

FUNCTIONAL EQUATIONS FOR CHARACTER SERIES ASSOCIATED WITH $n \times n$ MATRICES

BY

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ABSTRACT. Let A be either the ring of invariants or the trace ring of r generic $n \times n$ matrices. Then A has a character series $\chi(A)$ which is a symmetric rational function of commuting variables x_1, \dots, x_r . The main result is that if $r \geq n^2$, then $\chi(A)$ satisfies the functional equation

$$\chi(A)(x_1^{-1}, \dots, x_r^{-1}) = (-1)^d (x_1 \cdots x_r)^{n^2} \chi(A)(x_1, \dots, x_r),$$

where d is the Krull dimension of A .

1. Introduction. Let K be a field of characteristic zero, and let $U_r = (u_{ij}(r))$ ($1 \leq i, j \leq n$, $r = 1, 2, \dots$) be $n \times n$ generic matrices over K . Let

$$R(n, r) = K\text{-algebra generated by } U_1, \dots, U_r$$

$$= \text{ring of generic matrices.}$$

$$\overline{C}(n, r) = K\text{-algebra generated by the traces of elements of } R(n, r)$$

$$= \text{ring of invariants of } R(n, r).$$

$$\overline{R}(n, r) = R(n, r)\overline{C}(n, r) = \text{trace ring of } R(n, r).$$

Each of the above rings $A(n, r)$ is multigraded by \mathbf{N}^r and has a *character* (or *Hilbert* or *Poincaré*) *series*

$$\chi(A(n, r))(x_1, \dots, x_r) = \sum a_\alpha x_1^{\alpha_1} \cdots x_r^{\alpha_r} \in \mathbf{Z}[[x_1, \dots, x_r]],$$

where α varies over \mathbf{N}^r and a_α is the K -dimension of the homogeneous component of $A(n, r)$ of multidegree $\alpha = (\alpha_1, \dots, \alpha_r)$.

The above rings are all $\text{GL}(r, K)$ -modules, where the action of $\text{GL}(r, K)$ is induced by its standard action on the K -vector space spanned by the generic matrices U_1, \dots, U_r . We use the name *character series* because $\chi(A(n, r))$ is also a formal power series $1 + \chi(A_1) + \chi(A_2) + \cdots$, where $\chi(A_i)$ is the character, as a $\text{GL}(r, K)$ -module, of the homogeneous component of $A(n, r)$ of total degree i . For more details, see [1, §§4–6].

This article was motivated by a recent result of Le Bruyn [4], who showed that for $r \geq 3$, the character series of $\overline{R}(2, r)$ satisfies the functional equation

$$\chi(\overline{R}(2, r))(x_1^{-1}, \dots, x_r^{-1}) = -(x_1 \cdots x_r)^4 \chi(\overline{R}(2, r))(x_1, \dots, x_r).$$

Our main results are Theorems 13 and 22, which assert that

$$\chi(\overline{C}(n, r))(x_1^{-1}, \dots, x_r^{-1}) = (-1)^d (x_1 \cdots x_r)^{n^2} \chi(\overline{C}(n, r))(x_1, \dots, x_r)$$

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if $r \geq n^2 - n$, and that

$$\chi(\overline{R}(n,r))(x_1^{-1}, \dots, x_r^{-1}) = (-1)^d (x_1 \cdots x_r)^{n^2} \chi(\overline{R}(n,r))(x_1, \dots, x_r)$$

if $r \geq n^2$. In both formulas $d = (r - 1)n^2 + 1$, the Krull dimension of $\overline{C}(n,r)$ or $\overline{R}(n,r)$.

The proof, in outline, is as follows. Since $\overline{C}(n,r)$ is the fixed ring of $\mathrm{SL}(n,K)$ acting homogeneously on a polynomial ring over K , results of Hochster-Roberts [2] and Murthy [6] imply that $\overline{C}(n,r)$ is a Cohen-Macaulay, Gorenstein UFD. Since $\overline{C}(n,r)$ is Gorenstein, a result of Stanley [10] implies that its character series satisfies a functional equation of the above form but with an unspecified exponent on $(x_1 \cdots x_r)$. The determination of $\chi(\overline{C}(n,r))$ as a formal power series of Schur functions [1] is then used to show that the exponent is n^2 , provided that $r \geq n^2 - n$. Finally, the functional equation for the character series of $\overline{R}(n,r)$ is obtained by exploiting the close relationship between $\overline{C}(n,r)$ and $\overline{R}(n,r)$.

These results suggest that there are canonical presentations for $\overline{C}(n,r)$ and $\overline{R}(n,r)$, and I hope that they provide some hints toward finding them.

2. $\overline{C}(n,r)$ is a Cohen-Macaulay, Gorenstein UFD. Suppose that G is a reductive group acting homogeneously on a polynomial ring over K . Then the ring of invariants is Cohen-Macaulay by a fundamental theorem of Hochster and Roberts [2, Main Theorem]. They further observed [2, Corollary 1.9] that if G is connected (has no nontrivial finite homomorphic images) and semisimple (and thus has no nontrivial homomorphisms into K^*), then the ring of invariants is a UFD, and hence Gorenstein, using a theorem of Murthy [6] that a Cohen-Macaulay UFD which is a homomorphic image of a Noetherian regular local ring is Gorenstein.

Let $K[u_{ij}(r)]$ be the commutative polynomial ring generated by the rn^2 entries of the $n \times n$ generic matrices U_1, \dots, U_r . There is a homogeneous action of $\mathrm{GL}(n,K)$ on $K[u_{ij}(r)]$ induced by the simultaneous conjugation of U_1, \dots, U_r by $g \in \mathrm{GL}(n,K)$, $(U_1, \dots, U_r) \mapsto (gU_1g^{-1}, \dots, gU_rg^{-1})$, and $\overline{C}(n,r)$ is the ring of invariants of this action [7, Theorem 1.3]. Although $\mathrm{GL}(n,K)$ is not semisimple, $\mathrm{SL}(n,K)$ is connected, semisimple and reductive, and has the same ring of invariants since $K^* \cdot \mathrm{SL}(n,K)$ is Zariski dense in $\mathrm{GL}(n,K)$ and K^* acts trivially on $K[u_{ij}(r)]$. Thus we have

THEOREM 1. *$\overline{C}(n,r)$, the ring of invariants of r generic $n \times n$ matrices, is Cohen-Macaulay, Gorenstein, and a UFD. \square*

3. Stanley’s Theorem. Let A be a K -algebra multigraded by \mathbf{N}^r , with $\dim_K(A_\alpha)$ finite for all $\alpha \in \mathbf{N}^r$. Then A has a *Hilbert series*

$$F(A) = \sum \{ \dim_K(A_\alpha) x_1^{\alpha_1} \cdots x_r^{\alpha_r} \mid \alpha = (\alpha_1, \dots, \alpha_r) \in \mathbf{N}^r \}$$

which lies in the formal power series ring $\mathbf{Z}[[x_1, \dots, x_r]]$. We use the name *character series* and write $\chi(A)$ when each A_α is a $\mathrm{GL}(r,K)$ -module, as in the introduction. In that case $\chi(A)$ is a formal power series of symmetric polynomials in x_1, \dots, x_r .

DEFINITION. Let $f \in \mathbf{Z}[[x_1, \dots, x_r]]$. One says that f *satisfies a functional equation* if

(1) f is a nonzero rational function (i.e., there exist nonzero $g, h \in \mathbf{Z}[x_1, \dots, x_r]$ such that $hf = g$).

(2) $(g/h)(x_1^{-1}, \dots, x_r^{-1}) = \pm (x_1^{\alpha_1} \cdots x_r^{\alpha_r})(g/h)(x_1, \dots, x_r)$ for some $\alpha_1, \dots, \alpha_r \in \mathbf{Z}$ (as an equation in $\mathbf{Q}(x_1, \dots, x_r)$).

We have made this definition explicit to emphasize the point that the substitution of x_i^{-1} for x_i must take place in $\mathbf{Q}(x_1, \dots, x_n)$. A term-by-term substitution in a formal power series does not make sense.

THEOREM 2 (STANLEY [10, THEOREM 6.1(i)]). *Let A be a commutative K -algebra, multigraded by \mathbf{N}^r , with $A_0 = K$ and $\dim_K(A_\alpha)$ finite for all $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbf{N}^r$. Suppose that A is Gorenstein and has Krull dimension d . Then its Hilbert series satisfies a functional equation*

$$F(A)(x_1^{-1}, \dots, x_r^{-1}) = (-1)^d (x_1^{\alpha_1} \cdots x_r^{\alpha_r}) F(A)(x_1, \dots, x_r)$$

for some $\alpha_1, \dots, \alpha_r \in \mathbf{Z}$. \square

If A is assumed to be a Cohen-Macaulay domain, Stanley proves a converse [10, Theorem 6.1(iii)]: A is Gorenstein if and only if $F(A)$ satisfies such a functional equation.

By Theorem 1, $\overline{C}(n, r)$ is Gorenstein, so Stanley's theorem applies to it. As noted above, we write $\chi(\overline{C}(n, r))$ instead of $F(\overline{C}(n, r))$, and it is a formal power series of symmetric functions, so the monomial $x_1^{\alpha_1} \cdots x_r^{\alpha_r}$ is symmetric and therefore a power of $x_1 \cdots x_r$. Thus we have

THEOREM 3. *The character series of the ring of invariants of r generic $n \times n$ matrices satisfies a functional equation*

$$\chi(\overline{C}(n, r))(x_1^{-1}, \dots, x_r^{-1}) = (-1)^d (x_1 \cdots x_r)^\alpha \chi(\overline{C}(n, r))(x_1, \dots, x_r)$$

for some $\alpha \in \mathbf{Z}$, where d is the Krull dimension of $\overline{C}(n, r)$. \square

The Krull dimension of $\overline{C}(n, r)$ is $(r-1)n^2 + 1$ (Kirillov, Procesi, see [8, p. 197]), but this is never a major factor in any argument.

4. Polarization of symmetric functions. We follow the notation of [5, Chapter I] for symmetric functions; a resume can be found in [1, §2]. Thus x_1, x_2, \dots are commuting indeterminates, and for each $r = 1, 2, \dots$, the symmetric group S_r permutes x_1, \dots, x_r and so acts on $\mathbf{Z}[x_1, \dots, x_r]$. Then

$$\Lambda_r = \mathbf{Z}[x_1, \dots, x_r]^{S_r} = \mathbf{Z}[e_1, \dots, e_r]$$

is the ring of symmetric functions in r variables, where $e_i = e_i(x_1, \dots, x_r)$ is the i th elementary symmetric function.

For $s \geq r$, the obvious retraction $\rho(s, r): \mathbf{Z}[x_1, \dots, x_s] \rightarrow \mathbf{Z}[x_1, \dots, x_r]$ induces a surjection $\rho(s, r): \Lambda_s \rightarrow \Lambda_r$. The inverse limit, as graded rings, of the inverse system $\{\Lambda_r, \rho(s, r)\}$ is called the ring of symmetric functions in infinitely many variables and is denoted Λ ; the canonical projections are denoted $\rho(r): \Lambda \rightarrow \Lambda_r$; but if $f \in \Lambda$ we will usually write $f(x_1, \dots, x_r)$ rather than $\rho(r)(f)$ for the image of f in Λ_r . The ring Λ is a polynomial ring in countably many variables e_1, e_2, \dots , and

$$\rho(r)(e_i) = \begin{cases} e_i(x_1, \dots, x_r), & 1 \leq i \leq r, \\ 0, & i > r. \end{cases}$$

We use Λ only as a notational device which permits symmetric functions to be expressed independently of the number of variables involved.

To avoid excessive notation, we adopt the convention that if it can be inferred from the context that f is a function of x_1, \dots, x_r , then the variables x_1, \dots, x_r will usually be omitted; for example $\rho(s, r)(e_1) = e_1$ means that $\rho(s, r)(x_1 + \dots + x_s) = x_1 + \dots + x_r$.

With each partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is associated a *Schur function* $S_\lambda \in \Lambda$, and the Schur functions form a \mathbf{Z} -basis for Λ . If λ is a partition, the *length* of λ , denoted $l(\lambda)$, is the largest r such that $\lambda_r \neq 0$. For any nonnegative integer r , $S_\lambda(x_1, \dots, x_r) = 0$ if and only if $l(\lambda) > r$, and the Schur functions $\{S_\lambda(x_1, \dots, x_r) | l(\lambda) \leq r\}$ form a \mathbf{Z} -basis for Λ_r . The notations $(\lambda_1, \dots, \lambda_r, 0, \dots)$, $(\lambda_1, \dots, \lambda_r, 0, \dots, 0)$ and $(\lambda_1, \dots, \lambda_r)$ will all denote the same partition.

We let $[\Lambda_r]$ and $[\Lambda]$ denote, respectively, the completions of Λ_r and Λ relative to the gradings which give each x_i degree one and each e_i degree i . The canonical projections induce projections $\rho(r): [\Lambda] \rightarrow [\Lambda_r]$. Elements of $[\Lambda]$ will be expressed as formal power series of Schur functions; note that if $\lambda = (\lambda_1, \dots, \lambda_r)$, then S_λ is homogeneous of degree $|\lambda|$, where $|\lambda| = \lambda_1 + \dots + \lambda_r$. Elements of $[\Lambda_r]$ will be expressed variously as formal power series of Schur functions, as elements of $\mathbf{Z}[[x_1, \dots, x_r]]$, or (if possible) as rational functions of x_1, \dots, x_r .

One definition of the Schur functions is via generalized Vandermonde determinants [5, p. 24]. Suppose that $\lambda = (\lambda_1, \dots, \lambda_r)$ is a partition of length $\leq r$. Set

$$\begin{aligned} a_\lambda &= a_\lambda(x_1, \dots, x_r) = \sum_{\sigma \in S_r} (\text{sign } \sigma) x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(r)}^{\lambda_r} \\ &= \det \begin{pmatrix} x_1^{\lambda_1} & \cdots & x_r^{\lambda_1} \\ \vdots & & \vdots \\ x_1^{\lambda_r} & \cdots & x_r^{\lambda_r} \end{pmatrix}, \end{aligned}$$

and let $\delta = \delta(r) = (r-1, r-2, \dots, 1, 0)$. Then

$$S_\lambda(x_1, \dots, x_r) = a_{\lambda+\delta}(x_1, \dots, x_r) / a_\delta(x_1, \dots, x_r).$$

If $s \geq r$, $S_\lambda(x_1, \dots, x_s)$ is obtained by identifying λ with $(\lambda_1, \dots, \lambda_r, 0, \dots, 0)$ ($s-r$ zeros) and using $\delta(s) = (s-1, s-2, \dots, 1, 0)$ in place of $\delta(r)$. If $q \leq r$, $S_\lambda(x_1, \dots, x_q) = S_\lambda(x_1, \dots, x_q, 0, \dots, 0)$ ($r-q$ zeros).

One consequence of this definition is a formula for $S_\lambda(x_1, \dots, x_{r+1})$ in terms of $S_\lambda(x_1, \dots, x_r)$ provided, of course, that $l(\lambda) \leq r$:

$$S_\lambda(x_1, \dots, x_{r+1}) = \sum_{l=1}^{r+1} \left(\prod_{\substack{i=1 \\ i \neq l}}^{r+1} \frac{x_i}{x_i - x_l} \right) S_\lambda(x_1, \dots, \hat{x}_l, \dots, x_{r+1}).$$

(As usual, $f(x_1, \dots, \hat{x}_l, \dots, x_{r+1})$ stands for $f(x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_{r+1})$.) We use this formula to motivate a definition.

DEFINITION. Let $Q([\Lambda_r])$ be the field of quotients of $[\Lambda_r]$. The map $\Phi_r: Q([\Lambda_r]) \rightarrow Q([\Lambda_{r+1}])$ given by

$$\Phi_r(f) = \sum_{l=1}^{r+1} \left(\prod_{\substack{i=1 \\ i \neq l}}^{r+1} \frac{x_i}{x_i - x_l} \right) f(x_1, \dots, \hat{x}_l, \dots, x_{r+1})$$

is called the *polarization map*.

The polarization map is not a ring homomorphism. Note that since Φ_r is degree preserving on Λ_r it can be computed term by term on $[\Lambda_r]$:

$$\Phi_r(f_0 + f_1 + f_2 + \cdots) = \Phi_r(f_0) + \Phi_r(f_1) + \Phi_r(f_2) + \cdots.$$

It is also clear that Φ_r carries rational functions to rational functions; i.e., $\Phi_r(Q(\Lambda_r)) \subseteq Q(\Lambda_{r+1})$. Thus we have the following observation, which is used implicitly in several future calculations.

LEMMA 4. Suppose that $P = \sum p(\lambda)S_\lambda$ is a formal power series of Schur functions in $[\Lambda]$ and $p(\lambda) = 0$ if $l(\lambda) > r$. For each s , let $\rho(s)(P) = P_s = P_s(x_1, \dots, x_s)$. Then $P_{r+1} = \Phi_r(P_r)$, and if P_r is a rational function, so is P_{r+1} . \square

5. The action of polarization on $\Lambda_r[e_r^{-1}]$. The purpose of this section is to prove some technical lemmas needed later. They involve calculating the action of Φ_r on

$$\bar{\Lambda}_r = \mathbf{Z}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]^{S_r} = \Lambda_r[e_r^{-1}].$$

We will sometimes use an alternative notation for partitions [5, p. 1]. The *parts* of $\lambda = (\lambda_1, \dots, \lambda_r)$ are $\lambda_1, \dots, \lambda_r$, and writing $\lambda = (1^{m_1} 2^{m_2} \dots)$ means that exactly m_i of the parts are equal to i . In particular, $(a^r) = (a, \dots, a)$ (r times).

Recall the following facts about Λ_r :

(1) The Schur functions $S_\lambda(x_1, \dots, x_r)$, as λ varies over all partitions $(\lambda_1, \dots, \lambda_r)$ of length $\leq r$, form a \mathbf{Z} -basis for Λ_r .

(2) $S_{(1^r)} = x_1 \cdots x_r = e_r$; $S_{(a^r)} = (x_1 \cdots x_r)^a = (e_r)^a$.

(3) For any $\lambda = (\lambda_1, \dots, \lambda_r)$, $e_r S_\lambda = S_{(\lambda + (1^r))}$. If $\mu = (\lambda_1 - \lambda_r, \dots, \lambda_{r-1} - \lambda_r, 0)$, then $S_\lambda = (e_r)^{\lambda_r} S_\mu$.

(4) The set of all $(e_r)^a S_\mu$, $a = 0, 1, 2, \dots$, $\mu = (\mu_1, \dots, \mu_{r-1}, 0)$, $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{r-1} \geq 0$, forms a \mathbf{Z} -basis for Λ_r .

It is clear that a \mathbf{Z} -basis for $\bar{\Lambda}_r$ is obtained by allowing $a \in \mathbf{Z}$ in (4) above. However, future computations involving $\bar{\Lambda}_r$ are best facilitated by introducing *semipartitions* and *semi-Schur functions*.

DEFINITION. A *semipartition* of length r is an r -tuple $\lambda = (\lambda_1, \dots, \lambda_r)$, where $\lambda_i \in \mathbf{Z}$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$. The *semi-Schur function* associated with λ , denoted S_λ or $S_\lambda(x_1, \dots, x_r)$, is $(e_r)^{\lambda_r} S_\mu$, where $\mu = (\lambda_1 - \lambda_r, \dots, \lambda_{r-1} - \lambda_r, 0)$.

Note that a semi-Schur function is only defined as a function of exactly r variables. With this new terminology the analogue of (4) for $\bar{\Lambda}_r$ takes the following form.

LEMMA 5. The set of all semi-Schur functions $S_\lambda = S_\lambda(x_1, \dots, x_r)$, as λ varies over all semipartitions of length r , forms a \mathbf{Z} -basis for $\bar{\Lambda}_r$. \square

If $\lambda = (\lambda_1, \dots, \lambda_r)$ is a semipartition of length r , the determinant definition of S_λ remains valid: $S_\lambda = a_{\lambda+\delta}/a_\delta$, where a_λ has the obvious definition. We cannot expect that $\Phi_r(S_\lambda) = S_\lambda$, essentially because $(\lambda_1, \dots, \lambda_r, 0)$ is not a semipartition unless $\lambda_r \geq 0$ (λ is a partition). Instead

$$\Phi_r(S_\lambda) = \det \begin{pmatrix} x_1^{\lambda_1+r} & \cdots & x_{r+1}^{\lambda_1+r} \\ x_1^{\lambda_2+r-1} & \cdots & x_{r+1}^{\lambda_2+r-1} \\ \vdots & & \vdots \\ x_1^{\lambda_r+1} & \cdots & x_{r+1}^{\lambda_r+1} \\ 1 & \cdots & 1 \end{pmatrix} a_\delta(x_1, \dots, x_{r+1})^{-1},$$

and by putting the sequence $(\lambda_1 + r, \lambda_2 + r - 1, \dots, \lambda_r + 1, 0)$ into descending order, the value of $\Phi_r(S_\lambda)$ is easily ascertained. The conclusion is

THEOREM 6. *Let $\Phi_r: Q(\Lambda_r) \rightarrow Q(\Lambda_{r+1})$. Suppose that $\lambda = (\lambda_1, \dots, \lambda_r)$ is a semipartition of length r and consider the strictly decreasing sequence*

$$\lambda_1 + r > \lambda_2 + r - 1 > \lambda_3 + r - 2 \cdots > \lambda_{r-1} + 2 > \lambda_r + 1.$$

(1) *If zero occurs in the sequence (i.e., if $\lambda_i + r - i + 1 = 0$ for some i), then $\Phi_r(S_\lambda) = 0$.*

(2) *If $\lambda_r + 1 > 0$, then $\Phi_r(S_\lambda) = S_\mu$, where $\mu = (\lambda_1, \dots, \lambda_r, 0)$.*

(3) *If $\lambda_i + r - i + 1 > 0 > \lambda_{i+1} + r - i$ for some i ($1 \leq i \leq r - 1$), then $\Phi_r(S_\lambda) = (-1)^{r-i} S_\mu$, where $\mu = (\lambda_1, \dots, \lambda_i, i - r, \lambda_{i+1} + 1, \dots, \lambda_r + 1)$.*

(4) *If $0 > \lambda_1 + r$, then $\Phi_r(S_\lambda) = (-1)^r S_\mu$, where $\mu = (-r, \lambda_1 + 1, \dots, \lambda_r + 1)$. \square*

If $\lambda = (\lambda_1, \dots, \lambda_r)$ is a semipartition, let $|\lambda| = \lambda_1 + \dots + \lambda_r \in \mathbf{Z}$. Then S_λ is homogeneous of degree $|\lambda|$, where $\bar{\Lambda}_r$ is \mathbf{Z} -graded in the obvious way. The following consequences of Theorem 6 are immediate.

COROLLARY 7. *Under the same hypothesis as Theorem 6*

(1) *$\Phi_r(S_\lambda)$ is either zero or $\pm S_\mu$ for some semipartition μ of length $r + 1$.*

(2) *For each semipartition μ of length $r + 1$ there is at most one semipartition λ of length r such that $\Phi_r(S_\lambda) = \pm S_\mu$.*

(3) *If $a \geq 0$, $\Phi_r((e_r)^{-a} \Lambda_r) \subseteq (e_{r+1})^{-a} \Lambda_{r+1}$. Hence $\Phi_r(\bar{\Lambda}_r) \subseteq \bar{\Lambda}_{r+1}$ and $\Phi_r([\Lambda_r][e_r^{-1}]) \subseteq [\Lambda_{r+1}][e_{r+1}^{-1}]$.*

(4) *$\Phi_r: \bar{\Lambda}_r \rightarrow \bar{\Lambda}_{r+1}$ preserves the degree of homogeneous elements. \square*

LEMMA 8. *Suppose $r \geq m$, and $\lambda = (\lambda_1, \dots, \lambda_m, 0, \dots, 0)$ is a partition of length $\leq m$, written as a semipartition of length r (there are no zeros if $r = m$). Then*

$$\Phi_r((e_r)^{m-r-1} S_\lambda) = \begin{cases} 0, & \text{if } \lambda_m = 0, \\ (-1)^{r-m} (e_{r+1})^{m-r} S_{\lambda^*}, & \text{if } \lambda_m \neq 0, \end{cases}$$

where $\lambda^* = (\lambda_1 - 1, \dots, \lambda_m - 1, 0, \dots, 0)$.

PROOF. $(e_r)^{m-r-1} S_\lambda = S_\eta$, where

$$\eta = (\eta_1, \dots, \eta_m, \dots, \eta_r) = (\lambda_1 + m - r - 1, \dots, \lambda_m + m - r - 1, m - r - 1, \dots, m - r - 1).$$

Note that

$$(*) \quad \begin{aligned} \eta_m + r - m + 1 &= \lambda_m \geq 0, \\ \eta_{m+1} + r - (m + 1) + 1 &= -1 < 0 \quad (\text{if } r > m). \end{aligned}$$

If $r = m$, apply parts (1) and (2) of Theorem 6 to $(*)$; if $r > m$ apply parts (1) and (3). In either case the conclusion is

$$\Phi_r((e_r)^{m-r-1} S_\lambda) = \begin{cases} 0, & \text{if } \lambda_m = 0, \\ (-1)^{r-m} S_\mu, & \text{if } \lambda_m > 0, \end{cases}$$

where $\mu = (\lambda_1 + m - r - 1, \dots, \lambda_m + m - r - 1, m - r, \dots, m - r)$. Since $S_\mu = (e_{r+1})^{m-r} S_{\lambda^*}$, where $\lambda^* = (\lambda_1 - 1, \dots, \lambda_m - 1, 0, \dots, 0)$, the lemma follows. \square

The next two theorems are the main technical results which will be applied to obtain conclusions about the character series of $\bar{C}(n, r)$ and $\bar{R}(n, r)$. If $P = \sum p(\lambda) S_\lambda$ is a formal power series of (semi-) Schur functions in $[\Lambda_r]$ or $[\Lambda]$, the support of P is the set of S_λ for which $p(\lambda) \neq 0$.

THEOREM 9. Let $P = \sum p(\lambda)S_\lambda$ be a formal power series of Schur functions in $[\Lambda]$, and for each integer $r = 1, 2, \dots$ let $P_r = \sum p(\lambda)S_\lambda(x_1, \dots, x_r) \in [\Lambda_r]$. Suppose that

- (A) If $l(\lambda) > m$, $p(\lambda) = 0$.
- (B) For all partitions $\lambda = (\lambda_1, \dots, \lambda_m)$ of length $\leq m$, $p(\lambda_1 + 1, \dots, \lambda_m + 1) = p(\lambda_1, \dots, \lambda_m)$.
- (C) There are arbitrarily large integers a such that $S_{(a^{m-1})}$ lies in the support of P .
- (D) Each P_r is a rational function which satisfies a functional equation

$$P_r(x_1^{-1}, \dots, x_r^{-1}) = \pm(x_1 \cdots x_r)^{\alpha(r)} P_r(x_1, \dots, x_r)$$

for some $\alpha(r) \in \mathbf{Z}$.

Then $\alpha(m) = m$.

PROOF. Hypotheses (A) and (B) imply that

$$(1 - e_m)P_m = \sum \{p(\lambda)S_\lambda(x_1, \dots, x_m) | l(\lambda) \leq m - 1\}$$

and hence that

$$(1) \quad (1 - e_m)P_m = \Phi_{m-1}(P_{m-1}).$$

Since both sides of (1) are rational functions (hypothesis (D)), we can substitute x_i^{-1} for x_i and simplify using the functional equations of (D) as follows:

$$\begin{aligned} (1 - e_m)P_m(x_1, \dots, x_m) &= \Phi_{m-1}(P_{m-1})(x_1, \dots, x_m) \\ &= \sum_{l=1}^m \left(\prod_{\substack{i=1 \\ i \neq l}}^m \frac{x_i}{x_i - x_l} \right) P_{m-1}(x_1, \dots, \hat{x}_l, \dots, x_m), \\ (1 - e_m^{-1})P_m(x_1^{-1}, \dots, x_m^{-1}) &= \sum_{l=1}^m \left(\prod_{\substack{i=1 \\ i \neq l}}^m \frac{x_i^{-1}}{x_i^{-1} - x_l^{-1}} \right) P_{m-1}(x_1^{-1}, \dots, \hat{x}_l^{-1}, \dots, x_m^{-1}), \\ (1 - e_m^{-1})(e_m)^{\alpha(m)}P_m(x_1, \dots, x_m) &= \pm(e_m)^{m-1} \sum_{l=1}^m \left(\prod_{\substack{i=1 \\ i \neq l}}^m \frac{x_i}{x_i - x_l} \right) \\ &\quad \times (x_1 \cdots \hat{x}_l \cdots x_m)^{\alpha(m-1)-m} P_{m-1}(x_1, \dots, \hat{x}_l, \dots, x_m). \end{aligned}$$

$$(2) \quad (1 - e_m)P_m = \pm(e_m)^{m-\alpha(m)} \Phi_{m-1}((e_{m-1})^{\alpha(m-1)-m} P_{m-1}).$$

Combining (1) and (2),

$$(3) \quad \Phi_{m-1}(P_{m-1}) = \pm(e_m)^{m-\alpha(m)} \Phi_{m-1}((e_{m-1})^{\alpha(m-1)-m} P_{m-1}).$$

We now regard (3) as an equation of formal power series in $[\Lambda_m][e_m^{-1}]$, where Φ_{m-1} is applied term by term. This is justified by Corollary 7(3). By hypothesis (C) there is an integer $a \geq |\alpha(m-1) - m|$ such that $p((a^{m-1})) \neq 0$. Then for $w = a + \alpha(m-1) - m \geq 0$, $S_{(w^{m-1})}$ is a (true) Schur function which lies in the support of $(e_{m-1})^{\alpha(m-1)-m} P_{m-1}$. Since $S_{(w^{m-1})}$ is a Schur function,

$\Phi_{m-1}(S_{((w^{m-1}))}) = S_{((w^{m-1}))}$ (Theorem 6(2)), and so the semi-Schur function $(e_m)^{m-\alpha(m)} S_{((w^{m-1}))} = S_\mu$ lies in the support of the right-hand side of (3), where

$$(4) \quad \mu = (\mu_1, \dots, \mu_m) = (w + m - \alpha(m), \dots, w + m - \alpha(m), m - \alpha(m)).$$

But the support of the left-hand side of (3) contains only Schur functions of partitions of length $\leq m - 1$. Thus (4) implies that $0 = \mu_m = m - \alpha(m)$, or $\alpha(m) = m$. \square

THEOREM 10. *Let $P = \sum p(\lambda) S_\lambda$ be a formal power series of Schur functions in $[\Lambda]$, and let $P_r = \sum p(\lambda) S_\lambda(x_1, \dots, x_r) \in [\Lambda_r]$. Suppose that*

(A) *If $l(\lambda) > m$, $p(\lambda) = 0$.*

(B) *For all partitions $\lambda = (\lambda_1, \dots, \lambda_m)$ of length $\leq m$, $p(\lambda_1 + 1, \dots, \lambda_m + 1) = p(\lambda_1, \dots, \lambda_m)$.*

(E) *P_m is a rational function which satisfies the functional equation*

$$P_m(x_1^{-1}, \dots, x_m^{-1}) = (-1)^a (x_1 \cdots x_m)^m P_m(x_1, \dots, x_m).$$

Then for all $r \geq m$, P_r is a rational function which satisfies the functional equation

$$P_r(x_1^{-1}, \dots, x_r^{-1}) = (-1)^{a+(r-m)m} (x_1 \cdots x_r)^m P_r(x_1, \dots, x_r).$$

PROOF. The proof is by induction on $r \geq m$, with the case $r = m$ given by hypothesis (E). The inductive hypothesis is that

$$(1) \quad P_r(x_1^{-1}, \dots, x_r^{-1}) = (-1)^{a+(r-m)m} (x_1 \cdots x_r)^m P_r(x_1, \dots, x_r).$$

Note that since $r \geq m$, P_{r+1} has no partitions of length $\geq r + 1$ in its support by hypothesis (A), so

$$(2) \quad P_{r+1} = \Phi_r(P_r).$$

As in the proof of Theorem 9, we substitute x_i^{-1} for x_i in (2) and simplify, using (1):

$$\begin{aligned} P_{r+1}(x_1^{-1}, \dots, x_{r+1}^{-1}) &= \sum_{l=1}^{r+1} \left(\prod_{\substack{i=1 \\ i \neq l}}^{r+1} \frac{x_i^{-1}}{x_i^{-1} - x_l^{-1}} \right) P_r(x_1^{-1}, \dots, \hat{x}_l^{-1}, \dots, x_{r+1}^{-1}) \\ &= (-1)^{a+(r-m)m+r} \sum_{l=1}^{r+1} \left(\prod_{\substack{i=1 \\ i \neq l}}^{r+1} \frac{1}{x_i - x_l} \right) \\ &\quad \times x_l^r (x_1 \cdots \hat{x}_l \cdots x_{r+1})^m P_r(x_1, \dots, \hat{x}_l, \dots, x_{r+1}) \\ &= (-1)^{a+(r-m)m+r} (x_1 \cdots x_{r+1})^r \sum_{l=1}^{r+1} \left(\prod_{\substack{i=1 \\ i \neq l}}^{r+1} \frac{x_i}{x_i - x_l} \right) \\ &\quad \times (x_1 \cdots \hat{x}_l \cdots x_{r+1})^{m-r-1} P_r(x_1, \dots, \hat{x}_l, \dots, x_{r+1}). \end{aligned}$$

$$(3) \quad P_{r+1}(x_1^{-1}, \dots, x_{r+1}^{-1}) = (-1)^{a+(r-m)m+r} (e_{r+1})^r \Phi_r(e_r^{m-r-1} P_r).$$

Let $\lambda = (\lambda_1, \dots, \lambda_m, 0, \dots, 0)$ be a partition of length $\leq m$, written with r coordinates (i.e., there are $r - m$ zeros to the right of λ_m), and let $\lambda^* = (\lambda_1 -$

$1, \dots, \lambda_m - 1, 0, \dots, 0$), written with $r + 1$ coordinates (i.e., there are $r - m + 1$ zeros to the right of $\lambda_m - 1$). By Lemma 8

$$(4) \quad \Phi_r((e_r)^{m-r-1} S_\lambda) = \begin{cases} 0, & \text{if } \lambda_m = 0, \\ (-1)^{r-m} (e_{r+1})^{m-r} S_{\lambda^*}, & \text{if } \lambda_m \geq 1. \end{cases}$$

By hypothesis (B),

$$(5) \quad \sum \{p(\lambda) S_\lambda | l(\lambda) \leq m\} = \sum \{p(\lambda) S_{\lambda^*} | l(\lambda) = m\}.$$

Using (4), (5) and the fact that $\Phi_r((e_r^{m-r-1}) P_r)$ can be computed term by term gives

$$\begin{aligned} \Phi_r((e_r)^{m-r-1} P_r) &= \sum_{l(\lambda) \leq m} p(\lambda) \Phi_r((e_r)^{m-r-1} S_\lambda) \\ &= (-1)^{r-m} \sum_{l(\lambda)=m} p(\lambda) (e_{r+1})^{m-r} S_{\lambda^*} \\ &= (-1)^{r-m} (e_{r+1})^{m-r} \sum_{l(\lambda)=m} p(\lambda) S_\lambda \\ &= (-1)^{r-m} (e_{r+1})^{m-r} P_{r+1}. \end{aligned} \quad (6)$$

Finally, substituting (6) into (3) gives

$$\begin{aligned} (7) \quad P_{r+1}(x_1^{-1}, \dots, x_{r+1}^{-1}) &= (-1)^{a+(r+1-m)m} (e_{r+1})^m P_{r+1} \\ &= (-1)^{a+(r+1-m)m} (x_1 \cdots x_{r+1})^m P_{r+1}(x_1, \dots, x_{r+1}), \end{aligned}$$

which completes the inductive step. \square

6. The functional equation for $\chi(\overline{C}(n))$. By Theorem 3, the character series of $\overline{C}(n, r)$ satisfies a functional equation with an unspecified exponent α on $(x_1 \cdots x_r)$. The goal of this section is to show that if $r \geq n^2 - n$, then $\alpha = n^2$. The condition $r \geq n^2 - n$ comes from the fact that as an element of $[\Lambda]$,

$$\chi(\overline{C}(n)) = \prod_{i=1}^{\infty} (1 - x_i)^{-n} \sum p(\lambda) S_\lambda = \left(\sum_{i \geq 0} S_{(i)} \right)^n \sum p(\lambda) S_\lambda,$$

where $\sum p(\lambda) S_\lambda$ satisfies the hypotheses of Theorem 9 with $m = n^2 - n$.

We will now derive a formula for $\chi(\overline{C}(n))$ which is a slight modification of the formula of [1, Theorem 12]. The derivation will only be sketched since it closely follows [1, pp. 199–203].

We introduce a new set of commuting variables y_1, \dots, y_n . There is an inner product $\langle \cdot, \cdot \rangle$ defined on $\overline{\Lambda}_n(y) = \mathbf{Z}[y_1^{\pm 1}, \dots, y_n^{\pm 1}]^{S_n}$, relative to which the semi-Schur functions form an orthonormal basis [see 1, pp. 184, 188–189]. The formula of [1, Theorem 12] is

$$(1) \quad \chi(\overline{C}(n, r)) = \sum_{\lambda} \langle S_{\lambda}(y_i y_j^{-1}), 1 \rangle S_{\lambda}(x_1, \dots, x_r),$$

where $S_{\lambda}(y_i y_j^{-1})$ signifies the substitution of the n^2 values $\{y_i y_j^{-1} | 1 \leq i, j \leq n\}$ in $S_{\lambda}(z_{ij})$, and $\{z_{ij} | 1 \leq i, j \leq n\}$ is a set of n^2 independent variables. The summation

is over all partitions λ , but only partitions of length $\leq n^2$ make a contribution since $S_\lambda(z_{ij}) = 0$ if $l(\lambda) > n^2$. Formula (1) is obtained from the expansion

$$(2) \quad \prod_{\substack{1 \leq i, j \leq n \\ 1 \leq k \leq r}} (1 - y_i y_j^{-1} x_k)^{-1} = \sum_{\substack{1 \leq i, j \leq n \\ 1 \leq k \leq r}} S_\lambda(y_i y_j^{-1}) S_\lambda(x_k)$$

by “integrating over y ”—i.e., by replacing $S_\lambda(y_i y_j^{-1})$ with $\langle S_\lambda(y_i y_j^{-1}), 1 \rangle$. However, if $i = j$, $y_i y_j^{-1} = 1$, so there is a total of nr factors $(1 - x_k)^{-1}$ ($k = 1, \dots, r$) in the left-hand side of (2) which do not involve y_1, \dots, y_n . They can be “pulled out from under the integral sign” to give a slightly different formula for $\chi(\overline{C}(n, r))$, namely,

THEOREM 11.

$$\chi(\overline{C}(n, r)) = \prod_{k=1}^r (1 - x_k)^{-n} \sum_{\lambda} \langle S_\lambda(y_i y_j^{-1}), 1 \rangle S_\lambda(x_1, \dots, x_r),$$

where $S_\lambda(y_i y_j^{-1})$ denotes the evaluation of $S_\lambda(z_{ij} | 1 \leq i, j \leq n, i \neq j)$ at $z_{ij} = y_i y_j^{-1}$. Consequently, only Schur functions of partitions of length $\leq n^2 - n$ are in the support of the rightmost summation. \square

LEMMA 12. Let $P = \sum_{\lambda} p(\lambda) S_{\lambda} \in [A]$, where $p(\lambda) = \langle S_{\lambda}(y_i y_j^{-1}), 1 \rangle$ ($1 \leq i, j \leq n, i \neq j$) as in Theorem 11, $n \geq 2$, and for each $r = 1, 2, \dots$, let $P_r = \sum_{\lambda} p(\lambda) S_{\lambda}(x_1, \dots, x_r)$. Then P satisfies hypotheses (A)–(D) of Theorem 9 with $m = n^2 - n$, and thus also satisfies hypothesis (E) of Theorem 10. Hence for all $r \geq n^2 - n$, P_r satisfies a functional equation

$$P_r(x_1^{-1}, \dots, x_r^{-1}) = \pm (x_1 \cdots x_r)^{n^2 - n} P_r(x_1, \dots, x_r).$$

PROOF. The number of $y_i y_j^{-1}$ is $n^2 - n$, so $p(\lambda) = \langle S_{\lambda}(y_i y_j^{-1}), 1 \rangle = 0$ if $l(\lambda) > n^2 - n$, and hypothesis (A) holds.

Since $S_{(1^{n^2 - n})}(y_i y_j^{-1}) = e_{n^2 - n}(y_i y_j^{-1}) = 1$,

$$S_{\lambda + (1^{n^2 - n})}(y_i y_j^{-1}) = S_{\lambda}(y_i y_j^{-1})$$

and $p(\lambda + (1^{n^2 - n})) = p(\lambda)$ for any partition λ of length $\leq n^2 - n$, so hypothesis (B) holds.

For $t = 0, \dots, n^2 - n$, $e_t(y_i y_j^{-1}) = e_{n^2 - n - t}(y_i y_j^{-1})$; for $t < 0$ or $t > n^2 - n$, $e_t(y_i y_j^{-1}) = e_{n^2 - n - t}(y_i y_j^{-1}) = 0$. The formula for S_{λ} as a determinant of e_i 's [5, p. 25] shows that

$$S_{(a)} = \det(e_{1+i-j} | 1 \leq i, j \leq a),$$

$$S_{(a^{n^2 - n - 1})} = \det(e_{n^2 - n - 1 + i - j} | 1 \leq i, j \leq a).$$

Hence $S_{(a)}(y_i y_j^{-1}) = S_{(a^{n^2 - n - 1})}(y_i y_j^{-1})$ and $p((a)) = p((a^{n^2 - n - 1}))$ for all $a = 0, 1, 2, \dots$

Note that $p((a))$ is the coefficient of x_1^a in $P_1(x_1)$. It is classical (see [9, p. 10]) that $\overline{C}(n, 1)$ (the ring of invariants of a single $n \times n$ generic matrix U_1) is a polynomial ring in n variables of degrees $1, \dots, n$, the coefficients of the characteristic

polynomial of U_1 . Hence

$$\chi(\overline{C}(n, 1)) = \frac{1}{1-x_1} \frac{1}{1-x_1^2} \cdots \frac{1}{1-x_1^n},$$

$$P_1(x_1) = (1-x_1)^n \chi(\overline{C}(n, 1)) = \frac{1-x_1}{1-x_1^2} \cdots \frac{1-x_1}{1-x_1^n}.$$

By hypothesis $n \geq 2$, so $P_1(x_1)$ is not a polynomial, and there are arbitrarily large integers a such that $p((a))$ —and hence $p((a^{n^2-n-1}))$ —is not zero. This establishes hypothesis (C).

Finally, $\chi(\overline{C}(n, r))$ satisfies a functional equation by Theorem 3, and $\prod\{(1-x_k)^n | 1 \leq k \leq r\}$ satisfies a functional equation by inspection, so $P_r = \prod(1-x_k)^n \chi(\overline{C}(n, r))$ also satisfies a functional equation, which is hypothesis (D). \square

The final conclusion of Lemma 12, that P_r satisfies a functional equation with exponent n^2-n on $(x_1 \cdots x_r)$, is also valid for $n=1$ (i.e., for 1×1 generic matrices), since then $P_r = 1$.

Since $\chi(\overline{C}(n, r)) = \prod\{(1-x_k)^{-n} | 1 \leq k \leq r\} P_r$, and $\prod\{(1-x_k)^{-n} | 1 \leq k \leq r\}$ satisfies a functional equation with factor $(x_1 \cdots x_r)^n$ for all r , while P_r satisfies a functional equation with factor $(x_1 \cdots x_r)^{n^2-n}$ for all $r \geq n^2-n$, we have

THEOREM 13. *If $r \geq n^2-n$, the character series of the ring of invariants of r generic $n \times n$ matrices satisfies the functional equation*

$$\chi(\overline{C}(n, r))(x_1^{-1}, \dots, x_r^{-1}) = (-1)^d (x_1 \cdots x_r)^{n^2} \chi(\overline{C}(n, r))(x_1, \dots, x_r),$$

where d is the Krull dimension of $\overline{C}(n, r)$. \square

(Theorem 3 implies that the sign is $(-1)^d$.)

7. The functional equation for $\chi(\overline{R}(n))$. In this section we use the functional equation satisfied by $\chi(\overline{C}(n, n^2))$ to show that $\chi(\overline{R}(n, n^2))$ satisfies the same functional equation. The proof is based on the fact that the generic matrices U_1, \dots, U_{n^2} generate a free $\overline{C}(n, n^2)$ -module, as does their dual basis. This is the only place where the structure of $\overline{R}(n, n^2)$ as a noncommutative ring plays a role. The functional equation for $\chi(\overline{R}(n, r))$, $r \geq n^2$, then follows by applying Theorem 10.

From here until Theorem 21, we specialize to the case of n^2 generic $n \times n$ matrices U_1, \dots, U_{n^2} , and write $\overline{C}, \overline{R}, \chi(\overline{C}), \chi(\overline{R})$ instead of $\overline{C}(n, n^2)$, etc. We let $Q(\overline{C})$ denote the field of fractions of \overline{C} , and $Q(R) = Q(\overline{C})\overline{R}$. The discriminant of U_1, \dots, U_{n^2} is denoted Δ . It is the determinant of the $n^2 \times n^2$ matrix whose rows are U_1, \dots, U_{n^2} , written as $1 \times n^2$ row vectors (see [1, pp. 209–211] for a discussion of Δ). Our point of view is that $\chi(\overline{C})$ and $\chi(\overline{R})$ are character series which encode the structures of \overline{C} and \overline{R} as $\text{GL}(n^2, K)$ -modules, where the action of $\text{GL}(n^2, K)$ is induced by its standard linear action on the K -vector space spanned by U_1, \dots, U_{n^2} . Thus if $\overline{C} = K + \overline{C}_1 + \overline{C}_2 + \cdots$ as an \mathbf{N} -graded ring, then $\chi(\overline{C}) = 1 + \chi(\overline{C}_1) + \chi(\overline{C}_2) + \cdots$ in $[\Lambda_{n^2}] \subseteq \mathbf{Z}[[x_1, \dots, x_{n^2}]]$, where $\chi(\overline{C}_i)$ is a homogeneous symmetric functions of degree i .

Let V denote the standard module for $\mathrm{GL}(n^2, K)$, let $V^* = \mathrm{Hom}(V, K)$ be its dual (or contragredient) module, and let $(\det)^i$ ($i \in \mathbf{Z}$) denote K as a $\mathrm{GL}(n^2, K)$ -module with action $g(v) = (\det g)^i v$. Their respective characters are

$$\begin{aligned}\chi(V) &= x_1 + \cdots + x_{n^2} = e_1, \\ \chi(V^*) &= x_1^{-1} + \cdots + x_{n^2}^{-1} = e_{n^2-1}/e_{n^2}, \\ \chi((\det)^i) &= (x_1 \cdots x_{n^2})^i = (e_{n^2})^i.\end{aligned}$$

THEOREM 14. (1) $Q(\overline{R})$ is a central simple algebra of dimension n^2 over $Q(\overline{C})$.
 (2) U_1, \dots, U_{n^2} are a basis for $Q(\overline{R})$ over $Q(\overline{C})$.
 (3) Let W_1, \dots, W_{n^2} be a dual basis to U_1, \dots, U_{n^2} relative to the trace (i.e., $T(U_i W_j) = \delta_{ij}$), and let $Y_i = \Delta W_i$, where Δ is the discriminant of U_1, \dots, U_{n^2} . Then $Y_1, \dots, Y_{n^2} \in \overline{R}$.
 (4) As $\mathrm{GL}(n^2, K)$ -modules, $K\Delta \cong \det$, $\mathrm{span}\{U_i\} \cong V$, $\mathrm{span}\{W_i\} \cong V^*$, and $\mathrm{span}\{Y_i\} \cong (\det) \otimes V^*$. Hence their respective characters are $e_{n^2}, e_1, e_{n^2-1}/e_{n^2}$, and e_{n^2-1} .

PROOF. (1) is a special case of Posner's theorem (see [8, pp. 53, 176]).

(2) is clear since U_1, \dots, U_{n^2} are generic matrices.

(3) The $n^2 \times n^2$ matrix which expresses W_1, \dots, W_{n^2} as linear combinations of U_1, \dots, U_{n^2} is $(T(U_i U_j))^{-1}$, which has entries in $Q(\overline{C})$. Hence W_1, \dots, W_{n^2} lie in $Q(\overline{R})$. Let \mathcal{U} be the $n^2 \times n^2$ matrix whose rows are U_1, \dots, U_{n^2} , written as $1 \times n^2$ row vectors. Then W_1, \dots, W_{n^2} , written as $n^2 \times 1$ column vectors, are the columns of \mathcal{U}^{-1} . Since $\Delta = \det \mathcal{U}$, the entries of $\Delta \mathcal{U}^{-1}$ lie in $K[u_{ij}(r)]$, the polynomial ring generated by the entries of U_1, \dots, U_{n^2} . By [1, Lemma 9],

$$\overline{R} = Q(R) \cap M_n(K[u_{ij}(r)]).$$

Hence $\Delta W_1, \dots, \Delta W_{n^2}$ lie in \overline{R} .

(4) [1, pp. 209–210] shows that $K\Delta \cong \det$. The remaining assertions are clear.

□

Given formal power series of Schur functions we will write $\sum p(\lambda) S_\lambda \leq \sum q(\lambda) S_\lambda$ if $p(\lambda) \leq q(\lambda)$ for all partitions λ . Thus $\chi(M) \leq \chi(N)$ if M is a $\mathrm{GL}(n^2, K)$ -submodule of N . In particular, $\chi(\overline{C}) \leq \chi(\overline{R})$.

Now consider the bases U_i, W_i, Y_i for $Q(\overline{R})$ over $Q(\overline{C})$. If $Z \in R$,

$$\begin{aligned}(1) \quad Z &= T(ZW_1)U_1 + \cdots + T(ZW_{n^2})U_{n^2}, \\ \Delta Z &= T(ZY_1)U_1 + \cdots + T(ZY_{n^2})U_{n^2} \in \overline{C}U_1 + \cdots + \overline{C}U_{n^2}.\end{aligned}$$

Hence there are inclusions

$$(2) \quad \overline{R}\Delta \subseteq \overline{C}U_1 + \cdots + \overline{C}U_{n^2} \subseteq \overline{R}.$$

By reversing the roles of the U_i and Y_i we obtain the dual equation

$$(2) \quad \overline{R}\Delta \subseteq \overline{C}Y_1 + \cdots + \overline{C}Y_{n^2} \subseteq \overline{R}.$$

Moreover, (2) and (3) are inclusions of $\mathrm{GL}(n^2, K)$ -modules. In fact, as $\mathrm{GL}(n^2, K)$ -modules

$$\begin{aligned}(4) \quad \overline{R}\Delta &\cong \overline{R} \otimes \det, \quad \overline{C}U_1 + \cdots + \overline{C}U_{n^2} \cong \overline{C} \otimes V, \\ \overline{C}Y_1 + \cdots + \overline{C}Y_{n^2} &\cong \overline{C} \otimes (\det) \otimes V^*.\end{aligned}$$

Translating (2)–(4) into inequalities of character series using Theorem 14(4),

we have

LEMMA 15. *As formal power series of Schur functions in $[\Lambda_{n^2}]$, $\chi(\overline{C}) = \chi(\overline{C}(n, n^2))$ and $\chi(\overline{R}) = \chi(\overline{R}(n, n^2))$ satisfy the inequalities*

$$e_{n^2}\chi(\overline{R}) \leq e_1\chi(\overline{C}) \leq \chi(\overline{R}), \quad e_{n^2}\chi(\overline{R}) \leq e_{n^2-1}\chi(\overline{C}) \leq \chi(\overline{R}). \quad \square$$

In the statement of Lemma 15 we have repeated the standing hypothesis that the number of $n \times n$ generic matrices is exactly n^2 . The lemma is false for $r > n^2$ generic matrices since then $e_1\chi(\overline{C}(n, r))$ has Schur functions of partitions of length $n^2 + 1$ in its support, but $\chi(\overline{R}(n, r))$ does not. The difficulty is related to the fact that U_1, \dots, U_r are linearly dependent over $\overline{C}(n, r)$ if $r > n^2$. Thus, although $\overline{C}(n, r)U_1 + \dots + \overline{C}(n, r)U_r$ is a $\text{GL}(r, K)$ -submodule of $\overline{R}(n, r)$, it is not isomorphic to $\overline{C}(n, r) \otimes V(r)$, where $V(r)$ is the standard $\text{GL}(r, K)$ -module.

The first part of the next lemma is part of Theorem 12 of [1]; the second part is proved in the course of proving Theorem 16 of [1]. It is valid for any number of $n \times n$ generic matrices, and so will be stated in terms of $[\Lambda]$, without reference to the number of matrices. The notations $\overline{C}(n)$ and $\overline{R}(n)$ mean that the number of $n \times n$ generic matrices is arbitrary (or infinite).

LEMMA 16. *Let $\chi(\overline{C}(n)) = \sum c(\lambda)S_\lambda$ and $\chi(\overline{R}(n)) = \sum r(\lambda)S_\lambda$, both formal power series in $[\Lambda]$.*

(1) *If $l(\lambda) > n^2$, then $c(\lambda) = r(\lambda) = 0$.*

(2) *If $\lambda = (\lambda_1, \dots, \lambda_{n^2})$ has length $\leq n^2$, then*

$$c(\lambda_1 + 1, \dots, \lambda_{n^2} + 1) = c(\lambda_1, \dots, \lambda_{n^2}),$$

and

$$r(\lambda_1 + 1, \dots, \lambda_{n^2} + 1) = r(\lambda_1, \dots, \lambda_{n^2}). \quad \square$$

Recall the maps $\rho = \rho(n^2, n^2 - 1)$ and $\Phi = \Phi_{n^2-1}$ of §4:

$$[\Lambda_{n^2-1}] \begin{array}{c} \xrightarrow{\Phi} \\ \xleftarrow{\rho} \end{array} [\Lambda_{n^2}].$$

If $\lambda = (\lambda_1, \dots, \lambda_{n^2})$ is a partition of length $\leq n^2$, then

$$\Phi\rho(S_\lambda) = \begin{cases} S_\lambda, & \text{if } l(\lambda) < n^2, \\ 0, & \text{if } l(\lambda) = n^2. \end{cases}$$

Hence $\theta(f) = f - \Phi\rho(f)$ defines a projection operator on $[\Lambda_{n^2}]$ which satisfies

$$\theta \left(\sum_{l(\lambda) \leq n^2} p(\lambda) S_\lambda \right) = \sum_{l(\lambda) = n^2} p(\lambda) S_\lambda.$$

Consider the inequalities

$$(1) \quad e_{n^2}\chi(\overline{R}) \leq e_1\chi(\overline{C}) \leq \chi(\overline{R}), \quad e_{n^2}\chi(\overline{R}) \leq e_{n^2-1}\chi(\overline{C}) \leq \chi(\overline{R})$$

of Lemma 15. By Lemma 16,

$$(2) \quad \theta(e_{n^2}\chi(\overline{R})) = e_{n^2}\chi(\overline{R}) = \theta(\chi(\overline{R})).$$

Since θ preserves inequalities of power series of Schur functions, (1) and (2) yield

LEMMA 17.

- (1) $e_{n^2}\chi(\bar{R}) = \theta(e_1\chi(\bar{C})) = e_1\chi(\bar{C}) - \Phi\rho(e_1\chi(\bar{C})).$
 (2) $e_{n^2}\chi(\bar{R}) = \theta(e_{n^2-1}\chi(\bar{C})) = e_{n^2-1}\chi(\bar{C}) - \Phi\rho(e_{n^2-1}\chi(\bar{C})).$ \square

Since $\chi(\bar{R})$ and $\chi(\bar{C})$ are both rational functions of x_1, \dots, x_{n^2} , the substitution of x_i^{-1} for x_i is possible in Lemma 17(2). Then we can simplify, using the known form of the functional equations satisfied by $\chi(\bar{C})$ and $\rho(\chi(\bar{C}))$ given by Theorem 13.

$$\begin{aligned}
 e_{n^2}\chi(\bar{R}) &= e_{n^2-1}\chi(\bar{C}) - \Phi\rho(e_{n^2-1}\chi(\bar{C})) \\
 &= e_{n^2-1}\chi(\bar{C}) - \sum_{l=1}^{n^2} \left(\prod_{\substack{i=1 \\ i \neq l}}^{n^2} \frac{x_i}{x_i - x_l} \right) (x_1 \cdots \hat{x}_l \cdots x_{n^2}) \\
 &\quad \times \chi(\bar{C})(x_1, \dots, \hat{x}_l, \dots, x_{n^2}), \\
 (e_{n^2})^{-1}\chi(\bar{R})(x_1^{-1}, \dots, x_{n^2}^{-1}) &= e_{n^2-1}(x_1^{-1}, \dots, x_{n^2}^{-1})\chi(\bar{C})(x_1^{-1}, \dots, x_{n^2}^{-1}) \\
 &\quad - \sum_{l=1}^{n^2} \left(\prod_{\substack{i=1 \\ i \neq l}}^{n^2} \frac{x_i^{-1}}{x_i^{-1} - x_l^{-1}} \right) (x_1^{-1} \cdots \hat{x}_l^{-1} \cdots x_{n^2}^{-1}) \\
 &\quad \times \chi(\bar{C})(x_1^{-1}, \dots, \hat{x}_l^{-1}, \dots, x_{n^2}^{-1}) \\
 &= -e_1(e_{n^2})^{n^2-1}\chi(\bar{C}) \\
 &\quad - (e_{n^2})^{n^2-1} \sum_{l=1}^{n^2} \left(\prod_{\substack{i=1 \\ i \neq l}}^{n^2} \frac{x_i}{x_i - x_l} \right) (x_1^{-1} \cdots \hat{x}_l^{-1} \cdots x_{n^2}^{-1}) \\
 &\quad \times \chi(\bar{C})(x_1, \dots, \hat{x}_l, \dots, x_{n^2}) \\
 &= -e_1(e_{n^2})^{n^2-1}\chi(\bar{C}) - (e_{n^2})^{n^2-1}\Phi((e_{n^2-1})^{-1}\rho(\chi(\bar{C}))).
 \end{aligned}$$

Multiplying by e_{n^2} , we have

LEMMA 18.

$$\chi(\bar{R})(x_1^{-1}, \dots, x_{n^2}^{-1}) = -(e_{n^2})^{n^2} \{e_1\chi(\bar{C}) + \Phi((e_{n^2-1})^{-1}\rho(\chi(\bar{C})))\}. \quad \square$$

LEMMA 19. Let $\lambda = (\lambda_1, \dots, \lambda_{n^2-1})$ be a partition of length $\leq n^2 - 1$. Then in Λ_{n^2} ,

$$e_1\Phi(S_\lambda) - \Phi(e_1S_\lambda) = e_{n^2}\Phi((e_{n^2-1})^{-1}S_\lambda).$$

PROOF. By Theorem 6(1),(2),

$$(*) \quad e_{n^2}\Phi((e_{n^2-1})^{-1}S_\lambda) = \begin{cases} 0, & \text{if } \lambda_{n^2-1} = 0, \\ e_{n^2}S_{\lambda^*}, & \text{if } \lambda_{n^2-1} \geq 1, \end{cases}$$

where $\lambda^* = (\lambda_1 - 1, \dots, \lambda_{n^2-1} - 1)$.

In order to evaluate $e_1\Phi(S_\lambda) - \Phi(e_1S_\lambda)$ we use Young's rule [3, p. 88], which implies that in Λ_t (here $t = n^2 - 1$ or $t = n^2$), e_1S_λ is the sum of all Schur functions S_μ such that $l(\mu) \leq t$, $|\mu| = |\lambda| + 1$, and $\lambda \leq \mu$, where $\lambda \leq \mu$ means that each part of λ is \leq the corresponding part of μ .

Case I. $\lambda_{n^2-1} = 0$. Then exactly the same S_μ 's arise whether $e_1 S_\lambda$ is computed in Λ_{n^2-1} or in Λ_{n^2} . Hence $e_1 \Phi(S_\lambda) - \Phi(e_1 S_\lambda) = 0$, which agrees with (*).

Case II. $\lambda_{n^2-1} \geq 1$. Then exactly one more Schur function arises if $e_1 S_\lambda$ is computed in Λ_{n^2} instead of Λ_{n^2-1} , namely S_μ , where $\mu = (\lambda_1, \dots, \lambda_{n^2-1}, 1) = \lambda^* + (1^{n^2})$. Hence $e_1 \Phi(S_\lambda) - \Phi(e_1 S_\lambda) = S_\mu = e_{n^2} S_{\lambda^*}$, which again agrees with (*). \square

Suppose that λ is a partition of length $\leq n^2$. Then

$$\rho(S_\lambda) = \begin{cases} S_\lambda(x_1, \dots, x_{n^2-1}), & \text{if } l(\lambda) \leq n^2 - 1, \\ 0, & \text{if } l(\lambda) = n^2. \end{cases}$$

Using Lemma 19 in the first instance and the fact that $0 = 0$ in the second instance gives

$$(1) \quad e_1 \Phi \rho(S_\lambda) - \Phi(e_1 \rho(S_\lambda)) = e_{n^2} \Phi((e_{n^2-1})^{-1} \rho(S_\lambda)).$$

Applying (1) term by term to $\chi(\overline{C}) = \sum c(\lambda) S_\lambda$ gives

$$(2) \quad e_1 \Phi \rho(\chi(\overline{C})) - \Phi(e_1 \rho(\chi(\overline{C}))) = e_{n^2} \Phi((e_{n^2-1})^{-1} \rho(\chi(\overline{C}))).$$

By Lemma 16 and the subsequent remarks

$$(3) \quad \Phi \rho(\chi(\overline{C})) = (1 - e_{n^2}) \chi(\overline{C}).$$

Substituting (3) in (2) and simplifying gives

LEMMA 20.

$$e_1 \chi(\overline{C}) - \Phi(e_1 \rho(\chi(\overline{C}))) = e_{n^2} \{e_1 \chi(\overline{C}) + \Phi((e_{n^2-1})^{-1} \rho(\chi(\overline{C})))\}. \quad \square$$

Since $\rho: [\Lambda_{n^2}] \rightarrow [\Lambda_{n^2-1}]$ is a ring homomorphism and $\rho(e_1) = e_1$, $e_1 \rho(\chi(\overline{C})) = \rho(e_1 \chi(\overline{C}))$. Combining Lemmas 18, 20 and 17(1) yields

$$\begin{aligned} \chi(\overline{R})(x_1^{-1}, \dots, x_{n^2}^{-1}) &= -(e_{n^2})^{n^2} \{e_1 \chi(\overline{C}) + \Phi((e_{n^2-1})^{-1} \rho(\chi(\overline{C})))\} \\ &= -(e_{n^2})^{n^2-1} \{e_1 \chi(\overline{C}) - \Phi(e_1 \rho(\chi(\overline{C})))\} = -(e_{n^2})^{n^2} \chi(\overline{R}). \end{aligned}$$

Thus we have

THEOREM 21. $\chi(\overline{R}(n, n^2))$ satisfies the functional equation

$$\chi(\overline{R}(n, n^2))(x_1^{-1}, \dots, x_{n^2}^{-1}) = (-1)(x_1 \cdots x_{n^2})^{n^2} \chi(\overline{R}(n, n^2))(x_1, \dots, x_{n^2}). \quad \square$$

Note that the sign in Theorem 21 is correct since $\overline{R}(n, n^2)$ has Krull dimension $n^2(n^2 - 1) + 1$. Finally, for $r \geq n^2$, Theorem 10 applies to $\chi(\overline{R}(n, r))$. Hypotheses (A) and (B) follow from Lemma 16, while hypothesis (E) follows from Theorem 21. The sign is $(-1)^{1+(r-n^2)n^2} = (-1)^d$, where d is the Krull dimension of $\overline{R}(n, r)$, since

$$d = (r - 1)n^2 + 1 \equiv 1 + (r - n^2)n^2 \pmod{2}.$$

Therefore we have

THEOREM 22. If $r \geq n^2$, the character series of the trace ring of r generic $n \times n$ matrices satisfies the functional equation

$$\chi(\overline{R}(n, r))(x_1^{-1}, \dots, x_r^{-1}) = (-1)^d (x_1 \cdots x_r)^{n^2} \chi(\overline{R}(n, r))(x_1, \dots, x_r),$$

where d is the Krull dimension of $\overline{R}(n, r)$. \square

8. Remarks and examples. In case $r < n^2 - n$, $\chi(\overline{C}(n, r))$ still satisfies a functional equation, by Theorem 3. We have not shown that $\chi(\overline{R}(n, r))$ satisfies a functional equation for $r < n^2$, although it seems very likely that it does. This section consists mostly of speculation on the exponent of $(x_1 \cdots x_r)$ in the functional equations.

A few general statements are possible. As noted in the proof of Lemma 12, $\overline{C}(n, 1)$ is a polynomial ring in n variables of degrees $1, 2, \dots, n$, the coefficients of the characteristic polynomial of U_1 . Hence

$$(1) \quad \chi(\overline{C}(n, 1)) = \frac{1}{1-x_1} \frac{1}{1-x_1^2} \cdots \frac{1}{1-x_1^n},$$

and

$$(2) \quad \chi(\overline{C}(n, 1))(x_1^{-1}) = (-1)^n x_1^{\binom{n+1}{2}} \chi(\overline{C}(n, 1))(x_1).$$

Similarly, $\overline{R}(n, 1)$ is a polynomial ring in n variables of degrees $1, 1, 2, \dots, n-1$. It is an integral extension of $\overline{C}(n, 1)$ of degree n and has a transcendence basis consisting of U_1 and all the coefficients of the characteristic polynomial of U_1 except the determinant, as can be seen by examining the characteristic polynomial of U_1 . Hence,

$$(3) \quad \chi(\overline{R}(n, 1)) = \frac{1}{(1-x_1)^2} \frac{1}{1-x_1^2} \cdots \frac{1}{1-x_1^{n-1}},$$

$$(4) \quad \chi(\overline{R}(n, 1))(x_1^{-1}) = (-1)^n x_1^{\binom{n}{2}+1} \chi(\overline{R}(n, 1))(x_1).$$

Assuming that $\chi(\overline{R}(n, r))$ satisfies a functional equation, then for $n \geq 3$ the argument of §6 shows that for $r \geq n^2 - n$, the exponent of $(x_1 \cdots x_r)$ is n^2 . The argument breaks down for $n = 2$, and the exponent is actually smaller for $r = 2$, as we will see below.

For $n = 1$, the exponent of $(x_1 \cdots x_r)$ is always one for both $\chi(\overline{C}(1, r))$ and $\chi(\overline{R}(1, r))$. In fact $\overline{C}(1, r) = \overline{R}(1, r)$. The only nontrivial case for which the character series are known is $n = 2$ [1, p. 221], where

$$(5) \quad \chi(\overline{C}(2, 4)) = \frac{1 + e_3 - e_1 e_4 - e_4^2}{\prod (1 - x_i)(1 - x_i)^2(1 - x_i x_j)},$$

$$(6) \quad \chi(\overline{R}(2, 4)) = \frac{1 - e_4}{\prod (1 - x_i)^2(1 - x_i x_j)}.$$

In both denominators the products are over $1 \leq i \leq 4$ and $1 \leq i < j \leq 4$. The character series for $\overline{C}(2, r)$ and $\overline{R}(2, r)$ for $r = 1, 2, 3$ can be obtained from (5) and (6) by specializing the appropriate variables to zero. This amounts to applying $\rho(4, r): [\Lambda_4] \rightarrow [\Lambda_r]$. The exponent of $(x_1 \cdots x_r)$ in the functional equations of $\overline{C}(2, r)$ and $\overline{R}(2, r)$ can then be read off easily:

(7) Exponent of $(x_1 \cdots x_r)$ in the functional equation of

| | $x(\overline{C}(2, r))$ | $x(\overline{R}(2, r))$ |
|------------|-------------------------|-------------------------|
| $r = 1$ | 3 | 2 |
| $r = 2$ | 4 | 3 |
| $r = 3$ | 4 | 4 |
| $r \geq 4$ | 4 | 4 |

The experimental evidence given by $n = 1$ and $n = 2$ does not seem adequate to justify a conjecture on the exponent of $(x_1 \cdots x_r)$ in the functional equations of $\chi(\overline{C}(n, r))$ and $\chi(\overline{R}(n, r))$ for $n \geq 3, r < n^2 - n$.

ADDED IN PROOF. In a recent Nagoya University preprint. *The ring of invariants of matrices*, Y. Teranishi has independently obtained the functional equation for $\chi(\overline{C})$. His proof is entirely different and much shorter than the one given here. His method is to apply Cauchy's integral formula to the Molien-Weyl expression for $\chi(\overline{C})$ as a multiple integral. He shows that the exponent in the functional equation is n^2 as soon as the number of matrices is at least two, which answers the question raised in §8 of this paper. His method can no doubt be applied to give the same result for $\chi(\overline{R})$, except for the case of two generic 2×2 matrices, when the exponent in the functional equation is 3 instead of 4.

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