

## **$G$ -DEFORMATIONS AND SOME GENERALIZATIONS OF H. WEYL'S TUBE THEOREM**

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**ABSTRACT.** We prove an invariance theorem for the volumes of tubes about submanifolds in arbitrary analytic Riemannian manifolds under  $G$ -deformations of the second order. For locally symmetric spaces or two-point homogeneous spaces we give stronger invariance theorems using only  $G$ -deformations of the first order. All these results can be viewed as generalizations of the result of H. Weyl about isometric deformations and the volumes of tubes in spaces of constant curvature. They are derived from a new formula for the volume of a tube about a submanifold.

**Introduction.** In [13] H. Weyl proved his famous tube formula, which can be interpreted in the following way: Let  $M$  be a space of constant curvature, and let  $P \subset M$  be a topologically embedded submanifold with compact closure. Then the volumes of tubes about  $P$  for all sufficiently small radii  $r > 0$  remain invariant under isometric deformations of  $P$ .

The aim of this paper is to prove an analogous invariance theorem for the volumes of tubes when  $M$  is an *arbitrary* analytic Riemannian manifold and the isometric deformations of  $P \subset M$  are replaced by " $G$ -deformations of the second order" in the sense of E. Cartan.

For special Riemannian manifolds such as locally symmetric spaces or two-point homogeneous spaces, we are able to prove some stronger invariance theorems involving only  $G$ -deformations of the first order. The latter results can be interpreted as direct generalizations of H. Weyl's theorem.

The present paper is an extension of our previous article [8] in which we studied what happens with tubes under deformations of *curves*. As we shall see, our second-order deformation theorem for curves [8] can be extended directly to arbitrary submanifolds. As concerns the more special, first-order deformation theorems, our results about submanifolds are not as definitive, by far, as those for curves. Many questions for further research remain open.

In an appendix we consider the analogues of our deformation theorems for the total mean curvatures of the tubes.

**1. An explicit tube formula.** Let  $(M, g)$  be a Riemannian manifold of dimension  $n$  and let  $P$  be a  $p$ -dimensional connected submanifold topologically embedded in  $M$ . Further, we suppose that  $P$  has compact closure.

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Next, let  $P_r$  denote a tube of radius  $r$  about  $P$ ; i.e.

$$P_r = \{m \in M \mid \text{there exists a geodesic } \gamma \text{ with length } L(\gamma) = r \\ \text{from } m \text{ to } P \text{ meeting } P \text{ orthogonally}\}.$$

We always suppose that  $r$  is smaller than the distance from  $P$  to its nearest focal point.

In what follows we shall derive an integral formula for the volume  $V(P_r)$  of the hypersurface  $P_r$ . A feature of this formula is that it involves only the second fundamental form of  $P$  and some information about the geodesic spheres of  $M$  which are tangent to  $P$ . The method is based on the Jacobi vector field technique and the Sturm trick, and it is similar to that used in [12] for tubes about curves.

In §1 we recall briefly some facts about tubes. We follow closely [4], and we refer to that paper for more details. In §2 we derive the required formula.

1.1. *Preliminaries.* Let  $m \in P$  and let  $\{E_1, \dots, E_n\}$  be a local orthonormal frame field of  $M$  defined along  $P$  in a neighborhood of  $m$ . We specialize our moving frame so that  $E_1, \dots, E_p$  are tangent vector fields and  $E_{p+1}, \dots, E_n$  are normal vector fields of  $P$ . Next, let  $(x_1, \dots, x_n)$  be a system of Fermi coordinates with respect to  $m$  and  $\{E_1, \dots, E_n\}$ . These coordinates are defined in an open domain  $\mathcal{U}_m \subset M$ .

Let  $\omega$  be the Riemannian volume element in  $\mathcal{U}_m$  satisfying  $\omega(\partial/\partial x_1, \dots, \partial/\partial x_n) > 0$ .

Choose a fixed normal unit vector  $u$  at  $m$ ,  $u \in (T_m P)^\perp \subset T_m M$ , and consider the geodesic  $\gamma(t) = \exp_m(tu)$ . We have

$$\gamma(0) = m, \quad \gamma(r) = q \in P_r, \quad \gamma'(0) = u.$$

In what follows we always specialize the frame field  $\{E_1, \dots, E_n\}$  in such a way that

$$E_n(m) = u = \gamma'(0).$$

Next, define the function  $t \mapsto \theta_u(t)$  by

$$\theta_u(t) = \omega \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) (\gamma(t)) \quad \text{for } t \in [-r, r].$$

All vectors  $u$  as above form a fiber bundle over  $P$  with the unit sphere  $S^{n-p-1} \subset E^{n-p}$  as fiber. Then we have

$$(1) \quad V(P_r) = \int_P \int_{S^{n-p-1}} r^{n-p-1} \theta_u(r) du dP$$

where  $du$  denotes the volume element of  $S^{n-p-1}$ , and  $dP$  the volume element of  $P$ . Let us recall that the integral (1) exists even if  $P$  is a nonorientable manifold.

Now we shall find a more explicit form of (1) by using Jacobi vector fields. Let  $m \in P$  and  $u \in (T_m P)^\perp$  be again fixed. Consider the frame field  $\{e_1(t), \dots, e_n(t)\}$  along  $\gamma(t)$  obtained by parallel transport of  $\{E_1(m), \dots, E_n(m)\}$ . Further, let  $Y_i, Y_a, i = 1, \dots, p, a = p+1, \dots, n-1$ , denote the Jacobi vector fields along  $\gamma$  with the initial conditions

$$(2) \quad \begin{cases} Y_i(0) = E_i(m), & Y_i'(0) = \nabla_u^M \frac{\partial}{\partial x_i}, \\ Y_a(0) = 0, & Y_a'(0) = E_a(m), \end{cases}$$

where  $\nabla^M$  denotes the Riemannian connection of  $(M, g)$ . Note that

$$Y_i(t) = \left. \frac{\partial}{\partial x_i} \right|_{\gamma(t)}, \quad Y_a(t) = t \left. \frac{\partial}{\partial x_a} \right|_{\gamma(t)}.$$

Define the endomorphism-valued function  $t \mapsto D_u(t)$  by

$$Y_\alpha(t) = D_u(t)e_\alpha, \quad \alpha = 1, \dots, n-1.$$

Then the Jacobi equation implies

$$(3) \quad D_u'' + \mathcal{R} \circ D_u = 0,$$

where  $t \mapsto \mathcal{R}(t)$  is the endomorphism-valued function on  $(\gamma'(t))^\perp \subset T_{\gamma(t)}M$  defined by

$$\mathcal{R}(t)x = R_{\gamma'(t)x\gamma'(t)}, \quad x \in (\gamma'(t))^\perp.$$

$R$  denotes the Riemann curvature tensor on  $M$ . It is easy to see that

$$(4) \quad \det D_u(t) = t^{n-p-1}\theta_u(t),$$

and hence we have

$$(5) \quad V(P_r) = \int_P \int_{S^{n-p-1}} \det D_u(r) du dP.$$

To write down the initial conditions for  $D_u(t)$  (where  $u$  is fixed), we need some facts about submanifolds. Denote by  $\nabla$  the Riemannian connection of  $P$ . Further, let  $X, Y$  be tangent vector fields and  $N$  a unit normal vector field along an open domain in  $P$ . Then we have the orthogonal decompositions

$$\nabla_X^M Y = \nabla_X Y + T_X Y, \quad \nabla_X^M N = T(N)X + \nabla_X^\perp N,$$

where  $T_X Y = T(X, Y)$  is the second fundamental form operator of  $P$ ,  $T(N)$  is the shape operator of  $P$  corresponding to the normal vector  $N$ , and  $\nabla^\perp$  is the normal connection along  $P$ . Note that

$$g(T(N)X, Y) = -g(T(X, Y), N).$$

We shall also use the operator  $\perp$  defined by

$$\perp_X N = \nabla_X^\perp N.$$

(This operator is connected with the torsion operator introduced in [4].) Using (2) we obtain easily the following initial conditions (in the matrix form with respect to the basis  $\{E_1, \dots, E_{n-1}\}_m$  of  $(u)^\perp \subset T_m M$ ):

$$(6) \quad D_u(0) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad D'_u(0) = \begin{bmatrix} T(u) & 0 \\ -t \perp(u) & I \end{bmatrix},$$

where

$$T(u)_{ij} = g(T(u)E_i, E_j)(m), \\ \perp(u)_{ia} = g(\perp_{E_i} E_a, E_n)(m).$$

(Recall that  $E_n(m) = u$ .)

**1.2. The required formula.** Let  $m \in P$  be fixed. Choose a unit normal vector  $u \in (T_m P)^\perp$  and a small number  $r > 0$ . Let  $A_{ru}(t)$  be the endomorphism-valued

function along the geodesic  $\gamma(t) = \exp_m(tu)$ ,  $t \in [0, r]$ ,  $A_{ru}(t): (\gamma'(t))^\perp \subset T_{\gamma(t)}M \rightarrow (\gamma'(t))^\perp$ , which satisfies the Jacobi equation

$$A'' + \mathcal{R} \circ A = 0$$

and the initial conditions

$$(7) \quad A_{ru}(r) = 0, \quad A'_{ru}(r) = -I.$$

Let  $G_q(r)$  denote the geodesic sphere with center  $q = \exp_m(ru)$  and radius  $r$ . Then the shape operator  $S_{ru}$  of  $G_q(r)$  at  $m$  (in the direction of  $-u$ ) is given by

$$S_{ru} = -A'_{ru}(0)A_{ru}^{-1}(0)$$

(see for example [3]).

To derive our integral formula we consider the Wronskian

$$W(D_u, A_{ru}) = {}^t D'_u A_{ru} - {}^t D_u A'_{ru}$$

of the solutions  $D_u$  and  $A_{ru}$  of the Jacobi equation along  $\gamma(t)$ ,  $t \in [0, r]$ . It is well known that  $W(D_u, A_{ru})$  is constant along  $\gamma$ . Hence

$$(8) \quad {}^t D'_u(0)A_{ru}(0) - {}^t D_u(0)A'_{ru}(0) = {}^t D'_u(r)A_{ru}(r) - {}^t D_u(r)A'_{ru}(r).$$

Using the initial conditions (6) and (7), we obtain from (8) that

$$(9) \quad {}^t D_u(r) = \begin{bmatrix} T(u) & -\perp(u) \\ 0 & I \end{bmatrix} A_{ru}(0) - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} A'_{ru}(0)$$

since  $T(u)$  is symmetric. Next, with  $A'_{ru}(0) = -S_{ru}A_{ru}(0)$ , (9) implies

$$(10) \quad {}^t D_u(r) = \left\{ \begin{bmatrix} T(u) & -\perp(u) \\ 0 & I \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S_{ru} \right\} A_{ru}(0).$$

Finally, put

$$B(ru)_{ij} = g(S_{ru}E_i(m), E_j(m)), \quad i, j = 1, \dots, p.$$

Then we have

$$(11) \quad {}^t D_u(r) = \begin{bmatrix} T(u) + B(ru) & -\perp(u) \\ 0 & I \end{bmatrix} A_{ru}(0)$$

and hence

$$(12) \quad \det D_u(r) = \det(T(u) + B(ru)) \det A_{ru}(0).$$

Now, let  $\theta_q$  denote the normal volume function with respect to a normal coordinate system centered at  $q = \exp_m(ru)$ . Then

$$\theta_q(m) = r^{-(n-1)} \det A_{ru}(0),$$

and because  $\theta_q(m)$  is symmetric,  $\theta_q(m) = \theta_m(q)$  (see [1, 10, 11]), we obtain from (12) that

$$(13) \quad \det D_u(r) = \det(T(u) + B(ru)) r^{n-1} \theta_m(q).$$

We can also introduce the volume density function  $\vartheta_m$  of the exponential map  $\exp_m$  by the formula

$$\exp_m^* \omega = \vartheta_m \omega_m,$$

where  $\omega_m$  is the flat volume element of  $T_m M$  (see [12]). We have, in fact,  $\vartheta_m = \theta_m \circ \exp_m$ , and hence  $\theta_m(q) = \vartheta_m(ru)$ . From (5) and (13) we obtain

$$(14) \quad V(P_r) = r^{n-1} \int_P \int_{S^{n-p-1}} \det(T(u) + B(ru)) \vartheta_m(ru) du dP.$$

## 2. G-deformations in a Riemannian manifold.

2.1. *k-equivalence of maps.* Let  $\mathcal{U}$  be an open domain in  $\mathbf{R}^p$  containing the origin 0. Two smooth maps  $\alpha, \beta: \mathcal{U} \rightarrow M$  of  $\mathcal{U}$  into a Riemannian manifold  $M$  are said to be *k-equivalent at 0* if there is a local isometry  $D$  of  $M$  such that  $\alpha(0) \in \text{dom}(D)$  and

$$j_0^k \beta = j_0^k (D \circ \alpha).$$

Here  $j_0^k$  denotes the  $k$ -jet of (the germ of) a given map at  $0 \in \mathbf{R}^p$ .

Obviously, if  $\varphi: \mathcal{V} \rightarrow \mathcal{U}$  is a diffeomorphism such that  $\mathcal{V} \subset \mathbf{R}^p$ ,  $0 \in \mathcal{V}$ ,  $\varphi(0) = 0$ , then the maps  $\alpha, \beta$  are  $k$ -equivalent at 0 if and only if  $\alpha \circ \varphi, \beta \circ \varphi$  are  $k$ -equivalent at 0.

LEMMA 2.1. *Let  $\alpha, \beta: \mathcal{U} \rightarrow M$  be 1-equivalent maps at 0. If  $\alpha$  is an embedding near 0, then  $\beta$  is also an embedding near 0, and  $(\beta \circ \alpha^{-1})_{*\alpha(0)}$  is a linear isometry of the tangent subspace  $\alpha_*(T_0 \mathbf{R}^p) \subset T_{\alpha(0)} M$  onto  $\beta_*(T_0 \mathbf{R}^p) \subset T_{\beta(0)} M$ .*

The proof is obvious.

PROPOSITION 2.2. *Let  $\alpha, \beta: \mathcal{U} \rightarrow M$  be two embeddings such that  $j_0^2 \alpha = j_0^2 \beta$ . Then  $(\alpha_*)_0 = (\beta_*)_0$ , and for any two vector fields  $X, Y$  on  $\mathcal{U}$  we have*

$$(15) \quad \nabla_{\alpha_*(X_0)} \alpha_*(Y) = \nabla_{\beta_*(X_0)} \beta_*(Y),$$

where  $\nabla$  denotes the Riemannian connection of  $(M, g)$ .

Further, for the second fundamental form operators  $T_\alpha, T_\beta$  of the embeddings  $\alpha, \beta$  at 0 we have

$$(16) \quad T_\alpha(\alpha_* u, \alpha_* t) = T_\beta(\beta_* u, \beta_* t), \quad u, t \in \mathbf{R}^p.$$

PROOF. Let  $u^1, \dots, u^p$  be standard coordinates in  $\mathbf{R}^p$ , and let us take coordinate vector fields  $X = \partial/\partial u^i, Y = \partial/\partial u^j$ . Then, with respect to a local chart  $(x^1, \dots, x^n)$  around  $m = \alpha(0) = \beta(0)$ , we have

$$\alpha_* \left( \frac{\partial}{\partial u^i} \right) = \sum_k \frac{\partial \alpha^k}{\partial u^i} \frac{\partial}{\partial x^k}, \quad \alpha_* \left( \frac{\partial}{\partial u^j} \right) = \sum_l \frac{\partial \alpha^l}{\partial u^j} \frac{\partial}{\partial x^l}$$

and similarly for  $\beta$ . Hence  $(\alpha_*)_0 = (\beta_*)_0$  and

$$\nabla_{\alpha_*(\partial/\partial u^i)} \alpha_* \left( \frac{\partial}{\partial u^j} \right) = \sum_{k=1}^n \frac{\partial^2 \alpha^k}{\partial u^i \partial u^j} \frac{\partial}{\partial x^k} + \sum_{u,k,l=1}^n \frac{\partial \alpha^k}{\partial u^i} \frac{\partial \alpha^l}{\partial u^j} \Gamma_{kl}^u \frac{\partial}{\partial x^u},$$

where  $\Gamma_{kl}^u$  are the components of the connection  $\nabla$  with respect to the local coordinates  $x^i$ . Hence, and from a similar formula for  $\beta$ , we see that (15) holds for the coordinate vector fields. The extension for arbitrary vector fields is easy.

Formula (16) follows at once from (15) because  $T$  is the normal component of the covariant derivative  $\nabla$  with respect to the tangent subspace  $\alpha_*(T_0 \mathbf{R}^p) = \beta_*(T_0 \mathbf{R}^p)$  of  $T_m M$ .

**2.2.  $k$ -equivalence of submanifolds.** Let  $P$  be a smooth manifold of dimension  $p$ . A *parametrization* of  $P$  at a point  $m \in P$  is a diffeomorphism  $\varphi$  of a neighborhood  $\mathcal{U}$  of 0 in  $\mathbf{R}^p$  onto a neighborhood  $\mathcal{U}'$  of  $m$  in  $P$  such that  $\varphi(0) = m$ .

Now, let  $P, P'$  be two submanifolds of a Riemannian manifold  $M$ , with  $\dim P = \dim P' = p$ .  $P$  and  $P'$  are said to be *locally  $k$ -equivalent* if there is a diffeomorphism  $\phi: P \rightarrow P'$  (called a  *$k$ -deformation*) with the following property: for any point  $m \in P$ , and for some parametrization  $\varphi$  of  $P$  at  $m$ , the maps  $\varphi$  and  $\phi \circ \varphi$  are  $k$ -equivalent at 0. (Obviously, the last property then holds for any parametrization at  $m \in P$ .)

Let us remark that our concept of  $k$ -deformation is a slight modification of that of a  $k$ -deformation in a homogeneous space as treated for example by Jensen [6]. See also [8].

In the following,  $M$  will always denote a Riemannian manifold.

**PROPOSITION 2.3.** *A 1-deformation between two submanifolds  $P, P' \subset M$  is always an isometry. Further, it preserves the sectional curvature of  $M$  at all 2-planes tangent to  $P$  (and  $P'$  respectively).*

**PROOF.** Let  $\phi: P \rightarrow P'$  be a 1-deformation. Then for any  $m \in P$  and for some parametrization  $\varphi$  at  $m$ ,  $\varphi$  and  $\phi \circ \varphi$  are 1-equivalent. According to Lemma 2.1, this means that  $\phi_{*m}$  is a linear isometry of  $T_m P$  onto  $T_{\phi(m)} P'$ .

The second part is obvious from the definition.

The converse of the first part of Proposition 2.3 is only valid in a special situation.

**PROPOSITION 2.4.** *In a space of constant curvature, two isometric submanifolds  $P, P' \subset M$  are locally 1-equivalent.*

**PROOF.** If  $M$  has constant curvature and  $\phi: P \rightarrow P'$  is an isometry, then any orthonormal tangent frame  $\{e_1, \dots, e_p\}$  of  $P$  at  $m \in P$  can be transformed into the tangent frame  $\{\phi_* e_1, \dots, \phi_* e_p\}$  of  $P'$  at  $\phi(m)$  using a local isometry of  $M$  around  $m$ . On the other hand, we can take a parametrization  $\varphi: \mathcal{U} \rightarrow P$  at  $m$  such that  $\varphi_*(\partial/\partial u^i)_0 = e_i$ ,  $i = 1, \dots, p$ .

**REMARK 2.5.** In the other symmetric spaces of rank one our conclusion is not necessarily true. For instance, in the complex projective space  $\mathbf{C}P^n$ ,  $n \geq 2$ , imagine two isometric real surfaces  $P$  and  $P'$  such that  $P$  is a holomorphic curve and  $P'$  is not a holomorphic curve. Then  $P$  and  $P'$  cannot be locally 1-equivalent as real surfaces in  $\mathbf{C}P^n$  because of the second part of Proposition 2.3.

Now, on the basis of Propositions 2.3 and 2.4, we can state Weyl's result from the introduction in a completely *equivalent* form which is more convenient for our generalizations in the next section.

**THEOREM 2.6 (COROLLARY OF THE TUBE FORMULA OF H. WEYL).** *Let  $M$  be a space of constant curvature, and  $P, P'$  two locally 1-equivalent submanifolds with compact closure. Then  $V(P_r) = V(P'_r)$  holds for each sufficiently small radius  $r > 0$ .*

We shall complete this section with two technical results. We only prove the second since a proof of the first result can easily be given.

**LEMMA 2.7.** *Let  $P \subset M$  be a submanifold with compact closure. Then there is a number  $r > 0$  such that, for each  $m \in P$ , there is a geodesic ball with center  $m$  and radius  $r$ , which is a normal neighborhood of all its points.*

Now, we have

**PROPOSITION 2.8.** *Let  $P, P' \subset M$  be two submanifolds with compact closure in an analytic Riemannian manifold  $M$ . Let  $\phi: P \rightarrow P'$  be a diffeomorphism. Then there is a number  $r_0 > 0$  with the following property: whenever  $m \in P$  and  $D_m$  is a local isometry in  $M$  such that  $D_m(m) = \phi(m)$ , then  $D_m$  can be extended (in a unique way) to a local isometry  $\tilde{D}_m$  defined on the open ball  $B_m(r_0)$ .*

**PROOF.** Let  $r_1, r_2 > 0$  be numbers corresponding to  $P, P'$  in Lemma 2.7 and put  $r_0 = \min\{r_1, r_2\}$ . Let  $\mathcal{U}_m \subset B_m(r_0)$  be a connected neighborhood of  $m \in P$  such that a local isometry  $D_m$  is defined on  $\mathcal{U}_m$ . Using the constructions of [7, I, Chapter VI, §6], one can construct a unique analytic continuation  $\tilde{D}_m$  of  $D_m$  from  $\mathcal{U}_m$  to  $B_m(r_0)$ , which is the required local isometry.

**3. Invariance theorems for volumes of tubes under deformations.** In what follows we only consider the volumes of the tubes as hypersurfaces, but all our results are also true for the volumes of the corresponding *solid* tubes (see [4, 8]).

**3.1. Submanifolds of general Riemannian manifolds.** We start with our main theorem.

**THEOREM 3.1.** *Let  $(M, g)$  be an analytic Riemannian manifold, and  $P, P'$  two locally 2-equivalent submanifolds with compact closure. Then  $V(P_r) = V(P'_r)$  holds for each sufficiently small radius  $r > 0$ .*

**PROOF.** We need the following obvious

**LEMMA 3.2.** *Let  $P \subset M$  be a submanifold and  $D$  a local isometry of  $M$  defined in a neighborhood  $\mathcal{U}$  of a point  $m \in P$ . Then, for any unit normal vector  $u \in (T_m P)^\perp$  and any sufficiently small  $r > 0$  we have*

$$(17) \quad \begin{aligned} \vartheta_{m^*}(ru^*) &= \vartheta_m(ru), \\ \det(T^*(u^*) + B^*(ru^*)) &= \det(T(u) + B(ru)), \end{aligned}$$

where the asterisks mark the corresponding quantities at  $m^* = D(m)$  of the submanifold  $P^* = D(P \cap \mathcal{U})$ .

Now, let  $\phi: P \rightarrow P'$  be a 2-deformation, and let  $r_0 > 0$  be the corresponding number from Proposition 2.8. Choose  $r > 0$  such that  $r < r_0$ . According to formula (14) we have

$$V(P_r) = r^{n-1} \int_P V_r(m) dP, \quad V(P'_r) = r^{n-1} \int_{P'} V'_r(m') dP',$$

where

$$\begin{aligned} V_r(m) &= \int_{S^{n-p-1}} \det(T(u) + B(ru))(m) \vartheta_m(ru) du, \\ V'_r(m') &= \int_{S'^{n-p-1}} \det(T'(u') + B'(ru'))(m') \vartheta_{m'}(ru') du', \end{aligned}$$

and  $S^{n-p-1} \subset (T_m P)^\perp$ ,  $S'^{n-p-1} \subset (T_{m'} P')^\perp$  are unit spheres. All we need to prove is that

$$(18) \quad V'_r(\phi(m)) = V_r(m), \quad m \in P,$$

because  $\phi$  is an isometry of  $P$  onto  $P'$ , hence  $dP = \phi^*dP'$ , and the equality  $V(P'_r) = V(P_r)$  then follows by a substitution in the integral.

Choose a fixed  $m \in P$ ; then we have a local isometry  $D_m: B_m(r) \rightarrow B_{\phi(m)}(r)$  in  $M$  such that, for some parametrization  $\varphi$  of  $P$  at  $m$ ,

$$j_0^2(D_m \circ \varphi) = j_0^2(\phi \circ \varphi).$$

Because of Proposition 2.2 we have  $\phi(m) = D_m(m)$ ,  $(D_m)_*m = (\phi_*)_m$  on  $T_mP$ , and for  $X', Y' \in T_{\phi(m)}P'$  we get

$$(19) \quad T^*(X', Y') = T'(X', Y'),$$

where  $T^*$ , or  $T'$ , respectively, denotes the second fundamental form operator at  $\phi(m)$  of the submanifold  $P^* = D_m(P \cap B_m(r))$ , or of  $P'$ , respectively.

For  $u \in S^{n-p-1} \subset (T_mP)^\perp$  we now have

$$D_{m*}(u) = u' \in S'^{n-p-1} \subset (T_{\phi(m)}P')^\perp,$$

and with respect to any orthonormal basis  $\{e'_1, \dots, e'_p\}$  of  $T_{\phi(m)}P'$ , (19) implies

$$(20) \quad T'(u')_{ij} = T^*(u')_{ij}.$$

Obviously, for  $r < r_0$ , the spherical shape operators  $S_{ru}$ ,  $S'_{ru}$ ,  $S^*_{ru}$ , make sense, and  $S'_{ru} = S^*_{ru}$ , because  $T_{\phi(m)}P' = T_{\phi(m)}P^*$ . Hence, with respect to  $\{e'_1, \dots, e'_p\}$ , we get

$$(21) \quad B'(ru')_{ij} = B^*(ru')_{ij}.$$

We obtain from (20) and (21) that

$$(22) \quad \det(T'(u') + B'(ru')) = \det(T^*(u') + B^*(ru')).$$

Now using Lemma 3.2 we see that

$$\det(T'(u') + B'(ru'))(\phi(m))\vartheta_{\phi(m)}(ru') = \det(T(u) + B(ru))(m)\vartheta_m(ru).$$

Because  $u \mapsto u' = D_{m*}(u)$  is an isometry of the sphere  $S^{n-p-1}$  onto  $S'^{n-p-1}$ , we get  $du = D_m^*du'$ , and using this as a substitution in the integral we obtain (18).

REMARK. We shall now explain intuitively the role of the analyticity in Theorem 3.1. Consider a bounded region of the Euclidean plane with a closed unit disk inside and perturb smoothly the Euclidean metric *outside* this unit disk. The corresponding  $C^\infty$ -manifold will be denoted by  $M$ . (A good model for  $M$  may be a wine glass in  $E^3$  bending smoothly from the bottom, which is a flat unit disk.)

Consider two *open* line segments  $P$ ,  $P'$  of the same length  $l(P) = l(P') < 2$  inside the disk such that the endpoints of  $P$  are both contained inside the disk, whereas both endpoints of  $P'$  lie on the boundary circle. Then  $P$  and  $P'$  are locally 2-equivalent submanifolds of  $M$  (open Euclidean segments), and the tubes  $P_r$ ,  $P'_r$  exist for sufficiently small  $r > 0$  (each “tube” is a couple of curves). But it may happen, if our perturbation is general enough, that  $l(P_r) \neq l(P'_r)$  for any  $r > 0$ .

Hence Theorem 3.1 may not hold if the manifold  $M$  is only  $C^\infty$ . Adding elementary topological arguments one can see that the  $C^\infty$ -version of Theorem 3.1 will still be true for *compact* submanifolds (with or without boundaries). But the analytic version also has the advantage that we are able to “construct” the number  $r_0 > 0$  such that  $V(P_r) = V(P'_r)$  for all  $r < r_0$  (see Lemma 2.7 and Proposition 2.8).



### 3.2. Submanifolds of special Riemannian manifolds.

**THEOREM 3.3.** *Let  $(M, g)$  be a locally symmetric Riemannian space, and let  $P, P' \subset M$  be two locally 1-equivalent hypersurfaces with compact closure. Then  $V(P_r) = V(P'_r)$  holds for any sufficiently small radius  $r > 0$ .*

**PROOF.** Recall first that  $M$  is analytic (see [5]).

For  $r < r_0$  we have

$$V(P_r) = r^{n-1} \int_P V_r(m) dP,$$

where, for any fixed  $m \in P$ ,

$$(23) \quad V_r(m) = \sum_{\xi \in S^0} \det(T(\xi) + B(r\xi))(m) \vartheta_m(r\xi), \quad S^0 \subset (T_m P)^\perp.$$

Put  $S^0 = \{\xi, -\xi\}$  and choose an orthonormal basis  $\{e_1, \dots, e_{n-1}\}_m$  in  $T_m P$ . Due to the local symmetry with center  $m$  we have

$$\begin{aligned} \vartheta_m(-r\xi) &= \vartheta_m(r\xi), \\ B(-r\xi)_{ij} &= g(S_{-r\xi} e_i, e_j) = g(S_{-r\xi}(-e_i), -e_j) \\ &= g(S_{r\xi} e_i, e_j) = B(r\xi)_{ij}, \end{aligned}$$

and obviously  $T(-\xi)_{ij} = -T(\xi)_{ij}$ . Hence we get

$$(24) \quad V_m(r) = \frac{1}{2} \sum_{\xi \in S^0} [\det(T_{ij}(\xi) + B_{ij}(r\xi)) + \det(-T_{ij}(\xi) + B_{ij}(r\xi))](m) \vartheta_m(r\xi).$$

From elementary properties of determinants we obtain, by induction, that  $V_r(m)$  depends on the variables  $T_{ij}(\xi)$  only through the subdeterminants  $(T_{ik}T_{jl} - T_{il}T_{jk})(\xi)$  of the second order. Using the Gauss equations we see that  $V_r(m)$  depends only on the components  $R_{ijkl}^M, R_{ijkl}^P$  of the curvature tensors of  $M$  and  $P$ , respectively, on the components  $B_{ij}(r\xi)$  of the shape operator  $S_{r\xi}$  at  $m$ , and on the volume density function  $\vartheta_m(r\xi)$ . But, under a 1-deformation  $\phi: P \rightarrow P'$ , all the last quantities are mapped in the corresponding quantities of the hypersurface  $P'$  at the point  $\phi(m)$  with respect to the basis  $\{e'_1, \dots, e'_{n-1}\}_{\phi(m)}$ , where  $e'_i = \phi_* m(e_i)$ .

Hence  $V_r(m) = V'_r(\phi(m))$ , and our theorem follows.

**REMARK 3.4.** We see that our Theorem 3.3 is a direct generalization of “Weyl’s theorem” 2.6 for *hypersurfaces*.

For the “dual” case of curves, we proved in [8] a stronger result. Here the analogue of Theorem 3.3 holds if we replace the class of locally symmetric Riemannian spaces by the bigger class of all analytic Riemannian manifolds with *volume-preserving* local geodesic symmetries.

We now look at the two-point homogeneous spaces. As for curves and hypersurfaces, we have here just a special case of Theorem 3.3 and of its analogue for curves. Moreover, for the curves we see that in these spaces local 1-equivalence has the same meaning as isometry and hence we get an earlier result of A. Gray and the second author, saying that the volume of a tube about a curve in a two-point homogeneous space only depends on the length of the curve and the radius of the tube (see [4, 8]).

The next case to consider is two-dimensional surfaces. In this case we also have a positive answer.

**THEOREM 3.5.** *Let  $M$  be a two-point homogeneous space and  $P, P' \subset M$  locally 1-equivalent surfaces (with compact closure) in  $M$ . Then  $V(P_r) = V(P'_r)$  holds for any sufficiently small radius  $r > 0$ .*

**PROOF.** We again have

$$V(P_r) = r^{n-1} \int_P V_r(m) dP,$$

where now

$$(25) \quad V_r(m) = \int_{S^{n-3}} \det(T(u) + B(ru))(m) \vartheta_m(ru) du.$$

According to [9], the shape operator  $S_{ru}$  at  $m$  of the geodesic sphere  $G_q(r)$ ,  $q = \exp_m(ru)$ , in a two-point homogeneous space can be expressed in the form

$$(26) \quad g(S_{ru}X, Y) = a(r)g(X, Y) + b(r)R_{uXuY}^M, \quad X, Y \in T_m G_q(r),$$

where  $a(r)$ ,  $b(r)$  depend only on the space  $M$  as a whole and on the radius  $r$ .

Further,  $M$  being a harmonic space (see [1]), the volume density function  $\vartheta_m(ru)$  for  $u \in S_m M \subset T_m M$  depends only on  $r$ ; i.e.  $\vartheta_m(ru) = \vartheta(r)$ . Choosing now an orthonormal basis  $\{e_1, e_2\}$  of  $T_m P$ , we get for any  $u \in (T_m P)^\perp$  that

$$\det(T_{ij}(u) + B_{ij}(ru)) = C_{11}(u)C_{22}(u) - (C_{12}(u))^2,$$

where

$$(27) \quad C_{ij}(u) = T_{ij}(u) + a(r)\delta_{ij} + b(r)R_{uiuj}^M, \quad u \in (T_m P)^\perp.$$

Moreover, (25) takes the form

$$(28) \quad V_r(m) = \vartheta(r) \int_{S^{n-3}} [C_{11}(u)C_{22}(u) - (C_{12}(u))^2](m) du.$$

Here the  $T_{ij}(u)$  are linear functions of  $u$  and the  $R_{uiuj}^M$  are quadratic functions of  $u$ . If we substitute (27) in (28), and abandon all terms which are odd functions of  $u$  (their integrals vanish), we obtain

$$V_r(m) = \vartheta(r) \left\{ \int_{S^{n-3}} [T_{11}(u)T_{22}(u) - (T_{12}(u))^2](m) du + \int_{S^{n-3}} [a(r)^2 + a(r)b(r) \sum_i R_{uiui}^M + b(r)^2 \det(R_{iuj}^M)](m) du \right\}.$$

The second integral is obviously preserved under a 1-deformation of  $P$  (see Proposition 2.3), and the first integral depends only on the Gauss curvature of  $P$  at  $m$  and the sectional curvature of  $M$  restricted to  $P$ , as follows from Weyl's invariant theory [13] (see also [4, §7]). Thus  $V_r(m)$  is preserved under a 1-deformation of  $P$ . Hence our theorem follows.

**REMARK 3.6.** The property of local 1-equivalence in Theorem 3.5 cannot be replaced by the weaker property of isometry. As shown in [4], if  $P, P'$  are two isometric surfaces in  $\mathbb{C}P^n$  such that  $P$  is a holomorphic curve and  $P'$  is not a holomorphic curve (cf. Remark 2.5), then we always have  $V(P_r) \neq V(P'_r)$ .

The problem seems to be much more difficult for higher-dimensional submanifolds in this general situation. But we can still get some more results if we restrict ourselves to submanifolds of a special geometrical nature.

Let  $M$  again be a two-point homogeneous space. A tangent element  $L \subset T_m M$ ,  $m \in M$ , is said to be *umbilical* if the following holds: for any geodesic sphere  $S_q(r)$  tangent to  $L$ , the shape operator of  $S_q(r)$  at  $m$  restricted to  $L$  is a multiple of the identity operator.

A submanifold  $P \subset M$  is said to be of the *umbilical type* if all tangent spaces  $T_m P$ ,  $m \in P$ , are umbilical.

In  $E^n$ ,  $S^n$  and  $H^n$  all submanifolds are of the umbilical type. According to [9] (or [2]) we see easily that a tangent element of  $M = \mathbf{C}P^n$ ,  $\mathbf{H}P^n$ , or  $\text{Cay } P^2$  is umbilical if and only if it is preserved by the system of corresponding complex structures of the space  $M$ . (We have one complex structure on  $\mathbf{C}P^n$ , three basic local almost complex structures on  $\mathbf{H}P^n$  and seven local almost complex structures on  $\text{Cay } P^2$ .) Hence a submanifold of the umbilical type in  $\mathbf{C}P^n$ , or  $\mathbf{H}P^n$ , or  $\text{Cay } P^2$ , respectively, is the same as a complex submanifold or a quaternionic submanifold or a "Cayley submanifold" (of dimension 8), respectively.

The same holds for the noncompact duals of the last spaces. Now, we have

**THEOREM 3.7.** *Let  $P$ ,  $P'$  be two submanifolds of the umbilical type (and with compact closure) in a two-point homogeneous space. If  $P$ ,  $P'$  are locally 1-equivalent, then  $V(P_r) = V(P'_r)$  holds for any sufficiently small radius  $r > 0$ .*

**PROOF.** We see easily that the integrand of the tube formula for  $V(P_r)$  is equal to

$$V_r(m) = \vartheta(r) \int_{S^{n-p-1}} \det(T_{ij}(u) + \lambda \delta_{ij}) du,$$

where  $\lambda$  is a constant depending only on the space  $M$  and on  $r$ . Now, one can prove that  $V_r(m)$  depends only on the Riemann curvature tensor of  $P$  and on the Riemann curvature tensor of  $M$  restricted to  $T_m P$ . This follows exactly in the same way as in the original paper of Weyl [13], i.e., by using the Gauss equation for the submanifold  $P \subset M$  and Weyl's invariant theory. (See also [4, §7].) Hence  $V_r(m)$  is preserved under a 1-deformation.

**REMARK 3.8.** The last theorem again generalizes Weyl's Theorem 2.6, and we can also make from it the same conclusion about holomorphically isometric Kähler submanifolds in  $\mathbf{C}P^n$ , which follows from the explicit formula (7.10) in [4].

**Appendix. Total mean curvatures of tubes.** In [4] several theorems are given for the *integrated mean curvatures* of a tube about a submanifold. Here we shall derive a formula for the *total mean curvature* (the first integrated mean curvature) of a tube, and we derive from it a new result for 2-equivalent submanifolds.

Let  $q = \exp_m(ru)$  be a point of the tube  $P_r$  about the  $p$ -dimensional submanifold  $P$  and denote by  $h_{P_r}(q)$  the mean curvature of  $P_r$  at  $q$ . Then we have (see [4] and formula (4))

$$(29) \quad h_{P_r}(q) = \text{trace } D'_u(r) D_u^{-1}(r) = \frac{n-p-1}{r} + \frac{\theta'_u(r)}{\theta_u(r)}.$$

Now we again put  $C(u) = T(u) + B(ru)$ . From (4) and (13) we obtain

$$\theta_u(r) = r^p \theta_m(q) \det C(u),$$

and, hence, differentiating with respect to  $r$ ,

$$(30) \quad \frac{\theta'_u(r)}{\theta_u(r)} = \frac{p}{r} + \frac{\theta'_m(q)}{\theta_m(q)} + \frac{(\det C(u))'}{\det C(u)}.$$

Next, putting  $p = 0$  in (29) we obtain the well-known formula for the mean curvature  $h_m(q)$  of the geodesic sphere  $G_m(r)$  at  $q$  (see for example [1]):

$$(31) \quad h_m(q) = \frac{n-1}{r} + \frac{\theta'_m(q)}{\theta_m(q)}.$$

Combining (29), (30) and (31) we have

$$(32) \quad h_{P_r}(q) = h_m(q) + \frac{(\det C(u))'}{\det C(u)}.$$

PROPOSITION A.1. *Let  $(M, g)$  be an analytic Riemannian manifold and  $P, P'$  two locally 2-equivalent submanifolds with compact closure. Then the total mean curvatures of  $P_r$  and  $P'_r$  are the same for all sufficiently small  $r > 0$ .*

PROOF. The total mean curvature  $H_{P_r}$  of  $P_r$  is given by

$$H_{P_r} = \int_{P_r} h_{P_r}(q) dq.$$

As a generalization of (5) we obtain

$$H_{P_r} = \int_P \int_{S^{n-p-1}} h_{P_r}(q) \det D_u(r) du dP$$

or, with (13),

$$\begin{aligned} H_{P_r} &= r^{n-1} \int_P \int_{S^{n-p-1}} h_{P_r}(q) \theta_m(q) \det C(u) du dP \\ &= r^{n-1} \int_P \int_{S^{n-p-1}} h_{P_r}(q) \vartheta_m(ru) \det C(u) du dP. \end{aligned}$$

Finally, with (32), we have

$$H_{P_r} = r^{n-1} \int_P \int_{S^{n-p-1}} \{h_m(q) \det C(u) + (\det C(u))'\} \vartheta_m(ru) du dP.$$

The remaining part of the proof is similar to the proof of Theorem 3.1.

REMARK A.2. We note that analogues of Theorems 3.3, 3.5 and 3.7 can be proved for total mean curvature. (The analogue of Theorem 3.7 is essentially contained in [4].)

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