

DEFORMATIONS OF COMPLETE MINIMAL SURFACES

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ABSTRACT. A notion of deformation is defined and studied for complete minimal surfaces in R^3 and R^3/G , G a group of translations. The catenoid, Enneper's surface, and the surface of Meeks-Jorge, modelled on a 3-punctured sphere, are shown to be isolated. Minimal surfaces of total curvature 4π in R^3/Z and R^3/Z^2 are studied. It is proved that the helicoid and Scherk's surface are isolated under periodic perturbations.

Let M be a submanifold of a Riemannian manifold N and let $T_\epsilon(M)$ be a tubular neighborhood of M in N of radius ϵ . A $C^{1-\epsilon}$ variation of M in N is a submanifold $M_1 \subset T_\epsilon(M)$ which is a graph over M and is ϵ - C^1 -close to M . This means M_1 is pointwise ϵ -close to M in each fibre of $T_\epsilon(M)$ and the tangent planes as well. We are interested in complete minimal submanifolds M of N (abbreviated c.m.s.) and $C^{1-\epsilon}$ variations which are also c.m.s.'s, and henceforth we always assume M and M_1 are c.m.s.'s. We say M is isolated if for some $\epsilon > 0$, the only ϵ - C^1 -variations of M differ from M by an ambient isometry of N . In this paper we shall study this question when N is R^3 or a translation space: R^3 modulo a group of translations.

A flat plane in R^3 is isolated. This follows immediately from Bernstein's theorem: a function of two variables on R^2 whose graph is a c.m.s. is linear. We shall prove Enneper's surface, the catenoid, and a certain 3-punctured sphere, discovered by Meeks and Jorge, are isolated in R^3 . We prove the helicoid and Scherk's surface are isolated under periodic deformations, i.e., the helicoid of total curvature 4π in R^3 modulo one translation is isolated in this translation space and Scherk's surface of total curvature 4π is isolated in R^3 modulo two translations. This Scherk surface is conformally a 4-punctured sphere and the helicoid a 2-punctured sphere.

We shall study minimal submanifolds of translation spaces. Many of the techniques developed by R. Osserman can be adapted to this context and yield information about periodic minimal surfaces in R^3 . For example, we classify c.m.s.'s of total curvature 4π in translation space. In R^3 , Osserman has proved the catenoid and Enneper's surface are the only c.m.s.'s of total curvature 4π . We obtain an analogous classification in R^3/G . In §VI, we describe these surfaces for $G = Z$.

We study c.m.s.'s M of finite total curvature in translation spaces. We prove a deformation (i.e., an ϵ - C^1 -variation) M_1 of a finite total curvature c.m.s. in a translation space is also of finite total curvature, and is conformally a compact

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Riemann surface $\bar{M}_1$ punctured in a finite number of points. The Gauss map g_1 of M_1 and the analytic differential ω_1 of the Weierstrass representation of M_1 , extend to a meromorphic function $\bar{g}_1$ on $\bar{M}_1$ and a meromorphic differential $\bar{\omega}_1$ on $\bar{M}_1$. The degree of $\bar{g}_1$ is the same as the degree of $\bar{g}$, the extended Gauss map of the unperturbed surface M . This reduces the study of deformations of finite total curvature M to the study of pairs $(\bar{g}, \bar{\omega})$ on $\bar{M}_1$ with relations between the poles of $\bar{g}$ and the zeros of $\bar{\omega}$ and other relations coming from closeness in space and residue conditions. In general, this study seems difficult.

The Gauss map g of a nonflat c.m.s. of finite total curvature in R^3 can miss at most 3 points of the Riemann sphere S [4]. It is not known if 3 is sharp. Osserman's proof shows that g can miss at most 4 points for a nonflat c.m.s. of finite total curvature in a translation space. Four is sharp as Scherk's surface shows. This has already been observed by Gacksatter [1]; he calls such surfaces abelian (Lawson calls them algebraic [2]). We show the Gauss maps of deformations of these surfaces have the same image. Xavier has proved the Gauss map of a c.m.s. in R^3 can miss at most 7 points (unless it is flat) [6]. It is not known if 7 can be reduced to 4.

We would very much like to know if the helicoid is isolated in R^3 . We prove in §VII that certain deformations of the helicoid do not exist. It seems plausible that any c.m.s. conformally parametrised by $\mathbb{C}$ is isolated.

We originally proved the catenoid is isolated in collaboration with Remi Langevin. Discussions with Mike Beeson, Bill Meeks, Rich Schoen and Dennis Sullivan have been very helpful in the preparation of this paper.

Some remarks are in order concerning the definition of an isolated surface. When trying to understand something, it is natural to study a neighborhood of the object in question. Complete minimal surfaces are particularly mysterious (alas, their deformations as well). Weierstrass introduced a deformation of minimal surfaces (known to specialists as the Weierstrass deformation) and this has certainly proved useful. The one parameter family of (locally) isometric surfaces joining the helicoid to the catenoid can be realised as a Weierstrass deformation. However, the convergence here is analogous to the manner by which a family of circles can be made to converge to a line; certainly no circle is in a tubular neighborhood of the line of fixed radius. It seems natural to consider neighborhoods defined by tubular neighborhoods of a fixed radius. C^1 -close is perhaps superfluous but our techniques require this for the time being.

When this paper was written, we had no example of a c.m.s. in R^3 which was not isolated. Subsequently, the author, in collaboration with E. Toubiana, found such an example. In principal, our paper describing this example is in the same issue of the Transactions as the present paper. Robert Bryant has brought to our attention that this deformable example occurs in his paper on higher critical values of the Willmore functional (preprint). He was not studying deformations but his examples are very interesting in the context of deformations.

I. The Weierstrass representation in translation spaces. Let M be a minimal surface in R^3 . M admits local parametrisations $X: D \rightarrow M$, D an open subset of $\mathbb{C}$ and X a conformal map with harmonic coordinate functions; such parametrisations

are equivalent to minimality. Write $z = u + iv$, $X(z) = (x_1(z), x_2(z), x_3(z))$. X conformal means $X_u \perp X_v$ (X_u means partial derivative with respect to u) and $\|X_u\| = \|X_v\|$. Harmonicity means $(\partial^2 x_k)/(\partial u^2) + (\partial^2 x_k)/(\partial v^2) = 0$, $k = 1, 2, 3$. Let $\phi_k(z) = (\partial x_k)/(\partial u) - i(\partial x_k)/(\partial v)$, so that ϕ_k are analytic in D , (i) $\phi_1^2 + \phi_2^2 + \phi_3^2 \equiv 0$ and (ii) $|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 \neq 0$. Clearly if we are given analytic functions ϕ_k in D , satisfying (i) and (ii), then we obtain a parametrised (by D) minimal surface by defining $x_k(z) = \operatorname{Re} \int_{z_0}^z \phi_k(z) dz$, $z_0 \in D$, $k = 1, 2, 3$. The forms $\phi_k(z) dz$ do not depend on the local parameter, i.e., if $\tilde{X}: \tilde{D} \rightarrow M$ is a conformal map with harmonic coordinate functions, then $\phi_k(z) dz = \tilde{\phi}_k(\tilde{z}) d\tilde{z}$, $z \rightarrow \tilde{z}$ the change of coordinates. So a minimal surface determines three global analytic forms $\phi_k(z) dz$ satisfying (i) and (ii). Moreover the forms have no real periods on M since the coordinate functions are single valued on M . Conversely, if we are given three forms on M satisfying (i), (ii) and the period relations, then we obtain a conformal minimal immersion $M \xrightarrow{X} R^3$ by integration, i.e., $x_k = \operatorname{Re} \int \phi_k(z) dz$. Changing the base point in the integral changes the immersion by a translation. Now (assuming M is not a flat plane) the data: three analytic forms on M satisfying (i) and (ii) are equivalent to a pair (g, ω) , where g is a meromorphic function on M to S , ω is an analytic differential (i.e. an analytic one-form) satisfying: the poles of g are precisely the zeros of ω and the order of a pole of g is one half the order of this zero of ω . The relation is

$$\begin{aligned} \phi_1 &= \frac{1}{2}(1 - g^2)\omega, & \omega &= \phi_1 - i\phi_2, \\ & & \text{or} & \\ \phi_2 &= \frac{i}{2}(1 + g^2)\omega, & g &= \frac{\phi_3}{\phi_1 - i\phi_2}, \\ \phi_3 &= g\omega. \end{aligned}$$

Thus a minimal surface M in R^3 is determined by such pairs (g, ω) , provided the residue conditions are satisfied for the ϕ_k , i.e. they have no real periods on M . This is the usual Weierstrass representation of minimal surfaces in R^3 . We mention that the metric on M is given by $ds = \lambda|dz|$, where $\lambda = (1 + |g|^2)|\omega|/2$, and g is the Gauss map.

Now let G be a group of translations of R^3 and let $N = R^3/G$, $p: R^3 \rightarrow N$ the projection. Let M be a minimal submanifold of N and M_1 be a connected component of $p^{-1}(M)$. Let (g_1, ω_1) be the pair associated to M_1 via the Weierstrass representation. Since parallel translation in R^3 leaves invariant the normals, and g_1 is the Gauss map of M_1 , g passes to a meromorphic map $g: M \rightarrow S$. Notice that the forms ϕ_k^1 on M_1 , $k = 1, 2, 3$, also pass to the quotient. To see this, let $p \in M_1$, $\gamma \in G$ and $\xi: D \rightarrow M_1$ be a local parameter at p . So

$$\phi_k^1(z) dz = \left(\frac{\partial x_k^1}{\partial u} - i \frac{\partial x_k^1}{\partial v} \right) dz$$

in D . This does not depend on the local parameter at p , so to check that $\phi_k^1(p) = \phi_k^1(\gamma(p))$ we can choose any local parameter at $\gamma(p)$. We use $\gamma \cdot \xi: D \rightarrow M_1$

as a local parameter at $\gamma(p)$. Then

$$\phi_k^1(\gamma(p)) = \left(\frac{\partial x_k^1(\gamma(p))}{\partial u} - i \frac{\partial x_k^1(\gamma(p))}{\partial v} \right) dz.$$

Since γ is a translation, the partial derivatives are the same at p and $\gamma(p)$. Thus ω_1 passes to an analytic differential ω on M . Thus M also has a Weierstrass representation (g, ω) and the poles of g correspond to the zeros of ω with the proper multiplicities as before. However, the coordinate functions $x_k = \operatorname{Re} \int \phi_k$ need not be single valued on M . Many examples will be given in §VI; periodic minimal surfaces in R^3 are the most interesting. Any minimal surface in R^3 yields a minimal surface in R^3/G by projection, however periodic minimal surfaces may be simpler in the quotient. Infinite total curvature may become finite, e.g. the helicoid in R^3/Z and Scherk's surface in R^3/Z^2 .

II. Finite total curvature. Let G be a group of translations of R^3 and let $M \subset R^3/G$ be a c.m.s. of finite total curvature. Then there is a compact Riemann surface $\bar{M}$ and M is conformally equivalent to $\bar{M}$ punctured at a finite number of points $p_1, \dots, p_r$ [2]. We observe that (as in R^3) the pair (g, ω) extends to a meromorphic map $g: \bar{M} \rightarrow S^2$ and a meromorphic differential ω on $\bar{M}$. To see that g extends to $\bar{M}$, notice that the total curvature of M is the area of its spherical image under g , counted with multiplicity; hence each value is assumed at most a finite number of times; cf. [1]. Now M is finitely connected and each end of M is conformally a once punctured disc. The puncture cannot be an essential singularity of g since then g would take on values infinitely often in a neighborhood of the puncture. Therefore g extends to a meromorphic map at the puncture, thus to $\bar{M}$. Now the analytic differential ω also extends to a meromorphic differential on $\bar{M}$, having a pole at least of order one at each puncture (Scherk's surface in R^3/Z^2 shows that the poles need not be of order 2 at the punctures as in R^3).

Let D be a punctured disc, representing an end of M , $D = \{z \in \mathbb{C}/0 < |z| < 1\}$ and the puncture is the origin. Write $\omega = f(z) dz$, f analytic in D . Then the metric on M in D takes the form $ds = \lambda |dz|$, where $\lambda = |f|(1 + |g|^2)/2$. We can assume the origin is not a pole or zero of g (by eventually making a preliminary rotation of M and replacing G by the conjugate group). Then completeness of M implies $\int_\gamma |f(z)| |dz| = \infty$ for every path γ tending to the origin. Then f has a pole at 0, at least of order one [4]. Thus ω also extends to $\bar{M}$.

III. Deformations of finite total curvature surfaces. Let M_0 be a c.m.s. in R^3/G and let M be an ε deformation of M_0 . Orthogonal projection of M to M_0 is a quasiconformal homeomorphism. Thus if M_0 is conformally parametrised by $\mathbb{C}$ then so is M , e.g. the helicoid, catenoid and Enneper's surface. This idea applies to finite total curvature minimal surfaces and we discuss this now.

Suppose M_0 is of finite total curvature, so that M_0 is conformally equivalent to a compact Riemann surface $\bar{M}_0$ punctured at a finite number of points $p_1, \dots, p_r$. We claim that the same holds for deformations M of M_0 . First we shall prove that M is also conformally a compact Riemann surface $\bar{M}$ punctured at r points ($\bar{M}$ is

homeomorphic to $\overline{M}_0$). Let E_0 be an end of M_0 ; E_0 is conformally a once punctured disc and g_0 extends across the puncture. Let E be the corresponding end of M ; we need to prove E is also conformally a once punctured disc. After a preliminary rotation of M_0 (hence of M and G) we may assume g_0 has no poles in E_0 , nor g in E . An open subset A of $\mathbb{C}$ is called hyperbolic if A admits Green's functions, or equivalently, if $\log(1 + |z|^2)$ has a harmonic majorant in A [4]; otherwise A is called parabolic. Clearly an open subset of a hyperbolic set is hyperbolic. Now g_0 and g are open maps, hence $g(E)$ is hyperbolic since $g_0(E)$ is contained in the complement of a closed disc (which is hyperbolic) and $g(E)$ is close to $g_0(E)$, hence $g(E)$ is an open subset of a hyperbolic set. Therefore it suffices to prove:

LEMMA. *An end of M , whose spherical image is hyperbolic, is conformally a once punctured disc.*

PROOF. Let $E = \{z \in \mathbb{C} / 0 \leq r < |z| < 1\}$ and $ds = \lambda|dz|$ be the induced metric in E . We know, by completeness of M , that for all paths $z(t)$ in E such that $\lim_{t \rightarrow 1} |z(t)| = r$, we have $\int_{z(t)} ds = \infty$. Thus by [4], we have $r = 0$ provided we can find a harmonic $h(z)$ such that $\log \lambda(z) \leq h(z)$ in E . Here is how we find $h(z)$. We know $\lambda = |f|(1 + |g|^2)/2$, and since $g(E)$ is hyperbolic, there is a harmonic h_0 on $g(E)$ such that

$$\log(1 + |g(z)|^2) \leq h_0(g(z)) \quad \text{for } z \in E.$$

Now $\lambda \leq \tilde{\lambda} = \frac{1}{2}|f|e^{h_0(g(z))}$ so

$$\log \lambda \leq \log |f|^2 + h_0(g(z)).$$

Since g has no poles in E , we have $f(z) \neq 0$ in E . Thus $h(z) = \log |f(z)|^2 + h_0(g(z))$ is a harmonic majorant of $\log \lambda$ in E and $r = 0$.

Now to prove M has finite total curvature it suffices to prove g extends to $\overline{M}$ [2]. Let E be a punctured disc end of M . If the puncture were an essential singularity of g , then g would take on almost every value in a neighborhood of the puncture. However $g(E)$ is contained in a small disc of S since it is close to the image of the unperturbed end. Thus the puncture is a removable singularity.

Next we shall prove the degree of $g: \overline{M} \rightarrow S$ is the same as the degree of $g_0: \overline{M}_0 \rightarrow S$. Suppose the degree of g_0 is l , so that the preimage of a point of S is l points of $\overline{M}_0$, counted with multiplicity. Let $z \in S$ be a regular point of g_0 , so that $g_0^{-1}(z)$ consists of l points $k_1, \dots, k_l$ and each k_j has a disc neighborhood D_j mapped homeomorphically onto a disc neighborhood D of z . We choose z so that no k_j is a puncture of M_0 . Let $\tilde{D}_j$ be the disc on M obtained by orthogonal projection of D_j onto M . For ε sufficiently small, $g(\partial \tilde{D}_j)$ can be made close enough to $g_0(\partial D_j)$ so that $g\tilde{D}_j$ contains z . Hence each $\tilde{D}_j$ contains one preimage of z and $\deg g \geq \deg g_0$. Now consider $F = \overline{M}_0 - \bigcup_{j=1}^l \text{int } D_j$. Since $g_0 F$ is far from z , we can make $g(\overline{M} - \bigcup_{j=1}^l \text{int } \tilde{D}_j)$ disjoint from z for small perturbations. Hence $\deg g = \deg g_0$.

IV. Some isolated minimal surfaces. Robert Osserman has proved that Enneper's surface and the catenoid are the only c.m.s.'s in R of total curvature 4π [4]. Now we know that a deformation of a finite total curvature c.m.s. has the same total curvature since the Gauss maps have the same degree. Therefore Enneper's surface and the catenoid are isolated. For completeness we prove directly that these surfaces are isolated.

The Weierstrass representation of Enneper's surface M_0 is (g_0, ω_0) , where M_0 is parametrised by C , $g_0(z) = z$, $\omega_0 = dz$. Now if M is a deformation of M_0 , the conformal type of M is also C and M is given by a conformal immersion $X: C \rightarrow M$ with harmonic coordinate functions. The limiting value of the normal to M at its unique end is the same as that of M_0 , i.e. $(0, 0, 1)$, so $g(\infty) = \infty$ and after reparametrising M by composing X with a Moebius transformation $S \rightarrow S$, leaving ∞ fixed, we may suppose $g(z) = z$ (since $g: S \rightarrow S$ is a bijective meromorphic map leaving ∞ fixed, so of the form $az + b$). Now ω is an analytic differential on C with no zeros or poles and ω extends to a meromorphic differential on S . Therefore $\omega = c dz$ for some $c \in C$. It is a remarkable fact that for $c = e^{i\theta}$ the surfaces $\{g(z) = z, \omega = dz\}$ and $\{g(z) = z, \omega = c dz\}$ are congruent in R^3 ; they differ by rotation by $\theta/2$. So for general $c \in C$, $g(z) = z$, $\omega = c dz$ differs from $\{g(z) = z, \omega = dz\}$ by a rotation followed by a homothety. Hence for M we must have $c = 1$ and $M = M_0$.

Next consider the catenoid M_0 . M_0 can be described by a conformal embedding $X_0: S - \{0, \infty\} \rightarrow R^3$ with $g_0(z) = z$, $\omega_0 = dz/z^2$. Let M be a deformation of M_0 . M is also conformally a 2-punctured sphere so we have a conformal embedding $X: C - 0 \rightarrow M$ with harmonic coordinate functions. The values of the Gauss map g are the same as g_0 at the punctures $0, \infty$, thus, after reparametrising, g may be taken to be $g(z) = z$. Now ω is a meromorphic differential on S with no zeros in C , and analytic in C , hence $\omega = cdz/z^k$ for some $c \in C$ and integer k . We have $ds = \lambda|dz| = |\omega|(1 + |g|^2)/2$. By completeness of M at 0 we have $k > 0$ and by completeness at ∞ we have $k < 4$. A residue calculation shows $k \neq 1$. Indeed, if $k = 1$, then $\text{Res}(\phi, 0) = c/2$ and $\text{Res}(\phi_2, 0) = ic/2$. Since x_1 and x_2 are single valued on M we conclude c is both real and pure imaginary, hence $c = 0$, a contradiction. A similar residue calculation shows $k \neq 3$. So $\omega = c dz/z^2$. Now $x_3 = \text{Re} \int^z c dz/z$ is single valued so $\text{Res}(\phi_3, 0) = c$ must be real. Thus M differs from M_0 by a dilation and since they are close in space we have $c = 1$. Therefore M is a catenoid.

Meeks and Jorge have discovered interesting examples of finite total curvature c.m.s.'s which are modelled on an n -punctured sphere [3] for any integer n . The Weierstrass representation of this surface is $g(z) = z^{n-1}$, $\omega = dz/(z^n - 1)^2$, where $M = S$ punctured at the n roots of unity. For $n = 2$, this gives a catenoid (rotated from our previous discussion). In general, this surface is immersed and has n embedded "catenoid type" ends.

THEOREM. *The c.m.s. M_0 : $g_0(z) = z^2$, $\omega_0 = dz/(z^3 - 1)^2$, is isolated.*

PROOF. Let M be a deformation of M_0 . M has total curvature 8π ; the Gauss map $\bar{M} \rightarrow S$ is a rational map of degree 2. So $g(z) = R(z)/S(z)$, R and S polynomials

and $\max \deg\{R, S\} = 2$. Since three punctured spheres are all conformally equivalent, we can parametrise M by $S - \{1, \alpha_1, \alpha_2\}$, α_k the cube roots of unity. We know g takes the same values as g_0 at the punctures, so $g(1) = 1$, $g(\alpha_1) = \alpha_2$, $g(\alpha_2) = \alpha_1$. There are five cases to consider: 1. g is a polynomial of degree 2; 2. R is constant and $\deg S = 2$; 3. $\deg R = 2$ and $\deg S = 1$; 4. $\deg R = 1$ and $\deg S = 2$; and 5. $\deg R = \deg S = 2$; this is the difficult one. Let $\omega = P(z) dz/Q(z)$, P, Q polynomials.

1. Clearly $g(z) = z^2$ since g takes the prescribed values at $1, \alpha_1, \alpha_2$. We know ω has zeros (in M) only at the poles of g so $P(z)$ is constant. Moreover ∞ is a regular point of M so a zero of ω of order 4. It follows that $\deg Q(z) = 6$. In R^3 , it is a general fact that ω has a pole at each puncture (where g takes a finite value) at least of order 2 [4]. Therefore

$$\omega = \frac{c dz}{(z-1)^2(z-\alpha_1)^2(z-\alpha_2)^2}$$

for some $c \in \mathbb{C}$. Now x_3 is a single valued function on M and a calculation of $\text{Res}(\phi_3, 1)$ shows c is real. Therefore M differs from M_0 by a dilation and since they are ε close we have $c = 1$, $M = M_0$.

We leave case 2 to the reader.

3. A calculation yields $g(z) = (z^2 - \alpha^2)/(-\alpha^2 z + 1)$ for some $\alpha \in \mathbb{C}$, $\alpha \neq \pm 1$. Hence the zeros of ω are precisely the points $\beta = 1/\alpha^2$ and ∞ and each is of order 2 since they are simple poles of g . Thus $P(z) = c(z - \beta)^2$ and $\deg Q(z) = 6$. Since ω has poles at least of order 2 at each puncture, we have

$$\omega = c(z - \beta)^2/(z^3 - 1)^2.$$

We will prove this is impossible by doing the residue calculations at $z = 1$. Since x_1, x_2 are single valued functions on M we have $\int_\gamma \omega = \int_\gamma \omega g^2$, where $\gamma = \{z/|z-1| = .2\}$. These equations yield:

$$(I) \quad c\beta(1 - \beta) = \bar{c}(1 - \bar{\beta})(1 + \bar{\beta}).$$

Now x_3 single valued on M means $g\omega$ has real residues. A calculation at $z = 1$ yields $c(1 - \beta)$ is real. Combining this with (I) we conclude $\beta = 1 + \bar{\beta}$ or $c(1 - \beta) = 0$. Both are clearly impossible.

4. We have $g(z) = (az + b)/(z - \alpha)(z - \beta)$. The equations $g(1) = 1$, $g(\alpha_1) = \alpha_2$, $g(\alpha_2) = \alpha_1$, imply $\alpha = 0$ or $\beta = 0$, and $a = 1 - \alpha\beta$, $b = 2\alpha\beta - \alpha - \beta$. This gives $g(z) = 1/z$ which contradicts $\deg Q = 2$.

Finally we must treat case 5, $\deg R(z) = \deg S(z) = 2$. Let α, β be the poles of g , i.e. the roots of Q . An easy calculation shows

$$g(z) = (\alpha\beta z^2 + z - (\alpha + \beta))/(z - \alpha)(z - \beta).$$

Clearly α, β are not cube roots of unity so ω has double zeros at α and β and no others. Also ω has poles at the punctures at least of order 2 hence $\deg Q(z) \geq 6$.

Since ∞ is not a zero of ω , we have $\deg Q(z) = 6$ ($\deg P(z) = 4$). Hence

$$\omega = \frac{c(z - \alpha)^2(z - \beta)^2}{(z^3 - 1)^2} dz.$$

We now list the residue calculations, and we leave the calculations to the courageous reader. The x_1, x_2 well-defined functions are equivalent to $\int \omega = \int \omega g^2$ on all loops.

Equations (I) and (II) are this integral equation at $z = 1$ and α_1 respectively:

$$(I) \quad c(1 - \alpha)(1 - \beta)(1 - \alpha\beta) = -\bar{c}(1 - \bar{\alpha})(1 - \bar{\beta})(\bar{\alpha} + \bar{\beta} + \bar{\alpha}\bar{\beta}),$$

(II)

$$c(\alpha_1 - \alpha)(\alpha_1 - \beta)(\alpha_2 - \alpha\beta)\alpha_1 = -\bar{c}(\bar{\alpha}\bar{\beta}\alpha_1 + \alpha_2 - (\bar{\alpha} + \bar{\beta}))(\bar{\alpha}\bar{\beta}\alpha_1 + (\bar{\alpha} + \bar{\beta})\alpha_2).$$

We use related equations obtained as follows. Expand $g(z)$ and $\omega = f(z) dz$ in power series at $z = 1$:

$$g(z) = g(1) + g'(1)w + o(w^2), \quad f_1(z) = f_1(1) + f'_1(1)w + o(w^2),$$

where

$$f_1(z) = \frac{c(z - \alpha)^2(z - \beta)^2}{(z - \alpha_1)^2(z - \alpha_2)^2} \quad \text{and} \quad w = z - 1.$$

Then

$$\begin{aligned} \phi_1 &= \frac{f}{2}(1 - g^2) = \frac{1}{2} \left(f(1) + \frac{f'_1(1)}{w} + o(1) \right) (-2g'(1)w + o(w^2)) \\ &= -\frac{f_1(1)g'(1)}{w} + o(1) \end{aligned}$$

and

$$x_1 = \operatorname{Re} \int \phi_1 = -\operatorname{Re}(f_1(1)g'(1)\log w + o(w)),$$

so x_1 single valued implies $f_1(1)g'(1)$ is real, i.e.

$$c(1 - \alpha)(1 - \beta)(2\alpha\beta - 1 + (\alpha + \beta))$$

is real. A similar calculation with $\phi_2 = if(1 + g^2)/2$ shows $f_1(1)g'(1) + f'_1(1)$ is pure imaginary, and this easily implies $c(1 - \alpha)(1 - \beta)(1 + \alpha + \beta)$ is pure imaginary.

The x_3 residue condition at $z = 1$ is $c(1 - \alpha)(1 - \beta)(1 + \alpha + \beta)$ is real. Thus $c(1 - \alpha)(1 - \beta)(1 + \alpha + \beta) = 0$, hence $\alpha + \beta = -1$ since neither α or β is 1 and $c \neq 0$.

Now consider ϕ_3 near α_1 . We have

$$\phi_3 = fg = \left(\frac{h(\alpha_1)}{u^2} + \frac{h'(\alpha_1)}{u} + o(1) \right) (\alpha_2 + g'(\alpha_1)u + o(u^2)),$$

where $u = z - \alpha_1$, $h(z) = c(z - \alpha)^2(z - \beta)^2/(z - 1)^2(z - \alpha_2)^2$. Then since $\operatorname{Res}(\phi_3, \alpha_1)$ is real we conclude: $h(\alpha_1)g'(\alpha_1) + h'(\alpha_1)\alpha_2$ is real. This yields $c(\alpha_1 - \alpha)(\alpha_1 - \beta)$ is pure imaginary. Now substitute this last relation into (II) and use the fact $\alpha\beta\alpha_2 + \alpha_1 + 1 = (\alpha_1 - \alpha)(\alpha_1 - \beta)\alpha_2$ to obtain $2\alpha_2 = \alpha\beta + \bar{\alpha}\bar{\beta}$. This is a contradiction and completes the proof of the theorem.

V. The image of the Gauss map.

PROPOSITION. *Let M_0 be a periodic minimal surface in R^3 having a fundamental domain of finite total curvature (nonzero) and let M be a deformation of M_0 , periodic for the same group. Then the Gauss maps of M and M_0 have the same image, the complement of at most 4 points.*

PROOF. Considering the quotient spaces and §III, we know $\deg \bar{g}_0 = \deg \bar{g}$ and g, g_0 have the same limiting values at the punctures. Therefore $g_0(M_0) = g(M)$. Now the proof follows Osserman's argument [4]; cf. [1]. We rotate $\bar{M}$ (and conjugate G) so that g takes finite nonzero values at the punctures and each pole of g is simple. Let $q_1, \dots, q_k \in S$ be the points missed by g , so $g^{-1}\{q_1, \dots, q_k\} \subset \{p_1, \dots, p_r\}$. Let $N = \deg g$ and at p_j let g assume its value with multiplicity $1 + a_j$. Then

$$K \cdot N \leq \sum_{j=1}^r (1 + a_j) = r + \sum_{j=1}^r a_j.$$

Let $\Omega = g'(z) dz$ and $n =$ the total order of branching of $g = \#$ zeros of Ω . At a simple pole of g , Ω has a double pole so $\#$ poles of $\Omega = 2N$. By Riemann's relation:

$$2N - n = 2 - 2S,$$

$S =$ the genus of $\bar{M}$. Now $\sum_{j=1}^r a_j$ is the order of branching at $p_1, \dots, p_r$, hence $\leq n$. Thus

$$K \cdot N \leq r + \sum_{j=1}^r a_j \leq r + n,$$

and since $n = 2(N + S - 1)$ we have

$$(I) \quad K \cdot N - r \leq 2(N + S - 1).$$

Now ω has a pole at each puncture p_j of order v_j (by completeness), so by Riemann's relation:

$$\sum_{j=1}^r v_j - \# \text{zeros of } \omega = 2 - 2S,$$

so

$$(II) \quad r \leq \sum_{j=1}^r v_j = 2 - 2S + 2N.$$

Combining (I) and (II) we have $K \leq 4$.

VI. The total curvature 4π minimal surfaces in R^3/Z . Consider $N = R^3/Z$ as R^3 modulo one vertical translation and let $M \subset N$ be a c.m.s. of $C(M) =$ the total curvature of $M = 4\pi$. The helicoid is an example and the projection of the catenoid and Enneper's surface are others. We will now describe how such M are obtained in general and exhibit new examples of periodic minimal surfaces in R^3 , invariant under a vertical translation.

Let (g, ω) be a Weierstrass pair associated to M . Since $g: \bar{M} \rightarrow S$ is bijective we know M is conformally a K -punctured sphere. We observe that $K \leq 4$. First perform a preliminary rotation of M (and conjugate G) so that g takes finite nonzero values at the punctures. Then g has one simple pole in M and no others in

$\bar{M}$. Riemann's relation for ω is: #poles of ω - #zeros of ω = 2. Completeness of M implies ω has poles at the punctures, and we know #zeros of ω = 2, hence #poles of ω = 4 and $K \leq 4$. Therefore M had at most 4 punctures before we rotated. Now we consider each case $K = 1, 2, 3, 4$.

When $K = 1$, M is parametrised by a conformal map $X: \mathbb{C} \rightarrow M$ with harmonic coordinate functions and X lifts to a conformal map $\tilde{X}: \mathbb{C} \rightarrow R^3$ which parametrises a c.m.s. $\tilde{M}$. The Gauss map of $\tilde{M}$ is also injective so $C(\tilde{M}) = 4\pi$ and $\tilde{M}$ is a catenoid or Enneper's surface. Thus M is the projection of one of these surfaces.

Consider now $K = 2$. We parametrise M by $\mathbb{C}^* = \mathbb{C} - 0$ so 0 and ∞ correspond to the punctures. Suppose first that g has a pole at ∞ . We can reparametrise M by composing X with a Mobius transformation leaving 0 and ∞ fixed so that $g(z) = z + a$ for some $a \in \mathbb{C}$. Now ω is a meromorphic differential on S , analytic on $\mathbb{C}^*$ and having a pole at 0, and no zeros in $\mathbb{C}$. Therefore $\omega = c dz/z^n$ for some integer $n \geq 1$ and $c \in \mathbb{C}^*$. The metric on M is

$$ds = \frac{|c|}{2|z|^n} (1 + |z^2 + 2az + a^2|).$$

By completeness of M we know that $\int_\gamma ds = \infty$ for every path γ tending to ∞ . This clearly implies $n < 4$.

Let us first look at $n = 1$, so $g(z) = z + a$, $\omega = cdz/z$. We have

$$\begin{aligned} \phi_1 &= \frac{c}{2z} (1 - z^2 - 2az - a^2), \quad \phi_2 = \frac{ic}{2z} (1 + z^2 + 2az + a^2), \\ \phi_3 &= c(z + a) dz/z. \end{aligned}$$

The coordinate functions $x_1 = \text{Re} \int \phi_1$ and $x_2 = \text{Re} \int \phi_2$ are single valued on M , hence the residues of ϕ_1 and ϕ_2 at 0 are real. This implies $c(1 - a^2)$ is real and $c(1 + a^2)$ is pure imaginary. It is natural to ask that

$$x_3 = \text{Re} \int \phi_3 = \text{Re} \left(c + ca \int dz/z \right)$$

be multivalued on M (otherwise the surface exists in R^3) hence the residue of ϕ_3 at 0 should have a nonzero imaginary part; i.e. $\text{Re}(2\pi aci) \neq 0$. Hence the conditions $c(1 - a^2)$ real, $c(1 + a^2)$ imaginary and $\text{Re}(aci) \neq 0$ determine all minimal surfaces in R^3/Z of the type considered. For example, take $c = 1$, $a = i$. We obtain

$$\begin{aligned} x_1 &= \text{Re} \left(\log z - \frac{z^2}{4} - iz \right), \\ x_2 &= \text{Re} \left(\frac{iz^2}{4} - z \right), \\ x_3 &= \text{Re}(z + i \log z). \end{aligned}$$

To obtain the lifting M of $\tilde{M}$ to R^3 we use the conformal covering map $e^z: \mathbb{C} \rightarrow \mathbb{C} - 0$ and

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\tilde{X}} & \tilde{M} \subset R^3 \\ e^z \downarrow & & \downarrow P \\ \mathbb{C} - 0 & \xrightarrow{X} & M \end{array}$$

so the coordinate functions $\tilde{x}_k$ of $\tilde{X}$ are given by $\tilde{x}_k(z) = x_k(e^z)$ and $\tilde{g} = g(e^z)$, $\tilde{\omega} = (e^z)^*\omega$. Hence $\tilde{M}$ is the surface parametrised by

$$\begin{aligned}\tilde{x}_1(z) &= \operatorname{Re}\left(z - \frac{e^{2z}}{4} - ie^z\right), \\ \tilde{x}_2(z) &= \operatorname{Re}\left(\frac{ie^{2z}}{4} - e^z\right), \\ \tilde{x}_3(z) &= \operatorname{Re}(e^z + iz).\end{aligned}$$

Writing $z = u + iv$, we obtain

$$\begin{aligned}\tilde{x}_1 &= u - \frac{e^{2u}\cos 2v}{4} + e^u\sin v, \\ \tilde{x}_2 &= -\frac{e^{2u}\sin 2v}{4} - e^u\sin v, \\ \tilde{x}_3 &= e^u\cos v - v.\end{aligned}$$

Notice that replacing v by $v + 2\pi$ and keeping u fixed, changes $\tilde{x}_3$ by -2π and leaves $\tilde{x}_1, \tilde{x}_2$ fixed. Thus $\tilde{M}$ is a periodic surface, invariant by the vertical translation by 2π .

Next consider $n = 2$, so $g(z) = z + a$, $\omega = cdz/z^2$. Letting Γ denote the unit circle in $\mathbb{C}$ we have

$$\int_{\Gamma} \phi_1 = -ac2\pi i, \quad \int_{\Gamma} \phi_2 = -ac\pi.$$

Since x_1, x_2 are single valued on M we have ac is real and pure imaginary, thus $a = 0$. Notice that $\int_{\Gamma} \phi_3 = 2\pi ic$, so if this surface is to exist genuinely in R^3/Z we must have $\operatorname{Im}(c) \neq 0$. Now a calculation yields:

$$\begin{aligned}\tilde{x}_1 &= -r\cos v \cosh u + s\sin v \sinh u, \\ \tilde{x}_2 &= -s\cos v \sinh u - r\sin v \cosh u, \\ \tilde{x}_3 &= ru - sv,\end{aligned}$$

where $c = r + is$, $z = u + iv$. For $r = 0$ this is a helicoid and for $s = 0$ a catenoid. For $c = e^{i\theta}$, $0 \leq \theta \leq \pi/2$, we obtain the usual deformation of the helicoid into its associate surfaces. Whenever $s \neq 0$, the surface is invariant by vertical translation by 2π ; changing v by 2π does not change $\tilde{x}_1, \tilde{x}_2$.

The case $n = 3$ is impossible since then $\int_{\Gamma} \phi_1 = -2\pi ic$ and $\int_{\Gamma} \phi_2 = -2\pi c$; therefore c would be both real and imaginary, hence 0.

To complete the study of two punctures it remains to consider the case when g takes a finite value at ∞ . We reparametrise M by $\mathbb{C} - 0$ so that

$$g(z) = \frac{az + b}{z + d},$$

so that g has a simple pole at $-d$. We know $\omega = (P(z)/Q(z))dz$, where P has a double zero at $-d$ if $d \neq 0$ and Q can vanish only at 0. Suppose first that $d = 0$ so $\omega = cz^m dz$ for some integer m . If $m \geq 2$, then ϕ_k , $k = 1, 2, 3$, are analytic at 0, so M is parametrised by $\mathbb{C}$ and we know by the previous case what M is, the projection

of a catenoid or Enneper's surface. Suppose $m = 0$. Then calculating residues of ϕ_1 and ϕ_2 at 0 and using the fact that x_1, x_2 are single valued we conclude abc is real and imaginary, hence $a = 0$ or $b = 0$. Clearly $b \neq 0$ since g is injective, hence $a = 0$. The residue of ϕ_3 at 0 is cb so $\text{Im}(cb) = 0$ if x_3 is to be multivalued; otherwise M is the projection of a catenoid since $\text{Res}(\phi_1) = \text{Res}(\phi_2) = 0$, so M lifts to R^3 , parametrised by $\mathbb{C} - 0$ and of total curvature 4π . A calculation yields

$$\begin{aligned}\tilde{x}_1 &= \text{Re}\left(\frac{c}{2}(e^z + b^2e^{-z})\right), \\ \tilde{x}_2 &= \text{Re}\left(\frac{ic}{2}(e^z - b^2e^{-z})\right), \\ \tilde{x}_3 &= \text{Re}(cbz).\end{aligned}$$

These are the "associate type" surfaces of the helicoid (the helicoid is $c = i, b = 1$). Now suppose $g(z) = b/z, \omega = czdz$. Then $\text{Res}\phi_1 = -cb^2/2$ and $\text{Res}(\phi_2) = icb^2$, so cb^2 is real and imaginary, hence $b = 0$ which is impossible.

Next we study $d \neq 0$. Since g is analytic at 0, ω must have a pole at 0 by completeness of M . Also ω has a double zero at $-d$ since g has a simple pole there. So $\omega = c(z+d)^2 dz/z^n$ for some $c \in \mathbb{C} - 0$ and $n \geq 1$. We have

$$ds = \left| \frac{c(z+d)^2}{2z^n} \right| \left| 1 + \frac{az+b}{z+d} \right|^2 |dz|.$$

Completeness at ∞ implies $n < 4$. If $n = 1$, then calculating residues of ϕ_1, ϕ_2 at 0 yield $c(d^2 - b^2)$ is real and $c(d^2 + b^2)$ is imaginary (using the fact that x_1, x_2 are single valued). We want x_3 to be multivalued which yields $\text{Im}(cbd) \neq 0$. Such surfaces are easily realised, e.g., $d^2 = \bar{b}^2$ and c imaginary. When $n = 2$ the analogous calculations yield $c(d - ab)$ real, $c(d + ab)$ imaginary and $\text{Im}(c(b + da)) \neq 0$. For $n = 3$, $c(1 - a^2)$ real, $c(1 + a^2)$ imaginary and $\text{Im}(ac) \neq 0$; e.g., $a = 1, c$ imaginary. This completes the study of two punctures.

We will not treat the 3- and 4-puncture case in detail; we will make some general remarks and give some examples. Consider the 3-puncture case. Since there is only one conformal structure on a 3-punctured sphere, we may parametrise M by $S - \{\theta_1, \theta_2, \theta_3\}$, where θ_k are the cube roots of unity. Now $g(z) = (az + b)/(z + d)$ and we suppose $d \neq -\theta_k, k = 1, 2, 3$. Then ω has a double zero at $-d$ and a pole at each θ_k , so

$$\omega = (c(z+d)^2)/((z-\theta_1)^{n_1}(z-\theta_2)^{n_2}(z-\theta_3)^{n_3}).$$

Since ∞ is a regular point of M and ω is not zero there we have $n_1 + n_2 + n_3 \leq 4$, so there are several cases to consider (we will not do it). In order that M exist, one must choose a, b, c, d so that the residues of ϕ_1, ϕ_2 are real at each puncture and the imaginary parts of the residues of ϕ_3 at each puncture must generate a nontrivial cyclic group. If we assume ∞ is a pole of g , the situation is simpler. Then $g(z) = az + b$ and

$$\omega = (cdz)/((z-\theta_1)^{n_1}(z-\theta_2)^{n_2}(z-\theta_3)^{n_3}).$$

Now ∞ is a regular point of M , so letting $u = 1/z$, $\omega(u)$ must have a double pole at

0. This yields $n_1 + n_2 + n_3 = 4$. One can analyse this situation completely. For example taking $a = 1$, $b = 0$, c real and $n_1 = 2$, $n_2 = n_3 = 1$, a (tedious) calculation shows this surface exists in R^3/Z .

When M is a 4-punctured sphere, the conformal structure may change. One can check the following example exists:

$$g(z) = z, \quad c = 4i, \quad \text{and} \quad \omega = \frac{cdz}{(z-1)(z-i)(z+1)(z+i)},$$

$$M \text{ is } S - \{1, -1, i, -i\}.$$

Can every conformal structure be realised? Let $z_0 \in \mathbb{C}$ and $g(z) = az + b$, $\omega = cdz/(z - z_0)(z_0 + 1)(z_0 - i)(z_0 + i)$. Can one choose a , b and c so that $M = S - \{z_0, -1, i, -i\}$ exists?

VII. Some isolated periodic surfaces.

PROPOSITION. *Let $\tilde{M}_0$ be the usual helicoid in R^3 : $\tilde{g}_0(z) = +ie^z$, $\tilde{\omega}_0 = e^{-z} dz$. Then $\tilde{M}_0$ is isolated among complete minimal surfaces invariant under vertical translation by 2π .*

PROOF. Let M_0 be the projection of $\tilde{M}_0$ to R^3/Z , so that $C(M_0) = 4\pi$, M_0 is a 2-punctured sphere and $g_0(z) = iz$, $\omega_0 = dz/z^2$. A deformation M of $\tilde{M}_0$ invariant under vertical translation by 2π gives a deformation M of M_0 in R^3/Z . M is also conformally a 2-punctured sphere and the Gauss map g of M is bijective. We parametrise M by $\mathbb{C} - 0$. The limiting values of g and g_0 are the same at the punctures 0 and ∞ so we can parametrise M so that $g(z) = iz$. Now ω is a meromorphic differential on S with no zeros in $\mathbb{C} - 0$ and a pole at 0, hence $\omega = c dz/z^k$ for some $c \in \mathbb{C}$, $k \geq 1$. The metric on M is

$$ds = \frac{|c|}{|z|^k} (1 + |z|^2)$$

so by completeness of M at ∞ we have $k \leq 3$. If $k = 1$, then $\phi_1 = c(1 + z^2)/2z$ and $\phi_2 = ic(1 - z^2)/2z$. Hence $\text{Res}(\phi_1, 0) = c/2$ and $\text{Res}(\phi_2, 0) = ic/2$. Since $x_1 = \text{Re} f \phi_1$, $x_2 = \text{Re} f \phi_2$ are single valued functions on M , c is both real and imaginary, so $c = 0$, a contradiction. A similar reasoning shows $k = 3$ is impossible, and $\omega = cdz/z^2$.

We have $\text{Res}(\phi_3, 0) = -c2\pi$ so $\text{Re}(c) = \pm 1$. Now lift M to $\tilde{M} \subset R^3$ by the conformal covering $e^z: \mathbb{C} \rightarrow \mathbb{C} - 0$. This yields $\tilde{g} = g(e^z)$, $\tilde{\omega} = (e^z)^*\omega$ and a calculation shows (writing $c = a + ib$)

$$\begin{aligned} \tilde{x}_1 &= a \cos v \sinh u - b \sin v \cosh u, \\ \tilde{x}_2 &= b \cos v \cosh u + a \sin v \sinh u, \\ \tilde{x}_3 &= -(bu + av). \end{aligned}$$

Considering the trace of $\tilde{M}$ in the plane $\tilde{x}_3 = 0$ it is clear that $b = 0$ since $\tilde{M}$ is a deformation of $\tilde{M}_0$.¹ Thus $\tilde{M}$ is also a helicoid.

¹The surface is a homothety of an associate surface of the helicoid and is not embedded for $b \neq 0$.

PROPOSITION. *The surface $M_0: g_0(z) = z$, $\omega_0 = 4idz/(z^4 - 1)$ (modelled on S punctured at the 4th roots of unity) is isolated in R^3/Z .*

PROOF. Let M be a deformation of M_0 . The total curvature of M is 4π so $\bar{g}$ is bijective. Since $\bar{g}$ at the punctures misses precisely the 4th roots of unity $1, -1, i, -i$, g is a conformal equivalence between M and $S - \{1, -1, i, -i\} = S_0$. Hence we may parametrise M by S_0 so that $g(z) = z$. Then ω is a meromorphic differential on S with no zeros in $\mathbb{C}$ and poles at $\{1, -1, i, -i\}$, so

$$\omega = \frac{cdz}{(z-1)^{n_1}(z+1)^{n_2}(z-i)^{n_3}(z+i)^{n_4}}.$$

Since ∞ is a simple pole of g , it is a double zero of ω , hence $n_1 + n_2 + n_3 + n_4 = 4$. Thus $\omega = cdz/(z^4 - 1)$. We have the residue at ϕ_3 at each puncture is $\pm c/4$, and since M is invariant by vertical translation by 2π , the real part of $\pi ic/2 = \pm 2\pi$, i.e. $\text{Im}(c) = \pm 4$. Now $\text{Res}(\phi_1, i) = ic/2$ and x_1 is single valued on M so c is pure imaginary. Hence $M = M_0$.

Similar arguments apply to other 4π total curvature minimal surfaces in translation spaces; e.g. Scherk's surface, $g(z) = z$, $\omega = 2dz/(z^4 - 1)$, is isolated in R^3/G , G the group generated by translation by $(2\pi, 0, 0)$ and $(0, 2\pi, 0)$.

We have a partial result concerning deformations of the helicoid which are close at ∞ .

PROPOSITION. *Let M be a deformation of the helicoid M_0 satisfying:*

1. M is parabolic,
2. M is transverse to every horizontal plane $x_3 = \text{constant}$,
3. the curve $M \cap \{x_3 = \text{constant}\}$ makes a bounded angle with $M_0 \cap \{x_3 = \text{constant}\}$, the bound independent of the plane $x_3 = \text{constant}$.

Then M is a helicoid.

PROOF. Let $\sigma: M \rightarrow R$ be the angle which the curve $M \cap \{x_3 = \text{constant}\}$ makes with the vector $(1, 0, 0)$. A priori, σ is a local function on M but since M is simply connected and M is transverse to every horizontal plane, we can extend σ to a continuous (unbounded) function on M . It is a general fact that σ is a harmonic function on M [5]. Now M is a minimal surface so x_3 is also a harmonic function on M . Therefore $x_3 - \sigma$ is a harmonic function on M which is bounded by condition 2. But by condition 1, M is conformally parametrised by $\mathbb{C}$, so $x_3 - \sigma$ is constant. Hence for $x_3 = \text{constant}$ we have $\sigma = \text{constant}$, i.e., $M \cap \{x_3 = \text{constant}\}$ is a straight line. Therefore M is a helicoid.

REMARKS. 1. If ε is small enough, then M is certainly parabolic since the orthogonal projection is a quasiconformal homeomorphism and M_0 is conformally parametrised by $\mathbb{C}$.

2. It is elementary that for any minimal surface M there is a positive function $\varepsilon(x)$ on M such that any deformation of M which is $\varepsilon(x)$ -close to M , is equal to M .

BIBLIOGRAPHY

1. F. Gacksatter, *Über Abelsche Minimalflächen*, Math. Nachr. **74** (1976), 157–165.
2. B. Lawson, *Lectures on minimal submanifolds*, vol. 1, Math. Lecture Series 9, Publish or Perish, Berkeley, Calif., 1980.
3. W. Meeks, *A survey of the geometric results in the classical theory of minimal surfaces*, Bol. Soc. Brasil. Mat. **12** (1981), 29–86.
4. R. Osserman, *A survey of minimal surfaces*, Van Nostrand-Reinhold, New York, 1969.
5. M. Shiffman, *On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes*, Ann. of Math. (2) **63** (1956), 77–90.
6. F. Xavier, *The Gauss map of a complete non flat minimal surface cannot omit 7 points of the sphere*, Ann. of Math. **113** (1981), 211–214.

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