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SPHERICAL POLYNOMIALS AND THE PERIODS OF A CERTAIN MODULAR FORM

BY

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ABSTRACT. The space of cusp forms on $\mathrm{SL}_2(\mathbf{Z})$ of weight $2k$ is spanned by certain modular forms with rational periods.

Kohnen and Zagier [5] have investigated those modular forms on $\mathrm{SL}_2(\mathbf{Z})$ whose periods (in the sense of Eichler-Shimura) are rational. As an example they consider the function

$$f_{k,D}(z) = C \sum_{(a,b,c)} \frac{1}{(az^2 + bz + c)^k}$$

where D is a positive integer, k an integer greater than one, C a constant, and the summation over all triples of integers (a, b, c) with $b^2 - 4ac = D$. This function was introduced by Zagier in [13], where it was shown to be a cusp form of weight $2k$ on $\mathrm{SL}_2(\mathbf{Z})$, and was later shown by Kohnen [4] to be the Dth Fourier coefficient of the holomorphic kernel function for the Shimura-Shintani correspondence between modular forms of half-integral and integral weight. In [6] it is shown that the $f_{k,D}$ span S_{2k}, the space of cusp forms on $\mathrm{SL}_2(\mathbf{Z})$ of weight $2k$, if and only if $L(f, k)$, the L-series associated to the modular form f, does not vanish at $s = k$, the center of the critical strip, for every Hecke eigenform f. It is at present only a conjecture, supported by some numerical evidence, that the $f_{k,D}$ span S_{2k} for all even k (for k odd, $f_{k,D} = 0$). We study the related functions $f_{k,D,A}(z)$ where now the summation is restricted to quadratic forms (a, b, c) belonging to the $\mathrm{SL}_2(\mathbf{Z})$ equivalence class A. We prove that certain finite sets of the $f_{k,D,A}$ span S_{2k}. The method of proof relies on the special structure of certain ideal classes and does not indicate how the conjecture that the $f_{k,D}$ span S_{2k} might be settled.

A similar result for modular forms on an arbitrary discrete cocompact subgroup of $\mathrm{SL}_2(\mathbf{R})$ has been obtained by Svetlana Katok [3].

Let $\mathscr{A}$ be the set of all narrow equivalence classes of primitive binary quadratic forms of positive discriminant. Throughout k will denote a positive integer, and we set $w = 2k - 2$. Let A be an $\mathrm{SL}_2(\mathbf{Z})$ equivalence class of quadratic forms. We define the following classes associated to A:

$$\theta A = \{(-c, -b, -a) \mid (a, b, c) \in A\}, \quad A' = \{(a, -b, c) \mid (a, b, c) \in A\}.$$

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Let P_k denote the **C**-vector space of homogeneous polynomials in two variables of degree w. For an equivalence class $A \in \mathscr{A}$ of forms of discriminant D, let

$$P_{k,D,A}(x,y) = \sum_{Q \in A} Q(x,y)^{k-1},$$

with the summation over the reduced forms of A (a binary quadratic form $Q(x,y) = ax^2 + bxy + cy^2$ is reduced if $a > 0$, $c > 0$, $b > a + c$), and then set

$$P_{k,D,A}^{\pm} = P_{k,D,A} \pm P_{k,D,\theta A}.$$

We shall study the span in P_k of $P_{k,D,A}^{\pm}$, $A \in \mathscr{A}$, and prove a theorem about the space of modular forms on $\mathrm{SL}_2(\mathbf{Z})$ of weight $2k$.

DEFINITION 1. A polynomial $\pi(a,b,c)$ is called *spherical* with respect to $b^2 - 4ac$ *if it satisfies*

$$\left(\frac{\partial^2}{\partial b^2} - \frac{\partial^2}{\partial a \partial c}\right)\pi(a,b,c) = 0.$$

PROPOSITION 1. *Let $d_{n,k}(a,b,c)$ be defined by*

$$\sum_{n=0}^{2k-2} d_{n,k}(a,b,c)x^n y^{2k-2-n} = \left(ax^2 + bxy + cy^2\right)^{k-1}.$$

The set $\{d_{n,k}(a,b,c),\ 0 \leqslant n \leqslant w\}$ is a basis for the space of homogeneous spherical polynomials of degree $k - 1$.

For a proof see [**8**, the corollary to Theorem 18].

THEOREM 1. *Let $\mathscr{W}_k^{\pm}$ be the subspaces of P_k spanned by the collections $\{P_{k,D,A}^{\pm} + P_{k,D,A'}^{\pm}\}$, $A \in \mathscr{A}$.*

I. *Then the dimensions of $\mathscr{W}_k^{\pm}$ are equal to $\eta_k^{\pm}$, where*

$$\eta_k^{+} = \begin{cases} k/2 + 1, & k \text{ even}, \\ (k+1)/2, & k \text{ odd}, \end{cases}$$

and

$$\eta_k^{-} = \begin{cases} k/2, & k \text{ even}, \\ (k+1)/2, & k \text{ odd}. \end{cases}$$

II. *There is an infinite subset of the set $\mathscr{A}$ of equivalence classes of forms, which we shall construct explicitly, that has the following property: any $2k$ classes in this set have corresponding $P_{k,D,A}^{\pm} + P_{k,D,A'}^{\pm}$ which span $\mathscr{W}_k^{\pm}$. Moreover, given any positive discriminant D, we can choose these classes such that each has discriminant f^2D for some integer f; i.e., all of the equivalance classes are associated to module classes in the same real quadratic field.*

NOTE. We consider $P_{k,D,A}^{\pm} + P_{k,D,A'}^{\pm}$ rather than $P_{k,D,A}^{\pm}$ alone, because in the course of the proof we need to assume that our summation includes the form

(c, b, a) whenever it contains (a, b, c). Let

$$\Sigma_A^{\pm} \quad \text{mean} \quad \sum_A + \sum_{A'} \pm \sum_{\theta A} \pm \sum_{\theta A'},$$

where the summation is over the reduced forms in the given class. Since the coefficients of $Q(x, y)^{k-1}$ are the $d_{n,k}(a, b, c)$, it follows from Proposition 1 that to establish the dimension of $\mathscr{W}_k^{\pm}$ it suffices to find $w + 1 - \eta_k^{\pm}$ **Q**-linearly independent linear combinations of the $d_{n,k}(a, b, c)$, for each of which $\Sigma_A^{\pm} = 0$ for all $A \in \mathscr{A}$, and $\eta_k^{\pm}$ **Q**-linearly independent linear combinations of the $d_{n,k}(a, b, c)$, of which no nontrivial linear combination sums $(\Sigma_A^{\pm})$ to zero for all cycles $A \in \mathscr{A}$.

We first establish the requisite number of relations. Let

$$(1) \qquad \phi_{n,k}(a, b, c) = d_{n,k}(a, b, c) - d_{w-n,k}(a, b, c), \qquad 0 \leqslant n \leqslant w,$$

and

(2)

$$\psi_{n,k}^{\pm}(a, b, c) = d_{n,k}(a, b - 2a, a - b + c) - d_{n,k}(c, 2c - b, a - b + c)$$
$$\mp(-1)^k[d_{n,k}(a - b + c, b - 2a, a) - d_{n,k}(c - b + a, 2c - b, c)],$$
$$0 \leqslant n \leqslant w.$$

We assert that for any $A \in \mathscr{A}$

(i) $0 = \Sigma_A^{\pm} \phi_{n,k}(a, b, c)$,

(ii) $0 = \Sigma_A^{\pm} \psi_{n,k}^{\pm}(a, b, c)$.

That (i) is true follows immediately from the facts that $d_{w-n,k}(a, b, c) = d_{n,k}(c, b, a)$ and $(a, b, c) \in A \Leftrightarrow (c, b, a) \in A'$. To establish (ii), let $R_{n,k}^{\pm}(a, b, c) = d_{n,k}(a, b, c) \mp (-1)^k d_{n,k}(c, b, a)$. Then $R_{n,k}^{\pm}(a, b, c) = \pm R_{n,k}^{\pm}(-c, -b, -a)$, and the result follows from [7, Proposition 3′]. From among the polynomials (1) and (2) we select $w + 1 - \eta_k^{\pm}$ that are independent.

LEMMA 1. *The collections*

$$\mathscr{R}_k^{+} = \{\phi_{n,k}, 0 \leqslant n < k - 1;\ \psi_{n,k}^{+}, 1 \leqslant n < k - 1,\ n \text{ odd}\}$$

and

$$\mathscr{R}_k^{-} = \{\phi_{n,k}, 0 \leqslant n < k - 1;\ \psi_{n,k}^{-}, 1 \leqslant n \leqslant k - 1,\ n \text{ odd}\}$$

are each composed of linearly independent polynomials.

PROOF. From Proposition 1 it is clear that the $\phi_{n,k}$, $0 \leqslant n < k - 1$, are linearly independent. Moreover, $\phi_{n,k}(a, b, c) = -\phi_{n,k}(c, b, a)$, while for n odd, $\psi_{n,k}^{\pm}(a, b, c) = +\psi_{n,k}^{\pm}(c, b, a)$. Thus it suffices now to show that the $\psi_{n,k}^{\pm}(a, b, c)$ in $\mathscr{R}_k^{\pm}$ are independent among themselves.

To establish this we introduce a change of basis for the space of spherical polynomials, namely the transformation

$$(a, b, c) \mapsto (A, B + 2A, A + B + C).$$

Then

$$\begin{aligned}\psi^{\pm}_{n,k}(a,b,c) &= \psi^{\pm}_{n,k}(A, B+2A, A+B+C)\\ &= d_{n,k}(A,B,C) - d_{n,k}(A+B+C, B+2C, C)\\ &\quad \mp(-1)^k d_{n,k}(A,B,C) - d_{n,k}(C, B+2C, A+B+C)\\ &= d_{n,k}(A,B,C) - \sum_{j=0}^{n}\binom{w-j}{n-j} d_{j,k}(A,B,C)\\ &\quad \mp(-1)^k\left(d_{w-n,k}(A,B,C) - \sum_{j=0}^{w-n}\binom{w-j}{n} d_{j,k}(A,B,C)\right),\end{aligned}$$

which for $1 \leqslant n \leqslant k-1$, n odd, is a linear combination of the $d_{j,k}(A,B,C)$ with $0 \leqslant j \leqslant w-n-1$ and the coefficient of $d_{w-n-1,j}$ not equal to zero (except for $\psi^{+}_{k-1,k}$, k even, which is zero). This triangularization of the $\psi^{\pm}_{n,k}$ in $\mathscr{R}^{\pm}_{k}$ demonstrates their linear independence.

The number of $\phi_{n,k}$ in $\mathscr{R}^{\pm}_{k}$ is $k-1$ and the number of $\psi^{\pm}_{n}$ in $\mathscr{R}^{\pm}_{k}$ is given in Table 1.

TABLE 1. The number of $\psi^{\pm}_{n,k}$ in $\mathscr{R}^{\pm}_{k}$

	$\mathscr{R}^{+}$	$\mathscr{R}^{-}$
k even	$k/2-1$	$k/2$
k odd	$(k-1)/2$	$(k-1)/2$

Thus in all cases we have, as asserted, $w+1-\eta_k$ independent relations.

We complete the proof of Theorem 1 by proving the following: Consider the family of discriminants $D_t = t^2 + 4t$, $t = 2, 3, 4, \ldots$. Belonging to D_t is a wide equivalence class of quadratic forms associated to the reduced (in the wide sense) number

$$[[1,t]] = 1 + \cfrac{1}{t + \cfrac{1}{1+\cdots}}.$$

(For a discussion of the relationship between continued fractions and reduced quadratic forms see **[12]**.) Since this "+"-continued fraction is of even length, the wide equivalence class consists of two distinct narrow equivalence classes, A_t and θA_t, whose cycles of reduced forms are generated by the "−"-continued fractions $\mu_0 = ((3,2,2,\ldots,2))$ and $\omega_0 = ((t+2))$, respectively.

PROPOSITION 2. *Set*

$$g^{\pm}_{n,k}(t) = \frac{1}{2}\sum_{A_t}^{\pm} d_{n,k}(a, b-2a, a-b+c). \tag{3}$$

Then any nontrivial linear combination of elements of the set (of cardinality $\eta^{\pm}_{k}$*)*

$$\left\{g^{\pm}_{n,k}(t),\ n \text{ odd}, 1 \leqslant n < k-1 \left(\leqslant k-1 \text{ if } (-1)^k = \pm 1\right);\ g^{\pm}_{w,k}(t)\right\}$$

is zero for at most $2k$ *values of* $t > 2$.

PROOF. The cycle of reduced forms of A_t is $\{Q_p = (a_p, b_p, c_p)\}_{0 \leqslant p \leqslant t-1}$ where $a_p = (p+1)t - p^2$, $b_p = (2p+3)t - 2(p^2+p)$, $c_p = (p+2)t - (p+1)^2$. θA_t contains the single reduced form $\overline{Q} = (1, t+2, 1)$. (Note that $A_t = A_t'$, so $\Sigma^{\pm}_{A_t} = 2(\Sigma_{A_t} \pm \Sigma_{\theta A_t})$.) Taking n odd, and recalling that

$$d_{n,k}(a,b,c) = (-1)^n d_{n,k}(a,-b,c),$$

we have

$$(4) \quad \sum_{A_t} d_{n,k}(a, b-2a, a-b+c) = \sum_{p=0}^{t-1} d_{n,k}((p+1)t - p^2, t-2p, -1).$$

Under the transformation $p \mapsto t - p$ the right-hand side of (4) becomes

$$\sum_{p=1}^{t} d_{n,k}((p+1)t - p^2, 2p - t, -1) = -\sum_{p=1}^{t} d_{n,k}((p+1)t - p^2, t-2p, -1)$$
$$= -\sum_{p=0}^{t-1} d_{n,k}((p+1)t - p^2, t-2p, -1) + d_{n,k}(t,t,-1) - d_{n,k}(t,-t,-1),$$

from which we conclude that

$$\sum_{A_t} d_{n,k}(a, b-2a, a-b+c) = \tfrac{1}{2}(d_{n,k}(t,t,-1) - d_{n,k}(t,-t,-1))$$
$$= d_{n,k}(t,t,-1).$$

We also have

$$\sum_{\theta A_t} d_{n,k}(a, b-2a, a-b+c) = d_{n,k}(1,t,-t).$$

Thus for n odd,

$$\Big(\sum_{A} \pm \sum_{\theta A}\Big) d_{n,k}(a, b-2a, a-b+c) = d_{n,k}(t,t,-1) \pm d_{n,k}(1,t,-t) = g^{\pm}_{n,k}(t).$$

Writing

$$d_{n,k}(a,b,c) = \sum_{\substack{2\alpha+\beta=n \\ \alpha+\beta \leqslant k+1}} \binom{k-1}{\beta}\binom{k-1-\beta}{\alpha} a^{\alpha} b^{\beta} c^{k-(1+\alpha+\beta)},$$

we have

$$(5) \quad g^{\pm}_{n,k}(t) = \sum_{\substack{2\alpha+\beta=n \\ \alpha+\beta \leqslant k-1}} \binom{k-1}{\beta}\binom{k-1-\beta}{\alpha} (-1)^{k-(1+\alpha+\beta)} (t^{\alpha+\beta} \pm t^{k-1-\alpha}).$$

If $n = k-1$, then $\alpha + \beta = k - 1 - \alpha$, and so $g^{\pm}_{n,k}(t) = 0$. Otherwise, for $n \leqslant k - 1$, n odd, we claim that the right-hand side of (5) is a polynomial in t of degree $\leqslant k-1$ whose $t^{(n+1)/2}$ coefficient is nonzero but all of whose lower powers of t vanish.

To see this, observe that $\alpha + \beta = n - \alpha$, and so $\alpha + \beta$ and $k - 1 - \alpha$ are each minimal for the largest possible choice of α, namely for $\beta = 1$, $\alpha = (n-1)/2$. Then $\alpha + \beta = (n+1)/2$, and $k - 1 - \alpha = k - ((n+1)/2) \geqslant (n+1)/2$, since $n \leqslant k - 1$.

We now consider

$$g_{w,k}^{\pm}(t) = \sum_{p=0}^{t-1} d_{w,k}\big((p+1)t - p^2, t - 2p, -1\big) \pm d_{w,k}(1, t, -t)$$

$$= (\pm 1) + \sum_{p=0}^{t-1} \big((p+1)t - p^2\big)^{k-1}.$$

We claim that the sum in this expression is a polynomial in t with no constant term, whence $g_{w,k}^{\pm}(t)$ does have a constant term, namely ± 1.

$$\sum_{p=0}^{t-1} \big((p+1)t - p^2\big)^{k-1} = \sum_{p=0}^{t-1} (\pm p^w + \text{higher powers of } t).$$

But

$$\sum_{p=0}^{t-1} p^w = \frac{B_{w+1}(t) - B_{w+1}}{w+1},$$

where $B_s(t)$ is the sth Bernoulli polynomial. Since $w+1$ is odd, this is a polynomial in t with no constant term, and our claim has been proved.

This triangularization of the polynomials in (3) shows them to be linearly independent.

The $g_{n,k}^{\pm}(t)$ for n odd have degree at most $k-1$, while the degree of $g_{w,k}^{\pm}(t)$ is $2k-1$. Thus any nontrivial linear combination of the $g_{n,k}^{\pm}$ is a polynomial in t of degree at most $w+1 = 2k-1$ and so can be zero for at most $2k-1$ values of t. Thus for any $2k$ values of $t \geqslant 2$, the corresponding set of the $P_{k,D,A_t}^{\pm}$ must span $\mathcal{W}_k^{\pm}$.

To show that we can choose the A_t to be associated to the same real quadratic field, fix a discriminant $D > 0$. The class A_t has discriminant $t^2 + 4t$, and since $t^2 + 4t = f^2 D$ is equivalent to $(t+2)^2 - f^2 D = 4$, which is just Pell's equation, we can find infinitely many values of t for which A_t has discriminant $f^2 D$ (for different values of f).

We note the following corollary, which we shall need in the proof of a later theorem.

COROLLARY. *The sets* $\{\psi_{n,k}^{+}(a,b,c)\}$, *$n$ odd*, $1 \leqslant n < k-1$, *and* $\{\psi_{n,k}^{-}(a,b,c)\}$, *$n$ odd*, $1 \leqslant n \leqslant k-1$, *are bases for the spaces of those spherical polynomials P of degree $k-1$ for which* $\Sigma_A^{\pm} P = 0$ *for all* $A \in \mathcal{A}$.

We now consider the function

$$f_{k,D}(z) = \frac{(-1)^{(k-1)/2} 2^{3k-2} D^{k-1/2}}{\pi \binom{2k-2}{k-1}} \sum_{(a,b,c)} \frac{1}{(az^2 + bz + c)^k},$$

where D is a positive integer, k an integer $\geqslant 2$, and where the summation is over all triples of integers (a, b, c) with $b^2 - 4ac = D$. This function was introduced by Zagier in [13], where it was shown to be a cusp form of weight $2k$ on $SL_2(\mathbf{Z})$. Then Kohnen showed in [4] that $f_{k,D}$ is the Dth Fourier coefficient of the holomorphic kernel function for the Shimura-Shintani correspondence between modular forms of weight $k + \frac{1}{2}$ and weight $2k$. We shall be interested in the functions $f_{k,D,A}(z)$, defined in the same way as $f_{k,D}$, except that here the summation is over only those forms (a, b, c) belonging to the equivalence class A. The periods of the functions $f^{\pm}_{k,D,A} =: f_{k,D,A} \pm f_{k,D,A'}$ were studied by Kohnen and Zagier in [**5**]. Let us here recall the basic facts about periods of modular forms.

Let f be a cusp form of weight $2k$. Then for any integer $0 \leqslant n \leqslant 2k - 2 = w$, we can define the nth *period* of f,

$$r_n(f) = \int_0^{i\infty} f(z) z^n \, dz.$$

The fundamental theorems on periods are the following, our exposition being essentially that in Lang [**9**].

THEOREM 2 (EICHLER - SHIMURA). *There are certain relations among the periods of modular forms*:

(i) $$r_s + (-1)^s \sum_{\substack{j=0 \\ j \text{ even}}}^{s} \binom{s}{j} r_{w-s+j} + (-1)^s \sum_{\substack{j=0 \\ j \equiv s \bmod 2}}^{w-s} \binom{w-s}{j} r_j = 0;$$

(ii) $$\sum_{\substack{j=1 \\ j \text{ odd}}}^{s} \binom{s}{j} r_{w-s+j} + \sum_{\substack{j=0 \\ j \not\equiv s \bmod 2}}^{w-s} \binom{w-s}{j} r_j = 0.$$

Since $r_j(f)$ is real if j is odd, and pure imaginary if j is even, these relations can be separated into two sets of relations: one involving only the even periods, the other only the odd. These relations then define two subspaces, V^+ of $\mathscr{P}_k^+$ and V^- of $\mathscr{P}_k^-$, where $\mathscr{P}_k^+$ (resp. $\mathscr{P}_k^-$) is the subspace of $\mathscr{P}_k$ consisting of polynomials even (resp. odd) in both variables.

THEOREM 3 (EICHLER - SHIMURA). *Let S_{2k} be the space of cusp forms of weight $2k$ on* $\mathrm{SL}_2(\mathbf{Z})$. *Define the mappings $r^{\pm}$ of S_{2k} into $\mathscr{P}_k^{\pm}$ by*

(6) $$r^{\pm}(f)(x, y) = \varepsilon_{\pm} \sum_{\substack{0 \leqslant n \leqslant w \\ (-1)^n = \pm 1}} \binom{w}{n} r_n(f) x^n y^{w-n},$$

where $\varepsilon_+ = -i$, $\varepsilon_- = 1$.

Then r^- is an isomorphism of S_{2k} and V^-, while r^+ is an isomorphism of S_{2k} with a subspace of V^+ of codimension 1, *not containing $x^w - y^w$.*

Recall that $P_{k,D,A}(x, y) = \Sigma_A Q(x, y)^{k-1}$.

In [**5**] the following theorem is proved.

THEOREM 4. *Let $k \geqslant 2$, and let A be an* $\mathrm{SL}_2(\mathbf{Z})$ *class of quadratic forms of discriminant $D > 0$, D not a square. Then*

$$\frac{(-1)^{(k+1)/2}}{2^{3k+1}} r^{\pm}(f_{k,D,A})$$
$$= P_{k,D,A}(x + y, -x) - P_{k,D,A}(x - y, y) + (-1)^k P_{k,D,\theta A}(x, -x + y)$$
$$- P_{k,D,\theta A}(-y, x + y) \frac{1 + (-1)^n}{B_{2k}} \zeta(A, 1 - k)(x^w - y^w),$$

or, equivalently, for $(-1)^n = \pm 1$,

$$\frac{(-1)^{(k+1)/2}}{2^{3k-1}}\binom{w}{n} r_n\left(f^{\pm}_{k,D,A}\right)$$
$$= \sum_{\substack{(a,b,c)\in A\\ \text{reduced}}} d_{n,k}(c-b+a, 2a-b, a) - d_{n,k}(a, b-2a, a-b+c)$$
$$+(-1)^k \sum_{\substack{(a,b,c)\in \theta A\\ \text{reduced}}} \left(d_{n,k}(a-b+c, b-2c, c) - d_{n,k}(c, 2c-b, c-b+a)\right)$$
$$+(\delta_{w,n}-\delta_{0,n})\frac{2k}{B_{2k}}\zeta(A, 1-k),$$

where B_s *is the sth Bernoulli number,* $\delta_{j,k}$ *the Kronecker delta, and* $\zeta(A,s)$ *the Dedekind zeta function of the ideal class associated to* A.

We can now state and prove our second result.

THEOREM 5. *Let* $f_{k,D,A}$ *be defined as above. For* k *an integer greater than* 1, *the collection of the* $f_{k,D,A}$, $A \in \mathscr{A}$, *spans* S_{2k}, *the space of cusp forms on* $\mathrm{SL}_2(\mathbf{Z})$ *of weight* $2k$.

PROOF. By Theorems 3 and 4, if suffices to show that the polynomials $r^+(f^+_{k,D,A})$ span the image of $r^+(S_{2k})$ as defined in Theorem 3.

LEMMA 2. $f^{\pm}_{k,D,A} = (-1)^k f^{\pm}_{k,D,\theta A}$.

PROOF. If $(a,b,c) \in A$, then $(-a,-b,-c) \in \theta A'$. Thus $f_{k,D,A} = (-1)^k f_{k,D,\theta A'}$, and the result follows at once from the definition of $f^{\pm}_{k,D,A}$.

From Lemma 2 and Theorem 4, as well as the fact that $(a,b,c) \in A \Rightarrow (c,b,a) \in A'$, we immediately deduce

LEMMA 3. *Let*

$$S_{n,k}(a,b,c) = -d_{n,k}(a-b+c, b-2a, a)$$
$$-d_{n,k}(c-b+a, b-2c, c) + d_{n,k}(c, b-2c, c-b+a).$$

The even periods of $f^+_{k,D,A}$ *are given for* $2 \leqslant n \leqslant w-2$ *by*

$$\frac{-(-1)^{(k+1)/2}\binom{w}{n}}{2^{3k-3}} r_n\left(f^+_{k,D,A}\right)$$
$$= \left(\sum_A + (-1)^k \sum_{\theta A}\right)\Big(-d_{n,k}(a-b+c, b-2a, a) - d_{n,k}(c-b+a, b-2c, c)$$
$$+ d_{n,k}(c, b-2c, c-b+a)\Big)$$
$$=: \left(\sum_A + (-1)^k \sum_{\theta A}\right) S_{n,k}(a,b,c).$$

Now, by the corollary to Theorem 1 and Lemma 3, there is a relation among these even periods of $f^+_{k,D,A}$ if and only if some linear combinations of the $S_{n,k}$, n even, equals a linear combination of the $S_{n,k}$, n odd. Thus to prove Theorem 5, it suffices to show that the dimension of the subspace of the spherical polynomials of degree $k-1$ spanned by the $S_{n,k}$ has dimension equal to the span of the $S_{n,k}$, n odd (which is $[(k-1)/2]$) plus the dimension of S_{2k} (which is $[k/6]$, unless $2k \equiv 2 \pmod{12}$, in which case it is $[k/6]-1$), plus 1 (for the zeroth period), which totals $[2k/3]$ in all cases. We proceed now to calculate this dimension. For $0 \leqslant n \leqslant w$ define polynomials $\kappa_{n,k}(x, y)$ in $\mathcal{P}_k$ by

$$\kappa_{n,k}(x, y) = (x-y)^n(-x)^{w-n} + (-x)^n(x-y)^{w-n} - (x-y)^n y^{w-n} - y^n(x-y)^{w-n}.$$

LEMMA 4. $\sum_{n=0}^{w} S_{n,k}(a, b, c) = \sum_{n=0}^{w} d_{n,k}(a, b, c)\kappa_{n,k}(x, y)$.

PROOF.

$$\begin{aligned}
\sum_{n=0}^{w} S_{n,k}(a,b,c)x^n y^{w-n}
&= \sum_{n=0}^{w} \big(d_{n,k}(a, b-2a, a-b+c) + d_{n,k}(c, b-2c, c-b+a) \\
&\qquad - d_{n,k}(a-b+c, b-2a, a) - d_{n,k}(c-b+a, b-2c, c)\big)x^n y^{w-n} \\
&= \big((a-b+c)x^2 + (b-2a)xy + ay^2\big)^{k-1} \\
&\quad + \big((c-b+a)x^2 + (b-2c)xy + cy^2\big)^{k-1} \\
&\quad - \big(ax^2 + (b-2a)xy + (a-b+c)y^2\big)^{k-1} \\
&\quad - \big(cx^2 + (b-2c)xy + (c-b+a)y^2\big)^{k-1} \\
&= \sum_{n=0}^{w} \big(d_{n,k}(a,b,c)\big)\big((x-y)^n(-x)^{w-n} + (-x)^n(x-y)^{w-n} \\
&\qquad - (x-y)^n y^{w-n} - y^n(x-y)^{w-n}\big) \\
&= \sum_{n=0}^{w} d_{n,k}(a,b,c)\kappa_{n,k}(x, y).
\end{aligned}$$

We now assert that the dimension of the span of the $\mathcal{P}_{n,k}$ in the space of spherical polynomials of degree $k-1$ is equal to the dimension of the subspace of $\mathcal{P}_k$ spanned by the $\kappa_{n,k}(x, y)$.

PROPOSITION 3. *Let V and A be vector spaces over a field, of dimension N, with bases $v_1, \ldots, v_N$ and $a_1, \ldots, a_N$ respectively, and suppose that*

$$\sum_{i=1}^{N} w_i \otimes a_i = \sum_{j=1}^{N} v_j \otimes b_j,$$

for some w_i, $1 \leqslant j \leqslant N$, in V and b_j, $1 \leqslant j \leqslant N$, in A.

Then the dimensions of the subspace of V generated by the w_i, and of the subspace of A generated by the b_j are equal.

PROOF. Write, for $1 \leqslant i \leqslant N$,

$$w_i = \sum_{j=1}^{N} \alpha_{ij} v_j \quad \text{and} \quad b_i = \sum_{j=1}^{N} \beta_{ij} a_j.$$

Then $\sum_{i=1}^{N} w_i \otimes a_i = \sum_{i=1}^{N} v_i \otimes b_i$ implies $\alpha_{ij} = \beta_{ji}$. Hence the matrices (α_{ij}) and ${}^t(\beta_{ij})$ are equal, whence (α_{ij}) and (β_{ij}) have the same rank. Applying Proposition 3 to A, the space of spherical polynomials, $V = \mathscr{P}_k$, $a_n = S_{n,k}(a, b, c)$, and $v_n = \kappa_{n,k}(x, y)$ proves the assertion.

For a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbf{Z})$ we define an endomorphism $\Pi(\gamma)$ on $\mathscr{P}_k$ as follows: for any polynomial $P(x, y) \in \mathscr{P}_k$ let

$$\Pi(\gamma)P(x, y) - \phi_{n,k}((x, y)\gamma),$$

where

$$(x, y)\gamma = (x, y)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ax + cy, bx + dy).$$

Now let

$$t = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\mathscr{U}$ be the image of $\mathscr{P}_k$ under $\Psi = (1 - \sigma)(t - t^2)$. Then a simple calculation shows that $\mathscr{U}$ is precisely the span of the $\kappa_{n,k}$. In calculating the dimension of $\mathscr{U}$, it makes no difference if we consider the action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathscr{P}_k(\mathbf{C})$ instead of on $\mathscr{P}_k(\mathbf{Q})$. The operator $\Pi(t)$ has two eigenspaces on the homogeneous polynomials of degree 1: $\mathbf{C}(x + \rho y)$ with eigenvalue $-\rho$, and $\mathbf{C}(x + \bar{\rho} y)$ with eigenvalue $-\bar{\rho}$ ($\rho = e^{(2\pi i/3)}$), while $\Pi(\sigma)$ takes $(x + \rho y)$ to $\rho(x + \bar{\rho} y)$ and $(x + \bar{\rho} y)$ to $\bar{\rho}(x + \rho y)$.

Let $A = x + \rho y$, $B = x + \bar{\rho} y$. We have the direct sum decomposition

$$\mathscr{P}_w(\mathbf{C}) = \bigoplus_{j=0}^{k-1} \mathbf{C}(A^j B^{w-j} \oplus A^{w-j} B^j)$$

into ψ-invariant subspaces, with

$$\Pi(\Psi)(A^j B^{w-j} + A^{w-j} B^j) = \rho^j \bar{\rho}^{w-j}(1 - \rho^j \bar{\rho}^{w-j})(1 - \rho^j \bar{\rho}^{w-j}) A^j B^{w-j} + \rho^{w-j} \bar{\rho}^j (1 - \rho^j \bar{\rho}^{w-j})(1 - \rho^{w-j} \bar{\rho}^j) A^{w-j} B^j,$$

which is zero if and only if $\rho^j \bar{\rho}^{w-j} = 1$. But

$$\rho^j \bar{\rho}^{w-j} = \rho^{2w-j} = 1 \Leftrightarrow j \equiv 2w \pmod 3.$$

Thus the dimension of $\mathscr{U}$ is

$$\#\{j \mid 0 \leqslant j \leqslant k - 1; 2w - j \equiv 1 \text{ or } 2 \pmod 3\} = (2k/3),$$

as was to be shown.

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