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CONTRIBUTIONS FROM CONJUGACY CLASSES OF REGULAR ELLIPTIC ELEMENTS IN HERMITIAN MODULAR GROUPS TO THE DIMENSION FORMULA OF HERMITIAN MODULAR CUSP FORMS[1]

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ABSTRACT. The dimension of the vector space of hermitian modular cusp forms on the hermitian upper half plane can be obtained from the Selberg trace formula; in this paper we shall compute the contributions from conjugacy classes of regular elliptic elements in hermitian modular groups by constructing an orthonomal basis in a certain Hilbert space of holomorphic functions. A generalization of the main Theorem can be applied to the dimension formula of cusp forms of $SU(p,q)$. A similar theorem was given for the case of regular elliptic elements of $\mathrm{Sp}(n,\mathbf{Z})$ in [**5**] via a different method.

1. Introduction and notation. Denote by E the unit matrix and by 0 the zero matrix in the matrix ring $M_n(\mathbf{C})$. Put $J = \left[\begin{smallmatrix} 0 & E \\ -E & 0 \end{smallmatrix}\right]$. The hermitian symplectic group of degree n, Ω_n, is then defined as the group of matrices in $M_{2n}(\mathbf{C})$; it satisfies ${}^t\overline{M}JM = J$; i.e.

$$\Omega_n = \{M \in M_{2n}(\mathbf{C}) \mid {}^t\overline{M}JM = J\}.$$

Here ${}^t\overline{M}$ is the transpose complex conjugate to M.

Let $\mathcal{H}_n$ be the hermitian upper half plane; specifically,

$$\mathcal{H}_n = \{Z \in M_n(\mathbf{C}) | Z = X + iY,\ X = {}^t\overline{X},\ Y = {}^t\overline{Y} > 0\}.$$

The hermitian symplectic group Ω_n operates on $\mathcal{H}_n$ transitively by the action

$$M: Z \to M(Z) = (AZ+B)(CZ+D)^{-1}, \qquad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Omega_n.$$

For a given imaginary quadratic number field $\mathbf{F}$, we denote by $\mathbf{K}$ its ring of integers. The hermitian modular group of degree n, $\Gamma_n(\mathbf{K})$, is defined as

$$\Gamma_n(\mathbf{K}) = \Omega_n \cap M_{2n}(\mathbf{K}).$$

An element M in $\Gamma_n(\mathbf{K})$ is regular elliptic if M has an isolated fixed point on $\mathcal{H}_n$, i.e. the equation $M(Z) = Z$ has a unique solution on $\mathcal{H}_n$. A similar argument as in [**5**] shows that the following statements are equivalent:

(1) M is a regular elliptic element in $\Gamma_n(\mathbf{K})$ and its characteristic polynomial $\varphi(X)$ is in $Z[X]$.

(2) $M \in \Gamma_n(\mathbf{K})$ and is conjugate in Ω_n to $\mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n, \overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_n]$; λ_i $(i = 1, 2, \ldots, n)$ are roots of unity and $\lambda_i\lambda_j \neq 1$ for all i, j.

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Let $S(k;\Gamma_n(\mathbf{K}))$ denote the space of holomorphic functions $f(Z)$ on $\mathcal{H}_n$; $f(Z)$ satisfies the following conditions:

(1) $f(M(Z)) = \det(CZ+D)^k f(Z)$ for all $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ in $\Gamma_n(\mathbf{K})$, $Z \in \mathcal{H}_n$.

(2) $(\det Y)^{k/2} f(Z)$ is bounded on $\mathcal{H}_n, Z = X+iY$.

A function in $S(k;\Gamma_n(\mathbf{K}))$ is called a hermitian modular cusp form of weight k and degree n.

For fixed degree n and certain k, the first condition may be satisfied only for $f(Z) = 0$. However we shall exclude these trivial cases. For example, we assume $kn \equiv 0 \pmod 4$ when $\mathbf{K} = \mathbf{Z}[i]$ and $\mathbf{F} = \mathbf{Q}[i]$. It is well known that $S(k,\Gamma_n(\mathbf{K}))$ is a finite dimensional Hilbert space [**6**]. Furthermore, its dimension can be written as an integral of a Bergman kernel function on a certain Hilbert space over the fundamental domain in $\mathcal{H}_n$ with respect to $\Gamma_n(\mathbf{K})$, when k is sufficiently large (for example $k > (4n-2)$; see also [**9**]). This is the so-called Selberg trace formula.

More precisely, let $K(Z_1, Z_2)$ be a kernel function of the space $H(k;\mathcal{H}_n)$ which consists of a holomorphic function on $\mathcal{H}_n$ and satisfies

$$\int_{\mathcal{H}_n} (\det Y)^{k-2n} |f(Z)|^2 \, dZ < \infty.$$

Then

$$\dim_{\mathbf{C}} S(k;\Gamma_n(\mathbf{K})) = \int_{\mathcal{F}_n} \sum_{\gamma \in \overline{\Gamma_n(\mathbf{K})}} K(Z,\gamma(Z)) \overline{j(\gamma, Z)}^{-k} (\det Y)^{k-2n} \, dZ,$$

where

(1) $\overline{\Gamma_n(\mathbf{K})}$ is the quotient group $\Gamma_n(\mathbf{K})/U$ with U the center of $\Gamma_n(\mathbf{K})$,

(2) $\mathcal{F}_n$ is a fundamental domain in $\mathcal{H}_n$ with respect to $\Gamma_n(\mathbf{K})$,

(3) $j(\gamma, Z) = \det(CZ+D)$ if $\gamma = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \Gamma_n(\mathbf{K})$,

(4) $Z = X+iY$ in $\mathcal{H}_n$ and $dZ = dXdY$ is the Euclidean measure on C^{n^2}.

In this paper, we shall consider the subseries with the summation ranging over all regular elliptic elements in $\Gamma_n(\mathbf{K})$; or consider the contributions from conjugacy classes of regular elliptic elements to $\dim_{\mathbf{C}} S(k;\Gamma_n(\mathbf{K}))$. We shall obtain the following

THEOREM. *Suppose $M \in \Gamma_n(\mathbf{K})$ and is conjugate in Ω_n to*

$$\operatorname{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n, \overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_n]$$

with λ_j $(j = 1, 2, \ldots, n)$ roots of unity and $\lambda_1\lambda_j \neq 1$ for all $1 \leq i, j \leq n$. Then the contribution to $\dim_{\mathbf{C}} S(k;\Gamma_n(\mathbf{K}))$ $(k > (4n-2))$ of regular elliptic elements in $\Gamma_2(\mathbf{K})$ which are conjugate in $\Gamma_n(\mathbf{K})/U$ to M is given by

$$N_{\{M\}} = |C_{M,\mathbf{Z}}|^{-1} \prod_{j=1}^{n} \overline{\lambda}_j^k \cdot \prod_{j,k=1}^{n} (1 - \overline{\lambda}_j \overline{\lambda}_k)^{-1}.$$

Here $C_{M,\mathbf{Z}}$ is the centralizer of M in $\Gamma_n(\mathbf{K})/U$ and $|C_{M,\mathbf{Z}}|$ is its order.

REMARK. Here we shall exclude those integers k such that

$$f(M(Z)) = \det(CZ+D)^k f(Z), \qquad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_n(\mathbf{K}),$$

is satisfied only for $f(Z) = 0$.

2. The Selberg trace formula. Since $\mathcal{H}_n$ is mapped biholomorphically onto the bounded domain

$$D_n : W \in M_n(\mathbf{C}), \quad E - W{}^t\overline{W} > 0,$$

under the Cayley transform $W = (Z - iE)(Z + iE)^{-1}$, it suffices to consider $H(k; D_n)$ instead of $H(k; \mathcal{H}_n)$. $H(k; D_n)$ consists of holomorphic function $f(W)$ satisfying

$$\int_{D_n} \det(E - W{}^t\overline{W})^{k-2n} |f(W)|^2 \, dW < \infty.$$

The Bergmann kernel function for $H(k; D_n)$ is given by the following propositions.

PROPOSITION 1 [**11**, THEOREM 3.3]. *Let $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ be any orthonormal basis of the Hilbert space $H(k; D_n)$. Then the series*

$$\sum_{n=1}^{\infty} \varphi_n(W)\overline{\varphi_n(W_1)}$$

converges uniformly on each compact subset of $D_n \times D_n$. The sum, denoted by $K(W, W_1)$, is independent of the choice of orthonomal basis and

$$f(W) = \int_{D_n} \det(E - W_1{}^t\overline{W}_1)^{k-2n} K(W, W_1) f(W_1) \, dW_1$$

for each $f \in H(k; D_n)$.

PROPOSITION 2 [**9**, LEMMA 2.1]. *Suppose that $k > (4n - 2)$. Then the function $K(W, W_1)$ is given by*

$$K(W, W_1) = C(k, n) \det(E - W{}^t\overline{W}_1)^{-k}$$

with

$$C(k, n) = \pi^{-n^2} \prod_{0 \le i,j \le n-1} (k - 2n + 1 + i + j).$$

PROOF. The kernel function is a constant multiple of $\det(E - W{}^t\overline{W}_1)^{-k}$ by arguments similar to I of [**7**]. The constant $C(k, n)$ is determined by

$$C(k, n)^{-1} = \int_{D_n} \det(E - W{}^t\overline{W})^{k-2n} \, dW.$$

3. Convergence of the series. Let $\Lambda = \operatorname{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$ be a unitary matrix. As an element of a hermitian symplectic group, the operation of Λ on D_n is given by

$$\Lambda : W \to \Lambda W \Lambda, \qquad W \in D_n,$$

and

$$K(W, \Lambda W \Lambda)\overline{j(\Lambda, W)}^{-k} = C(k, n)(\det \Lambda)^{-k} \det(E - \overline{\Lambda} W \overline{\Lambda}{}^t\overline{W})^{-k}.$$

Now we shall prove this function is absolutely integrable on D_n with respect to the measure $\det(E - W{}^t\overline{W})^{k-2n} \, dW$ when $k > (2n - 1)$.

LEMMA 1 [**8**, THEOREM 1, P. 266]. *If $E - Z^t\overline{Z} \geq 0$ and $E - W^t\overline{W} \geq 0$, then*

$$\det(E - Z^t\overline{Z})\det(E - W^t\overline{W}) + |\det(Z - W)|^2 \leq |\det(E - Z^t\overline{W})|^2.$$

Equality holds only when $Z = W$.

LEMMA 2. *If $\Lambda = \operatorname{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$ with λ_i $(i = 1, 2, \ldots, n)$ roots of unity and $1 - \lambda_i\lambda_j \neq 0$ for all i, j, then*

$$\det(E - \overline{\Lambda}W\overline{\Lambda}^t\overline{W}) \neq 0$$

for all $W \in \overline{D}_n$.

PROOF. Applying the previous lemma with $Z = \overline{\Lambda}W\overline{\Lambda} = [\overline{\lambda}_i\overline{\lambda}_j w_{ij}]$, we get

$$[\det(E - W^t\overline{W})]^2 + |\det(\overline{\Lambda}W\overline{\Lambda} - W)|^2 \leq [\det(E - \overline{\Lambda}W\overline{\Lambda}^t\overline{W})]^2.$$

Now suppose $\det(E - \overline{\Lambda}W\overline{\Lambda}^t\overline{W}) = 0$. Then it forces

$$\det(E - W^t\overline{W}) = 0 \quad \text{and} \quad \overline{\Lambda}W\overline{\Lambda} = W.$$

From $\overline{\Lambda}W\overline{\Lambda} = W$ and our assumption on Λ, we get $W = 0$, which contradicts $\det(E - W^t\overline{W}) = 0$. This proves our assertion.

PROPOSITION 3. *Let $M \in \Gamma_n(\mathbf{K})$ and be conjugate in Ω_n to*

$$\operatorname{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n, \overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_n],$$

λ_i $(i = 1, 2, \ldots, n)$ roots of unity and $\lambda_i\lambda_j \neq 1$ for all i, j. Then we have

(1) $\int_{D_n} \det(E - W^t\overline{W})^{k-2n}|\det(E - \overline{\Lambda}W\overline{\Lambda}^t\overline{W})|^{-k}\,dW < \infty$ *for $k > (2n - 1)$,*

(2) *the contribution $N_{\{M\}}$ in the Theorem is given by*

$$N_{\{M\}} = C(k, n)(\det\Lambda)^{-k}|C_{M,Z}|^{-1} \times \int_{D_n} \det(E - W^t\overline{W})^{k-2n}\det(E - \overline{\Lambda}W\overline{\Lambda}^t\overline{W})^{-k}\,dW.$$

PROOF. (1) follows since $\det(E - W^t\overline{W})^{k-2n}\,dW$ is a bounded measure on D_n if $k > (2n - 1)$ and $\det(E - \overline{\Lambda}W\overline{\Lambda}^t\overline{W}) \neq 0$ for all W in D_n.

To prove (2), we let $\{M\}$ denote the conjugacy class in $\Gamma_n(\mathbf{K})/U$ and which can be represented by M. Then we have

$$N_{\{M\}} = \int_{\mathcal{F}_n} (\det Y)^{k-2n} \sum_{\gamma \in \{M\}} K(Z, \gamma(Z))\overline{j(\gamma, Z)}^{-k}\,dZ.$$

Note that the integral

$$N = \int_{\mathcal{H}_n} (\det Y)^{k-2n} K(Z, M(Z))\overline{j(M, Z)}^{-k}\,dZ$$

is transformed into

$$(\det\Lambda)^{-k}C(k, n)\int_{D_n} \det(E - W^t\overline{W})^{k-2n}\det(E - \overline{\Lambda}W\overline{\Lambda}^t\overline{W})^{-k}\,dW$$

under the Cayley transform $W = (Z - iE)(Z + iE)^{-1}$. Now with (1), we know the integral N is absolutely convergent. Hence we have

$$N = \int_{\mathcal{F}_n} (\det Y)^{k-2n} \sum_{\gamma \in \overline{\Gamma_n(K)}} K(Z, \gamma^{-1}M\gamma(Z))\overline{j(\gamma^{-1}M\gamma, Z)}^{-k}\, dZ$$
$$= |C_{M,\mathbf{Z}}| \int_{\mathcal{F}_n} \sum_{\gamma \in \{M\}} K(Z, \gamma(Z))\overline{j(\gamma, Z)}^{-k}\, dZ = |C_{M,\mathbf{Z}}| \cdot N_{\{M\}}.$$

This proves our assertion in (2).

REMARK. Here we use the fact that the centralizer $C_{M,\mathbf{Z}}$ of M in $\overline{\Gamma_n(\mathbf{K})}$ is a group of finite order since it is discrete and is conjugate in Ω_n to a subgroup of a unitary group which is compact.

4. Proof of the Theorem. To prove our Theorem, by Proposition 3 it suffices to evaluate the integral

$$C(k,n) \int_{D_n} \det(E - W{}^t\overline{W})^{k-2n} \det(E - \overline{\Lambda} W \overline{\Lambda}{}^t\overline{W})^{-k}\, dW.$$

But this is not easy when $n \geq 2$. Here we shall first construct a new orthonomal basis in $H(k; D_n)$.

LEMMA 3 [**15**, LEMMA 1, P. 27]. *Let $S = [s_{ij}]$ be an $n \times n$ hermitian matrix, i.e. $s_{ij} = \bar{s}_{ji}$ for all i,j, and let S_j ($j = 1, 2, \ldots, n-1$) be the submatrix consisting of $j \times j$ entries on the upper left block of S. Then S is positively definite if and only if*

$$\det S > 0 \quad \textit{and} \quad \det S_j > 0 \qquad (j = 1, 2, \ldots, n-1).$$

PROPOSITION 4. *Let θ_{1j}, θ_{j1} ($j = 1, 2, 3, \ldots, n$) be $2n - 1$ real numbers and $W' = [w'_{jk}] \in D_n$. Suppose $W = [w_{jk}] \in M_n(\mathbf{C})$ is defined by*

$$\begin{cases} w_{jk} = w'_{jk} e^{i\theta_{jk}}, & j = 1 \text{ or } k = 1, i = \sqrt{-1}, \\ w_{jk} = w'_{jk} e^{i(\theta_{j1} + \theta_{1k} - \theta_{11})}, & j \neq 1 \text{ and } k \neq 1. \end{cases}$$

Then we have $W \in D_n$.

PROOF. Let $E - W{}^t\overline{W} = [a_{jk}]$ and $E - W'{}^t\overline{W}' = [b_{jk}]$. A direct calculation shows

(1) $a_{jk} = \overline{a_{kj}}$, $b_{jk} = \overline{b_{kj}}$ for all k, j;
(2) $a_{jj} = b_{jj} > 0$, $j = 1, 2, \ldots, j$;
(3) $a_{jk} = b_{jk} e^{i(\theta_{j1} - \theta_{k1})}$ for all j, k.

If $W' \in D_n$, then the submatrix W'_{n-1} obtained from cancellation of the nth row and nth column of W is in D_{n-1}. Thus by Lemma 3 and an induction on n, it suffices to prove $\det(E - W{}^t\overline{W}) > 0$. But it is easy to show

$$\det(E - W{}^t\overline{W}) = \det(E - W'{}^t\overline{W}')$$

by properites (1)–(3) and elementary properties of the determinant. This proves our assertion.

For each n^2-tuple of nonnegative integers $\alpha = [\alpha_{jk}]$, $1 \le j,k \le n$; we shall let W^α denote the monomial

$$\prod_{j,k=1}^{n} w_{jk}^{\alpha_{jk}}$$

in the variable W, and let $|\alpha| = \sum_{j,k=1}^{n} \alpha_{jk}$ be the degree of W^α.

PROPOSITION 5. *Let W^α and W^β be monomials in w_{jk} $(j,k = 1,\dots,n)$. Then*

$$\int_{D_n} \det(E - W{}^t\overline{W})^{k-2n} W^\alpha \overline{W}^\beta \, dW = 0$$

unless

$$\begin{cases} \alpha_{11} - \beta_{11} + \sum_{j,k\ge 2}(-\alpha_{jk} + \beta_{jk}) = 0, \\ \sum_{j=1}^{n}(\alpha_{jk} - \beta_{jk}) = 0, \ \sum_{j=1}^{n}(\alpha_{kj} - \beta_{kj}) = 0, \qquad (k = 2,3,\dots,n). \end{cases}$$

Under the above conditions, we have $|\alpha| = |\beta|$ and

$$\sum_{j=1}^{n}(\alpha_{jk} + \alpha_{kj}) = \sum_{j=1}^{n}(\beta_{jk} + \beta_{kj}), \qquad (k = 1,2,\dots,n).$$

PROOF. By the previous proposition, we can use polar coordinates on certain entries of W as follows:

$$\begin{cases} w_{jk} = r_{jk} e^{i\theta_{jk}}, & j = 1 \text{ or } k = 1, \ r_{jk} \ge 0, 0 \le \theta_{jk} < 2\pi; \\ w_{jk} = w'_{jk} e^{i(\theta_{j1} + \theta_{1k} - \theta_{11})}, & j \ne 1 \text{ and } k \ne 1. \end{cases}$$

Let D'_n be the subset of D_n and be defined by

$$D'_n \colon W = [w_{jk}] \in D_n, \qquad w_{j1}, w_{1j} \ge 0 \ (j = 1,2,\dots,n).$$

With these new coordinates, we have

$$\begin{aligned} &\int_{D_n} \det(E - W{}^t\overline{W})^{k-2n} W^\alpha \overline{W}^\beta \, dW \\ &\quad = \int_{D'_n} \det(E - W'{}^t\overline{W}')^{k-2n} \prod_{j=1}^{n} r_{1j}^{\alpha_{1j}+\beta_{1j}+1} \, dr_{1j} \\ &\qquad \times \prod_{j=2}^{n} r_{j1}^{\alpha_{j1}+\beta_{j1}+1} \, dr_{j1} \prod_{j,k\ge 2} w_{jk}^{\alpha_{jk}} \overline{w}_{jk}^{\beta_{jk}} \, dw_{jk} \\ &\qquad \times \int_0^{2\pi} \exp[i\theta_{11}(\alpha_{11} - \beta_{11} + \sum_{j,k\ge 2}(-\alpha_{jk} + \beta_{jk}))] \, d\theta_{11} \\ &\qquad \times \prod_{k=2}^{n} \int_0^{2\pi} \int_0^{2\pi} \exp\left[i\theta_{1k} \left(\sum_{j=1}^{n} (\alpha_{jk} - \beta_{jk}) \right) \right] \\ &\qquad\qquad \times \exp\left[i\theta_{k1} \left(\sum_{j=1}^{n} (\alpha_{kj} - \beta_{kj}) \right) \right] d\theta_{1k} \, d\theta_{k1}. \end{aligned}$$

The above integral will vanish unless

$$\begin{cases} \alpha_{11} - \beta_{11} + \sum_{j,k\ge 2}(-\alpha_{jk} + \beta_{jk}) = 0, \\ \sum_{j=1}^{n}(\alpha_{jk} - \beta_{jk}) = 0, \ \sum_{j=1}^{n}(\alpha_{kj} - \beta_{kj}) = 0, \qquad (k = 2,\dots,n). \end{cases}$$

This proves our first assertion. Multiplying the first eqution by 2 and adding all together, we get

$$\sum_{j=1}^{n}(\alpha_{j1}+\alpha_{1j})=\sum_{j=1}^{n}(\beta_{j1}+\beta_{1j}).$$

For $k=2,3,\ldots,n$, we note that

$$\begin{aligned}\sum_{j=1}^{n}(\alpha_{jk}+\alpha_{kj})-\sum_{j=1}^{n}(\beta_{jk}+\beta_{kj})\\ =\sum_{j=1}^{n}(\alpha_{jk}-\beta_{jk})+\sum_{j=1}^{n}(\alpha_{kj}-\beta_{kj})=0\end{aligned}$$

and

$$2|\alpha|=\sum_{k,j=1}^{n}(\alpha_{jk}+\alpha_{kj})=\sum_{j,k=1}^{n}(\beta_{jk}+\beta_{kj})=2|\beta|.$$

Thus the proof is completed.

COROLLARY. *Suppose α,β are two n^2-tuples of nonnegative integers satisfying the conditions in Proposition* 5. *Then*

$$\prod_{j,k=1}^{n}(\lambda_j\lambda_k)^{\alpha_{jk}}=\prod_{j,k=1}^{n}(\lambda_j\lambda_k)^{\beta_{jk}}$$

for any numbers $\lambda_1,\lambda_2,\ldots,\lambda_n$.

PROOF OF THE THEOREM. Let

$$N_{\{M\}}(t\Lambda)=C(k,n)\int_{D_n}\det(E-W\,{}^t\overline{W})^{k-2n}\det(E-t^2\overline{\Lambda}W\overline{\Lambda}\,{}^t\overline{W})^{-k}\,dW$$

with $0<t<1$. If we can prove

$$N_{\{M\}}(t\Lambda)=\prod_{j,k=1}^{n}(1-t^2\overline{\lambda}_j\overline{\lambda}_k)^{-1}, \tag{A}$$

then we get

$$\begin{aligned}C(k,n)\textstyle\int_{D_n}\det(E-W\,{}^t\overline{W})^{k-2n}\det(E-\overline{\Lambda}W\overline{\Lambda}\,{}^t\overline{W})^{-k}\,dW\\ =\textstyle\prod_{j,k}^{n}(1-\overline{\lambda}_j\overline{\lambda}_k)^{-1}\end{aligned} \tag{B}$$

by letting t approach 1. Now, we shall prove (A).

Let S be the index set of all n^2-tuples of integers $\alpha=[\alpha_{jk}]$, $\alpha_{jk}\geq 0$. Consider all monomials $a_\alpha(W)=W^\alpha$, $\alpha\in S$, which are arranged in such order that their degrees are nondecreasing. By an argument similar to [**7**, p. 188], we can prove that $\{a_\alpha(W)|\alpha\in S\}$ is a complete system in $H(k,D_n)$ in the sense that if $f\in H(k,D_n)$ and

$$\int_{D_n}\det(E-W\,{}^t\overline{W})^{k-2n}a_\alpha(W)\overline{f(W)}\,dW=0\quad\forall\alpha\in S,$$

then $f(W) = 0$. This system precisely consists of all terms in the power series expansion

$$\prod_{j,k=1}^{n} (1 - w_{jk})^{-1} = \prod_{j,k=1}^{n} (1 + w_{jk} + \cdots + w_{jk}^{m} + \cdots), \qquad |w_{jk}| < 1.$$

Of course, $\{a_\alpha(W) = W^\alpha | \alpha \in S\}$ is a linear independent set in $H(k, D_n)$. By the well-known Gram-Schmidt orthogonalization process, we can construct an orthonormal basis $\{\psi_\alpha(W)|\alpha \in S\}$ from $\{a_\alpha(W)|\alpha \in S\}$. Proposition 5 and its corollary then imply that the basis $\{\psi_\alpha(W)|\alpha \in S\}$ has the following properties:

(1) $\psi_\alpha(W)$ is a finite linear combination of monomials of degree $|\alpha|$.

(2) $\psi_\alpha(t^2\overline{\Lambda}W\overline{\Lambda}) = t^{2|\alpha|} \prod_{j,k=1}^{n} (\overline{\lambda}_j \overline{\lambda}_k)^{\alpha_{jk}} \cdot \psi_\alpha(W)$.

Choose $\{\psi_\alpha(W)|\alpha \in S\}$ as an orthonormal basis of $H(k, D_n)$ and note that

$$C(k, n) \det(E - W\,{}^t\overline{W}_1)^{-k} = K(W, W_1)$$

is a kernel function of $H(k, D_n)$. By Proposition 1 we then have

$$\begin{aligned} C(k,n) \det(E - t^2\overline{\Lambda}W\overline{\Lambda}\,{}^t\overline{W})^{-k} &= \sum_{\alpha \in S} \psi_\alpha(t^2\overline{\Lambda}W\overline{\Lambda})\overline{\psi_\alpha(W)} \\ &= \sum_{\alpha \in S} t^{2|\alpha|} \prod_{p,q=1}^{n} (\overline{\lambda}_p\overline{\lambda}_q)^{\alpha_{pq}} \psi_\alpha(W)\overline{\psi_\alpha(W)}. \end{aligned}$$

Multiply both sides with $\det(E - W\,{}^t\overline{W})^{k-2n}$ and integrate on D_n to get

$$\begin{aligned} N_{\{M\}}(t\Lambda) &= \sum_{\alpha \in S} t^{2|\alpha|} \prod_{j,k=1}^{n} (\overline{\lambda}_j\overline{\lambda}_k)^{\alpha_{jk}} \\ &= \prod_{j,k=1}^{n} (1 - t^2\overline{\lambda}_j\overline{\lambda}_k)^{-1} \end{aligned}$$

by the orthnormality of $\{\psi_\alpha(W)|\alpha \in S\}$. This proves our assertion in (A) and hence completes our proof.

REMARK 1. Note that for $0 < t < 1$, the integrand in $N_{\{M\}}(t\Lambda)$ is absolutely integrable and it can be integrated term by term after its decomposition as a Bergmann kernel function. However, it is not permissible for the integrand of $N_{\{M\}}(\Lambda)$ to do so.

REMARK 2. The Gram-Schmidt orthogonalization process is applied to monomials of the same degree since monomials of different degrees are orthogonal to each other by Proposition 5. Furthermore, we assume $\psi_\alpha(W)$ is the function obtained from W^α by this process.

5. Generalizations and applications. We shall generalize the evaluation of the integral is our Theorem to cases as follows:

(1) The integrand $\det(E - \overline{\Lambda}W\overline{\Lambda}\,{}^t\overline{W})^{-k}$ is changed into a general form $\det(E - \overline{\Lambda}_1 W\overline{\Lambda}_2\,{}^t\overline{W})^{-k}$ with Λ_1, Λ_2 in $U(n)$, the unitary group.

(2) The domain D_n is changed into the hyperbolic space of $p \times q$ matrices defined by

$$D_{p,q}: W \in M_{p,q}(\mathbf{C}), \quad E_q - {}^t\overline{W}W > 0.$$

Here $M_{p,q}(\mathbf{C})$ is the set of all $p \times q$ matrices over $\mathbf{C}$ and E_q is the unit matrix of $M_q(\mathbf{C})$.

For the first generalization, we then have the following

PROPOSITION 6. *Let* $\Lambda_1 = \operatorname{diag}[\lambda_1, \ldots, \lambda_n], \Lambda_2 = \operatorname{diag}[\lambda_{n+1}, \ldots, \lambda_{2n}]$, *with* λ_j $(j = 1, 2, \ldots, 2n)$ *roots of unity and* $\lambda_j \lambda_{n+k} \neq 1$ *for all* $1 \leq j, k \leq n$, *and let*

$$I = C(k,n) \int_{D_n} \det(E - W{}^t\overline{W})^{k-2n} \det(E - \overline{\Lambda}_1 W \overline{\Lambda}_2 {}^t\overline{W})^{-k} \, dW \qquad (k > 4n-2).$$

Then

$$I = \prod_{j,k=1}^{n} (1 - \overline{\lambda}_j \overline{\lambda}_{n+k})^{-1}.$$

PROOF. The proof follows from a slight change in our proof of the Theorem. Conditions in Proposition 5 imply

$$\sum_{j=1}^{n} (\alpha_{jk} - \beta_{jk}) = \sum_{j=1}^{n} (\alpha_{kj} - \beta_{kj}) = 0 \qquad (k = 1, 2, \ldots, n).$$

Let $\psi_\alpha(W)$, $\alpha \in S$, be the function obtained from W^α with a Gram-Schmidt orthogonalization process. Then we have

$$\begin{aligned} \psi_\alpha(\overline{\Lambda}_1 W \overline{\Lambda}_2) &= \prod_{j=1}^{n} \overline{\lambda}_j^{a(j)} \prod_{k=1}^{n} \overline{\lambda}_{n+k}^{b(k)} \cdot \psi_\alpha(W) \\ &= \prod_{j,k=1}^{n} (\overline{\lambda}_j \overline{\lambda}_{n+k})^{\alpha_{jk}} \cdot \psi_\alpha(W) \end{aligned}$$

with $a(j) = \sum_{k=1}^{n} \alpha_{jk}$ and $b(k) = \sum_{j=1}^{n} \alpha_{jk}$. It follows

$$\begin{aligned} I &= \lim_{t \to 1} \sum_{\alpha \in S} (t^2 \overline{\lambda}_j \overline{\lambda}_{n+k})^{\alpha_{jk}} \\ &= \lim_{t \to 1} \prod_{j,k=1}^{n} (1 - t^2 \overline{\lambda}_j \overline{\lambda}_{n+k})^{-1} = \prod_{j,k=1}^{n} (1 - \overline{\lambda}_j \overline{\lambda}_{n+k})^{-1}. \end{aligned}$$

Now we consider the second generalization. Let $\mathbf{F}$ be any imaginary quadratic field and define an algebraic group $G_{p,q}$ over Q as follows:

$$(G_{p,q})_Q = \left\{ M \in SL_{p+q}(F) \mid {}^t\overline{M} R M = R, \; R = \begin{bmatrix} E_p & 0 \\ 0 & -E_q \end{bmatrix} \right\},$$

and $(G_{p,q})_{\mathbf{R}} = SU(p,q)$. The group $SU(p,q)$ operates on the bounded domain $D_{p,q}$ by the action

$$M: Z \to M(Z) = (AZ + B)(CZ + D)^{-1}, \qquad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ in } SU(p,q).$$

Here M is so decomposed that A, B, C and D are $p \times p, p \times q, q \times p$ and $q \times q$ matrices respectively.

Let Γ be a discrete subgroup of $G_{\mathbf{R}}$ such that $\Gamma \backslash G_{\mathbf{R}}$ has definite volume with respect to the invariant measure $\det(E_q - {}^t\overline{W} W)^{-p-q} \, dW$. For positive integer

k, we let $S(k;\Gamma)$ be the vector space of the holomorphic function $f(W)$ on $D_{p,q}$ satisfying the conditions:

(1) $f(\gamma(W)) = \det(CW+D)^k f(W)$ for all $\gamma = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \Gamma$.

(2) $[\det(E_q - {}^t\overline{W}W)]^{k/2} f(W)$ is bounded in $D_{p,q}$.

A function f in $S(k;\Gamma)$ is called a cusp form of weight k. A standard argument [**6**] shows that if $k > 2(p+q-1)$, then $S(k;\Gamma)$ is a finite dimensional vector space. Furthermore, its dimension can be calculated via the Selberg trace formula. An element M in Γ is regular elliptic if M is conjugate in $G_{\mathbf{R}}$ to $\Lambda_p \times \overline{\Lambda}_q = \operatorname{diag}[\lambda_1, \ldots, \lambda_p, \overline{\lambda}_{p+1}, \ldots, \overline{\lambda}_{p+q}] \in S(U_p \times U_q)$ and $\lambda_j \lambda_{p+q} \neq 1$ for all $1 \leq j \leq p$, $1 \leq k \leq q$. This is equivalent to saying that M has an isolated fixed point on $D_{p,q}$.

With these preparations, we now have the following

PROPOSITION 7. *Suppose $M \in \Gamma$ and is conjugate in $G_{\mathbf{R}}$ to $\Lambda_p \times \overline{\Lambda}_q = \operatorname{diag}[\lambda_1, \ldots, \lambda_p, \overline{\lambda}_{p+1}, \ldots, \overline{\lambda}_{p+q}] \in S(U_p \times U_q)$ with $\lambda_j \lambda_{p+r} \neq 1$ for all $1 \leq j \leq p$, $1 \leq r \leq q$. Then the contribution of elements in Γ which are conjugate in Γ to M, to $\dim_{\mathbf{C}} S(k;\Gamma)$ $(k > 2(p+q-1))$, is given by*

$$N_{\{M\}} = |C_{M,\mathbf{Z}}|^{-1} \prod_{s=1}^{q} \overline{\lambda}_{p+s}^{k} \cdot \prod_{j=1}^{p} \prod_{r=1}^{q} (1 - \overline{\lambda}_j \overline{\lambda}_{p+r})^{-1}.$$

Here $|C_{M,\mathbf{Z}}|$ is the order of $C_{M,\mathbf{Z}}$ which is the centralizer of M in $\overline{\Gamma}$, the quotient of Γ by its center.

PROOF. Let $H(k;D_{p,q})$ be the vector space of holomorphic functions which are square integrable on $D_{p,q}$ with respect to the measure $\det(E_q - {}^t\overline{W}W)^{k-p-q}\,dW$. From the argument of [**7**] or the explicit formula given in [**9**], we get that

$$K(W_1, W_2) = C(k;p,q) \det(E_q - {}^t\overline{W}_2 W_1)^{-k}$$

with

$$C(k;p,q) = \pi^{-pq} \prod_{j=0}^{p-1} \prod_{r=0}^{q-1} (k-p-q+1+j+r),$$

the kernel function of $H(k;D_{p,q})$. Also we note that the set of monomials in $W = [w_{jr}]$ $(j = 1, \ldots, p,\ r = 1, \ldots, q)$ is an independent set as well as a complete system in $H(k;D_{p,q})$. Hence we can apply the Gram-Schmidt orthogonalization process to this set and get an orthonormal basis of $H(k;D_{p,q})$. The orthogonal relations in Proposition 5 still exist if we introduce the same coordinates for $D_{p,q}$ as we have done for D_n in Proposition 4. Consequently, we prove that

$$\begin{aligned} I_{p,q} &= C(k;p,q) \int_{D_{p,q}} \det(E_q - {}^t\overline{W}W)^{k-p-q} \det(E_q - \overline{\Lambda}_q {}^t\overline{W}\, \overline{\Lambda}_p W)^{-k}\, dW \\ &= \prod_{j=1}^{p} \prod_{r=1}^{q} (1 - \overline{\lambda}_j \overline{\lambda}_{p+r})^{-1}. \end{aligned}$$

On the other hand, a standard argument (to change the order of integration and summation) [**13**] shows that the contribution $N_{\{M\}}$ is given by

$$\begin{aligned} N_{\{M\}} &= |C_{M,\mathbf{Z}}|^{-1}(\det \Lambda_q)^{-k} I_{p,q} \\ &= |C_{M,\mathbf{Z}}|^{-1} \prod_{s=1}^{q} \overline{\lambda}_{p+s}^{k} \cdot \prod_{j=1}^{p} \prod_{r=1}^{q} (1 - \overline{\lambda}_j \overline{\lambda}_{p+r})^{-1}. \end{aligned}$$

This proves our assertion.

REMARK. Proposition 6 can be applied to cases which may be left out by our main Theorem.

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