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APPROXIMATION ORDER FROM CERTAIN SPACES OF SMOOTH BIVARIATE SPLINES ON A THREE-DIRECTION MESH[1]

RONG-QING JIA

ABSTRACT. Let Δ be the mesh in the plane obtained from a uniform square mesh by drawing in the north-east diagonal in each square. Let $\pi^{\rho}_{k,\Delta}$ be the space of bivariate piecewise polynomial functions in C^{ρ}, of total degree $\leq k$, on the mesh Δ. Let $m(k,\rho)$ denote the approximation order of $\pi^{\rho}_{k,\Delta}$. In this paper, an upper bound for $m(k,\rho)$ is given. In the space $3 \leq 2k-3\rho \leq 7$, the exact values of $m(k,\rho)$ are obtained:

$$m(k,\rho) = 2k-2\rho-1 \quad \text{for } 2k-3\rho = 3 \text{ or } 4,$$
$$m(k,\rho) = 2k-2\rho-2 \quad \text{for } 2k-3\rho = 5, 6 \text{ or } 7.$$

In particular, this result answers negatively a conjecture of de Boor and Höllig.

1. Introduction. In this paper we study the approximation order from certain spaces of smooth bivariate splines on a three-dimension mesh. The work in this respect has been initiated by de Boor and DeVore [**BD**], and de Boor and Höllig [$\mathbf{BH_1}$–$\mathbf{BH_3}$]. Here we follow them and introduce some notation. Let

$$\Delta := \bigcup_{n\in Z} \{(x_1,x_2) \in R^2; x_1 = n,\ x_2 = n \text{ or } x_2 - x_1 = n\}.$$

Namely, the mesh Δ is obtained from a uniform square mesh by drawing in the north-east diagonal in each square. Let

$$S := \pi^{\rho}_{k,\Delta} := \pi_{k,\Delta} \cap C^{\rho}$$

be the space of bivariate pp (piecewise polynomial) functions in C^{ρ}, of total degree $\leq k$, on the mesh Δ. The approximation order of S is, by definition, the integer m for which the following holds: For any $f \in C^m$,

$$\operatorname{dist}(f, S_h) \leq (\text{const})h^m |f|_{m,\infty},$$

while, for some C^{m+1}-function f with $\|f\|_{m+1,\infty} < \infty$,

$$\operatorname{dist}(f, S_h) \neq o(h^m).$$

Here the scale (S_h) of approximating spaces is generated from S by simple scaling:

$$S_h := \sigma_h(S)$$

with

$$(\sigma_h f)(x) := f(x/h), \quad \text{all } f, x, h.$$

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Further,

$$\operatorname{dist}(f, S) := \inf_{s \in S} \|f - s\|,$$

and $\|\cdot\|$ is the sup norm on $\mathbf{R}^2$:

$$\|f\| := \|f\|_\infty := \sup_{x \in R^2} |f(x)|.$$

Moreover,

$$|f|_{m,\infty} := \sum_{|\alpha|=m} \|D^\alpha f\|, \quad \|f\|_{m,\infty} := \sum_{|\alpha|\le m} \|D^\alpha f\|.$$

We denote by $m(k,\rho)$ the approximation order of $\pi^\rho_{k,\Delta}$.

Only a few results about $m(k,\rho)$ are known:

$$m(k,\rho) = \begin{cases} k+1 & \text{for } k \ge 4\rho + 1 \text{ (see [}\mathbf{BZ}\text{])}, \\ 0 & \text{for } 2k - 3\rho \le 1 \text{ (see [}\mathbf{BD}\text{])}, \\ 2k - 2\rho & \text{for } 2k - 3\rho = 2 \text{ (see [}\mathbf{BH_1}\text{])}, \end{cases}$$

$$m(3,1) = 3 \quad \text{(see [}\mathbf{BH_2}\text{])}.$$

An upper bound for $m(k,\rho)$ has been obtained by de Boor and Höllig (see [**BH**$_3$, Theorem 3]):

$$m(k,\rho) \le \min\{2(k-\rho), k+1\}.$$

Also, they raised the following

CONJECTURE. $m(k,\rho) \ge \min\{2(k-\rho), k+1\} - 1$.

By using a quasi-interpolant scheme, [**J**] gives

$$m(k,\rho) \ge \min\{2(k-\rho), k+1\} - 2. \tag{1}$$

A question naturally arises: Can the lower bound given by (1) be improved? This paper shows that this lower bound is sharp. More precisely, we will prove the following results:

$$m(k,\rho) = 2k - 2\rho - 1 \quad \text{for } 2k - 3\rho = 3 \text{ or } 4. \tag{2}$$

$$m(k,\rho) = 2k - 2\rho - 2 \quad \text{for } 2k - 3\rho = 5, 6 \text{ or } 7. \tag{3}$$

In particular,

$$m(k,\rho) = \min\{2(k-\rho), k+1\} - 2 \text{ for } 2k - 3\rho = 5, 6 \text{ or } 7 \text{ and } k \le 2\rho + 1.$$

This answers negatively the conjecture of de Boor and Höllig.

Here is an outline of this paper. §§2–4 treat the algebra generated by the shift operators, the box splines and the jump operators, respectively. Those three sections are tools, and they prepare for the core of this paper, §5, which reduces the approximation problem to a determinant problem and hence gives an upper bound for the approximation order. In §6, the result of §5 is applied to obtain (2) and (3).

Before proceeding with the proofs of (2) and (3), we need to introduce more notation. For a set E, we denote by $|E|$ the cardinality of E. Let $\mathbf{Z}_+$ be the set of nonnegative integers. Let e_1, e_2 be the unit coordinate vectors in the plane; i.e.,

$$e_1 = (1,0) \quad e_2 = (0,1).$$

As usual, D_i denotes the derivative with respect to the ith argument ($i = 1, 2$). Let

$$e_3 = e_1 + e_2 \quad \text{and} \quad D_3 = D_1 + D_2.$$

For a bivariate function $f\colon \mathbf{R}^2 \to \mathbf{R}$ and a real number a, the difference operators $\nabla_{r,a}$ are given by

$$\nabla_{r,a} f := f - f(\cdot - a e_r) \qquad (r = 1, 2, 3).$$

If $a = 1$, $\nabla_{r,a}$ is abbreviated to ∇_r. Let π be the space of all bivariate polynomials, π_k the space of all bivariate polynomials of total degree $\leq k$. For a polynomial p, its degree is denoted by $\deg p$.

2. The algebra generated by the shift operators. A mapping from $\mathbf{Z}^2$ to $\mathbf{R}$ is called a bivariate sequence. The set of all the bivariate sequences equipped with addition and scalar multiplication forms a linear space, which we denote by $l(\mathbf{Z}^2)$. All the linear operators on $l(\mathbf{Z}^2)$ form a *noncommutative* algebra. We want to consider one of its subalgebras. Let T_r be the shift operators given by

$$T_r g := g(\cdot - e_r), \quad \text{all } g \in l(\mathbf{Z}^2) \ (r = 1, 2, 3).$$

Clearly

$$T_3 = T_2 T_1 = T_1 T_2.$$

Let A be the subalgebra generated by T_1, T_1^{-1}, T_2 and T_2^{-1}. Then A is *commutative.* Let I be the identity operator on $l(\mathbf{Z}^2)$. Then the difference operators can be represented as

$$\nabla_r = I - T_r \qquad (r = 1, 2, 3).$$

In particular,

$$\begin{aligned} \nabla_3 &= I - T_3 = I - T_1 T_2 \\ &= I - (I - \nabla_1)(I - \nabla_2) = \nabla_1 + \nabla_2 - \nabla_1 \nabla_2. \end{aligned}$$

If we impose a sup norm on $l(\mathbf{Z}^2)$, then we get the normed linear space $l_\infty(\mathbf{Z}^2)$. Thus we can talk about the norm of an operator on $l_\infty(\mathbf{Z}^2)$ in the usual sense. Moreover, we can talk about positive operators. A sequence $g \in l(\mathbf{Z}^2)$ is called positive, and denoted by $g \geq 0$, if

$$g(j) \geq 0 \quad \text{for any } j \in \mathbf{Z}^2.$$

An operator L is called positive if

$$Lg \geq 0 \quad \text{whenever } g \geq 0.$$

A sequence $g \in l(\mathbf{Z}^2)$ is called constant if there exists a real number b such that

$$g(j) = b \quad \text{for any } j \in \mathbf{Z}^2.$$

We denote by $\mathbf{1}$ the constant sequence which takes value 1. If L is a positive operator on $l(\mathbf{Z}^2)$, then we have $\|L\| = \|L\mathbf{1}\|$. Indeed, since $-\|g\|\mathbf{1} \leq g \leq \|g\|\mathbf{1}$, we have

$$-\|g\|(L\mathbf{1}) \leq Lg \leq \|g\|(L\mathbf{1}),$$

and so

$$\|Lg\| \leq \|g\| \, \|L\mathbf{1}\|.$$

This shows that $\|L\| \leq \|L\mathbf{1}\|$. The other direction holds because $\|\mathbf{1}\| = 1$. To give an example of positive operators, we consider

$$H_r := \sum_{t=0}^{N-1} T_r^t \qquad (r = 1, 2, 3), \tag{4}$$

where N is a positive integer. Then $H_r\mathbf{1} = N$; hence

$$\|H_r\| = N \quad \text{for } r = 1, 2, 3.$$

3. Box splines. Box splines were introduced in [**BD** and **BH$_1$**] and have proved useful in approximation problems. The key point is that the approximation order of S is determined by all the box splines contained in S (see [**BH$_3$**]). Here we specify the definition of box splines from [**BH$_1$**] to suit our discussion. For $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbf{Z}_+^3$ with $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3$ as usual, let $\Xi = (\xi_i)_1^{|\lambda|}$ be the sequence given by

$$\xi_1 = \cdots = \xi_{\lambda_1} = e_1, \quad \xi_{\lambda_1+1} = \cdots = \xi_{\lambda_1+\lambda_2} = e_2, \quad \xi_{\lambda_1+\lambda_2+1} = \cdots = \xi_{|\lambda|} = e_3.$$

Then the box spline $M_\lambda := M_\Xi$ is defined as the distribution given by the rule

$$M_\Xi \colon \phi \to \int \phi\left(\sum_{i=1}^{|\lambda|} u(i)\xi_i\right)\, du,$$

where the integral is taken over the cube $[0,1]^{|\lambda|}$. Let

$$d := |\lambda| - \max\{\lambda_r\} - 1.$$

Then $M_\lambda \subset L_\infty^{(d)} \cap C^{(d-1)}$. In addition, d is the largest integer such that this relation is true. In particular, a box spline M_λ belongs to L_∞ if and only if $|\lambda| - \max\{\lambda_r\} \geq 1$. In what follows, all the box splines are assumed to be in L_∞.

A box spline series has the following nice property with respect to derivatives:

$$D_i\left(\sum_{j\in Z^2} a(j) M_\Xi(\cdot - j)\right) = \sum_{j\in Z^2} (\nabla_i a)(j) M_{\Xi\setminus e_i}(\cdot - j), \quad \text{if } e_i \in \Xi.$$

Let $S_\lambda :=$ the linear span of $M_\lambda(\cdot - j)$, $j \in Z^2$; that is,

$$S_\lambda := \left\{\sum a(j) M_\lambda(\cdot - j);\, a \in l(Z^2)\right\}.$$

LEMMA 1. *The following inclusion relations hold:*

(1°) $\quad S_{\lambda_1,\lambda_2,\lambda_3} \subset S_{\lambda_1+1,\lambda_2,\lambda_3-1} + S_{\lambda_1+1,\lambda_2-1,\lambda_3}$ *if* $\min\{\lambda_2, \lambda_3\} \geq 1$;

(2°) $\quad S_{\lambda_1,\lambda_2,\lambda_3} \subset S_{\lambda_1,\lambda_2+1,\lambda_3-1} + S_{\lambda_1-1,\lambda_2+1,\lambda_3}$ *if* $\min\{\lambda_3, \lambda_1\} \geq 1$;

(3°) $\quad S_{\lambda_1,\lambda_2,\lambda_3} \subset S_{\lambda_1-1,\lambda_2,\lambda_3+1} + S_{\lambda_1,\lambda_2-1,\lambda_3+1}$ *if* $\min\{\lambda_1, \lambda_2\} \geq 1$.

PROOF. We need only prove (1°). Any $s \in S_\lambda$ can be expressed as a series $\sum a(j) M_\lambda(\cdot - j)$. Set, for $j = (j_1, j_2) \in Z^2$,

$$b(j_1, j_2) := \begin{cases} a(0, j_2) + \cdots + a(j_1, j_2) & \text{for } j_1 \geq 0, \\ 0 & \text{for } j_1 = -1, \\ -a(-1, j_2) - \cdots - a(j_1 + 1, j_2) & \text{for } j_1 < -1. \end{cases}$$

Then $\nabla_1 b = a$; hence

$$\begin{aligned} s &= \sum (\nabla_1 b)(j) M_\lambda(\cdot - j) = D_1\left(\sum b(j) M_{\lambda_1+1,\lambda_2,\lambda_3}(\cdot - j)\right) \\ &= (D_3 - D_2)\left(\sum b(j) M_{\lambda_1+1,\lambda_2,\lambda_3}(\cdot - j)\right) \\ &= \sum (\nabla_3 b)(j) M_{\lambda_1+1,\lambda_2,\lambda_3-1}(\cdot - j) - \sum (\nabla_2 b)(j) M_{\lambda_1+1,\lambda_2-1,\lambda_3}(\cdot - j). \end{aligned}$$

This shows that

$$s \in S_{\lambda_1+1,\lambda_2,\lambda_3-1} + S_{\lambda_1+1,\lambda_2-1,\lambda_3}.$$

The proof of Lemma 1 is complete.

4. Jump operators. We denote by $\mathbf{S}_\Delta$ the space of all splines (piecewise polynomials) on the mesh Δ. For $s \in \mathbf{S}_\Delta$, we think of s as defined on $\mathbf{R}^2 \backslash \Delta$. To describe the jump of a given spline s in the direction e_r ($r = 1,2,3$), we introduce the jump operators J_r as follows:

$$J_r s := \lim_{\varepsilon \to +0} [s(\cdot + \varepsilon e_r) - s(\cdot - \varepsilon e_r)].$$

On each component of $\mathbf{R}^2 \backslash \Delta$, s is a polynomial; hence the above limit always exists. Clearly, if s is continuous at x, then $J_r s(x) = 0$ for all $r = 1,2,3$. Thus the support of $J_r s$ is included in Δ. Since we think of s as defined on $\mathbf{R}^2 \backslash \Delta$, $J_r s$ is thought of as defined on $\Delta \backslash \mathbf{Z}^2$. The operators J_r are linear and bounded: $\|J_r\| \le 2$.

If g is defined on $\Delta \backslash \mathbf{Z}^2$, and if g is a polynomial in each component of $\Delta \backslash \mathbf{Z}^2$, then we can define

$$K_r g(j) := \lim_{\delta \to +0} g(j + \delta e_r), \qquad j \in Z^2,\ r = 1,2,3.$$

The operators K_r given by the above are linear and bounded: $\|K_r\| \le 1$.

We also want to give a description for the jump of the derivatives of a given spline. To this end we introduce the operators $R_{r,n}$ on $l(\mathbf{Z}^2)$ given by the rule

$$R_{r,n} a := \sum_{t=0}^{n} a(\cdot - t e_r) M_n(t), \qquad r = 1,2,3;\ n \ge 1.$$

(Recall that M_n is the univariate B-spline with support $[0,n]$ on the uniform mesh $\mathbf{Z}$.) When $n < 1$, we interpret $R_{r,n}$ as zero. Since $\sum_{t=0}^{n} M_n(t) = 1$, the operator $R_{r,n}$ is bounded by 1.

We make some convention about the combination notation $\binom{m}{n}$. Whatever m and n might be, we agree that

$$\binom{m}{n} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m < n. \end{cases}$$

Now we are ready to state the main result of this section.

LEMMA 2. *The following formulae hold:*

$$(1^\circ) \quad K_1 J_2 D_2^{k-1}\left(\sum a(j) M_\lambda(\cdot - j)\right) = R_{1,|\lambda|-k} \binom{k - \lambda_2 - 1}{\lambda_3 - 1} (-\nabla_1)^{k-(\lambda_2+\lambda_3)} \nabla_2^{\lambda_2} \nabla_3^{\lambda_3} a$$

$$(2^\circ) \quad K_2 J_3 D_3^{k-1}\left(\sum a(j) M_\lambda(\cdot - j)\right) = R_{2,|\lambda|-k} \binom{k - \lambda_3 - 1}{\lambda_1 - 1} \nabla_1^{\lambda_1} \nabla_2^{k-(\lambda_3+\lambda_1)} \nabla_3^{\lambda_3} a$$

$$(3^\circ) \quad K_3 J_1 D_1^{K-1}\left(\sum a(j) M_\lambda(\cdot - j)\right) = R_{3,|\lambda|-k} \binom{k - \lambda_1 - 1}{\lambda_2 - 1} \nabla_1^{\lambda_1} (-\nabla_2)^{\lambda_2} \nabla_3^{k-(\lambda_1+\lambda_2)} a.$$

PROOF. We need only prove (1°), because (2°) and (3°) can be proved in a similar way. For simplicity, write

$$s = \sum a(j) M_\lambda(\cdot - j).$$

The proof of (1°) will go case by case.

Case 1. $k \le \lambda_2 - 1$.

In this case,

$$D_2^{k-1} s = \sum (\nabla_2^{k-1} a)(j) M_{\lambda_1, \lambda_2 - k + 1, \lambda_3}(\cdot - j).$$

Since $\lambda_2 - k + 1 \ge 2$, $M_{\lambda_1, \lambda_2 - k+1, \lambda_3}$ is continuous in the e_2 direction. This shows that $J_2 D_2^{k-1} s = 0$. On the other hand, $k - \lambda_2 - 1 < \lambda_3 - 1$; hence the right-hand side of (1°) is also zero in this case.

Case 2. $k = \lambda_2$.

This case is divided into three subcases.

Subcase 1. $\min\{\lambda_1, \lambda_3\} \ge 1$.

In this case,

$$D_2^{k-1} s = \sum (\nabla_2^{k-1} a)(j) M_{\lambda_1, 1, \lambda_3}(\cdot - j).$$

Since $M_{\lambda_1, 1, \lambda_3} \in C(R^2)$, we obtain the desired conclusion.

Subcase 2. $\lambda_1 = 0$, $\lambda_3 \ge 1$.

In this case, M_λ is continuous in the e_2 direction at the points $j + \delta e_1$, $j \in \mathbf{Z}^2$, $0 < \delta < 1$; hence $K_1 J_2 D_2^{k-1} s = 0$, while the right-hand side of (1°) is also zero, because $k - \lambda_2 - 1 < \lambda_3 - 1$.

Subcase 3. $\lambda_1 \ge 1$, $\lambda_3 = 0$.

In this case

$$D_2^{k-1} s = \sum (\nabla_2^{k-1} a)(j) M_{\lambda_1, 1, 0}(\cdot - j).$$

We have

$$M_{\lambda_1, 1, 0}(x_1, x_2) = M_{\lambda_1}(x_1) M_1(x_2).$$

It follows that, for $i = (i_1, i_2) \in Z^2$ and $0 < \delta < 1$,

$$J_2 M_{\lambda_1, 1, 0}(i + \delta e_1) = \begin{cases} M_{\lambda_1}(i_1 + \delta) & \text{if } i_2 = 0, \\ -\, M_{\lambda_1}(i_1 + \delta) & \text{if } i_2 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned}
&(K_1 J_2 D_2^{k-1} s)(i) \\
&\quad = \sum_{j_1 \in Z} [(\nabla_2^{k-1} a)(j_1, i_2) M_{\lambda_1}(i_1 - j_1) - (\nabla_2^{k-1} a)(j_1, i_2 - 1) M_{\lambda_1}(i_1 - j_1)] \\
&\quad = \sum_{j_1 \in Z} (\nabla_2^k a)(j_1, i_2) M_{\lambda_1}(i_1 - j_1) \\
&\quad = \sum_{t=0}^{\lambda_1} (\nabla_2^k a)(i_1 - t, i_2) M_{\lambda_1}(t) \\
&\quad = R_{1, |\lambda| - k} \nabla_2^{\lambda_2} a(i).
\end{aligned}$$

This proves (1°) in this case, since $k - \lambda_2 - 1 = -1 = \lambda_3 - 1$ and $k - (\lambda_2 + \lambda_3) = 0$.

Case 3. $k > \lambda_2$.
In this case,

$$D_2^{k-1}s = D_2^{k-1-\lambda_2}\left(\sum(\nabla_2^{\lambda_2}a)(j)M_{\lambda_1,0,\lambda_3}(\cdot - j)\right).$$

By the binomial theorem, we have

$$D_2^{k-1-\lambda_2} = (D_3 - D_1)^{k-1-\lambda_2} = \sum_{p=0}^{k-1-\lambda_2}\binom{k-1-\lambda_2}{p}D_3^p(-D_1)^{k-1-\lambda_2-p}.$$

If $p \geq \lambda_3$ or $k - 1 - \lambda_2 - p \geq \lambda_1$, then

$$D_3^p(-D_1)^{k-1-\lambda_2-p}M_{\lambda_1,0,\lambda_3}(x) = 0 \quad \text{for } x \notin \Delta;$$

hence

$$J_2(D_3^p(-D_1)^{k-1-\lambda_2-p}M_{\lambda_1,0,\lambda_3}) = 0.$$

Assume $p < \lambda_3$ and $k - 1 - \lambda_2 - p < \lambda_1$. Then

$$\begin{aligned}&D_3^p(-D_1)^{k-1-\lambda_2-p}\left(\sum_j(\nabla_2^{\lambda_2}a)(j)M_{\lambda_1,0,\lambda_3}(\cdot - j)\right)\\ &\quad = \sum_j\left((-\nabla_1)^{k-1-\lambda_2-p}\nabla_2^{\lambda_2}\nabla_3^p a\right)(j)M_{\lambda_1+\lambda_2+p+1-k,0,\lambda_3-p}(\cdot - j).\end{aligned}$$

Note that

$$M_{\lambda_1+\lambda_2+p+1-k,0,\lambda_3-p}(x_1,x_2) = M_{\lambda_1+\lambda_2+p+1-k}(x_1 - x_2)M_{\lambda_3-p}(x_2).$$

If $\lambda_3 - p \geq 2$, then M_{λ_3-p} is continuous everywhere. Moreover, for fixed δ, $0 < \delta < 1$, and $i = (i_1, i_2) \in Z^2$,

$$\begin{aligned}\lim_{\varepsilon\to+0}&[M_{\lambda_1+\lambda_2+p+1-k}(i_1 - i_2 + \delta e_1 + \varepsilon e_2)\\ &- M_{\lambda_1+\lambda_2+p+1-k}(i_1 - i_2 + \delta e_1 - \varepsilon e_2)] = 0.\end{aligned}$$

This shows that

$$J_2\left[D_3^p(-D_1)^{k-1-\lambda_2-p}\left(\sum(\nabla_2^{\lambda_2}a)(j)M_{\lambda_1,0,\lambda_3}(\cdot - j)\right)\right] = 0$$

unless $p = \lambda_3 - 1$ and $k < |\lambda|$. Thus

$$\begin{aligned}&J_2D_2^{k-1}s\\ &\quad = J_2\left[\sum\binom{k-\lambda_2-1}{\lambda_3-1}(-\nabla_1)^{k-(\lambda_2+\lambda_3)}\nabla_2^{\lambda_2}\nabla_3^{\lambda_3-1}a(j)M_{|\lambda|-k,0,1}(\cdot - j)\right].\end{aligned}$$

By straightforward calculation we have

$$J_2M_{|\lambda|-k,0,1}(i + \delta e_1) = \begin{cases} M_{|\lambda|-k}(i_1 + \delta) & \text{if } i_2 = 0,\\ -M_{|\lambda|-k}(i_1 - 1 + \delta) & \text{if } i_2 = 1,\\ 0 & \text{otherwise.}\end{cases}$$

For simplicity, write

$$b = \binom{k-\lambda_2-1}{\lambda_3-1}(-\nabla_1)^{k-\lambda_2-\lambda_3}\nabla_2^{\lambda_2}\nabla_3^{\lambda_3-1}a.$$

Then the above calculation yields

$$\begin{aligned} J_2 D_2^{k-1} s(i) &= \sum_{j_1 \in Z} [b(j_1, i_2) M_{|\lambda|-k}(i_1 - j_1) - b(j_1, i_2 - 1) M_{|\lambda|-k}(i_1 - j_1 - 1)] \\ &= \sum_{j_1 \in Z} \nabla_3 b(j_1, i_2) M_{|\lambda|-k}(i_1 - j_1) \\ &= R_{1,|\lambda|-k} \nabla_3 b(i). \end{aligned}$$

This proves (1°) in Case 3. The proof of Lemma 2 is complete.

For simplicity, we denote by $U_{k,r,\lambda}$ the operator appearing on the right-hand side of Lemma 2(r°), $r = 1, 2, 3$, respectively. Further, let

$$\begin{aligned} L_{k,1,\lambda} &:= \binom{k - \lambda_2 - 1}{\lambda_3 - 1} (-\nabla_1)^{k-\lambda_2-\lambda_3} \nabla_2^{\lambda_2} (\nabla_1 + \nabla_2)^{\lambda_3}, \\ L_{k,2,\lambda} &:= \binom{k - \lambda_3 - 1}{\lambda_1 - 1} \nabla_1^{\lambda_1} \nabla_2^{k-\lambda_3-\lambda_1} (\nabla_1 + \nabla_2)^{\lambda_3}, \\ L_{k,3,\lambda} &:= \binom{k - \lambda_1 - 1}{\lambda_2 - 1} \nabla_1^{\lambda_1} (-\nabla_2)^{\lambda_2} (\nabla_1 + \nabla_2)^{k-\lambda_1-\lambda_2}. \end{aligned}$$

5. An upper bound for the approximation order. Let E be a finite subset of $\mathbf{Z}_+^3$. Let S be the span of $\{M_\lambda(\cdot - j); \lambda \in E,\ j \in \mathbf{Z}^2\}$. In this section, we will give an upper bound for the approximation order of S.

We want to put the operators $U_{k,r,\lambda}$ and $L_{k,r,\lambda}$ into a two-dimensional array. Note that $U_{k,r,\lambda} = 0$ if $k \le d = \min_\lambda\{\lambda_1+\lambda_2, \lambda_2+\lambda_3, \lambda_3+\lambda_1\} - 1$ or $k \ge |\lambda|$. Thus the only interesting case is $d < k < |\lambda|$. Assume $|E| = n$. There is a one-to-one mapping from $\{1, \ldots, n\}$ onto E. The image of q under this mapping is denoted by $\lambda(q)$. Let

$$\left.\begin{aligned} U_{3(k-d-1)+r,q} &:= U_{k,r,\lambda(q)}, \\ L_{3(k-d-1)+r,q} &:= L_{k,r,\lambda(q)}, \\ R_{3(k-d-1)+r,q} &:= R_{r,\lambda(q)-k}, \end{aligned}\right\} \qquad r = 1, 2, 3.$$

We observe that $L_{k,r,\lambda}$ are homogeneous polynomials in ∇_1 and ∇_2 of degree k. Let

$$\beta_{3(k-d-1)+r} = k, \qquad r = 1, 2, 3.$$

We are now in a position to prove the main result of this paper.

THEOREM 1. *If $3(m - d) \ge n$, and if the determinant of $L = (L_{pq})_{p,q=1}^n$ is nonzero, then the approximation order of S does not exceed m.*

PROOF. Suppose to the contrary that to any $h > 0$ and any $f \in C^{(m+1)}$ with $\|f\|_{m+1,\infty} < \infty$ there corresponds $u_h \in S_h$ such that $\|f - u_h\| \le \varepsilon_h h^m$ with $\varepsilon_h \to 0$ as $h \to +0$. It follows that

$$\|\sigma_{1/h} f - \sigma_{1/h} u_h\| \le \varepsilon_h h^m. \tag{5}$$

(Recall that $\sigma_{1/h}$ is a scaling operator. See §1.) Assume

$$u_h = \sum_\lambda \sum_j a_{\lambda,h}(j) M_\lambda\left(\frac{\cdot}{h} - j\right).$$

Then

$$\sigma_{1/h}u_h = \sum_\lambda \sum_j a_{\lambda,h}(j) M_\lambda(\cdot - j).$$

Suppose that f is a polynomial on a square Q. Then $\sigma_{1/h}f$ is a polynomial on the square Q/h. In each component of $R^2\backslash\Delta$ included in this square, $\sigma_{1/h}(f-u_h)$ is a polynomial, so we can invoke Markov's inequailty and obtain

$$\|D_2^{k-1}\sigma_{1/h}f - D_2^{k-1}\sigma_{1/h}u_h\| \le \operatorname{const}\varepsilon_h h^m.$$

Moreover, since K_1 and J_2 both are bounded operators, we have

$$\|K_1J_2D_2^{k-1}\sigma_{1/h}f - K_1J_2D_2^{k-1}\sigma_{1/h}u_h\| \le \operatorname{const}\varepsilon_h h^m.$$

But $f \in C^{(m+1)}$; hence

$$K_1J_2D_2^{k-1}\sigma_{1/h}f = 0 \quad \text{for } k \le m.$$

Thus

$$\|K_1J_2D_2^{k-1}\sigma_{1/h}u_h\| \le \operatorname{const}\varepsilon_h h^m.$$

By Lemma 2,

$$K_1J_2D_2^{k-1}\sigma_{1/h}u_h = \sum_\lambda U_{k,1,\lambda}a_{\lambda,h};$$

hence

$$\left\|\sum_\lambda U_{k,1,\lambda}a_{\lambda,h}\right\| \le \operatorname{const}\varepsilon_h h^m.$$

The above estimate is also true for $r = 2$ or 3:

$$\left\|\sum_\lambda U_{k,r,\lambda}a_{\lambda,h}\right\| \le \operatorname{const}\varepsilon_h h^m. \tag{6}$$

Let

$$a_q := a_{\lambda(q),h}, \qquad q = 1,\dots,n,$$

and

$$\xi_p := \sum_{q=1}^n U_{pq}a_q. \tag{7}$$

Then (6) reads

$$\|\xi_p\| \le \operatorname{const}\varepsilon_h h^m, \qquad p = 1,\dots,n. \tag{8}$$

Let

$$\mathbf{a} = (a_1,\dots,a_n)^\tau, \qquad \boldsymbol{\xi} = (\xi_1,\dots,\xi_n)^\tau.$$

Here τ means "transpose". Equation (7) can be written

$$U\mathbf{a} = \boldsymbol{\xi}, \tag{9}$$

where U is the matrix $(U_{pq})_{p,q=1}^n$.

Let I_n be the $n \times n$ identity matrix. Let $\operatorname{adj}(U)$ be the adjugate matrix of U. Then

$$U(\operatorname{adj} U) = (\operatorname{adj} U)U = (\det U)I_n.$$

By (9), we have

$$(\det U)\mathbf{a} = (\operatorname{adj} U)U\mathbf{a} = (\operatorname{adj} U)\boldsymbol{\xi}. \tag{10}$$

Take h to be $1/N$, where N is a positive integer. Let $\beta := \sum_{p=1}^{n} \beta_p$. Then $\det U$ has the form

$$\det U = \sum_{\alpha_1+\alpha_2=\beta} R_{\alpha_1,\alpha_2} \nabla_1^{\alpha_1} \nabla_2^{\alpha_2},$$

where $R_{\alpha_1,\alpha_2} \in A$, the algebra generated by the shift operators (see §2), and $\|R_{\alpha_1,\alpha_2}\| \le \text{const}$. Assume $\operatorname{adj} U = (V_{pq})_{p,q=1}^{n}$. Then each V_{pq} has the form

$$V_{pq} = \sum_{\alpha_1+\alpha_2=\beta-\beta_p} R_{\alpha_1,\alpha_2}^{(p,q)} \nabla_1^{\alpha_1} \nabla_2^{\alpha_2}$$

with $R_{\alpha_1,\alpha_2}^{(p,q)} \in A$ and $\|R_{\alpha_1,\alpha_2}^{(p,q)}\| \le \text{const}$. Let

$$W := h^\beta H_1^\beta H_2^\beta (\det U),$$

where H_1 and H_2 are the operators defined in (4). We observe that

$$\nabla_r H_r = (I - T_r)\left(\sum_{t=0}^{N-1} T_r^t\right) = I - T_r^N = \nabla_{r,N} \qquad (r = 1, 2, 3).$$

Hence

$$W = \sum_{\alpha_1+\alpha_2=\beta} R_{\alpha_1,\alpha_2} h^\beta H_1^{\beta-\alpha_1} H_2^{\beta-\alpha_2} \nabla_{1,N}^{\alpha_1} \nabla_{1,N}^{\alpha_2}.$$

Since $\|H_r\| \le N$, we have

$$\|h^\beta H_1^{\beta-\alpha_1} H_2^{\beta-\alpha_2}\| \le (1/N)^\beta N^{2\beta-(\alpha_1+\alpha_2)} = 1.$$

In addition, $\|R_{\alpha_1,\alpha_2}\| \le \text{const}$ and $\|\nabla_{1,N}^{\alpha_1} \nabla_{2,N}^{\sigma_2}\| \le \text{const}$; therefore

$$\|W\| \le \text{const}. \tag{11}$$

Next, we want to estimate $W\mathbf{a}$. It follows from (10) that

$$W\mathbf{a} = h^\beta H_1^\beta H_2^\beta (\det U)\mathbf{a} = h^\beta H_1^\beta H_2^\beta (\operatorname{adj} U)\boldsymbol{\xi}.$$

Consider $h^\beta H_1^\beta H_2^\beta V_{pq}$. We have

$$h^\beta H_1^\beta H_2^\beta V_{pq} = \sum_{\alpha_1+\alpha_2=\beta-\beta_p} R_{\alpha_1,\alpha_2}^{(p,q)} h^\beta H_1^{\beta-\alpha_1} H_2^{\beta-\alpha_2} \nabla_{1,N}^{\alpha_1} \nabla_{2,N}^{\alpha_2}.$$

Note that, for $\alpha_1 + \alpha_2 = \beta - \beta_p$,

$$\|h^\beta H_1^{\beta-\alpha_1} H_2^{\beta-\alpha_2}\| \le N^{\beta_p} \le N^m.$$

Also $\|R_{\alpha_1,\alpha_2}^{(p,q)}\| \le \text{const}$ and $\|\nabla_{1,N}^{\alpha_1} \nabla_{2,N}^{\alpha_2}\| \le \text{const}$. Therefore

$$\|h^\beta H_1^\beta H_2^\beta V_{pq}\| \le \text{const}\, N^m.$$

This combined with (9) enables us to conclude that

$$\|W\mathbf{a}\| \le \text{const}\, N^m \|\boldsymbol{\xi}\| \le \text{const}\, \varepsilon_h. \tag{12}$$

We restrict the domain of $\sigma_{1/h}f$ and $\sigma_{1/h}u_h$ to $\mathbf{Z}^2$. Thus they become elements of $l(\mathbf{Z}^2)$. Let $G_\lambda \in A$ be defined by the rule

$$G_\lambda a = \sum a(j)M_\lambda(\cdot - j).$$

Since $\sum M_\lambda(\cdot - j) = 1$, we have

$$\|G_\lambda\| \le 1.$$

Recall that $q \to \lambda(q)$ is a one-to-one map from $\{1, \dots, n\}$ onto E. Let $G_q = G_{\lambda(q)}$. Then

$$\sigma_{1/h}u_h = \sum_{q=1}^{n} G_q a_q. \tag{13}$$

Substitute (13) into (5). Let W act on both sides of this inequality. Since $\|W\| \le$ const by (11), we obtain

$$\left\| W\sigma_{1/h}f - \sum_{q=1}^{n} G_q W a_q \right\| \le \text{const}\, \varepsilon_h h^m.$$

Invoking estimate (12), we have

$$\left\| \sum_{q=1}^{n} G_q W a_q \right\| \le \text{const}\, \|Wa\| \le \text{const}\, \varepsilon_h.$$

From the foregoing two inequalities we conclude that

$$\|W\sigma_{1/h}f\| \le \text{const}\, \varepsilon_h. \tag{14}$$

Suppose now $\det L \ne 0$. Then in the expression

$$\det L = \sum_{\gamma_1+\gamma_2=\beta} C_{\gamma_1,\gamma_2} \nabla_1^{\gamma_1} \nabla_2^{\gamma_2},$$

there exists some (δ_1, δ_2) such that $\delta_1 + \delta_2 = \beta$ and $C_{\delta_1,\delta_2} \ne 0$. We can find a function $f \in C^{m+1}$ such that f has compact support and

$$f(x_1, x_2) = x_1^{\delta_1} x_2^{\delta_2} / (\delta_1! \delta_2!) \quad \text{for } (x_1, x_2) \in [-\alpha, \alpha] \times [-\alpha, \alpha],$$

where α is a sufficiently large real number.

Recall that

$$R_{r,n} = \sum_{t=0}^{n} M_n(t) T_r^t.$$

Since $\sum M_n(t) = 1$, we have

$$\begin{aligned} I - R_{r,n} &= \sum_{t=0}^{n} M_n(t)(I - T_r^t) \\ &= \sum_{t=0}^{n} M_n(t) \nabla_r (I + \dots + T_r^{t-1}). \end{aligned}$$

We also have observed that $\nabla_3 - (\nabla_1 + \nabla_2) = -\nabla_1 \nabla_2$. Now think of $\det U$ as a polynomial in ∇_1 and ∇_2. Decompose $\det U$ into homogeneous components. Then the above facts tell us that $\det L$ is its component of the lowest degree. Therefore we may write

$$\det U = \det L + \sum_{\gamma_1+\gamma_2>\beta} c_{\gamma_1,\gamma_2} \nabla_1^{\gamma_1} \nabla_2^{\gamma_2}.$$

Let

$$\gamma = \max\{\gamma_1 + \gamma_2; c_{\gamma_1,\gamma_2} \neq 0\}.$$

Since $\sigma_{1/h} f$ is a monomial of degree β on the square $[-N\alpha, N\alpha] \times [-N\alpha, N\alpha]$, and since $\det U - \det L$ is a polynomial in ∇_1 and ∇_2 of degree bigger than β, we have

$$(\det U)\sigma_{1/h} f = (\det L)\sigma_{1/h} f \quad \text{on } \mathbf{Z}^2 \cap [-N(\alpha-\gamma),\ N(\alpha-\gamma)]^2.$$

Moreover,

$$\nabla_1^{\gamma_1} \nabla_2^{\gamma_2} \sigma_{1/h} f = 0 \quad \text{if } (\gamma_1, \gamma_2) \neq (\delta_1, \delta_2).$$

Hence

$$(\det L)\sigma_{1/h} f = c_{\delta_1,\delta_2} \nabla_1^{\delta_1} \nabla_2^{\delta_2} \sigma_{1/h} f.$$

Furthermore,

$$\begin{aligned} H_1^{\delta_1} H_2^{\delta_2} (\det L)\sigma_{1/h} f &= c_{\delta_1,\delta_2} (H_1\nabla_1)^{\delta_1} (H_2\nabla_2)^{\delta_2} \sigma_{1/h} f \\ &= c_{\delta_1,\delta_2} \nabla_{1,N}^{\delta_1} \nabla_{2,N}^{\delta_2} \sigma_{1/h} f = c_{\delta_1,\delta_2} \nabla_1^{\delta_1} \nabla_2^{\delta_2} f = c_{\delta_1,\delta_2}. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} W\sigma_{1/h} f &= h^\beta H_1^\beta H_2^\beta (\det U)\sigma_{1/h} f \\ &= h^\beta H_1^{\beta-\delta_1} H_2^{\beta-\delta_2} (H_1^{\delta_1} H_2^{\delta_2} (\det U)\sigma_{1/h} f) \\ &= c_{\delta_1,\delta_2} h^\beta H_1^{\beta-\delta_1} H_2^{\beta-\delta_2} \mathbf{1} \\ &= c_{\delta_1,\delta_2} \quad \text{on } [-N(\alpha-\gamma-2\beta), N(\alpha-\gamma-2\beta)]^2. \end{aligned}$$

Therefore (14) becomes

$$|c_{\delta_1,\delta_2}| \leq \text{const}\, \varepsilon_h.$$

But c_{δ_1,δ_2} does not depend on h. Letting $h \to +0$ in the above inequality, we obtain $c_{\delta_1,\delta_2} = 0$. This contradiction shows that the approximation order of S does not exceed m. The proof of Theorem 1 is complete.

6. The approximation order of $\pi_{k,\Delta}^\rho$ in the case $3 \leq 2k - 3\rho \leq 7$. De Boor and Höllig have shown that $\pi_{k,\Delta}^\rho$ has the same approximation order that S_{loc} does. Here

$$S_{\text{loc}} := \text{the span of } \{M_\lambda(\cdot - j); M_\lambda \in \pi_{k,\Delta}^\rho \text{ and } j \in Z^2\}$$

(see [**BH₃**]). This fact enables us to apply Theorem 1 to obtain the approximation order of $\pi_{k,\Delta}^\rho$ in the case $3 \leq 2k - 3\rho \leq 7$.

Let

$$E' := \{\lambda; \rho + 2 \leq \min\{\lambda_1 + \lambda_2, \lambda_2 + \lambda_3, \lambda_3 + \lambda_1\} < |\lambda| \leq k + 2\}.$$

Then $M_\lambda \in \pi_{k,\Delta}^\rho$ is equivalent to $\lambda \in E'$. By Lemma 1, we may reduce E' to its subset E such that

$$S_{\text{loc}} = \text{the span of } \{M_\lambda(\cdot - j); \lambda \in E \text{ and } j \in Z^2\}.$$

Then we form the matrix L as in §5 and check whether $\det L \neq 0$. In this way we can prove the following theorem.

THEOREM 2.

$$(1°) \qquad m(k,\rho) = 2k - 2\rho - 1 \quad \textit{for } 2k - 3\rho = 3 \textit{ or } 4.$$

$$(2°) \qquad m(k,\rho) = 2k - 2\rho - 2 \quad \textit{for } 2k - 3\rho = 5, 6 \textit{ or } 7.$$

PROOF. (i) The case $2k - 3\rho = 3$.

In this case ρ must be an odd number. There exists some integer $\mu \geq 1$ such that $\rho = 2\mu - 1$. Then $k = 3\mu$ and $2k - 2\rho - 1 = 2\mu + 1 = \rho + 2$. By Lemma 1,

$$S_{\mathrm{loc}} = S_{\mu,\mu+1,\mu+1} + S_{\mu+1,\mu,\mu+1}.$$

Then $E = \{(\mu, \mu+1, \mu+1), (\mu+1, \mu, \mu+1)\}$ and $n = |E| = 2$. It is known from [$\mathbf{BH_1}$] that $m(k,\rho) \geq \rho + 2 = 2\mu + 1$. We want to prove $m(k,\rho) = 2\mu + 1$.

We have, for $m = 2\mu + 1$, that

$$L = \begin{bmatrix} 0 & \nabla_2^\mu(\nabla_1 + \nabla_2)^{\mu+1} \\ \nabla_1^\mu(\nabla_1 + \nabla_2)^{\mu+1} & 0 \end{bmatrix}.$$

Clearly, $\det L \neq 0$. By Theorem 1 we obtain $m(k,\rho) \leq 2\mu + 1$. Thus

$$m(k,\rho) = 2\mu + 1 = 2k - 2\rho - 1 \quad \text{in the case } 2k - 3\rho = 3.$$

(ii) The case $2k - 3\rho = 4$.

If $\rho = 0$, then $m(2,0) = 3$ is a well-known fact. Assume $\rho \geq 1$. There exists an integer $\mu \geq 2$ such that $\rho = 2\mu - 2$. Then $k = 3\mu - 1$ and $2k - 2\rho - 1 = 2\mu + 1 = \rho + 3$. It is known from [**DM**] that $m(k,\rho) \geq 2k - 2\rho - 1$. We want to prove $m(k,\rho) \leq 2\mu + 1$. By Lemma 1,

$$S_{\mathrm{loc}} = S_{\mu+1,\mu,\mu} + S_{\mu,\mu+1,\mu} + S_{\mu,\mu,\mu+1} + S_{\mu,\mu,\mu}.$$

For $m = 2\mu + 1$, we have

$$L = \begin{bmatrix} \nabla_2^\mu(\nabla_1+\nabla_2)^\mu & 0 & 0 & \nabla_2^\mu(\nabla_1+\nabla_2)^\mu \\ 0 & \nabla_1^\mu(\nabla_1+\nabla_2)^\mu & 0 & \nabla_1^\mu(\nabla_1+\nabla_2)^\mu \\ 0 & 0 & \nabla_1^\mu(-\nabla_2)^\mu & \nabla_1^\mu(-\nabla_2)^\mu \\ \mu(-\nabla_1)\nabla_2^\mu(\nabla_1+\nabla_2)^\mu & \nabla_2^{\mu+1}(\nabla_1+\nabla_2)^\mu & \nabla_2^\mu(\nabla_1+\nabla_2)^{\mu+1} & \mu(-\nabla_1)\nabla_2^\mu(\nabla_1+\nabla_2)^\mu \end{bmatrix}.$$

Then

$$\det L = (-1)^{\mu+1}\nabla_1^{2\mu}\nabla_2^{3\mu}(\nabla_1+\nabla_2)^{3\mu}(\nabla_1 + 2\nabla_2) \neq 0.$$

This shows that

$$m(k,\rho) = 2k - 2\rho - 1 \quad \text{in the case } 2k - 3\rho = 4.$$

(iii) The case $2k - 3\rho = 5$.

There exists an integer $\mu \geq 1$ such that $\rho = 2\mu - 1$. Then $k = 3\mu + 1$ and $2k - 2\rho - 2 = 2\mu + 2$. It is shown by [**J**] that $m(k,\rho) \geq 2k - 2\rho - 2$. We want to prove $m(k,\rho) \leq 2k - 2\rho - 2$. By Lemma 1,

$$S_{\mathrm{loc}} = \text{the span of } \{M_\lambda(\cdot - j); \lambda \in E,\ j \in Z^2\},$$

where

$$E = \{(\mu+2, \mu+1, \mu), (\mu+2, \mu, \mu+1), (\mu+1, \mu+2, \mu), \\ (\mu+1, \mu, \mu+2), (\mu+1, \mu+1, \mu), (\mu+1, \mu, \mu+1)\}.$$

Let $m = 2\mu + 2$. To check whether $\det L$ is nonzero, we may use the following technique to simplify the computation. We observe that each entry of the matrix L is a polynomial of ∇_1 and ∇_2, so we may assign values to ∇_1 and ∇_2. Write $L = L(\nabla_1, \nabla_2)$. If $\det L(1,1) \neq 0$, then $\det L \neq 0$. Let us now look at $L(1,1)$:

$$L(1,1) = \begin{bmatrix} 2^\mu & 2^{\mu+1} & 0 & 0 & 2^\mu & 2^{\mu+1} \\ 0 & 0 & 2^\mu & 0 & 2^\mu & 0 \\ 0 & 0 & 0 & (-1)^\mu & 0 & (-1)^\mu \\ -\mu 2^\mu & -(1+\mu)2^{\mu+1} & 2^\mu & 2^{\mu+2} & -\mu 2^\mu & -(1+\mu)2^{\mu+1} \\ 2^\mu & 0 & (\mu+1)2^\mu & 0 & (\mu+1)2^\mu & 2^{\mu+1} \\ 0 & (-1)^\mu & 0 & 2\mu(-1)^\mu & (-1)^{\mu+1} & 2\mu(-1)^\mu \end{bmatrix}.$$

By straightforward computation, we conclude that $\det L(1,1) \neq 0$. This shows that

$$m(k,\rho) = 2k - 2\rho - 2 \quad \text{in the case } 2k - 3\rho = 5.$$

(iv) The case $2k - 3\rho = 6$ or 7.

The process goes as before. Since the computation is tedious, we omit the details. The proof of Theorem 2 is complete.

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MATHEMATICS RESEARCH CENTER, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53705

Current address: Department of Mathematics, Zhejiang University, Hangzhou, Zhejiang, The People's Republic of China