so

$$-\max\{l(\omega)\} = \min\{-l(\omega)\} = \min\{l(\omega\omega_0)\} - \dim G/B.$$

The formula then follows by observing that if $\tau = \omega\omega_0$ and $\mu \in \Lambda(S)$, then

$$\omega(\mu) = \tau\omega_0(\mu) \in \Lambda(E^*) \Leftrightarrow \tau\sigma_0(\mu) \in \Lambda(E). \qquad \square$$

Thus, to compute the ampleness of a homogeneous vector bundle $\mathbf{E} = G \times_P E$ (for which $\xi(\mathbf{E})^n$ is spanned), one need only find the smallest distance from the weights in $\sigma_0\Lambda(S) \subset \Lambda^+$ to those in the same W-orbit in $\Lambda(E)$ (measured by the number of hyperplanes H_α, $\alpha > 0$, between them; see §1.3) and subtract this from the dimension of the base space.

2.2 The proof of the Theorem is in two steps. The first is to examine the map ν: $\mathbf{E}^* \to V_n^*$ lifted from the ampleness map (§1) to obtain a formula for $a(\mathbf{E})$ in terms of T-stable lines in $\nu(E^*)$. (Here, we are identifying E^* with $\{[1, z] | z \in E^*\} \subset G \times_P E^* = \mathbf{E}^*$.) The second step is to translate this information into statements about the weights of E^*. The first step is contained in the following lemma. We let L_μ denote a T-stable line of weight μ, and for each $\omega \in W$ we fix a representative $\eta_\omega \in N(T)$.

LEMMA. *Let M be the maximum of $l(\omega)$ for all $\omega \in W$ which satisfy $\eta_\omega L_\mu \subset \nu(E^*)$ for some $L_\mu \subset \nu(E^*) \cap (V_n^*)^U$. Then*

$$a(\mathbf{E}) = M + \dim B - \dim P.$$

PROOF. Since α is G-equivariant, the fiber dimension of α is constant on G-orbits. Moreover, the fiber dimension is upper semicontinuous, so its maximum can be found by specializing within a G-orbit to a point on a *closed* G-orbit in $\mathbf{P}(V_n)$, say $G.[y]$ where $[y] = \alpha([z]) \in \alpha(\mathbf{P}(E))$. Since $\alpha(\mathbf{P}(E)) \cap G.[y]$ is closed and B-invariant, the B-orbits of minimal dimension in $\alpha(\mathbf{P}(E)) \cap G.[y]$ are closed. However, any closed B-orbit in projective space must be a B-fixed point. Therefore, we may assume that y lies in a B-stable line $L_\mu \subset V_n^* \cap \nu(E^*)$ for some $\mu \in \Lambda((V_n^*)^U)$. Then

$$\begin{aligned}\dim \alpha^{-1}[y] &= \dim \nu^{-1}(y) = \dim\{[g, z] \,|\nu([g, z]) = y\} \\ &= \dim\{(g, z) \,|\nu(z) = g^{-1}.y\} - \dim P \\ &= \dim\{g \in G \,|g.y \in \nu(E^*)\} - \dim P.\end{aligned}$$

Since $\nu(E^*)$ is a cone, we must find the dimension of$\{g \in G | gL_\mu \subset \nu(E^*)\}$. By the Bruhat decomposition, each $g \in G$ lies in some $U\eta_\omega B$, $\omega \in W$, with $\dim U\eta_\omega B = \dim B + l(\omega)$ (§1.3). Now,

$$gL_\mu = u\eta_\omega b L_\mu \subset \nu(E^*) \Leftrightarrow \eta_\omega L_\mu \subset \nu(E^*).$$

Therefore,

$$\begin{aligned}\dim \nu^{-1}(y) &= \max\{\dim U\eta_\omega B \,|\omega \in W, \text{ and } \eta_\omega L_\mu \subset \nu(E^*)\} - \dim P \\ &= \max\{l(\omega) \,|\omega \in W, \text{ and } \eta_\omega L_\mu \subset \nu(E^*)\} + \dim B - \dim P.\end{aligned}$$

The maximum fiber dimension is then found by maximizing this number over $L_\mu \subset \nu(E^*) \cap (V_n^*)^U$. □

The remainder of the proof of the Theorem now consists of showing that:

(a) $\Lambda(S) \subset \Lambda^+$; and

(b) there exists a B-stable line $L_\mu \subset \nu(E^*) \cap (V_n^*)^U$ with $\eta_\omega L_\mu \subset \nu(E^*)$ $(\omega \in W)$ if and only if $\mu = n\lambda$ for some $\lambda \in \Lambda(S)$ with $\omega(\lambda) \in \Lambda(E^*)$.

(a) Let $\lambda \in \Lambda(S)$ and consider the surjective B-module homomorphism $S^n(E) \to -n\lambda$. We then obtain a G-module homomorphism of induced G-modules β: $V_n = S^n(E)|^G \to -n\lambda|^G$. Now, β cannot be the zero map, since $\nu|L_\lambda \to \nu(L_\lambda)$ is *finite*, taking the B-stable line $L_\lambda \subset E^*$ to a B-stable *line* of weight $n\lambda$ in $\varepsilon_n^*(S^n(E)^*) \subset V_n^*$ (ε_n: $V_n \to S^n(E)$ is the evaluation map). Furthermore, $-n\lambda|^G$ is either 0 or irreducible, the latter occurring precisely when $\omega_0(-n\lambda)$ is a dominant weight, i.e., when $\lambda \in \Lambda^+$. This proves (a).

(b) Sufficiency ($\Leftarrow$) is obvious so we only prove Necessity ($\Rightarrow$). Let $\nu_0 := \nu | E^*$. Then ν_0: $E^* \to V_n$ is finite, P-equivariant, and satisfies $\nu_0(az) = a^n \nu_0(z)$ for all $a \in \mathbf{C}$, $z \in E^*$ (§1.1). Therefore, the preimage $\nu_0^{-1} L_\gamma$ of any B-stable line $L_\gamma \subset \nu(E^*) \cap (V_n^*)^U$ is a 1-dimensional cone in $(E^*)^U = S$; hence each irreducible component of $\nu_0^{-1} L_\gamma$ is a B-stable line in S. Thus, if L_μ is a B-stable line in $\nu(E^*) \cap (V_n^*)^U$ with $\eta_\omega L_\mu \subset \nu(E^*)$ for some $\omega = W$, there is a B-stable line $L_\lambda \subset S$, $\lambda \in \Lambda(S)$, such that for all $t \in T$, $z \in L_u$:

$$\mu(t)\nu(z) = t.\nu(z) = \nu(t.z) = \nu(\lambda(t)z) = \lambda(t)^n \nu(z).$$

Therefore, $\mu = n\lambda$ (additive notation). Now, $\eta_\omega L_\mu$ has weight $\omega(\mu) = n\omega(\lambda)$, so we write $\eta_\omega L_\mu = L_{n\omega(\lambda)}$. As above, each irreducible component of $\nu_0^{-1}(L_{n\omega(\lambda)})$ is a B-stable line in E^*; and we have, for all $t \in T$, $z_1 \in \nu_0^{-1}(L_{n\omega(\lambda)})$,

$$\nu(t.z_1) = t.\nu(z_1) = (\omega\lambda(t))^n \nu(z_1) = \nu(\omega\lambda(t)z_1).$$

We may thus conclude that $\omega(\lambda) \in \Lambda(E^*)$. This proves (b) and finishes the proof of the Theorem. □

3. Tangent bundles.

3.1 An important class of homogeneous vector bundles is provided by the tangent bundles of homogeneous projective manifolds G/P, where G is a semisimple complex Lie group and P is a parabolic subgroup. The ampleness of such bundles was first worked out by Goldstein [**5**]. We now give a summary of his results to illustrate how the ampleness formula §2 can be applied.

Let E be the vector space quotient of the Lie algebra of G by the Lie algebra of P. Then, under the adjoint representation, E is a rational P-module, and the tangent bundle $T(G/P)$ is isomorphic to $\mathbf{E} = G \times_P E$. This bundle is clearly spanned by global sections. If we decompose G into simple factors $G = G_1 \times \cdots \times G_m$, then G/P is isomorphic to $G_1/P_1 \times \cdots \times G_m/P_m$, P_k parabolic in G_k, $k = 1, \ldots, m$. The tangent bundle splits correspondingly into a direct sum of G-stable subbundles $\mathbf{E} = \mathbf{E}_1 \oplus \cdots \oplus \mathbf{E}_m$, and E splits into a direct sum of P-submodules $E = E_1 \oplus \cdots \oplus E_m$. Now,

$$\Lambda(E) = \Lambda(E_1) \cup \cdots \cup \Lambda(E_m) \quad \text{and} \quad S = (E^*)^U = (E_1^*)^U \oplus \cdots \oplus (E_m^*)^U.$$

Inspecting the coadjoint representation we find $\Lambda((E_k^*)^U) = \{\beta_k\}$, where β_k is the *highest root* of G_k (always invariant under σ_0), $k = 1, \ldots, m$. Thus, $\Lambda(S) = \sigma_0 \Lambda(S) = \{\beta_1, \ldots, \beta_k\}$. Therefore, by Corollary 2.1,

$$\begin{aligned} a(\mathbf{E}) &= \dim G/P - d(\sigma_0 \Lambda(S), \Lambda(E)) \\ &= \dim G/P - \min\{ d(\beta_k, \Lambda(E_k)) \mid k = 1, \ldots, m \}. \end{aligned}$$

Goldstein worked out a procedure for determining $d(\beta_k, \Lambda(E_k))$ which we shall now outline (see also [**11**, Theorem 4.6]). To simplify notation, we shall drop the subscript k in the following. On a case by case basis, it is not difficult to compute the orbital distance of β to the negative roots [**5**, p. 370]:

G	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$d(\beta, \Phi^-)$	n	$2n-2$	n	$2n-3$	11	17	29	8	3

Moreover, for any *long* simple root α_i one can find an $\omega \in W$ such that $d(\beta, \Phi^-) = l(\omega)$ with $\omega(\beta) = -\alpha_i$ [5, (4.3)]. Now, if I is the subset of simple roots which defines P, then the weights of E are the negative roots of the form

$$-\sum n_j \alpha_j, \quad \text{where } n_j > 0 \text{ for some } \alpha_j \in \Delta \setminus I.$$

Therefore, if some $\alpha_j \in \Delta \setminus I$ is *long*, then by the above remark

$$d(\beta, \Lambda(E)) = d(\beta, \Phi^-).$$

If all $\alpha_j \in \Delta \setminus I$ are *short*, then find an $\omega \in W$ such that $d(\beta, \Phi^-) = l(\omega)$, with $\omega(\beta) = -\alpha_i$, α_i long. Again, a case by case computation shows that for $G = G_2$ or B_n we need exactly one more reflection to get $\omega(\beta)$ to involve the short root α_j (there is only one short root for G_2 and B_n so $\Delta \setminus I = \{\alpha_j\}$). If $G = C_n$ or F_4, we need a minimum of $d(I)$ additional reflections to get $\omega(\beta)$ to involve some $\alpha_j \in \Delta \setminus I$, where

$$d(I) = \text{minimum number of nodes on the Dynkin diagram from an element } \alpha \in \Delta \setminus I \text{ to a long root.}$$

Using the same definition of $d(I)$ for all the different types of simple Lie groups, we can summarize the above cases in the formula

$$d(\beta, \Lambda(E)) = d(\beta, \Phi^-) + d(I).$$

4. Irreducible homogeneous vector bundles.

4.1 Let P be a parabolic subgroup of a semisimple complex Lie group G. We shall call a homogeneous vector bundle $\mathbf{E} = G \times_P E$ *irreducible* if E is an irreducible P-module. Any line bundle over G/P is necessarily homogeneous and irreducible. If $\mathbf{E}$ is irreducible, there is a simple criterion for $\xi(\mathbf{E})^n$ to be spanned.

PROPOSITION. *Let E be an irreducible P-module, and let λ be the weight of the unique B-stable line in E^*. Then the following are equivalent*:

(1) $\xi(\mathbf{E})^n$ *is spanned by global sections for some* $n \geqslant 1$;

(2) $\mathbf{E}$ *is spanned by global sections*;

(3) λ *is a dominant weight*, $\lambda \in \Lambda^+$.

PROOF. Recall that (1) $\Rightarrow$ (3) was already proved in Theorem 2.1, and (2) $\Rightarrow$ (1) is immediate. Thus, it remains to show (3) $\Rightarrow$ (2). Since E is irreducible, it is induced from the B-character $-\lambda$, i.e., $E = -\lambda_B|^P$ (see [8]). By transitivity of induction [4, 2] we have

$$E_P|^G = \left(-\lambda_B|^P\right)|^G = -\lambda_B|^G.$$

Recall also that $E_P|^G = E_B|^G$. If $\lambda \in \Lambda_+$, then the P-module (or B-module) homomorphism ε: $E|^G \to E$ cannot be the zero map since the B-module homomorphism $E|^G = -\lambda_B|^G \to -\lambda_B$ is surjective and factors (via ε) through the B-module epimorphism $E \to -\lambda_B$. Therefore ε must be surjective. $\square$

4.2 The ampleness of *irreducible* homogeneous vector bundles (which are spanned) can be easily computed using the formulas of §2.

THEOREM. *Let E be an irreducible P-module and let μ (resp. λ) be the weight of the unique B-stable line in E (resp. E^*). If λ is dominant, $\lambda \in \Lambda^+$, then the ampleness of $\mathbf{E} = G \times_P E$ is given by*

$$a(\mathbf{E}) = \dim G/P - \operatorname{index} \mu.$$

PROOF. Let I be the subset of simple roots which defines P and let W_I be the corresponding subgroup of W (= Weyl group of G). Because E is irreducible, $\sigma_0\Lambda(S) = \sigma_0\lambda$ and

$$W.(\sigma_0\lambda) \cap \Lambda(E) = W.(-\lambda) \cap \Lambda(E) = W_I.(-\lambda); \quad W_I(-\lambda) \cap \Lambda^+ = \sigma_0\lambda.$$

Therefore, $\omega.\sigma_0\lambda \in \Lambda(E)$ for $\omega = W$ if and only if $\omega^{-1}\mu' \in \Lambda^+$ for some $\mu' \in W_I.(-\lambda)$. Then

$$\begin{aligned} d(\sigma_0\Lambda(S), \Lambda(E)) &= \min\{l(\omega) \mid \omega \in W \text{ and } \omega.\mu' \in \Lambda^+ \text{ for some } \mu' \in W_I.(-\lambda)\} \\ &= \min\{\operatorname{index} \mu' \mid \mu' \in W_I.(-\lambda)\}. \end{aligned}$$

We claim that the minimum index is attained at μ which, by Corollary 2.1, proves the Theorem. To see this, note that $-\lambda$ is the lowest weight in E and satisfies $(\alpha, -\lambda) \leqslant 0$ for all positive roots $\alpha > 0$. Now, since μ is the highest weight in E, at least as many reflections σ_α, $\alpha > 0$, are needed to take $-\lambda$ to μ as to take $-\lambda$ to any other weight $\mu' \in W_I.(-\lambda)$. Thus, the number of $\alpha > 0$ which satisfy $(\alpha, \mu) \geqslant 0$ is at least as large as the number of $\alpha > 0$ which satisfy $(\alpha, \mu') \geqslant 0$ for any $\mu' \in W_I.(-\lambda)$. In particular,

$$\operatorname{index} \mu = \#\{\alpha > 0 \mid (\alpha, \mu) < 0\} \leqslant \#\{\alpha > 0 \mid (\alpha, \mu') < 0\} = \operatorname{index} \mu'. \quad \square$$

4.3 A direct application of Theorem 2.1 yields the following sufficient condition for 0-ampleness:

THEOREM. *Let E be an irreducible P-module and let λ be the weight of the unique B-stable line in E^*. If λ is regular and dominant, i.e., $(\lambda, \lambda_i) > 0$ for every fundamental dominant weight λ_i, then $\mathbf{E} = G \times_P E$ is 0-ample.*

PROOF. First we observe that the stabilizer of λ in W is trivial. Also, if I is the subset of simple roots which defines P, then $W.(\lambda) \cap \Lambda(E^*) = W_I.(\lambda)$. Therefore, if $\omega.\lambda \in \Lambda(E^*)$ then $\omega.\lambda = \omega'.\lambda$ for some $\omega' \in W_I$, so that $\omega = \omega'$. Thus,

$$\max\{l(\omega) \mid \omega.\lambda \in \Lambda(E^*)\} = \max\{l(\omega') \mid \omega' \in W_I\} = \dim P/B.$$

By Theorem 2.1, we obtain $a(\mathbf{E}) = 0$. $\square$

This Theorem shows that we need only apply Theorem 4.2 in the case where λ is singular.

4.4 We conclude with the relatively simple class of examples of line bundles over G/P. Such bundles are automatically homogeneous, and their "ampleness" was already computed in 1954 [**8**]. For completeness, we include this calculation here.

Given any weight $\lambda \in \Lambda$, let I be the subset of simple roots α orthogonal to λ, $(\alpha, \lambda) = 0$, and denote by P_λ the standard parabolic subgroup P_I (§1.3). Clearly, P_λ is the largest standard parabolic subgroup of G which can have λ as a character.

Now let $\mathbf{E} = G \times_P E$ be a line bundle over G/P and let λ be the weight of E. Then by Theorem 4.1, $\mathbf{E}$ (or any power of $\mathbf{E}$) is spanned if and only if $-\lambda$ is dominant. Note also that $P \subset P_\lambda$. To compute $a(\mathbf{E})$ we must find the maximum length $l(\omega)$ of a Weyl group element $\omega \in W$ such that $\omega.\lambda = \lambda$. This is clearly given by the maximum of $l(\omega)$ over $\omega = W_I$ (W_I is the stabilizer of λ in W). But,

$$\max\{\, l(\omega)\ |\omega \in W_I \} = \dim P_I/B.$$

Therefore, by Theorem 2.1,

$$a(\mathbf{E}) = \dim P_\lambda/P,$$

and $\mathbf{E}$ is 0-ample if and only if $P = P_\lambda$.

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