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THE COMPLEX EQUILIBRIUM MEASURE OF A SYMMETRIC CONVEX SET IN $\mathbf{R}^n$

BY

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ABSTRACT. We give a formula for the measure on a convex symmetric set K in $\mathbf{R}^n$ which is the Monge-Ampere operator applied to the extremal plurisubharmonic function L_K for the convex set. The measure is concentrated on the set K and is absolutely continuous with respect to Lebesgue measure with a density which behaves at the boundary like the reciprocal of the square root of the distance to the boundary. The precise asymptotic formula for $x \in K$ near a boundary point x_0 of K is shown to be of the form $c(x_0)/[\mathrm{dist}(x, \partial K)]^{-1/2}$, where the constant $c(x_0)$ depends both on the curvature of K at x_0 and on the global structure of K.

1. Introduction. Let us denote the family of plurisubharmonic (psh) functions on $\mathbf{C}^n$ of minimal growth by

$$\mathcal{L} = \{v \text{ psh on } \mathbf{C}^n, v(z) \leq \log(1 + |z|) + O(1)\}.$$

For K a compact subset of $\mathbf{C}^n$ the extremal function L_K^* for K with logarithmic singularity at infinity is defined by setting

$$L_K(z) = \sup\{v(z) \colon v \in \mathcal{L},\ v \leq 0 \text{ on } K\}$$

and

$$L_K^*(z) = \limsup_{\varsigma \to z} L_K(\varsigma)$$

(cf. Siciak [**13**] and Zaharjuta [**17**]). The function L_K^* is in general not smooth on $\mathbf{C}^n \setminus K$ when $n > 1$ and, in particular, it is not harmonic. It is a theorem of Siciak that either $L_K^* \equiv +\infty$, in which case the set K is pluripolar, or else $L_K^* \in \mathcal{L}$. If L_K is continuous on $\mathbf{C}^n$, then $L_K \equiv L_K^* \in \mathcal{L}$.

The extremal function L_K^* satisfies the complex Monge-Ampere equation

$$(dd^c L_K^*)^n = 0$$

in a generalized sense on $\mathbf{C}^n \setminus K$ [**2**, Corollary 9.4]. Thus, for nonpluripolar sets K,

$$\lambda_K := (dd^c L_K^*)^n \tag{1.1}$$

is a positive Borel measure supported on K. It has total mass equal to $(2\pi)^n$ (cf. [**16**]). We will call λ_K the *complex equilibrium measure* for K.

Here we consider compact sets $K \subset \mathbf{R}^n \subset \mathbf{C}^n$. In this case K is polynomially convex, and so $L_K > 0$ on $\mathbf{C}^n \setminus K$. There are rather reasonable hypotheses on

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K that will insure that L_K is continuous on $\mathbf{C}^n$, such as:

> for each $x_0 \in \partial K$, there exists a real analytic curve $t \to \gamma(t) \in \mathbf{R}^n$, $-1 < t < +1$, such that $\gamma(t) \in \operatorname{int}(K)$ for $-1 < t < 0$, and $\gamma(0) = x_0$.

(See Plesniak [**9**] and Sadullaev [**12**].)

Our first result concerning the extremal measure for subsets $K \subset \mathbf{R}^n$ is that on compact subsets of the interior of K, λ_K is equivalent to n-dimensional Lebesgue measure.

THEOREM 1.1. *If $K \subset \mathbf{R}^n$ is compact, then there exist constants $0 < c_1 < c_2 < \infty$ such that*

$$\int_E c_1\,dx \le \lambda_K(E) \le \int_E \frac{c_2\,dx}{[\operatorname{dist}(x, \partial K)]^n} \tag{1.2}$$

holds for any Borel set $E \subset \operatorname{int}(K)$. We can take

$$c_1 = 2^n[\operatorname{diam}(K)]^{-n}, \quad c_2 = (2\sqrt{n})^n.$$

In case $K \subset \mathbf{R}^n$ is convex and symmetric about the origin, the extremal function L_K may be given in a relatively simple form. Starting with the case $n = 1$, we note that a convex $K \subset \mathbf{R}$ is an interval. The extremal function for $K = [-1, +1] \subset \mathbf{C}$ is given by the familiar Green function

$$L_{[-1,+1]}(z) = \psi(z) := \log|z + \sqrt{z^2 - 1}|.$$

For convex $K \subset \mathbf{R}^n$, we have the support function

$$\rho_K(\xi) := \sup_{x \in K} \xi \cdot x \tag{1.3}$$

defined for $\xi \in \mathbf{R}^n$. It is a theorem of Lundin that the extremal function may be obtained as

$$L_K(z) = \sup_{|\xi|=1, \xi \in \mathbf{R}^n} \psi\left(\frac{z \cdot \xi}{\rho_K(\xi)}\right) \tag{1.4}$$

where $z \cdot \xi = z_1\xi_1 + \cdots + z_n\xi_n$ is a projection to the complexification of the ξ-axis. Dividing by the support function $\rho_K(\xi)$ is a normalization so that K projects to the interval $[-1, +1]$. The representation (1.4) was given by Lundin in [**6**].

For the case of convex symmetric sets, the extremal measure can also be computed in an explicit form.

THEOREM 1.2. *Let $K \subset \mathbf{R}^n$ be a compact convex set, symmetric about the origin and with nonempty interior. Then $\lambda_K = n! \cdot \lambda(x)\,dx$, where $\lambda(x)$ is the n-dimensional volume of the convex hull of $S_x(K^*)$, where K^* is the convex set dual to K, and $S_x\colon \mathbf{R}^n \to \mathbf{R}^n$ is the map*

$$S_x(\eta) = \eta/(1 - (x \cdot \eta)^2)^{1/2}.$$

In case $S_x(K^*)$ is convex, then

$$\lambda(x) = \frac{1}{n}\int_{|\xi|=1} \frac{d\sigma(\varsigma)}{[\rho_K^2(\xi) - (x \cdot \xi)^2]^{n/2}}, \qquad x \in K$$

($d\sigma$ denotes surface area measure on the unit sphere).

This theorem extends a result of Lundin [**7**], who has shown that when $K = \mathbf{B}^n$ is the unit ball in $\mathbf{R}^n$, then

$$L_{\mathbf{B}^n}(z) = \sinh^{-1}[\tfrac{1}{2}(|z|^2 - 1 + |z^2 - 1|)]^{1/2}$$

and that

$$\lambda_{\mathbf{B}^n} = \frac{c_n}{(1-|x|^2)^{1/2}}\,dx, \quad \text{where } c_n = 2^n \cdot \Gamma\left(\frac{n+1}{2}\right) \cdot \pi^{(n-1)/2}.$$

A less precise but more general result is as follows.

THEOREM 1.3. *If $K \subset \mathbf{R}^n$ is compact and convex with nonempty interior and smooth boundary, then there are constants $0 < c_1 < c_2$ such that*

$$\frac{c_1\,dx}{[\mathrm{dist}(x,\partial K)]^{1/2}} \le \lambda_K \le \frac{c_2\,dx}{[\mathrm{dist}(x,\partial K)]^{1/2}}. \tag{1.5}$$

We also compute the asymptotic behavior of $\lambda_K(x)$ at ∂K.

THEOREM 1.4. *If K is a smoothly bounded symmetric convex set with non-vanishing curvature, then there is a smooth function $c(x)$ on ∂K such that*

$$\lambda_K(x) \approx c(\bar{x})|x - \bar{x}|^{-1/2}$$

where $\bar{x} \in \partial K$ is the boundary point closest to x.

This theorem is proved in §4, where a geometric construction is given for the quantity $c(\bar{x})$. It turns out that $c(\bar{x})$ is given by the volume of a certain set in $\mathbf{R}^n$ (see Figure 3), which depends both on the curvature of ∂K at $\bar{x}$ and a global geometric envelope, the "ellipsoidal hull", of K.

Our interest in knowing λ_K more precisely in these specific cases arises from the connection between L_K and some problems concerning polynomials. Theorem 1.3 gives a more concrete statement of the Leja polynomial condition in the case of a smoothly bounded convex set $K \subset \mathbf{R}^n$. The condition is as follows (cf. [**14**]). *If $\mathcal{F} = \{P_\alpha : \alpha \in A\}$ is any family of polynomials and if*

$$S_{\mathcal{F}} = \left\{x \in K\colon \sup_{\alpha \in A} |p_\alpha(x)| = +\infty\right\},$$

then the following are equivalent:

for any $\lambda > 1$ there exists an open set $U \supset K$ and an $M < \infty$ such that for all polynomials in the family $\mathcal{F}$

$$\sup_U |p_\alpha| \le M \cdot \lambda^{\deg(p_\alpha)} \tag{1.6}$$

and

the fine interior of $S_{\mathcal{F}}$ has Lebesgue measure zero.

Here the "fine interior" is taken with respect to the plurifine topology induced from $\mathbf{C}^n$, cf. [**3**].

It was shown by Siciak [**14**] that if a measure μ satisfies the Leja polynomial condition, in the sense that (1.6) holds whenever $\mu(S_{\mathcal{F}}) = 0$, then it also satisfies a version of the Bernstein-Markov condition:

for $0 < s < +\infty, \lambda > 1$, *there exist an open set* $U \supset K$, *and a constant* $C_s < \infty$ *such that for any polynomial* p,

$$\sup_U |p| \leq C_s \cdot \lambda^{\deg(p)} \left[\int |p|^s \, d\mu\right]^{1/s}. \tag{1.7}$$

This inequality is used in polynomial approximation and analytic continuation. It follows from the results above that any measure μ with $dx|_K \ll \mu$ will satisfy (1.7). There is evidence, however, that the measure λ_K should be in some sense optimal in these problems. For instance, it is conjectured in [**15**] that λ_K should reflect the asymptotic behavior of the extremal points of K.

2. Equivalence of λ_K and Lebesgue measure. In this section we will prove a comparison result which is closely related to a theorem of Levenberg [**5**]. This will lead to a proof of Theorem 1.1.

We will denote by $P(\Omega)$ the space of all psh functions on the domain Ω in $\mathbf{C}^n$.

LEMMA 2.1. *Let* $u_1, u_2 \in P(\Omega)$ *be given locally bounded functions, and let* $S \subset \Omega \cap \mathbf{R}^n$ *be a closed set containing the supports of the Borel measures* $(dd^c u_1)^n$ *and* $(dd^c u_2)^n$. *If the sets* $\{u_1 = 0\}$ *and* $\{u_2 = 0\}$ *differ from* S *by at most a pluripolar set, and if* $0 \leq u_1 \leq u_2$ *on* Ω, *then* $(dd^c u_1)^n \leq (dd^c u_2)^n$.

PROOF. Without loss of generality, we may assume that $B = \{|z| < 1\}$ is a compact subset of Ω. Let $u_j^\varepsilon = u_j * \chi_\varepsilon$ be smoothings decreasing to u_j as $\varepsilon \downarrow 0$. Let v_j denote the restriction of u_j to ∂B. Similarly, let v_j^ε denote the restriction of $u_j^\varepsilon + j \cdot \varepsilon$ to ∂B, so that $0 < v_1^\varepsilon < v_2^\varepsilon$ on ∂B. Given a subset S of the unit ball B and a function v on ∂B, let $\mathcal{F}(S, v)$ denote the family of all psh functions w on the unit ball B such that $w \leq 0$ on S, and $\limsup_{\varsigma \to z} w(\varsigma) \leq v(z)$ for all $z \in \partial B$. Let

$$S_\delta^\varepsilon = \{x \in \mathbf{R}^n \cap \Omega \colon \operatorname{dist}(x, \partial\Omega) \geq \varepsilon \text{ and } \operatorname{dist}(x, S) \leq \delta\}$$

and let

$$U_{\delta,j}^\varepsilon(z) := \sup\{w(z) \colon w \in \mathcal{F}(S_\delta^\varepsilon, v_j^\varepsilon)\}, \qquad z \in B.$$

The set S_j^ε is easily seen to be regular, so $U_{\delta,j}^\varepsilon$ is psh and continuous on the closure $\overline{B}$ of B. It therefore follows from a theorem of Levenberg [**5**] that

$$(dd^c U_{\delta,1}^\varepsilon)^n \leq (dd^c U_{\delta,2}^\varepsilon)^n.$$

As $\delta \downarrow 0$, the functions $U_{\delta,j}^\varepsilon$ increase almost everywhere to the functions $[U_{S,j}^\varepsilon]^*$, the uppersemicontinuous regularizations of the envelope functions for the families $\mathcal{F}(S, v_j^\varepsilon)$. Because the operator $(dd^c)^n$ is continuous on monotone limits of locally bounded psh functions [**2**], we can pass to the limit in the last inequality to obtain

$$(dd^c U_{S,1}^\varepsilon)^n \leq (dd^c U_{S,2}^\varepsilon)^n.$$

Now, let $\varepsilon \downarrow 0$. The functions $U_{S,j}^\varepsilon$ clearly decrease. And, they decrease to the uppersemicontinuous regularization of the envelope function of the family $\mathcal{F}(S, v_j)$, $U_{S,j}^*$. Thus, we have

$$(dd^c U_{S,1}^*)^n \leq (dd^c U_{S,2}^*)^n.$$

But, it is evident that $U^*_{S,j} \geq u_j$. On the other hand, we have $u_j \geq U^*_{S,j}$ by the domination principle (see [2, Corollary 4.5]). Thus, $u_j = U_{S,j}$ which proves the lemma.

PROPOSITION 2.2. *Let $E_1, \ldots, E_n$ be compact subsets of* $\mathbf{C}$. *Then*

$$\lambda_{E_1 \times \cdots \times E_n} = \lambda_{E_1} \otimes \cdots \otimes \lambda_{E_n}.$$

PROOF. By a result of Siciak [13] we know that

$$L^*_{E_1 \times \cdots \times E_n}(z_1, \ldots, z_n) = \max_{1 \leq j \leq n} L^*_{E_j}(z_j).$$

Let us assume that E_j is a Jordan domain in $\mathbf{C}$ with real analytic boundary. Then L_{E_j} may be extended from $\mathbf{C} \setminus E_j$ to a function $\tilde{L}_j$ which is harmonic in a neighborhood of ∂E_j. Thus we may write

$$L^*_{E_1 \times \cdots \times E_n}(z_1, \ldots, z_n) = \max_{1 \leq j \leq n} \{\tilde{L}_1, \ldots, \tilde{L}_n, 0\}.$$

However,

$$(dd^c L_{E_1 \times \cdots \times E_n})^n = d^c \tilde{L}_1 \wedge \cdots \wedge d^c \tilde{L}_n |_M$$

where $M = \{\tilde{L}_1 = \cdots = \tilde{L}_n = 0\}$ is given the orientation of $d\tilde{L}_1 \wedge \cdots \wedge d\tilde{L}_n$ (see the remark on p. 7 of [1]). Since

$$d^c \tilde{E}_j = \frac{\partial}{\partial n_j}(\tilde{L}_j)\, d\sigma_j$$

where $d\sigma_j$ is the arclength measure of ∂E_j and dn_j is the outward normal, we see that $d^c \tilde{L}_j$ may be naturally identified with λ_{E_j}. This proves the proposition in the smooth case.

For the general case, we take smooth E^ε_j,

$$E_j \subset E^\varepsilon_j \subset \{z \in \mathbf{C}\colon \operatorname{dist}(z, E_j) \leq \varepsilon\}.$$

If we take $E^\varepsilon_j \subset E^\delta_j$ for $\varepsilon \leq \delta$, then $L^*_{E^\varepsilon_1 \times \cdots \times E^\varepsilon_n}$ increases a.e. to $L_{E_1 \times \cdots \times E_n}$. Thus, by the convergence theorem of [2],

$$\lim_{\varepsilon \to 0} \lambda_{E^\varepsilon_1 \times \cdots \times E^\varepsilon_n} = \lambda_{E_1 \times \cdots \times E_n},$$

and the proposition is proved.

In the case of the interval $[-1, +1]$, it is well known that

$$\lambda_{[-1,+1]} = (2 \cdot dx)/(1 - x^2)^{1/2},$$

and so a product of intervals,

$$E = \{-\delta \leq x_1 \leq \delta, \ldots, -\delta \leq x_n \leq \delta\}$$

has extremal measure

$$\lambda_E = \frac{2^n \cdot dx_1 \cdots dx_n}{\delta^n \cdot [1 - (x_1/\delta)^2]^{1/2} \cdots [1 - (x_n/\delta)^2]^{1/2}}.$$

PROOF OF THEOREM 1.1. If r is the diameter of K, then without loss of generality we may assume that $K \subset (r \cdot I)^n \subset \mathbf{R}^n$ where $r \cdot I = [-r, +r]$. Thus, $L^*_K \geq L^*_{(r \cdot I)^n}$. If $x_0 \in \operatorname{int}(K)$, then $\operatorname{dist}(x_0, \partial K) = \eta > 0$. Without loss of generality, we can assume that $x_0 = 0$. Let $\Omega = \{z \in \mathbf{C}^n : |z| < \eta\}$ and $S = \Omega \cap \mathbf{R}^n$. It follows that $S = \{L_K^* = 0\} \cap \Omega = \{L_{(r\cdot I)^n}^* = 0\} \cap \Omega$ and so by Lemma 2.1,

$$\lambda_K|_S \geq \left.\frac{2^n \cdot dx_1 \cdots dx_n}{r^n \cdot [1-(x_1/r)^2]^{1/2} \cdots [1-(x_n/r)^2]^{1/2}}\right|_S \geq \left.\frac{2^n \cdot dx_1 \cdots dx_n}{r^n}\right|_S .$$

For the other inequality, we note that $\{x\colon -\eta/\sqrt{n} \leq x_j \leq +\eta/\sqrt{n}\} \subset \operatorname{int}(K)$, so again by Lemma 2.1

$$\frac{[2\sqrt{n}]^n \cdot dx_1 \cdots dx_n}{(\operatorname{dist}(x, \partial K))^{n/2}} \geq \lambda_K|_{\operatorname{int}(K)}.$$

This completes the proof.

PROOF OF THEOREM 1.3. First let us assume that ∂K is real analytic. Thus, it is pluripolar and $\lambda_K(\partial K) = 0$. Since ∂K is C^2, we may touch any $x_0 \in K$ by an internally tangent ball of fixed radius r. Thus, they may use an internally tangent ball and an externally tangent cube to obtain (1.5) as in Theorem 1.1, because there is no mass on ∂K.

If ∂K is merely C^2, we may consider a sequence $\{K_j\}$ of convex, symmetric domains with real analytic boundary such that ∂K_j approaches ∂K in C^2. Thus, we obtain (1.5) for each λ_{K_j}. Further, the constants c_1 and c_2 depend only on the diameter of K_j and the radius of an internally tangent ball. Thus, we may be chosen independent of j. It follows from the dominated convergence theorem, then, that λ_K puts no mass on ∂K, so that (1.5) holds.

3. Representation of $(dd^c L_K)^n$. In this section we will use Lundin's representation (1.4) of L_K and give the basic form of $(dd^c L_K)^n$ for convex sets in $\mathbf{R}^n$. We first give a geometrical construction of an auxillary function that will be used to describe $(dd^c L_K)^n$.

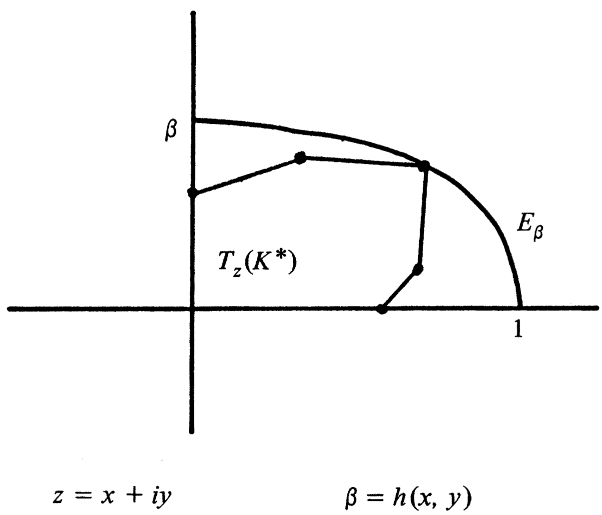


FIGURE 1

For $\beta > 0$, let E_β denote the ellipse

$$E_\beta: \quad s^2 + t^2/\beta^2 = 1. \tag{3.1}$$

Let K^* denote the dual convex set to K. That is,

$$K^* = \{\eta \in \mathbf{R}^n : x \cdot \eta \le 1 \text{ for all } x \in K\}.$$

It is readily checked that when K is convex and symmetric about the origin, so is K^* and

$$K^* = \{\eta = r \cdot \alpha \in \mathbf{R}^n : |\alpha| = 1, r \le 1/\rho(\alpha)\}.$$

We will use the function $h = h_K$ defined for $z = x + i \cdot y$ with $x \in \operatorname{int}(K)$, $y \in \mathbf{R}^n$ by

$$h_K(z) = h(x, y) = \inf\{\beta : T_z(K^*) \subset E_\beta\} \tag{3.2}$$

(see Figure 1) where T_z is the linear map from $\mathbf{R}^n$ to $\mathbf{R}^2$ defined by $\varsigma = s + i \cdot t = T_z(\xi) = z \cdot \xi$, or, more precisely by

$$\begin{bmatrix} s \\ t \end{bmatrix} = T_z(\xi) = \begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & & y_n \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}.$$

The function $h(x, y) = h_K(x, y)$ has the following properties.

PROPOSITION 3.1. *If K^* is the dual convex set to the compact convex set K, and if K^* is compact, then for $h(x, y)$ defined in (3.2) we have*

(i) *$h(x, y)$ is continuous for $x \in \operatorname{int}(K)$ and $y \in \mathbf{R}^n$;*

(ii) *$h(x, \lambda \cdot y) = \lambda \cdot h(x, y)$ for $\lambda > 0$;*

(iii) *$y \to h(x, y)$ is convex on R^n for each fixed $x \in \operatorname{int}(K)$;*

(iv) *$y \to h(x, y)$ is the support function of the convex hull of the set, $S_x(K^*)$, where S_x is the (nonlinear) transformation of $\mathbf{R}^n$ defined by*

$$S_x(\xi) = \xi/(1 - (x \cdot \xi)^2)^{1/2}.$$

PROOF. First, note that h is defined and finite when $x \in \operatorname{int}(K)$ because then $\xi \cdot x < 1$ for all $\xi \in K^*$, so the set $T_z(K^*)$ is contained in E_β for some $\beta > 0$. That h is continuous is clear because K^* is compact. Assertion (ii) follows from the obvious scaling property

$$T_{x+i\cdot\lambda\cdot y}(K^*) \subset E_\beta \quad \text{iff} \quad T_{x+i\cdot y}(K^*) \subset E_{\beta/\lambda}.$$

Both assertions also follow directly from (iv), as does assertion (iii). We will therefore prove assertion (iv).

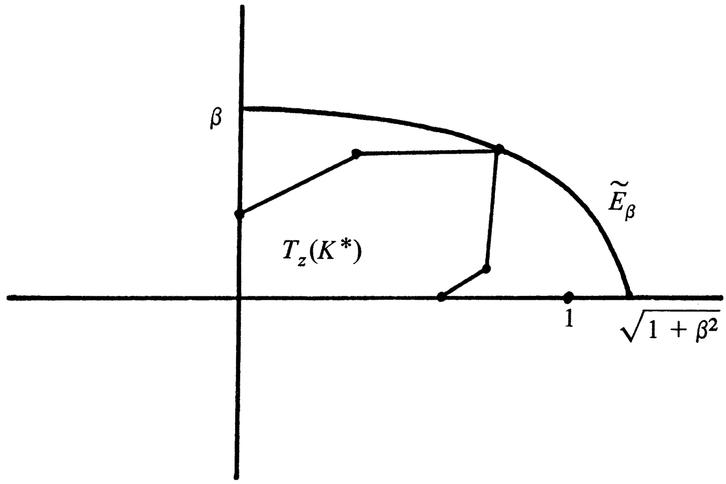
The number $\beta = h(x, y)$ is defined as the smallest number $\beta \ge 0$ such that

$$(x \cdot \xi)^2 + (y \cdot \xi)^2/\beta^2 \le 1, \qquad \xi \in K^*,$$

with equality holding for at least one ξ. Solving the inequality for β yields

$$\beta \ge y \cdot \xi/(1 - [x \cdot \xi]^2)^{1/2} = y \cdot S_x(\xi), \qquad \xi \in K^*, \tag{3.3}$$

with equality for at least one ξ. In other words, $h(x, y)$ is the maximum of the right-hand side of (3.3) over K^*, which is exactly assertion (iv). This completes the proof of the proposition.



$z = x + iy$ $\qquad \log\left(\beta + \sqrt{1+\beta^2}\right) = L_K(z)$

FIGURE 2

For f a convex function of $y \in \mathbf{R}^n$, let $M(f)$ denote the real Monge-Ampere operator applied to f. That is, if f is smooth, then $M(f)$ is the determinant of the Hessian of f,

$$M(f) = \det[\partial^2 f/\partial y_j \partial y_k].$$

The operator extends by continuity to be defined as a nonnegative Borel measure on the class of convex functions. The measure of a Borel set E is the volume of the set of all direction vectors λ of hyperplanes, $\lambda \cdot y + \text{const.}$, which support the graph of f at a point $x \in E$. See e.g. [**11**].

We will use two facts about this operator. First, if the convex function $f(y)$ defined on $\mathbf{R}^n$ is thought of as a psh function on $\mathbf{C}^n$, then we have the formula

$$(dd^c f)^n = n! \cdot M(f) \otimes dx.$$

Second, if f is the function $y \to h(x, y)$, then because h is homogeneous of degree 1, $M(f)$ must be supported at the origin. Further, this set of direction vectors which support the graph of the function is clearly the convex hull of the set $S_x(K^*)$. Thus, if x_0 is a fixed point of $\operatorname{int}(K)$, then

$$(3.4) \qquad (dd^c h(x_0, y))^n = n! \cdot \operatorname{vol}\{\operatorname{ch} S_x(K^*)\} \cdot dx.$$

We next show the relation between the function $h = h_K$ and the extremal function L_K.

THEOREM 3.2. *Let K be a convex compact symmetric set in $\mathbf{R}^n$ with $0 \in \operatorname{int}(K)$. Then*

$$(3.5) \qquad \lim_{\varepsilon \to 0^+} \frac{L_K(x + i \cdot \varepsilon \cdot y)}{\varepsilon} = h_K(x, y)$$

uniformly on compact subsets of $\operatorname{int}(K) \oplus i\mathbf{R}^n \subset \mathbf{C}^n$. *In particular, if $x_0 \in \operatorname{int}(K)$ and $\varepsilon > 0$ is given, there exists $\delta > 0$ such that, for all $|x - x_0| \le \delta$, $|y| \le \delta$,*

$$(3.6) \qquad h_K(x_0, y) - \varepsilon \cdot |y| \le L_K(x_0 + i \cdot y) \le h_K(x_0, y) + \varepsilon \cdot |y|.$$

PROOF. Recall that the level sets of the function

$$g(\varsigma) := |\varsigma + \sqrt{\varsigma^2 - 1}|$$

are the confocal ellipses

$$\tilde{E}_\beta: \quad s^2/(1+\beta^2) + t^2/\beta^2 = 1, \qquad \varsigma = s + i\cdot t,\ \beta > 0,$$

and $g = \beta + \sqrt{1+\beta^2}$ on $\tilde{E}_\beta$. According to Lundin's formula, (1.4), choosing $\beta = \beta(z)$ such that $L_K(z) = \log[\beta + \sqrt{1+\beta^2}]$ is the same as

$$\beta + \sqrt{1+\beta^2} = \sup\{g(z\cdot\xi/\rho(\xi)) \colon |\xi| = 1\}.$$

Equivalently, $\beta = \beta(z)$ is defined by

$$\beta(z) = \inf\{\beta \colon T_z(K^*) \subset \tilde{E}_\beta\} \tag{3.7}$$

(see Figure 2). Note also that $d\beta/dg = 1$ at $g = 1$, $\beta = 0$. The limit in (3.5) is therefore equivalent to calculating

$$\lim_{\varepsilon\to 0^+} \frac{\beta(x+i\varepsilon y)}{\varepsilon}. \tag{3.8}$$

We will calculate this limit by comparing it with the function h defined in terms of the ellipses E_β in (3.2). Notice that $E_\beta \subset \tilde{E}_\beta$ for all $\beta > 0$ so that for all $z = x + iy \in C^n$,

$$h(x,y) \geq \beta(x+iy). \tag{3.9}$$

On the other hand, if $x \in \operatorname{int}(K)$ then for sufficiently small $y \in \mathbf{R}^n$, the first components, $s = x\cdot\xi$, of $T_z(K^*)$ are contained in $[-r, +r]$ for some $r < 1$. However, if $\lambda \geq 1$, then $\lambda\cdot E_\beta \cap \{(s,t)\colon |s| \leq r\} \subset E_{a\cdot\beta}$ for $a \geq [(\lambda^2 - r^2)/(1-r^2)]^{1/2}$. Further, $\tilde{E}_\beta \subset \sqrt{1+\beta^2}\cdot E_\beta$. Consequently, if $T_z(K^*) \subset \tilde{E}_\beta$, then

$$T_z(K^*) = T_z(K^*) \cap \{(s,t)\colon |s| \leq r\} \subset E_{a\cdot\beta}$$

for $a = [1 + \beta^2/(1-r^2)]^{1/2} \approx 1 + \text{const.}\cdot\beta^2$. That is,

$$\beta(x+iy) \geq h(x,y)\cdot[1 + \text{const.}\cdot\beta^2]^{-1/2}. \tag{3.10}$$

From (3.9), (3.10), and the definition (3.2) of h, it therefore follows that the limit in (3.5) is equal to $h(x,y)$. All the set inclusions used above vary continuously with x, so it is easy to see that the limit is uniform on compact subsets of $\operatorname{int}(K) \oplus i\mathbf{R}^n$. This completes the proof.

PROOF OF THEOREM 1.2. First we note that it may be argued as in the proof of Theorem 1.3 to see that λ_K puts no mass on ∂K. Now the formula for $\lambda(x)$ follows directly from formula (3.4), inequality (3.6) of Theorem 3.2, and the comparison theorem of §2. The integral formula may be seen as follows. The set $S_x(K^*)$ is given by $\{\eta = r\cdot\xi\colon 0 \leq r \leq ([\rho_K(\xi)]^2 - [x\cdot\xi]^2)^{-1/2}\}$, and so the integral formula gives the volume of this region in polar coordinates.

REMARK. We note that $\operatorname{ch} S_x(K^*)$ is the dual of the set

$$\{y \in \mathbf{R}^n\colon (y\cdot\xi)^2 + (x\cdot\xi)^2 \leq \rho_K^2(\xi) \text{ for all } \xi \in \mathbf{R}^n\}.$$

Since the total mass of λ_K is independent of K, it follows that the constant

$$c_n = \int \operatorname{vol}(\operatorname{ch} S_x(K^*))\, dx$$

is the same for all symmetric convex sets $K \subset \mathbf{R}^n$. [$c_n = (2\pi)^n/n!$.] It would be interesting to know if there is a purely geometrical proof of this fact.

4. Asymptotic behavior of λ_K. Let $K \subset \mathbf{R}^n$ be smoothly bounded, convex, and symmetric. We will discuss the asymptotic behavior of $\lambda(x)$, and, in the process, prove Theorem 1.4. By Theorem 1.2, we will need to estimate the asymptotic behavior of the volume of the convex hull of $S_x(K^*)$ as x approaches $x_0 \in \partial K$.

First we establish some geometric notation. Let η_0 be a normal vector to ∂K at x_0, normalized so that $\eta_0 \cdot x_0 = 1$. Any ellipsoid symmetric about $0 \in \mathbf{R}^n$ and tangent to ∂K at x_0 has the form

$$(\eta_0 \cdot x)^2 + R(x) \leq 1 \tag{4.1}$$

where $R(x)$ is a positive semidefinite quadratic form on $\mathbf{R}^n$ such that $R(x_0) = 0$. By abuse of notation, R may be identified with a positive semidefinite quadratic form on the hypersurface $x_0^\perp$ orthogonal to x_0.

We define the *ellipsoidal hull* of K at x_0, written $\mathrm{EH}(K, x_0)$, as the intersection of all the ellipsoids of the form (4.1) which contain K. Similarly, we define the *ellipsoidal core* of K at x_0, written $\mathrm{EC}(K, x_0)$, as the union of all of the ellipsoids of the form (4.1) which are contained in K. The ellipsoidal hull and core are related as follows, which is proved in [**4**].

PROPOSITION 4.1. *If K and K^* are both smoothly bounded, then* $\mathrm{EC}(K, x_0)^* = \mathrm{EH}(K^*, \eta_0)$.

Next, we relate the ellipsoidal hull to the mapping

$$S_{x_0}(\eta) = \eta \cdot (1 - (x_0 \cdot \eta)^2)^{-1/2}.$$

LEMMA 4.2. $\mathrm{ch}\, S_{x_0}(K^*) = S_{x_0}\{\mathrm{EH}(K^*, \eta_0)\}$.

PROOF. We note that $\xi = S_{x_0}(\eta)$ carries the degenerate ellipsoid $(\xi_0 \cdot \eta)^2 + (x_0 \cdot \eta)^2 \leq 1$ to the "strip" $|\xi_0 \cdot \xi| \leq 1$. If $\xi_0 \perp \eta_0$, then these degenerate ellipsoids are tangent to K^* at η_0, and thus the set of these ellipsoids with $\xi_0 \perp \eta_0$ generates the ellipsoidal hull. Thus,

$$S_{x_0}\{\mathrm{EH}(K^*, \eta_0)\} = \bigcap \{|\xi_0 \cdot \xi| \leq 1\}$$

where the ξ_0 are chosen so that $\xi_0 \perp \eta_0$ and the corresponding degenerate ellipsoid contains K^*. Thus, $\{|\xi_0 \cdot \xi| \leq 1\} \supset S_{x_0}(K^*)$, and so $\mathrm{ch}\, S_{x_0}(K^*) \subset S_{x_0}\{\mathrm{EH}(K^*, \eta_0)\}$.

On the other hand, $x_0 \cdot \eta_0 = 1$, and so $S_{x_0}(K^*)$ contains the line $\{t\eta_0 : t \in \mathbf{R}\}$. It follows that $\mathrm{ch}\, S_{x_0}(K^*)$ is an intersection of half-spaces of the form $\{\xi_0 \cdot \xi \leq 1\}$ with $\xi_0 \cdot \eta_0 = 0$. This proves the lemma.

LEMMA 4.3. *For $x = tx_0$, $0 < t < 1$, let us set*

$$\omega_t = (\mathrm{ch}\, S_x(K^*)) \cap (\eta_0^\perp).$$

Then ω_t increases to $S_{x_0}(\mathrm{EH}(K^*, \eta_0)) \cap (\eta_0^\perp)$ as t increases to 1.

PROOF. This follows from Lemma 4.2 since ω_t is clearly increasing in t.

Now we extend η_0 to an orthogonal set of coordinate axes, $(\eta_0, \eta_1, \ldots, \eta_{n-1})$ such that $\eta_1, \ldots, \eta_{n-1}$ are the principal curvature directions of ∂K at x_0, and we let $\kappa_1, \ldots, \kappa_{n-1}$ be the principal curvatures of ∂K at x_0. Let $(\xi_0, \ldots, \xi_{n-1})$ denote coordinates with respect to these new axes, and let $W_t : \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation given by

$$W_t(\xi) = ((1 - t^2)^{1/2} \cdot \xi_0, \xi_1, \ldots, \xi_{n-1}).$$

Let Q be the quadratic form on $\mathbf{R}^n$ which in ξ-coordinates is

$$Q(\eta) = \sum_{j=1}^{n-1} \frac{\xi_j^2}{\kappa_j}$$

and define the ellipsoid

$$E = \{\eta \in \mathbf{R}^n : \xi_0^2 + Q(\xi) \leq 1\}. \tag{4.2}$$

LEMMA 4.4.

$$\lim_{t \to 1} W_t[S_{tx_0}(K^*) \cap \{\mathbf{R}^n \setminus \eta_0^\perp\}] = E. \tag{4.3}$$

PROOF. Since $\pm\eta_0$ are the only points of ∂K^* which get mapped to infinity under S_{x_0}, it follows that the left-hand side of (4.3) is determined only by a neighborhood of $\pm\eta_0$ in ∂K^*. That is, if $\tilde{\eta} \in \partial K^*$ and $\tilde{\eta} \neq \pm\eta_0$, then $\lim_{t\to 1} W_t S_{tx_0}(\tilde{\eta}) \in \eta_0^\perp$. Thus, as $t \to 1$, we must consider smaller and smaller neighborhoods in order to find the left-hand side of (4.3).

Consider the ellipsoid which is symmetric about 0 and which is tangential to ∂K^* at $\pm\eta_0$. The normal to K^* at η_0 is x_0, so this ellipsoid has the form $(x_0 \cdot \eta)^2 + R(\eta) \leq 1$, as in (4.1). Since $\eta_0 \cdot x_0 = 1$ and the curvatures of ∂K^* at η_0 are $\kappa_1^{-1}, \dots, \kappa_{n-1}^{-1}$, it follows that this ellipsoid coincides with

$$\tilde{E} = \{\eta \in \mathbf{R}^n : (\eta \cdot x_0)^2 + Q(\eta) \leq 1\}.$$

Without loss of generality, we may assume that ∂K^* coincides with $\partial\tilde{E}$ in a small neighborhood of $\pm\eta_0$. Now it suffices to show that

$$\lim_{t\to 1} W_t S_{tx_0}(\tilde{E}) = E.$$

To see this we consider first the ellipsoid

$$\tilde{E}_c = \{\eta : (\eta \cdot x_0)^2/c^2 + Q(\eta) \leq 1\}.$$

We see that

$$S_{tx_0}^{-1}(E_c) = \{\eta : (\eta \cdot x_0)^2/c^2 + Q(\eta) \leq 1 - t^2(\eta \cdot x_0)^2\}$$

is equal to $\tilde{E}$ if and only if $c^2 = (1-t^2)^{-1}$. Thus

$$W_t[S_{tx_0}(\tilde{E})] = W_t[\tilde{E}_c]$$

with $c^2 = (1-t^2)^{-1}$, and so we have, with $\eta = \xi_0\eta_0 + \cdots + \xi_{n-1}\eta_{n-1}$,

$$W_t[S_{tx_0}(\tilde{E})] = \left\{ \eta \in \mathbf{R}^n : (1-t^2) \left[\frac{\xi_0}{\sqrt{1-t^2}} + \sum_{j=1}^{n-1} \xi_j (\eta_j \cdot x_0) \right]^2 + Q(\eta) \leq 1 \right\}.$$

Taking the limit as $t \to 1$, we obtain E.

Now we define $\omega = (\eta_0^\perp) \cap \mathrm{EH}(K^*, \eta_0)$, $\Omega = \mathrm{ch}(\omega \cup E)$ (see Figure 3), and

$$c(x_0) = n\text{-dimensional volume of } \Omega.$$

LEMMA 4.5. *With the above notation*

$$\mathrm{volume}(\mathrm{ch}\, S_{tx_0}(K^*)) \approx \frac{|\eta_0|}{\sqrt{1-t^2}} \cdot \mathrm{vol}(\Omega)$$

holds as $t \to 1$.

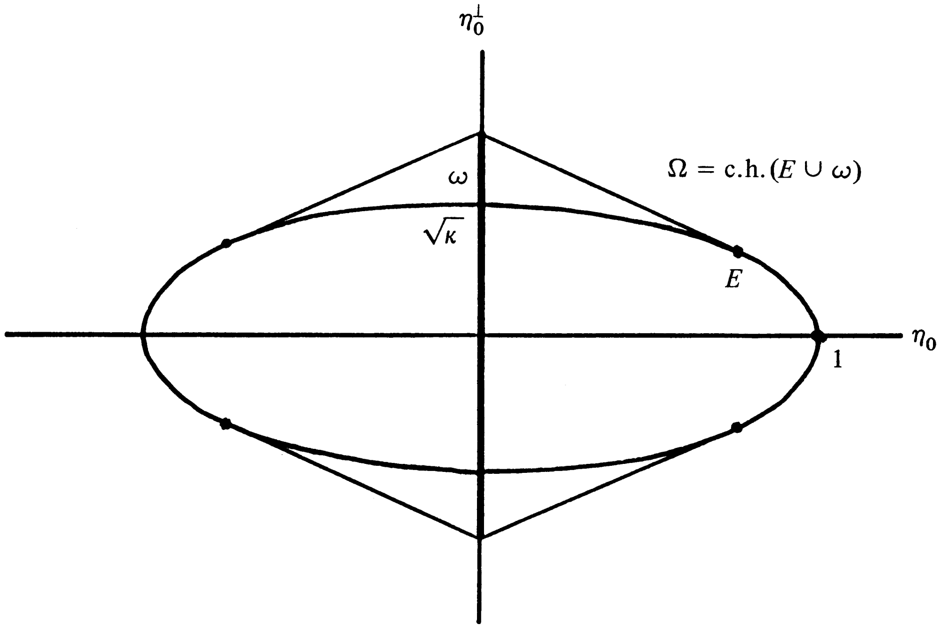


FIGURE 3

PROOF. We will show that

$$\lim_{t\to 1} W_t(\operatorname{ch} S_{tx_0}(K^*)) = \operatorname{ch}(\omega \cup E_1). \tag{4.4}$$

It follows, then, that the volume of $\operatorname{ch} S_{tx_0}(K^*)$ is $(1-t^2)^{-1/2}$ times

$$|\eta_0| \cdot \operatorname{vol}(\operatorname{ch}(\omega \cup E_1)),$$

where the factor $|\eta_0|$ enters because $|\eta_0|$ is the unit of length on the η_0-axis. Thus, the lemma follows from (4.4).

To show (4.4), we use the notation $\omega_t = (\eta_0^\perp) \cap S_{tx_0}(K^*)$. From Lemma 4.3 we have

$$W_t(\operatorname{ch} S_{tx_0}(K^*) \cap (\eta_0^\perp)) \supset \omega_t,$$

and so

$$\lim_{t\to 1} W_t(\operatorname{ch} S_{tx_0}(K^*) \cap (\eta_0^\perp)) \supset \operatorname{ch}(\omega_t \cup E).$$

Thus, $\supset$ holds in (4.4).

For the reverse inclusion, we note that for any $\varepsilon > 0$,

$$\begin{aligned} S_{tx_0}(K^*) &\subset S_{x_0}(K^* \cap \{|\eta - \eta_0| > \varepsilon\}) \cup S_{tx_0}(K^* \cap \{|\eta - \eta_0| > \varepsilon\}) \\ &= \overline{K}_1 \cup \overline{K}_2(t). \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{t\to 1} W_t(\operatorname{ch} S_{tx_0}(K^*)) &\subset \lim_{t\to 1} W_t(\operatorname{ch}(\overline{K}_1 \cup \overline{K}_2(t)) \\ &= \lim_{t\to 1} \operatorname{ch}(W_t\overline{K}_1 \cup W_t\overline{K}_2(t)) \subset \operatorname{ch}(\omega \cup E) = \Omega \end{aligned}$$

where the next-to-last inclusion follows from Lemma 4.4.

PROOF OF THEOREM 1.4. We will show that the quantity

$$c(x_0) = (\mathrm{vol}(\Omega))/\sqrt{2 \cdot \rho_K(x_0)} \tag{4.5}$$

is the asymptotic value in Theorem 1.4. First, we note that, with the notation of Theorem 1.4, $|x - \bar{x}| = \mathrm{dist}(x, \partial K)$ and for fixed $x_0 \in \partial K$

$$\mathrm{dist}(tx_0, \partial K) \approx |1-t|x_0 \cdot \frac{\eta_0}{|\eta_0|} = \frac{1-t}{|\eta_0|}$$

holds for $t \to 1$. Now by Lemma 4.5 we have, with $\Omega = \Omega_{x_0}$,

$$\begin{aligned}\lambda(tx_0) &= \mathrm{vol\,ch}\, S_{tx_0}(K^*) \approx \frac{|\eta_0|\mathrm{vol}(\Omega)}{\sqrt{1-t^2}} \approx \frac{|\eta_0|\mathrm{vol}(\Omega)}{\sqrt{2}\sqrt{1-t}} = \frac{|\eta_0|^{1/2}\mathrm{vol}(\Omega)}{\sqrt{2}(\mathrm{dist}(tx_0, \partial K))^{1/2}} \\ &= \frac{\mathrm{vol}(\Omega_{x_0})}{(2 \cdot \rho_K(x_0)|tx_0 - \bar{x}|)^{1/2}} \approx \frac{\mathrm{vol}(\Omega_{x_0})}{(2 \cdot \rho_K(\bar{x}) \cdot |tx_0 - \bar{x}|)^{1/2}}\end{aligned}$$

since η_0 was normalized so that $x_0 \cdot \eta_0 = 1$; i.e., $\rho_K(x_0) = |\eta_0|^{-1}$.

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