

CONGRUENCES ON REGULAR SEMIGROUPS

FRANCIS PASTIJN AND MARIO PETRICH

ABSTRACT. Let S be a regular semigroup and let ρ be a congruence relation on S . The kernel of ρ , in notation $\ker \rho$, is the union of the idempotent ρ -classes. The trace of ρ , in notation $\text{tr } \rho$, is the restriction of ρ to the set of idempotents of S . The pair $(\ker \rho, \text{tr } \rho)$ is said to be the congruence pair associated with ρ . Congruence pairs can be characterized abstractly, and it turns out that a congruence is uniquely determined by its associated congruence pair. The triple $((\rho \vee \mathcal{L})/\mathcal{L}, \ker \rho, (\rho \vee \mathcal{R})/\mathcal{R})$ is said to be the congruence triple associated with ρ . Congruence triples can be characterized abstractly and again a congruence relation is uniquely determined by its associated triple.

The consideration of the parameters which appear in the above-mentioned representations of congruence relations gives insight into the structure of the congruence lattice of S . For congruence relations ρ and θ , put $\rho T_l \theta$ [$\rho T_r \theta$, $\rho T \theta$] if and only if $\rho \vee \mathcal{L} = \theta \vee \mathcal{L}$ [$\rho \vee \mathcal{R} = \theta \vee \mathcal{R}$, $\text{tr } \rho = \text{tr } \theta$]. Then T_l , T_r and T are complete congruences on the congruence lattice of S and $T = T_l \cap T_r$.

Introduction and summary. After it was realized by Wagner that a congruence on an inverse semigroup S is uniquely determined by its idempotent classes, Preston provided an abstract characterization of such a family of subsets of S called the kernel normal system (see [2, Chapter 10]). This approach was the only usable means for handling congruences on inverse semigroups for two decades. A new approach to the problem of describing congruences on inverse semigroups was sparked by the work of Scheiblich [13] who described congruences in terms of kernels and traces. A systematic exposition of the achievements of this approach can be found in [10, Chapter III].

It was recognized by Feigenbaum [3] that every congruence ρ on a regular semigroup S is uniquely determined by its kernel, $\ker \rho$, equal to elements ρ -equivalent to idempotents, and its trace, $\text{tr } \rho$, equal to the restriction of ρ to the set $E(S)$ of idempotents of S . In the case of an inverse semigroup S , $\ker \rho$ and $\text{tr } \rho$, as well as their mutual relationship, can be described abstractly by means of simple conditions on a subset of S and an equivalence on $E(S)$ (see [10, Chapter III]). Following in the footsteps of Scheiblich, for orthodox and arbitrary regular semigroups, Feigenbaum [3] and Trotter [14] adopted the following approach: $\text{tr } \rho$ is characterized abstractly and to each such $\text{tr } \rho$ all matching kernels are described. This unbalances the symmetry of the kernel-trace approach by giving preference to the trace. Hence a balanced view relative to the kernel and the trace is evidently called for.

The unqualified success of the kernel-trace approach for inverse semigroups, including its diverse ramifications, gave a certain hope that this may also turn out to be the case for regular semigroups. Judging by the complexity of regular semigroups and the attempts made for both orthodox and general regular semigroups,

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the handling of congruences on regular semigroups must of necessity be more complicated, and thus less explicit, than on inverse semigroups. With this pessimistic forecast, we are actually in for a few pleasant surprises.

After taking care of some preparations in §1, in §2 a number of new concepts are introduced. These concepts amount to abstract characterizations of the kernel and the trace of a congruence on a regular semigroup and are used for defining a congruence pair. The climax of this section represents a description of congruences in terms of congruence pairs. Contrary to reasonable expectation that the axioms for a congruence pair ought to be complex, they are actually quite concise but not what one might call explicit.

Certain results in §2 yield expressions for the least and the greatest congruences with a given trace or kernel. Unlike in inverse semigroups, these do not seem to admit simple expressions in closed form. These deliberations lead in §3 to the concept of a min network whose basis, it turns out, is of lattice-theoretical nature.

In §4, the left and the right traces of a congruence are introduced as well as their abstract counterparts, the left and the right normal equivalences. Their mutual relationship and their connection with certain congruences on the congruence lattice are then explored in some detail. Classes of these congruences are intervals of the congruence lattice; their minimum and maximum elements are found explicitly. It is proved that the mapping $\rho \rightarrow \text{tr } \rho$ is a complete homomorphism for several classes of regular semigroups, but the general case remains open.

Various properties of the intervals mentioned above are explored in §5. Modularity of some of these intervals and commutativity of certain congruences are established.

In the final §6, congruence triples are introduced which are then used to describe an arbitrary congruence. These triples make use of the normal equivalences on the sets S/\mathcal{L} and S/\mathcal{R} as well as a normal subset of S , which is an abstraction of the kernel. These triples represent a close analogue of admissible triples which are used for describing congruences on a Rees matrix semigroup. For arbitrary regular semigroups they seem the natural setting for an elaborate study of congruences. They admit several natural ramifications thereby providing for unexpected diversity of congruences associated with a given congruence.

1. Preliminaries. In the following we shall use the notation and terminology of [2 and 5]. This will be supplemented with the following.

Throughout the entire paper, S stands for an arbitrary regular semigroup. If $a \in S$, then $V(a)$ denotes the set of inverses of a in S . The set of idempotents of S is denoted by $E(S)$. On $E(S)$ we shall consider the relations \leq_l, \leq_r and \leq given by, for $e, f \in E(S)$,

$$\begin{aligned} e \leq_l f &\Leftrightarrow ef = e, \\ e \leq_r f &\Leftrightarrow fe = e, \\ e \leq f &\Leftrightarrow ef = e = fe. \end{aligned}$$

The relation $\leq = \leq_l \cap \leq_r$ is the *natural partial order* on $E(S)$. For $e, f \in E(S)$,

$$S(e, f) = fV(ef)e$$

is the *sandwich set* of e and f .

1.1. RESULT [8, THEOREM 1.1, PROPOSITION 2.5]. *If e and f are idempotents of S , then*

$$\begin{aligned} S(e, f) &= \{h \in E(S) \mid fh = h = he \text{ and } ef = ehf\} \\ &= \{h \in E(S) \mid fh = h = he \text{ and } h \in V(ef)\}. \end{aligned}$$

If ρ is a congruence on S and $h \in S(e, f)$, then $h\rho \in S(e\rho, f\rho)$.

The nice behaviour of idempotents in connection with homomorphisms is further illustrated by the following.

1.2. RESULT [6] (see also [5, LEMMA II.2.7]). *If ρ is a congruence on S , then $a\rho \in S/\rho$ is an idempotent if and only if $a\rho e$ for some $e \in E(S)$.*

The following simple statements will be applied without further mention: for $e, f \in E(S)$,

$$\begin{aligned} e \leq_l f &\Rightarrow fe \in E(S) \text{ and } e \mathcal{L} fe \leq f, \\ e \leq_r f &\Rightarrow ef \in E(S) \text{ and } e \mathcal{R} ef \leq f, \\ e \mathcal{L} f &\Rightarrow S(e, f) = \{f\}, \\ e \mathcal{R} f &\Rightarrow S(e, f) = \{e\}. \end{aligned}$$

For a set X , $\text{Eq } X$ is the lattice of equivalence relations on X ordered by inclusion. Further, $\text{Con } S$ is the lattice of congruences on S . We remark that $\text{Con } S$ is a complete sublattice of $\text{Eq } S$. The least element of both $\text{Eq } S$ and $\text{Con } S$ is ε , the equality on S , and the greatest element of both $\text{Eq } S$ and $\text{Con } S$ is ω , the universal relation. We shall sometimes use the notation ε_S, ω_S to avoid confusion.

Let τ be a relation on S . The congruence generated by τ is denoted by τ^* : it is the intersection of all congruences on S which contain τ . If γ is an equivalence on S , then γ^0 is the greatest congruence on S contained in γ .

The remaining results in this section are true for arbitrary semigroups.

1.3. RESULT [2, LEMMA 10.3]. *If γ is an equivalence on S , then for $a, b \in S$,*

$$a\gamma^0 b \Leftrightarrow (xay)\gamma(xby) \quad \text{for all } x, y \in S^1.$$

1.4. RESULT. *If \mathcal{F} is a family of relations on S , then*

$$\bigvee_{\tau \in \mathcal{F}} \tau^* = \left(\bigcup \mathcal{F} \right)^*.$$

If \mathcal{F} is a family of equivalences on S , then

$$\bigcap_{\gamma \in \mathcal{F}} \gamma^0 = \left(\bigcap \mathcal{F} \right)^0.$$

PROOF. The straightforward argument is omitted.

Let $K \subseteq S$. A congruence ρ on S saturates K if $a \in K$ implies $a\rho \subseteq K$. The greatest congruence on S which saturates K is denoted by π_K .

1.5. RESULT. *Let $K \subseteq S$. Then for $a, b \in S$, $a\pi_K b$ if and only if*

$$xay \in K \Leftrightarrow xby \in K \quad (x, y \in S^1).$$

Define a relation θ_K on S by

$$a\theta_K b \Leftrightarrow a, b \in K \text{ or } a, b \in S \setminus K.$$

Then $\theta_K \in \text{Eq } S$ and $\pi_K = \theta_K^0$.

PROOF. This follows routinely from Result 1.3.

Let X be a set and $\alpha, \beta \in \text{Eq } X$ be such that $\alpha \subseteq \beta$. We define a relation β/α on X/α by

$$(a\alpha)(\beta/\alpha)(b\alpha) \Leftrightarrow a\beta b \quad (a, b \in X).$$

Then $\beta/\alpha \in \text{Eq}(X/\alpha)$.

The following results deal with commuting equivalences.

1.6. RESULT [6, PROPOSITION III.3.8]. *If $\lambda, \rho \in \text{Con } S$ are such that $\lambda \subseteq \mathcal{L}$ and $\rho \subseteq \mathcal{R}$, then $\lambda\rho = \rho\lambda$.*

1.7. RESULT. *Let X be a set and $\alpha, \beta, \gamma \in \text{Eq } X$ be such that $\alpha \subseteq \beta$ and $\alpha \subseteq \gamma$. Then*

$$(\beta/\alpha)(\gamma/\alpha) = (\gamma/\alpha)(\beta/\alpha) \Leftrightarrow \beta\gamma = \gamma\beta.$$

If this is the case, then $\beta \vee \gamma = \beta\gamma$.

PROOF. The equivalence of the above statements follows easily from the definitions. The final assertion is proved in [2, Lemma 1.4].

2. Congruence pairs. We begin our study of congruences on a regular semigroup by the consideration of the kernel and the trace of a congruence and their mutual relationship.

2.1. DEFINITION. For a congruence ρ on S ,

- (i) $\text{tr } \rho = \rho|_{E(S)}$ is the *trace* of ρ ,
- (ii) $\ker \rho = \{s \in S \mid s\rho e \text{ for some } e \in E(S)\}$ is the *kernel* of ρ .

Note that in view of Result 1.2 we have

$$\ker \rho = \{s \in S \mid s\rho s^2\}$$

for a congruence ρ .

We may thus associate to each congruence ρ on a regular semigroup the ordered pair $(\ker \rho, \text{tr } \rho)$. Our first task is to give an abstract characterization of such a pair and to prove that the pair $(\ker \rho, \text{tr } \rho)$ uniquely determines ρ . An abstract characterization of the trace of a congruence was given in [14]. We adopt a different approach here. To this end, we first introduce the following basic concepts.

2.2. DEFINITION. An equivalence τ on $E(S)$ is *normal* if $\tau = \text{tr } \tau^*$. A subset K of S is *normal* if $K = \ker \pi_K$.

2.3. LEMMA. *An equivalence τ on $E(S)$ is normal if and only if τ is the trace of a congruence on S . If this is the case, then τ^* is the least congruence on S with trace τ .*

PROOF. If ρ is a congruence on S with $\tau = \text{tr } \rho$, then $\tau \subseteq \rho$; thus $\tau^* \subseteq \rho$, whence $\text{tr } \tau^* \subseteq \text{tr } \rho = \tau$. Since $\tau \subseteq \text{tr } \tau^*$ trivially holds, it follows that $\text{tr } \tau^* = \tau$.

2.4. LEMMA. *A subset K of S is normal if and only if K is the kernel of a congruence on S . If this is the case, then π_K is the greatest congruence on S with kernel K .*

PROOF. If ρ is a congruence on S with $\ker \rho = K$, then ρ saturates K . Since π_K is the greatest congruence saturating K we have $\rho \subseteq \pi_K$, whence $K = \ker \rho \subseteq \ker \pi_K$. If $k \in \ker \pi_K$, then $k\pi_K e$ for some $e \in E(S)$. Since $e \in K$ we also have $k \in K$ since K is a union of π_K -classes. Thus $K = \ker \pi_K$.

The following simple results show that both the kernel and the trace are compatible with intersections.

2.5. LEMMA. Let \mathcal{F} be a nonempty family of congruences on S . Then

$$(i) \ker(\bigcap \mathcal{F}) = \bigcap_{\rho \in \mathcal{F}} \ker \rho,$$

$$(ii) \operatorname{tr}(\bigcap \mathcal{F}) = \bigcap_{\rho \in \mathcal{F}} \operatorname{tr} \rho.$$

PROOF. For any $s \in S$, we have

$$\begin{aligned} s \in \ker\left(\bigcap \mathcal{F}\right) &\Leftrightarrow s\left(\bigcap \mathcal{F}\right)s^2 \Leftrightarrow s\rho s^2 \quad \text{for all } \rho \in \mathcal{F} \\ &\Leftrightarrow s \in \bigcap_{\rho \in \mathcal{F}} \ker \rho, \end{aligned}$$

which proves part (i). Furthermore, for any $e, f \in E(S)$, we have

$$\begin{aligned} e \operatorname{tr}\left(\bigcap \mathcal{F}\right) f &\Leftrightarrow e\left(\bigcap \mathcal{F}\right) f \Leftrightarrow e\rho f \quad \text{for all } \rho \in \mathcal{F} \\ &\Leftrightarrow e\left(\bigcap_{\rho \in \mathcal{F}} \operatorname{tr} \rho\right) f, \end{aligned}$$

establishing part (ii).

The next lemma will be used many times.

2.6. LEMMA. Let ρ be a congruence on S with $\tau = \operatorname{tr} \rho$.

$$(i) (e\rho) \mathcal{R} (f\rho) \text{ in } S/\rho \Leftrightarrow e(\tau \mathcal{R} \tau) f \text{ in } S \ (e, f \in E(S)).$$

$$(ii) (a\rho) \mathcal{R} (b\rho) \text{ in } S/\rho \Leftrightarrow a(\mathcal{R} \tau \mathcal{R} \tau \mathcal{R}) b \text{ in } S \ (a, b \in S).$$

$$(iii) \rho \mathcal{R} \rho = \mathcal{R} \tau \mathcal{R} \tau \mathcal{R} = \rho \vee \mathcal{R},$$

$$(iv) \tau \mathcal{R} \tau \mathcal{R} \tau = \tau \mathcal{R} \tau = (\rho \vee \mathcal{R})|_{E(S)} = \tau \vee (\mathcal{R}|_{E(S)}).$$

$$(v) \tau = \operatorname{tr}(\mathcal{L} \tau \mathcal{L} \tau \mathcal{L} \cap \mathcal{R} \tau \mathcal{R} \tau \mathcal{R})^0 = \tau \mathcal{L} \tau \cap \tau \mathcal{R} \tau.$$

PROOF. (i) Let $e, f \in E(S)$ be such that $(e\rho) \mathcal{R} (f\rho)$ in S/ρ . If $h \in S(e, f)$, then $h\rho \in S(e\rho, f\rho) = \{e\rho\}$ by Result 1.1, so $e\rho h$. Further $hf \in E(R_h)$ and thus $(hf)\rho = (h\rho)(f\rho) = (e\rho)(f\rho) = f\rho$ since $(e\rho) \mathcal{R} (f\rho)$. Consequently $(hf)\rho f$. Therefore $e\tau h\mathcal{R}(hf)\tau f$ holds in S .

If conversely $e\tau g\mathcal{R}k\tau f$ in S for some $e, g, k, f \in E(S)$, then in S/ρ ,

$$e\rho = (g\rho) \mathcal{R} (k\rho) = f\rho.$$

(ii) If $(a\rho) \mathcal{R} (b\rho)$ in S/ρ for $a, b \in S$, then for $a' \in V(a)$, $b' \in V(b)$, we have

$$a \mathcal{R} (aa') = e \in E(S), \quad b \mathcal{R} (bb') = f \in E(S)$$

and

$$(a\rho) \mathcal{R} (e\rho) \mathcal{R} (f\rho) \mathcal{R} (b\rho) \quad \text{in } S/\rho.$$

Using part (i) we see that $e(\tau \mathcal{R} \tau) f$, whence $a(\mathcal{R} \tau \mathcal{R} \tau \mathcal{R}) b$.

If conversely $a \mathcal{R} e\tau g\mathcal{R}h\tau f \mathcal{R} b$ in S for $a, b \in S$ and $e, f, g, h \in E(S)$, then in S/ρ

$$(a\rho) \mathcal{R} (e\rho) = (g\rho) \mathcal{R} (h\rho) = (f\rho) \mathcal{R} (b\rho),$$

and thus $(a\rho) \mathcal{R} (b\rho)$.

(iii) If $a \mathcal{R} \tau \mathcal{R} \tau \mathcal{R} b$, then by part (ii) we have $(a\rho) \mathcal{R} (b\rho)$. Let $e \in E(R_a)$, $f \in E(R_b)$ and $h \in S(e, f)$. From Result 1.1 we have $h\rho \in S(e\rho, f\rho) = \{e\rho\}$ and thus $e\rho h$, whence $e\rho(eh)$. If $a' \in V(a)$ with $aa' = e$, then $(eha)a' = eh$ since $he = h$. Therefore $(eha) \mathcal{R} (eh)$, where $(eh)a\rho ea = a$. From Result 1.1 and since \mathcal{R} is a left

congruence, we also have $(eh) \mathcal{R} (ehf) = (ef) \mathcal{R} (eb)$, where $b = (fb) \rho (eb)$. We have thus proved that $a \rho (eha) \mathcal{R} (eh) \mathcal{R} (eb) \rho b$. We conclude that $\mathcal{R} \tau \mathcal{R} \tau \mathcal{R} \subseteq \rho \mathcal{R} \rho$.

Conversely, if $a \rho c \mathcal{R} d \rho b$, then $a \rho = (c \rho) \mathcal{R} (d \rho) = (b \rho)$, and thus by part (ii), $a \mathcal{R} \tau \mathcal{R} \tau \mathcal{R} b$.

It follows from part (ii) that $\rho \mathcal{R} \rho$ is an equivalence relation and thus clearly $\rho \mathcal{R} \rho = \rho \vee \mathcal{R}$ in $\text{Eq } S$.

(iv) By part (i), we have $\tau \mathcal{R} \tau \in \text{Eq } E(S)$, so that by transitivity

$$\tau \mathcal{R} \tau = (\tau \mathcal{R} \tau)^2 = \tau \mathcal{R} \tau^2 \mathcal{R} \tau = \tau \mathcal{R} \tau \mathcal{R} \tau.$$

It is easy to see that $\tau \mathcal{R} \tau = \tau \vee (\mathcal{R}|_{E(S)})$ in $\text{Eq } E(S)$, whereas $\tau \mathcal{R} \tau = (\rho \vee \mathcal{R})|_{E(S)}$ is an immediate consequence of parts (i), (ii) and (iii).

(v) Using part (iii) and its dual we have

$$\rho \subseteq (\rho \vee \mathcal{L}) \cap (\rho \vee \mathcal{R}) = \mathcal{L} \tau \mathcal{L} \tau \mathcal{L} \cap \mathcal{R} \tau \mathcal{R} \tau \mathcal{R},$$

hence

$$\rho \subseteq (\mathcal{L} \tau \mathcal{L} \tau \mathcal{L} \cap \mathcal{R} \tau \mathcal{R} \tau \mathcal{R})^0.$$

Therefore

$$\begin{aligned} \tau &= \text{tr } \rho \subseteq \text{tr}(\mathcal{L} \tau \mathcal{L} \tau \mathcal{L} \cap \mathcal{R} \tau \mathcal{R} \tau \mathcal{R})^0 \\ &= \text{tr}((\mathcal{L} \tau \mathcal{L} \tau \mathcal{L})^0 \cap (\mathcal{R} \tau \mathcal{R} \tau \mathcal{R})^0) \quad (\text{by Result 1.4}) \\ &= \text{tr}(\mathcal{L} \tau \mathcal{L} \tau \mathcal{L})^0 \cap \text{tr}(\mathcal{R} \tau \mathcal{R} \tau \mathcal{R})^0 \quad (\text{by Lemma 2.5}) \\ &\subseteq (\mathcal{L} \tau \mathcal{L} \tau \mathcal{L})|_{E(S)} \cap (\mathcal{R} \tau \mathcal{R} \tau \mathcal{R})|_{E(S)} \\ &= \tau \mathcal{L} \tau \cap \tau \mathcal{R} \tau \quad (\text{by (iii) and (iv)}). \end{aligned}$$

If $e \tau g \mathcal{R} h \tau f$ for some $e, f, g, h \in E(S)$, then $e \rho = (g \rho) \mathcal{R} (h \rho) = f \rho$, that is, $(e \rho) \mathcal{R} (f \rho)$. From this and its dual we see that $e(\tau \mathcal{L} \tau \cap \tau \mathcal{R} \tau) f$ implies that $(e \rho) \mathcal{X} (f \rho)$, whence $e \rho = f \rho$. We have proved that $\tau \mathcal{R} \tau \cap \tau \mathcal{L} \tau \subseteq \tau$. Therefore, the desired equalities hold.

We record two more equivalent characterizations of the equivalence $\tau \mathcal{R} \tau$.

2.7. LEMMA. *Let τ be the trace of a congruence on S . Then for $e, f \in E(S)$, the following are equivalent:*

- (i) $e \tau \mathcal{R} \tau f$,
- (ii) $\square \neq (e \tau)(f \tau) \cap E(S) \subseteq f \tau$, $\square \neq (f \tau)(e \tau) \cap E(S) \subseteq e \tau$,
- (iii) $S(e, f) \subseteq e \tau$, $S(f, e) \subseteq f \tau$.

PROOF. This is immediate from Lemma 2.6 and [14].

2.8. LEMMA. *An equivalence τ on $E(S)$ is normal if and only if $\tau \mathcal{L} \tau \mathcal{L} \tau = \tau \mathcal{L} \tau$, $\tau \mathcal{R} \tau \mathcal{R} \tau = \tau \mathcal{R} \tau$ and $\tau = \text{tr}(\mathcal{L} \tau \mathcal{L} \tau \mathcal{L} \cap \mathcal{R} \tau \mathcal{R} \tau \mathcal{R})^0$. If this is the case, then $(\mathcal{L} \tau \mathcal{L} \tau \mathcal{L} \cap \mathcal{R} \tau \mathcal{R} \tau \mathcal{R})^0$ is the greatest congruence on S with trace τ .*

PROOF. If τ is a normal equivalence, then $\tau = \text{tr } \rho$ for some congruence ρ on S by Lemma 2.3. By Lemma 2.6(iv), (v), τ satisfies the required conditions.

If $\tau \in \text{Eq } E(S)$ with $\tau \mathcal{L} \tau \mathcal{L} \tau = \tau \mathcal{L} \tau$, then

$$(\mathcal{L} \tau \mathcal{L} \tau \mathcal{L})^2 = \mathcal{L} \tau \mathcal{L} \tau \mathcal{L}^2 \tau \mathcal{L} \tau \mathcal{L} = \mathcal{L} \tau \mathcal{L} \tau \mathcal{L} \tau \mathcal{L} \tau \mathcal{L} = \mathcal{L} \tau \mathcal{L} \tau \mathcal{L}$$

so that $\mathcal{L} \tau \mathcal{L} \tau \mathcal{L}$ is transitive. It is easy to see that $\mathcal{L} \tau \mathcal{L} \tau \mathcal{L}$ is in fact an equivalence. If also $\tau \mathcal{R} \tau \mathcal{R} \tau = \tau \mathcal{R} \tau$ is satisfied, then by duality, $\mathcal{R} \tau \mathcal{R} \tau \mathcal{R}$ is an equivalence. Therefore $\mathcal{L} \tau \mathcal{L} \tau \mathcal{L} \cap \mathcal{R} \tau \mathcal{R} \tau \mathcal{R}$ is an equivalence so that it indeed makes sense to consider the

congruence $(\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^0$. If τ is the trace of this congruence, then τ is normal by Lemma 2.3.

If ρ is any congruence with $\tau = \text{tr } \rho$, then by Lemma 2.6(iii),

$$\rho \subseteq (\rho \vee \mathcal{L}) \cap (\rho \vee \mathcal{R}) = \mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R},$$

and thus

$$\rho \subseteq (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^0.$$

2.9. LEMMA. *A subset K of S is normal if and only if*

$$K = \ker\{(x, x^2) | x \in K\}^*.$$

If this is the case, then $\{(x, x^2) | x \in K\}^$ is the least congruence on S with kernel K .*

PROOF. If K is a normal subset of S , then $K = \ker \rho$ for some congruence ρ on S by Lemma 2.4. Clearly $K \subseteq \ker\{(x, x^2) | x \in K\}^*$. Further, $\{(x, x^2) | x \in K\} \subseteq \rho$, and thus $\{(x, x^2) | x \in K\}^* \subseteq \rho$, whence

$$\ker\{(x, x^2) | x \in K\}^* \subseteq \ker \rho = K,$$

and the equality prevails.

If conversely $K = \ker\{(x, x^2) | x \in K\}^*$, then K is normal by Lemma 2.4.

2.10. LEMMA. *For any congruence ρ on S , $K = \ker \rho$, $\tau = \text{tr } \rho$ and $a, b \in S$ we have*

$$a \rho b \Leftrightarrow a(\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})b, \quad ab' \in K \text{ for some } [all] \ b' \in V(b).$$

PROOF. *Necessity.* If $a \rho b$, then Lemma 2.6(ii) and its dual give $a(\mathcal{R}\tau\mathcal{R}\tau\mathcal{R})b$ and $a(\mathcal{L}\tau\mathcal{L}\tau\mathcal{L})b$. Further, if $b' \in V(b)$, then $ab' \rho bb'$, so $ab' \in K$.

Sufficiency. Let $a(\mathcal{R}\tau\mathcal{R}\tau\mathcal{R})b$, $a(\mathcal{L}\tau\mathcal{L}\tau\mathcal{L})b$ and $ab' \in K$ for some $b' \in V(b)$. Then by Lemma 2.6(ii) and its dual, we have $(a\rho) \mathcal{H} (b\rho)$. Hence $b'b \in L_b \cap R_{b'}$ implies that $(b'b)\rho \in L_{a\rho} \cap R_{b'\rho}$ so that by [2, Theorem 2.17],

$$(ab')\rho = (a\rho)(b'\rho) \in L_{b'\rho} \cap R_{a\rho} = L_{b'\rho} \cap R_{b\rho}.$$

Now $ab' \in K$ and the above show that $(ab')\rho$ is an idempotent in the \mathcal{H} -class $L_{b'\rho} \cap R_{b\rho}$. Obviously also $(bb')\rho$ is an idempotent in that \mathcal{H} -class, and thus $ab' \rho bb'$. Therefore,

$$\begin{aligned} a \rho ab'b & \quad (\text{since } (a\rho) \mathcal{L} (b'b)\rho) \\ \rho bb'b & \quad (\text{since } (ab')\rho (bb')) \\ & = b. \end{aligned}$$

2.11. COROLLARY [3]. *A congruence on S is uniquely determined by its kernel and its trace.*

Our next basic concept is the following.

2.12. DEFINITION. A pair (K, τ) is a *congruence pair* for S if

- (i) K is a normal subset of S ,
- (ii) τ is a normal equivalence on $E(S)$,
- (iii) $K \subseteq \ker(\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^0$
- (iv) $\tau \subseteq \text{tr } \pi_K$.

In such a case we define

$$\rho_{(K,\tau)} = \pi_K \cap (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^0.$$

Note that

$$\rho_{(K,\tau)} = (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \theta_K \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^0$$

in view of Results 1.4 and 1.5.

We are finally ready for the characterization of congruences on a regular semi-group in terms of congruence pairs.

2.13. THEOREM. *If (K, τ) is a congruence pair for S , then $\rho_{(K,\tau)}$ is the unique congruence ρ on S for which $\ker \rho = K$ and $\text{tr } \rho = \tau$. Conversely, if ρ is a congruence on S , then $(\ker \rho, \text{tr } \rho)$ is a congruence pair for S and $\rho = \rho_{(\ker \rho, \text{tr } \rho)}$.*

PROOF. If (K, τ) is a congruence pair for S , then

$$\begin{aligned} \ker \rho_{(K,\tau)} &= \ker (\pi_K \cap (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^0) \\ &= \ker \pi_K \cap \ker (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^0 \quad (\text{by Lemma 2.5}) \\ &= K \cap \ker (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^0 \quad (\text{since } K \text{ is normal}) \\ &= K \quad (\text{since } K \subseteq \ker (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^0), \end{aligned}$$

and

$$\begin{aligned} \text{tr } \rho_{(K,\tau)} &= \text{tr}(\pi_K \cap (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^0) \\ &= \text{tr } \pi_K \cap \text{tr}(\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^0 \quad (\text{by Lemma 2.5}) \\ &= \text{tr } \pi_K \cap \tau \quad (\text{by Lemma 2.8}) \\ &= \tau \quad (\text{since } \tau \subseteq \text{tr } \pi_K). \end{aligned}$$

The uniqueness of $\rho_{(K,\tau)}$ follows from Corollary 2.11.

If ρ is a congruence, $K = \ker \rho$, $\tau = \text{tr } \rho$, then K is a normal subset of S by Lemma 2.4 and τ is a normal equivalence on $E(S)$ by Lemma 2.3. Further, $\rho \subseteq \pi_K$ by Lemma 2.4, so $\tau = \text{tr } \rho \subseteq \text{tr } \pi_K$. Also $\rho \subseteq (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^0$ by Lemma 2.8, and thus $K = \ker \rho \subseteq \ker (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^0$. Hence (K, τ) is a congruence pair and by the above $\ker \rho = \ker \rho_{(K,\tau)}$, $\text{tr } \rho = \text{tr } \rho_{(K,\tau)}$. From Corollary 2.11 we thus have $\rho = \rho_{(K,\tau)}$.

Note that the condition $\tau \subseteq \text{tr } \pi_K$ can be written more explicitly as follows:

$$e\tau f, \quad x, y \in S^1, \quad xey \in K \Rightarrow xfy \in K.$$

A simpler expression for $\rho_{(K,\tau)}$ is provided by the following result.

2.14. COROLLARY. *If (K, τ) is a congruence pair for S , then for any $a, b \in S$,*

$$a\rho_{(K,\tau)}b \Leftrightarrow a(\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})b, \quad ab' \in K \text{ for some [all] } b' \in V(b).$$

PROOF. This follows directly from Lemma 2.10 and Theorem 2.13.

We now record an obvious consequence of the above deliberations.

2.15. PROPOSITION. *Let $\text{Cp } S$ be the poset of all congruence pairs for S under the partial order given by*

$$(K, \tau) \leq (K', \tau') \Leftrightarrow K \subseteq K', \quad \tau \subseteq \tau'.$$

Then the mappings

$$\rho \rightarrow (\ker \rho, \operatorname{tr} \rho), \quad (K, \tau) \rightarrow \rho_{(K, \tau)}$$

are mutually inverse isomorphisms of $\operatorname{Con} S$ and $\operatorname{Cp} S$.

3. The min network. In this section we show how our consideration of kernels and traces of congruences on S naturally gives rise to certain sublattices of $\operatorname{Con} S$.

3.1. NOTATION. If ρ is a congruence on S , $\ker \rho = K$, $\operatorname{tr} \rho = \tau$, then we put

$$\begin{aligned} \rho_T &= \tau^*, & \rho^T &= (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^0, \\ \rho_K &= \{(x, x^2) | x \in K\}^*, & \rho^K &= \pi_K. \end{aligned}$$

We summarize the main points of Lemmas 2.3, 2.4, 2.8 and 2.9 in

3.2. THEOREM. Let ρ be a congruence on S . Then ρ_T, ρ^T [ρ_K, ρ^K] are, respectively, the least and the greatest congruences with trace [kernel] equal to $\operatorname{tr} \rho$ [$\ker \rho$].

The same results concerning ρ_T and ρ^K were given in [9, §3], which also contains another characterization of ρ^T and ρ_K .

3.3. COROLLARY [9, THEOREM 3.2]. If ρ and θ are congruences on S , then

$$\begin{aligned} \operatorname{tr} \rho \subseteq \operatorname{tr} \theta &\Rightarrow \rho_T \subseteq \theta_T, & \rho^T &\subseteq \theta^T, \\ \ker \rho \subseteq \ker \theta &\Rightarrow \rho_K \subseteq \theta_K. \end{aligned}$$

Recall that a transformation $x \rightarrow x^*$ on a poset P is a *closure operation* if for all $x, y \in P$,

- (i) $x \leq y$ implies $x^* \leq y^*$,
- (ii) $x^* \leq x$,
- (iii) $(x^*)^* = x^*$.

The following result concerns closure operations on lattices.

3.4. LEMMA. Let L be a lattice and $x \rightarrow x^*$ and $x \rightarrow x^+$ be closure operations on L such that

$$x^* = y^*, \quad x^+ = y^+ \Rightarrow x = y \quad (x, y \in L).$$

Then for all $x \in L$,

$$x^* \wedge x^+ = (x^*)^+ \vee (x^+)^* \quad \text{and} \quad x = x^* \vee x^+.$$

PROOF. Since $x^* \leq x$, we have $(x^*)^+ \leq x^+$. Also $(x^+)^* \leq x^*$ and thus $(x^*)^+ \vee (x^+)^* \leq x^+$. Similarly, $(x^*)^+ \vee (x^+)^* \leq x^*$ and hence

$$(x^*)^+ \vee (x^+)^* \leq x^* \wedge x^+.$$

It follows that

$$(x^*)^+ \leq (x^*)^+ \vee (x^+)^* \leq x^* \wedge x^+ \leq x^*.$$

Since $^+$ is a closure operation, we obtain

$$(x^*)^+ = ((x^*)^+ \vee (x^+)^*)^+ = (x^* \wedge x^+)^+.$$

Also,

$$(x^+)^* \leq (x^*)^+ \vee (x^+)^* \leq x^* \wedge x^+ \leq x^+$$

and since $*$ is a closure operation,

$$(x^+)^* = ((x^+)^+ \vee (x^+)^*)^* = (x^* \wedge x^+)^*.$$

Hence $(x^+)^+ \vee (x^+)^* = x^* \wedge x^+$.

For $x \in L$, we have $x^+ \leq x^+ \vee x^* \leq x$ and since we deal with a closure operation, $x^+ = (x^+ \vee x^*)^+$. Similarly, we get $x^* = (x^+ \vee x^*)^*$. Thus $x = x^+ \vee x^*$.

We are now able to give a new proof of [9, Theorems 3.3 and 3.4].

3.5. THEOREM. *If ρ is a congruence on S , then*

$$\rho_T \cap \rho_K = \rho_{TK} \vee \rho_{KT}, \quad \rho = \rho_T \vee \rho_K = \rho^T \cap \rho^K.$$

PROOF. From Corollary 3.3 it easily follows that $\rho \rightarrow \rho_T$ and $\rho \rightarrow \rho_K$ are closure operations on the lattice $\text{Con } S$. If $\rho, \theta \in \text{Con } S$, then

$$\begin{aligned} \rho_T = \theta_T, \quad \rho_K = \theta_K &\Leftrightarrow \text{tr } \rho = \text{tr } \theta, \quad \ker \rho = \ker \theta \\ &\Leftrightarrow \rho = \theta \quad (\text{by Corollary 2.11}). \end{aligned}$$

We can thus apply Lemma 3.4 to obtain the results involving ρ_T and ρ_K . That $\rho = \rho^T \cap \rho^K$ follows from Theorem 2.13.

In view of the foregoing theorem we may, as in [9], consider the sublattice of $\text{Con } S$ which consists of the congruences $\rho, \rho_T, \rho_K, \rho_{TK}, \rho_{KT}, \dots$. This lattice is called the *min network* corresponding to ρ .

4. The left and the right traces. We shall introduce here some relations on the lattice $\text{Con } S$. They will turn out to be complete congruences induced by certain complete homomorphisms of $\text{Con } S$ into $\text{Eq } E(S)$.

4.1. DEFINITION. For a congruence ρ on S , $\text{ltr } \rho = \text{tr}(\rho \vee \mathcal{L})^0$ is the *left trace* of ρ , $\text{rtr } \rho = \text{tr}(\rho \vee \mathcal{R})^0$ is the *right trace* of ρ .

An abstract characterization of left and right traces will be given by means of the following concepts.

4.2. DEFINITION. An equivalence τ on $E(S)$ is

- (i) *left normal* if $\tau \mathcal{L} \tau \mathcal{L} \tau = \tau \mathcal{L} \tau$ and $\tau = \text{tr}(\mathcal{L} \tau \mathcal{L} \tau \mathcal{L})^0$,
- (ii) *right normal* if $\tau \mathcal{R} \tau \mathcal{R} \tau = \tau \mathcal{R} \tau$ and $\tau = \text{tr}(\mathcal{R} \tau \mathcal{R} \tau \mathcal{R})^0$.

The next result will be used several times.

4.3. LEMMA. *Let ρ be a congruence on S , $\tau = \text{tr } \rho$, $\tau_l = \text{ltr } \rho$ and $\tau_r = \text{rtr } \rho$.*

- (i) $\mathcal{R} \tau \mathcal{R} \tau \mathcal{R} = \rho \vee \mathcal{R} = (\rho \vee \mathcal{R})^0 \vee \mathcal{R} = \mathcal{R} \tau_r \mathcal{R} \tau_r \mathcal{R}$.
- (ii) $\tau \mathcal{R} \tau = \tau_r \mathcal{R} \tau_r$.
- (iii) $\tau_r = \text{tr}(\mathcal{R} \tau \mathcal{R} \tau \mathcal{R})^0 = \text{tr}(\mathcal{R} \tau_r \mathcal{R} \tau_r \mathcal{R})^0 = \text{rtr}(\mathcal{R} \tau \mathcal{R} \tau \mathcal{R})^0 = \text{rtr}(\mathcal{R} \tau_r \mathcal{R} \tau_r \mathcal{R})^0$.
- (iv) $\tau = \tau_l \cap \tau_r$.

PROOF. (i) In view of Lemma 2.6(iii), it suffices to show that $\rho \vee \mathcal{R} = (\rho \vee \mathcal{R})^0 \vee \mathcal{R}$. Since obviously $\rho \subseteq (\rho \vee \mathcal{R})^0$, we have $\rho \vee \mathcal{R} \subseteq (\rho \vee \mathcal{R})^0 \vee \mathcal{R}$. On the other hand $(\rho \vee \mathcal{R})^0 \subseteq \rho \vee \mathcal{R}$ and $\mathcal{R} \subseteq \rho \vee \mathcal{R}$ give $(\rho \vee \mathcal{R})^0 \vee \mathcal{R} \subseteq \rho \vee \mathcal{R}$.

(ii) From Theorem 2.6(iv) and part (i), we find that $\tau \mathcal{R} \tau = (\rho \vee \mathcal{R})|_{E(S)} = ((\rho \vee \mathcal{R})^0 \vee \mathcal{R})|_{E(S)} = \tau_r \mathcal{R} \tau_r$.

(iii) This follows directly from part (i).

(iv) From Result 1.4, Lemma 2.5 and Lemma 2.6(v) we have

$$\begin{aligned} \tau &= \text{tr}(\mathcal{L} \tau \mathcal{L} \tau \mathcal{L} \cap \mathcal{R} \tau \mathcal{R} \tau \mathcal{R})^0 \\ &= \text{tr}(\mathcal{L} \tau \mathcal{L} \tau \mathcal{L})^0 \cap \text{tr}(\mathcal{R} \tau \mathcal{R} \tau \mathcal{R})^0 = \tau_l \cap \tau_r. \end{aligned}$$

4.4. LEMMA. *An equivalence τ_r on $E(S)$ is right normal if and only if it is the right trace of a congruence on S . If this is the case, then $(\mathcal{R}\tau_r\mathcal{R}\tau_r\mathcal{R})^0$ is the greatest congruence on S with right trace τ_r .*

PROOF. If τ_r is the right trace of a congruence ρ , then $\tau_r = \text{tr}(\rho \vee \mathcal{R})^0$, so that by Lemma 2.6(iv), we have $\tau_r \mathcal{R}\tau_r \mathcal{R}\tau_r = \tau_r \mathcal{R}\tau_r$. Using Lemma 4.3(iii), we may conclude that τ_r is a right normal equivalence.

Suppose conversely that τ_r is a right normal equivalence on $E(S)$. Then $\tau_r \mathcal{R}\tau_r \mathcal{R}\tau_r = \tau_r \mathcal{R}\tau_r$ so that $\mathcal{R}\tau_r \mathcal{R}\tau_r \mathcal{R}$ is an equivalence on S and $\tau_r = \text{tr}(\mathcal{R}\tau_r \mathcal{R}\tau_r \mathcal{R})^0$. By Lemma 2.6(iii), we obtain

$$(\mathcal{R}\tau_r \mathcal{R}\tau_r \mathcal{R})^0 \vee \mathcal{R} = \mathcal{R}\tau_r \mathcal{R}\tau_r \mathcal{R},$$

so that

$$\tau_r = \text{tr}((\mathcal{R}\tau_r \mathcal{R}\tau_r \mathcal{R})^0 \vee \mathcal{R})^0 = \text{rtr}(\mathcal{R}\tau_r \mathcal{R}\tau_r \mathcal{R})^0.$$

If ρ is any congruence with $\text{rtr} \rho = \tau_r$, then $\rho \subseteq (\rho \vee \mathcal{R})^0 = (\mathcal{R}\tau_r \mathcal{R}\tau_r \mathcal{R})^0$, $\text{tr}(\mathcal{R}\tau_r \mathcal{R}\tau_r \mathcal{R})^0 = \tau_r$ by Lemma 4.3. Thus $(\mathcal{R}\tau_r \mathcal{R}\tau_r \mathcal{R})^0$ is the greatest congruence with right trace τ_r .

4.5. COROLLARY. *An equivalence τ on $E(S)$ is normal if and only if it is the intersection of a left normal equivalence and a right normal equivalence.*

PROOF. If τ is a normal equivalence, then by Lemma 2.3 there exists a congruence ρ on S with $\text{tr} \rho = \tau$. By Lemma 4.4 and its dual, $\eta = \text{ltr} \rho$ is a left normal equivalence and $\tau_r = \text{rtr} \rho$ is a right normal equivalence. By Lemma 4.3(iv), we have $\tau = \eta \cap \tau_r$.

If $\tau = \eta \cap \tau_r$ where η is a left normal and τ_r a right normal equivalence, then $\mathcal{L}\eta \mathcal{L}\eta \mathcal{L}$ and $\mathcal{R}\tau_r \mathcal{R}\tau_r \mathcal{R}$ are equivalences on S and

$$\begin{aligned} \tau &= \text{tr}(\mathcal{L}\eta \mathcal{L}\eta \mathcal{L})^0 \cap \text{tr}(\mathcal{R}\tau_r \mathcal{R}\tau_r \mathcal{R})^0 \\ &= \text{tr}(\mathcal{L}\eta \mathcal{L}\eta \mathcal{L} \cap \mathcal{R}\tau_r \mathcal{R}\tau_r \mathcal{R})^0 \quad (\text{by Result 1.4 and Lemma 2.5}), \end{aligned}$$

so that τ is a normal equivalence by Lemma 2.3.

We now introduce several relations on $\text{Con } S$.

4.6. NOTATION. For any $\rho, \theta \in \text{Con } S$, let

$$\begin{aligned} \rho T_l \theta &\Leftrightarrow \text{ltr} \rho = \text{ltr} \theta, \\ \rho T_r \theta &\Leftrightarrow \text{rtr} \rho = \text{rtr} \theta, \\ \rho T \theta &\Leftrightarrow \text{tr} \rho = \text{tr} \theta, \\ \rho K \theta &\Leftrightarrow \ker \rho = \ker \theta. \end{aligned}$$

4.7. THEOREM. *The mapping*

$$\rho \rightarrow (\rho \vee \mathcal{R})|_{E(S)} \quad (\rho \in \text{Con } S)$$

is a complete lattice homomorphism of $\text{Con } S$ into $\text{Eq } E(S)$ which induces T_r .

PROOF. Let \mathcal{F} be a family of congruences on S , and suppose that $e, f \in E(S)$ with $e((\bigvee_{\rho \in \mathcal{F}} \rho) \vee \mathcal{R})f$. Then $e(\bigvee_{\rho \in \mathcal{F}} (\rho \vee \mathcal{R}))f$ and there exist $x_0, \dots, x_n \in S$ and $\rho_1, \dots, \rho_n \in \mathcal{F}$ such that

$$e = x_0(\rho_1 \vee \mathcal{R})x_1(\rho_2 \vee \mathcal{R}) \cdots x_{n-1}(\rho_n \vee \mathcal{R})x_n = f.$$

For every $1 \leq i \leq n$ we know from Lemma 2.6(iii) that

$$\rho_i \vee \mathcal{R} = \mathcal{R}(\text{tr } \rho_i) \mathcal{R}(\text{tr } \rho_i) \mathcal{R},$$

so there exist idempotents $g_{i-1}, h_i \in E(S)$ such that

$$x_{i-1} \mathcal{R} g_{i-1} (\text{tr } \rho_i) \mathcal{R}(\text{tr } \rho_i) h_i \mathcal{R} x_i$$

and we find

$$\begin{aligned} & e \mathcal{R} g_0 (\text{tr } \rho_1) \mathcal{R}(\text{tr } \rho_1) h_1 \mathcal{R} g_1 (\text{tr } \rho_2) \mathcal{R}(\text{tr } \rho_2) \\ & \cdots h_{n-1} \mathcal{R} g_{n-1} (\text{tr } \rho_n) \mathcal{R}(\text{tr } \rho_n) h_n \mathcal{R} f. \end{aligned}$$

Thus $e(\bigvee_{\rho \in \mathcal{F}} (\text{tr } \rho \vee (\mathcal{R}|_{E(S)}))) f$, which by Lemma 2.6(iv) yields

$$e \left(\bigvee_{\rho \in \mathcal{F}} (\rho \vee \mathcal{R})|_{E(S)} \right) f.$$

We have proved that

$$\left(\left(\bigvee_{\rho \in \mathcal{F}} \rho \right) \vee \mathcal{R} \right)|_{E(S)} \subseteq \bigvee_{\rho \in \mathcal{F}} (\rho \vee \mathcal{R})|_{E(S)}$$

and the reverse inclusion obviously holds.

Let $e, f \in E(S)$ be such that $e(\bigcap_{\rho \in \mathcal{F}} (\rho \vee \mathcal{R})) f$. Then for every $\rho \in \mathcal{F}$, we have $e(\rho \vee \mathcal{R}) f$. Thus, if $g \in S(e, f)$ we have from Theorem 2.6(iv) and Theorem 2.7 that $g\rho e$ for all $\rho \in \mathcal{F}$. Further, $gf \in E(R_g)$ and $(gf)\rho = (g\rho)(f\rho) = (e\rho)(f\rho) = f\rho$ for all $\rho \in \mathcal{F}$ since $(e\rho) \mathcal{R} (f\rho)$ by Lemma 2.6(ii), (iii). Consequently

$$e \left(\bigcap_{\rho \in \mathcal{F}} \rho \right) g \mathcal{R} gf \left(\bigcap_{\rho \in \mathcal{F}} \rho \right) f,$$

that is,

$$e \left(\left(\bigcap_{\rho \in \mathcal{F}} \rho \right) \vee \mathcal{R} \right)|_{E(S)} f.$$

Hence

$$\left(\left(\bigcap_{\rho \in \mathcal{F}} \rho \right) \vee \mathcal{R} \right)|_{E(S)} \supseteq \bigcap_{\rho \in \mathcal{F}} (\rho \vee \mathcal{R})|_{E(S)}$$

holds and the reverse inclusion obviously holds.

We have proved that the mapping $\rho \rightarrow (\rho \vee \mathcal{R})|_{E(S)}$ is a complete lattice homomorphism of $\text{Con } S$ into $\text{Eq } E(S)$. For $\rho, \theta \in \text{Con } S$, we have

$$\begin{aligned} (\rho \vee \mathcal{R})|_{E(S)} &= (\theta \vee \mathcal{R})|_{E(S)} \\ &\Leftrightarrow (\text{tr } \rho) \mathcal{R}(\text{tr } \rho) = (\text{tr } \theta) \mathcal{R}(\text{tr } \theta) \quad (\text{by Lemma 2.6(iv)}) \\ &\Rightarrow \text{tr}(\mathcal{R}(\text{tr } \rho) \mathcal{R}(\text{tr } \rho) \mathcal{R})^0 = \text{tr}(\mathcal{R}(\text{tr } \theta) \mathcal{R}(\text{tr } \theta) \mathcal{R})^0 \\ &\Rightarrow \rho T_r \theta \quad (\text{by Lemma 2.6(iii)}) \\ &\Rightarrow (\text{tr } \rho) \mathcal{R}(\text{tr } \rho) = (\text{tr } \theta) \mathcal{R}(\text{tr } \theta) \quad (\text{by Lemma 4.3(ii)}) \\ &\Leftrightarrow (\rho \vee \mathcal{R})|_{E(S)} = (\theta \vee \mathcal{R})|_{E(S)} \quad (\text{by Lemma 2.6(iv)}). \end{aligned}$$

Hence

$$(\rho \vee \mathcal{R})|_{E(S)} = (\theta \vee \mathcal{R})|_{E(S)} \Leftrightarrow \rho T_r \theta.$$

4.8. COROLLARY. *We have $T = T_l \cap T_r$ and*

$$\rho \rightarrow ((\rho \vee \mathcal{L})|_{E(S)}, (\rho \vee \mathcal{R})|_{E(S)}) \quad (\rho \in \text{Con } S)$$

is a complete lattice homomorphism of $\text{Con } S$ into $(\text{Eq } E(S))^2$ which induces T .

PROOF. For $\rho, \theta \in \text{Con } S$,

$$\begin{aligned} \rho T \theta &\Leftrightarrow \text{tr } \rho = \text{tr } \theta \\ &\Rightarrow \text{rtr } \rho = \text{tr}(\mathcal{R}(\text{tr } \rho) \mathcal{R}(\text{tr } \rho) \mathcal{R})^0 \\ &\quad = \text{tr}(\mathcal{R}(\text{tr } \theta) \mathcal{R}(\text{tr } \theta) \mathcal{R})^0 = \text{rtr } \theta, \\ \text{ltr } \rho &= \text{tr}(\mathcal{L}(\text{tr } \rho) \mathcal{L}(\text{tr } \rho) \mathcal{L})^0 \\ &\quad = \text{tr}(\mathcal{L}(\text{tr } \theta) \mathcal{L}(\text{tr } \theta) \mathcal{L})^0 = \text{ltr } \theta \\ &\quad \text{(by Lemma 4.3(iii) and its dual)} \\ &\Rightarrow \rho(T_l \cap T_r) \theta \\ &\Rightarrow \rho = \text{rtr } \rho \cap \text{ltr } \rho = \text{rtr } \theta \cap \text{ltr } \theta = \text{tr } \theta \\ &\quad \text{(by Lemma 4.3(iv))} \\ &\Rightarrow \rho T \theta. \end{aligned}$$

Hence $T = T_l \cap T_r$. The remaining part in the statement of the corollary now follows easily from Theorem 4.7.

4.9. COROLLARY (CF. [4, 7, 12]). *The relations T_l, T_r and T are complete congruences on $\text{Con } S$.*

We will need a result concerning complete congruences on complete lattices.

4.10. LEMMA. *Let L be a complete lattice and ρ be a complete congruence on L . For $x \in L$,*

$$\bar{x} = \bigwedge_{z \in x\rho} z$$

is the least element in $x\rho$ and for any $A \subseteq L$,

$$\bigvee_{x \in A} \bar{x} = \overline{\bigvee_{x \in A} x}.$$

PROOF. That \bar{x} is the least element in $x\rho$ is clear from the fact that ρ is a complete congruence. Therefore also

$$\bigvee_{x \in A} \bar{x} \in \left(\bigvee_{x \in A} x \right) \rho$$

and thus

$$\overline{\bigvee_{x \in A} x} \leq \bigvee_{x \in A} \bar{x}.$$

If $y \in A$, then in L/ρ

$$y\rho \leq \left(\bigvee_{x \in A} x \right) \rho = \overline{\left(\bigvee_{x \in A} x \right) \rho}$$

so that

$$y\rho = \left(y \wedge \overline{\left(\bigvee_{x \in A} x \right)} \right) \rho.$$

It follows that

$$\bar{y} \leq y \wedge \overline{\left(\bigvee_{x \in A} x \right)},$$

whence

$$\bar{y} \leq \overline{\bigvee_{x \in A} x}$$

and the equality

$$\bigvee_{x \in A} \bar{x} = \overline{\bigvee_{x \in A} x}$$

holds.

The following supplements the notation introduced in 3.1.

4.11. NOTATION. If ρ is a congruence on S , we put

$$\begin{aligned} \rho_{T_l} &= (\leq_r \cap \rho)^*, & \rho_{T_r} &= (\leq_l \cap \rho)^*, \\ \rho^{T_l} &= (\rho \vee \mathcal{L})^0, & \rho^{T_r} &= (\rho \vee \mathcal{R})^0. \end{aligned}$$

The next two results describe the classes of the congruences T_r and T .

4.12. THEOREM. *If $\rho \in \text{Con } S$, then ρ_{T_r} is the interval $[\rho_{T_r}, \rho^{T_r}]$ of $\text{Con } S$. For any family \mathcal{F} of congruences on S , we have*

$$\left(\bigvee_{\rho \in \mathcal{F}} \rho \right)_{T_r} = \bigvee_{\rho \in \mathcal{F}} \rho_{T_r}, \quad \left(\bigcap_{\rho \in \mathcal{F}} \rho \right)^{T_r} = \bigcap_{\rho \in \mathcal{F}} \rho^{T_r}.$$

PROOF. Clearly $(\leq_l \cap \rho)^* \subseteq \rho$, so that

$$((\leq_l \cap \rho)^* \vee \mathcal{R})|_{E(S)} \subseteq (\rho \vee \mathcal{R})|_{E(S)}.$$

Let $e, f \in E(S)$ be such that $e\rho \vee \mathcal{R} f$. Let $h \in S(e, f)$. By Lemmas 2.6(iv) and 2.7, we have $h \in e\rho$. Thus $e(\leq_l \cap \rho) h$. Further, $h \mathcal{R} hf \leq f$ and

$$(hf)\rho = (h\rho)(f\rho) = (e\rho)(f\rho) = f\rho$$

since $(e\rho) \mathcal{R} (f\rho)$ by Lemma 2.6(i), (iii). We thus have

$$e(\leq_l \cap \rho) h \mathcal{R} (hf)(\leq_l \cap \rho) f$$

and hence

$$e(\text{tr}(\leq_l \cap \rho)^*) \mathcal{R} (\text{tr}(\leq_l \cap \rho)^*) f,$$

so by Lemma 2.6(iv), we have

$$e((\leq_l \cap \rho)^* \vee \mathcal{R}) f.$$

We conclude that the equality

$$((\leq_l \cap \rho)^* \vee \mathcal{R})|_{E(S)} = (\rho \vee \mathcal{R})|_{E(S)}$$

holds. By Theorem 4.7 we have $(\leq_l \cap \rho)^* T_r \rho$. Hence $(\leq_l \cap \rho)^*$ is indeed the least element in ρ_{T_r} . The remaining statements of the theorem are now clear from Lemmas 4.3, 4.4, Corollary 4.9 and Lemma 4.10.

With the notation of §3 we obtain in a similar way the following result.

4.13. THEOREM. *If ρ is a congruence on S , then ρT is the interval $[\rho T, \rho^T]$ of $\text{Con } S$. For any family \mathcal{F} of congruences on S ,*

$$\left(\bigvee_{\rho \in \mathcal{F}} \rho \right)_T = \bigvee_{\rho \in \mathcal{F}} \rho T, \quad \left(\bigcap_{\rho \in \mathcal{F}} \rho \right)^T = \bigcap_{\rho \in \mathcal{F}} \rho^T.$$

The following result establishes certain relationships among ρT and ρ^T with their one-sided counterparts.

4.14. THEOREM. *Let ρ be a congruence on S . Then*

$$\rho T = \rho_{T_l} \vee \rho_{T_r}, \quad \rho^T = \rho^{T_l} \cap \rho^{T_r}.$$

PROOF. By Corollaries 4.8 and 4.9, $(\text{Con } S)/T$ is a complete lattice and T_l/T , T_r/T are complete lattice congruences on $(\text{Con } S)/T$ such that $T_l/T \cap T_r/T$ is the equality on $(\text{Con } S)/T$. Clearly, for $\rho \in \text{Con } S$, $\rho_{T_l} T$ $[\rho_{T_r} T]$ is the least element in the (T_l/T) -class $[(T_r/T)$ -class] of ρT .

For $\rho, \theta \in \text{Con } S$, we obtain

$$\begin{aligned} \rho T \leq \theta T \quad \text{in } (\text{Con } S)/T &\Leftrightarrow (\rho \cap \theta) T \rho \\ &\Rightarrow \rho T \subseteq \rho \cap \theta \subseteq \theta \\ &\Rightarrow \rho_{T_l} T \subseteq \theta_{T_l} \quad (\text{by Theorem 4.12}) \\ &\Rightarrow \rho_{T_r} T \subseteq \theta_{T_r} \quad (\text{since } \rho T \subseteq \theta T) \\ &\Rightarrow \rho_{T_r} T \leq \theta_{T_r} T. \end{aligned}$$

Therefore $\rho T \rightarrow \rho_{T_r} T$ is a closure operation on $(\text{Con } S)/T$. Similarly for the mapping $\rho T \rightarrow \rho_{T_l} T$.

If $\rho, \theta \in \text{Con } S$ are such that $\rho_{T_l} T = \theta_{T_l} T$ and $\rho_{T_r} T = \theta_{T_r} T$, then

$$\rho_{T_l} \rho_{T_l} T \theta_{T_l} T_l \theta, \quad \rho_{T_r} \rho_{T_r} T \theta_{T_r} T_r \theta$$

and hence $\rho T = \theta T$ since $T = T_l \cap T_r$ by Corollary 4.8. We can thus use Lemma 3.4 to obtain

$$\rho T = \rho_{T_l} T \vee \rho_{T_r} T,$$

so that

$$(\rho_{T_l} \vee \rho_{T_r}) T \rho,$$

and thus by Theorem 4.13 and $T \subseteq T_l$, $T \subseteq T_r$, we get

$$\rho T = (\rho_{T_l} \vee \rho_{T_r}) T = \rho_{T_l} T \vee \rho_{T_r} T = \rho_{T_l} \vee \rho_{T_r}.$$

We further have

$$\begin{aligned} \rho^{T_l} \cap \rho^{T_r} &= (\rho \vee \mathcal{L})^0 \cap (\rho \vee \mathcal{R})^0 \\ &= ((\rho \vee \mathcal{L}) \cap (\rho \vee \mathcal{R}))^0 \quad (\text{by Result 1.4}) \\ &= \rho^T \quad (\text{by Lemma 2.6(iii)}). \end{aligned}$$

The next two results concern commutativity of certain congruences.

4.15. THEOREM. *Let ρ be a congruence on S . Then every congruence in $[\rho_T, \rho^{T_l}]$ commutes with every congruence in $[\rho_T, \rho^{T_r}]$.*

PROOF. Let $\alpha \in [\rho_T, \rho^{T_l}]$ and $\beta \in [\rho_T, \rho^{T_r}]$. We have $\rho_T T \rho T_r \rho^{T_r}$ and thus $\rho_T T_r \rho^{T_r}$ in view of $T \subseteq T_r$. Therefore $\rho_T T_r \beta$ so that, in view of Lemma 4.3(ii),

$$(\text{tr } \rho_T) \mathcal{R}(\text{tr } \rho_T) = (\text{tr } \beta) \mathcal{R}(\text{tr } \beta)$$

and hence

$$\mathcal{R}(\text{tr } \rho_T) \mathcal{R}(\text{tr } \rho_T) \mathcal{R} = \mathcal{R}(\text{tr } \beta) \mathcal{R}(\text{tr } \beta) \mathcal{R}.$$

Consequently Lemma 2.6(ii) gives

$$\begin{aligned} a \beta b &\Rightarrow a \mathcal{R}(\text{tr } \beta) \mathcal{R}(\text{tr } \beta) \mathcal{R} b \\ &\Leftrightarrow a \mathcal{R}(\text{tr } \rho_T) \mathcal{R}(\text{tr } \rho_T) \mathcal{R} b \\ &\Leftrightarrow (a \rho_T) \mathcal{R} (b \rho_T) \end{aligned}$$

for any $a, b \in S$. Thus β/ρ_T is a congruence contained in the \mathcal{R} -relation on S/ρ_T . Dually, α/ρ_T is contained in the \mathcal{L} -relation on S/ρ_T . By Results 1.6 and 1.7, we have $\alpha\beta = \beta\alpha$.

4.16. THEOREM. *Let ρ be a congruence on S . Then ρ^{T_l} commutes with ρ^{T_r} and ρ_{T_l} commutes with ρ_{T_r} .*

PROOF. The first statement is a direct consequence of Theorem 4.15. As we have seen in the proof of Theorem 4.14, the mappings

$$\rho T \rightarrow \rho_{T_l} T, \quad \rho T \rightarrow \rho_{T_r} T \quad (\rho \in \text{Con } S)$$

are closure operations on $(\text{Con } S)/T$ which satisfy the conditions of Lemma 3.4. Thus by Lemma 3.4, we get

$$\rho_{T_l} T \wedge \rho_{T_r} T = \rho_{T_r T_l} T \wedge \rho_{T_l T_r} T,$$

whence

$$\rho_{T_r T_l} \subseteq \rho_{T_l} \cap \rho_{T_r} \subseteq \rho_{T_r}, \quad \rho_{T_l T_r} \subseteq \rho_{T_l} \cap \rho_{T_r} \subseteq \rho_{T_l}.$$

Therefore

$$\rho_{T_r} T_l (\rho_{T_l} \cap \rho_{T_r}) T_r \rho_{T_l}$$

so that

$$\rho_{T_r} \in [(\rho_{T_l} \cap \rho_{T_r})_T, (\rho_{T_l} \cap \rho_{T_r})^{T_l}]$$

and dually

$$\rho_{T_l} \in [(\rho_{T_l} \cap \rho_{T_r})_T, (\rho_{T_l} \cap \rho_{T_r})^{T_r}].$$

The desired result now follows from Theorem 4.15.

4.17. COROLLARY. *Let ρ be a congruence on S . Then $\rho_T = \rho_{T_l} \rho_{T_r} = \rho_{T_r} \rho_{T_l}$.*

PROOF. This is immediate from Theorems 4.14 and 4.16.

4.18. COROLLARY. *An equivalence τ on $E(S)$ is normal if and only if*

$$(\leq_l \cap \tau)^* (\leq_r \cap \tau)^* (\leq_r \cap \tau)^* (\leq_l \cap \tau)^*$$

and

$$\tau = \text{tr}(\leq_l \cap \tau)^* (\leq_r \cap \tau)^*.$$

PROOF. If τ is normal, then $\tau = \text{tr } \rho$ for some $\rho \in \text{Con } S$ by Lemma 2.3. The first part of the statement of the corollary holds by Corollary 4.17. Also the second part follows from Corollary 4.17 since $\tau = \text{tr } \rho_T$.

Conversely, if $\tau \in \text{Eq } E(S)$ is such that $(\leq_l \cap \tau)^*$ and $(\leq_r \cap \tau)^*$ commute, then $(\leq_l \cap \tau)^*(\leq_r \cap \tau)^* \in \text{Con } S$. If, moreover, τ is the trace of this congruence, then it is normal by Lemma 2.3.

With a congruence ρ on S we have associated several new ones. The situation is depicted by Diagram 1.

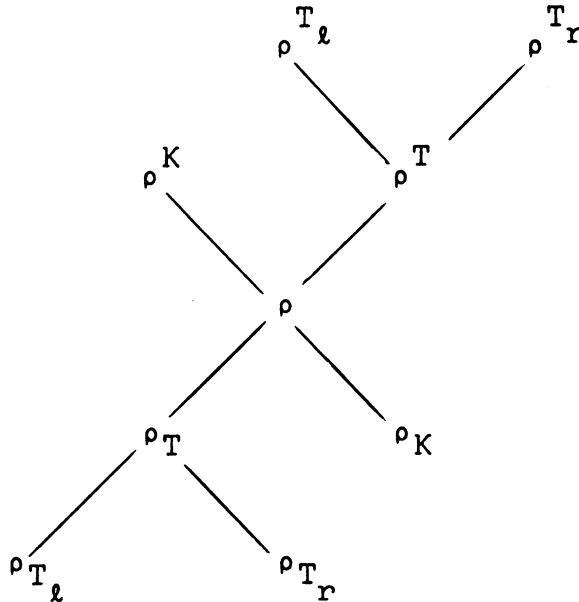


Diagram 1

4.19. *Problem.* Obviously

$$\rho \rightarrow \text{tr } \rho \quad (\rho \in \text{Con } S)$$

is an order-preserving mapping of $\text{Con } S$ into $\text{Eq } E(S)$. By Lemma 2.5, this mapping is a complete \cap -homomorphism inducing T on $\text{Con } S$. The question arises whether this mapping is a complete lattice homomorphism, or equivalently, whether the normal equivalences on $E(S)$ form a complete sublattice of $\text{Eq } E(S)$. In order for this to be true it suffices that for every nonempty family \mathcal{F} of congruences

$$\text{tr} \left(\bigvee_{\rho \in \mathcal{F}} \rho \right) = \bigvee_{\rho \in \mathcal{F}} \text{tr } \rho$$

holds, where the latter join is to be taken in $\text{Eq } E(S)$. As we shall see below this property is satisfied in some large classes of regular semigroups.

Recall that a semigroup S is called *group bound* if for every $a \in S$ there exists a natural number n such that a^n belongs to a subgroup of S . We note that group bound regular semigroups include completely regular semigroups and finite regular semigroups. A regular semigroup S is said to be *locally inverse* if for every $e \in E(S)$, eSe is an inverse semigroup.

4.20. THEOREM. Let S be either locally inverse, group bound or orthodox. Then

$$\rho \rightarrow \text{tr } \rho \quad (\rho \in \text{Con } S)$$

is a complete homomorphism of $\text{Con } S$ into $\text{Eq } E(S)$ which induces T .

PROOF. Let \mathcal{F} be a family of congruences on S . Clearly $\bigvee_{\rho \in \mathcal{F}} \text{tr } \rho \leq \text{tr } \bigvee_{\rho \in \mathcal{F}} \rho$ always holds. If for $e, f \in E(S)$ we have $e(\text{tr } \bigvee_{\rho \in \mathcal{F}} \rho)f$, then there exist $\rho_1, \dots, \rho_n \in \mathcal{F}$ and $x_0, \dots, x_n \in S$ such that

$$e = x_0 \rho_1 x_1 \rho_2 x_2 \cdots x_{n-1} \rho_n x_n = f.$$

We first assume that S is locally inverse. For $i = 0, \dots, n$, we choose $x'_i \in V(x_i)$. Since S is locally inverse, sandwich sets are singletons [8, Theorem 7.6]. Thus, there exist elements h_0, \dots, h_n such that $S(x'_i x_i, x_i x'_i) = \{h_i\}$ for $i = 0, \dots, n$. We note that $h_0 = e$ and $h_n = f$. For any i we have $x_{i-1} \rho_i = x_i \rho_i$, so

$$((x'_{i-1} x_{i-1}) \rho_i) \mathcal{L} (x_{i-1} \rho_i) = (x_i \rho_i) \mathcal{L} ((x'_i x_i) \rho_i)$$

and

$$((x_{i-1} x'_{i-1}) \rho_i) \mathcal{R} (x_{i-1} \rho_i) = (x_i \rho_i) \mathcal{R} ((x_i x'_i) \rho_i).$$

Using Result 1.1 we have

$$S((x'_{i-1} x_{i-1}) \rho_i, (x_{i-1} x'_{i-1}) \rho_i) = S((x'_i x_i) \rho_i, (x_i x'_i) \rho_i),$$

where

$$h_{i-1} \rho_i \in S((x'_{i-1} x_{i-1}) \rho_i, (x_{i-1} x'_{i-1}) \rho_i)$$

and

$$h_i \rho_i \in S((x'_i x_i) \rho_i, (x_i x'_i) \rho_i).$$

Since locally inverse semigroups are closed for taking homomorphic images, S/ρ_i is locally inverse. From [8, Theorem 7.6], it thus follows that in S/ρ_i sandwich sets are singletons. From the above we infer $h_{i-1} \rho_i = h_i \rho_i$. Hence there exists a sequence

$$e = h_0 \rho_1 h_1 \rho_2 \cdots \rho_{n-1} h_{n-1} \rho_n h_n = f,$$

thus $(e, f) \in \text{tr } \bigvee_{\rho \in \mathcal{F}} \rho$. We conclude that the equality $\bigvee_{\rho \in \mathcal{F}} \text{tr } \rho = \text{tr } \bigvee_{\rho \in \mathcal{F}} \rho$ holds.

Suppose next that S is group bound. For $i \in \{0, \dots, n\}$, let m_i be a positive integer such that $x_i^{m_i}$ belongs to a maximal subgroup of S . Let $m = \max\{m_0, \dots, m_n\}$. Then x_i^m belongs to a maximal subgroup of S for every $i = 0, \dots, n$, and $x_{i-1}^m \rho_i x_i^m$ for each $i = 1, \dots, n$. Let e_i be the identity element of the maximal subgroup to which x_i^m belongs. Then $x_{i-1}^m \rho_i x_i^m$ implies $e_{i-1} \rho_i e_i$. Further $e_0 = x_0^m = e^m = e$ and $e_n = x_n^m = f^m = f$. Hence there exists a sequence

$$e = e_0 \rho_1 e_1 \rho_2 e_2 \cdots \rho_{n-1} e_{n-1} \rho_n e_n = f$$

and we again conclude that $\bigvee_{\rho \in \mathcal{F}} \text{tr } \rho = \text{tr } \bigvee_{\rho \in \mathcal{F}} \rho$ holds.

We finally consider the case where S orthodox. Let $x'_i \in V(x_i)$, $i = 0, \dots, n$. Then for $1 < i \leq n$ we find $(x'_{i-1} x_{i-1} x'_i x_i) \rho_i (x'_{i-1} x_{i-1})$ since $(x'_{i-1} x_{i-1}) \rho_i$ and $(x'_i x_i) \rho_i$ are \mathcal{L} -related idempotents in S/ρ_i . Therefore

$$\begin{aligned} & x'_0 x_0 \rho_1 x'_0 x_0 x'_1 x_1 \rho_2 x'_0 x_0 x'_1 x_1 x'_2 x_2 \rho_3 \\ & \cdots \rho_n x'_0 x_0 x'_1 x_1 x'_2 x_2 \cdots x'_{n-2} x_{n-2} x'_{n-1} x_{n-1} x'_n x_n \\ & \rho_{n-1} x'_0 x_0 x'_1 x_1 x'_2 x_2 \cdots x'_{n-2} x_{n-2} x'_n x_n \\ & \cdots \rho_2 x'_0 x_0 x'_1 x_1 x'_n x_n \rho_1 x'_0 x_0 x'_n x_n \end{aligned}$$

so

$$x'_0 x_0 \left(\bigvee_{\rho \in \mathcal{F}} \text{tr } \rho \right) x'_0 x_0 x'_n x_n.$$

Dually, from $(x_{i-1} x'_{i-1} x_i x'_i) \rho_i (x_i x'_i)$ for all $1 < i \leq n$ we find

$$x_n x'_n \left(\bigvee_{\rho \in \mathcal{F}} \text{tr } \rho \right) x_0 x'_0 x_n x'_n.$$

If in particular $x'_0 = e$ and $x'_n = f$ the above expressions yield

$$e \left(\bigvee_{\rho \in \mathcal{F}} \text{tr } \rho \right) e f \left(\bigvee_{\rho \in \mathcal{F}} \text{tr } \rho \right) f,$$

thus $e(\bigvee_{\rho \in \mathcal{F}} \rho) f$. We conclude that $\text{tr}(\bigvee_{\rho \in \mathcal{F}} \rho) = \bigvee_{\rho \in \mathcal{F}} \text{tr } \rho$.

5. Properties of traces and kernels. In this section we investigate the T_l -, T_r -, T - and K -classes of a congruence ρ . As we shall see, the classes of the first three congruences exhibit many nice properties. It appears that not much could be said about K -classes. We first consider the traces.

5.1. THEOREM [11, §3]. *Let ρ be a congruence on S . Then ρT consists of commuting congruences.*

PROOF. This is immediate from Theorem 4.15.

5.2. COROLLARY [11, THEOREM 3.4]. *If ρ is a congruence on S , then ρT is a complete modular lattice.*

PROOF. This is immediate from Theorem 5.1 and [1, Corollary 1 to IV.9].

Later in this section we shall show another way to derive the above result. We first introduce some new concepts.

5.3. DEFINITION. A regular semigroup S is *left [right] fundamental* if ε is the only congruence on S contained in $\mathcal{L} [\mathcal{R}]$; S is *fundamental* if ε is the only congruence on S contained in \mathcal{H} .

5.4. NOTATION. Let

$$\mu_l = \varepsilon^{T_l}, \quad \mu_r = \varepsilon^{T_r}, \quad \mu = \varepsilon^T.$$

In order to avoid confusion we shall sometimes use the notation $\mu_S, \mu_{lS}, \mu_{rS}$.

5.5. LEMMA. *The following hold in S .*

- (i) $\mu_l = \mathcal{L}^0$, $\mu_r = \mathcal{R}^0$, $\mu = \mathcal{H}^0$.
- (ii) $\mu = \mu_l \cap \mu_r$.
- (iii) S is left [right] fundamental if and only if $\varepsilon = \mathcal{L}^0$ [$\varepsilon = \mathcal{R}^0$].
- (iv) S is fundamental if and only if $\varepsilon = \mathcal{H}^0$.

PROOF. That $\mu_l = \mathcal{L}^0$ and $\mu_r = \mathcal{R}^0$ follows immediately from Theorem 4.12 and its dual. Theorem 4.14 gives

$$\mu = \varepsilon^T = \varepsilon^{T_l} \cap \varepsilon^{T_r} = \mu_l \cap \mu_r = \mathcal{L}^0 \cap \mathcal{R}^0 = (\mathcal{L} \cap \mathcal{R})^0 = \mathcal{H}^0.$$

The remaining statements are obvious from the definitions.

We remark that from Lemma 2.8 it follows that $\mu = \mathcal{K}^0$ is the greatest congruence on S with trace the equality on $E(S)$. In other words μ is the greatest idempotent separating congruence on S . Accordingly, S is fundamental if and only if ε is the only idempotent separating congruence on S .

5.6. LEMMA. *For a congruence ρ on S we have*

$$\rho^{T_r}/\rho = \mu_{rS/\rho}, \quad \rho^T/\rho = \mu_{S/\rho}.$$

PROOF. If $a\rho^{T_r}b$, then $a\mathcal{R}(\text{tr } \rho)\mathcal{R}(\text{tr } \rho)\mathcal{R}b$ by Lemma 4.3 and Theorem 4.12. Thus there exist $e, f, g, h \in E(S)$ such that $a\mathcal{R}e(\text{tr } \rho)f\mathcal{R}g(\text{tr } \rho)h\mathcal{R}b$, whence $(a\rho)\mathcal{R}(e\rho) = (f\rho)\mathcal{R}(g\rho) = (h\rho)\mathcal{R}(b\rho)$. We have proved that

$$(a\rho)(\rho^{T_r}/\rho)(b\rho) \Rightarrow (a\rho)\mathcal{R}(b\rho).$$

Consequently $\rho^{T_r}/\rho \subseteq \mu_{rS/\rho}$.

Conversely, if $(a\rho)\mu_{rS/\rho}(b\rho)$, choose $e \in E(R_a)$, $f \in E(R_b)$ and $g \in S(e, f)$. Then $(e\rho)\mathcal{R}(a\rho)\mathcal{R}(b\rho)\mathcal{R}(f\rho)$ in S/ρ and thus

$$h\rho \in S(e, f)\rho \subseteq S(e\rho, f\rho) = e\rho$$

by Result 1.1 and

$$(hf)\rho = (h\rho)(f\rho) = (e\rho)(f\rho) = f\rho.$$

We thus find $a\mathcal{R}e(\text{tr } \rho)h\mathcal{R}hf(\text{tr } \rho)f\mathcal{R}b$. Consequently, if θ is the congruence on S given by

$$a\theta b \Leftrightarrow (a\rho)\mu_{rS/\rho}(b\rho),$$

then $\theta \subseteq \mathcal{R}(\text{tr } \rho)\mathcal{R}(\text{tr } \rho)\mathcal{R}$. It follows that $\theta \subseteq (\mathcal{R}(\text{tr } \rho)\mathcal{R}(\text{tr } \rho)\mathcal{R})^0 = \rho^{T_r}$, where the latter equality follows from Lemma 4.3 and Theorem 4.12. Therefore

$$\mu_{rS/\rho} = \theta/\rho \subseteq \rho^{T_r}/\rho$$

and the required equality prevails. Hence also

$$\begin{aligned} \mu_{S/\rho} &= \mu_{lS/\rho} \cap \mu_{rS/\rho} \quad (\text{by Lemma 5.5(ii)}) \\ &= (\rho^{T_l}/\rho) \cap (\rho^{T_r}/\rho) \quad (\text{by the first part of the proof and its dual}) \\ &= (\rho^{T_l} \cap \rho^{T_r})/\rho = \rho^T/\rho \quad (\text{by Theorem 4.14}). \end{aligned}$$

5.7. COROLLARY. *Let ρ be a congruence on S . Then $\rho = \rho^{T_r}$ if and only if S/ρ is right fundamental; $\rho = \rho^T$ if and only if S/ρ is fundamental.*

PROOF. This follows easily from Lemmas 5.5 and 5.6.

5.8. PROPOSITION. *For any congruence ρ on S , the mapping*

$$\varphi: \theta \rightarrow \theta/\rho_{T_r} \quad (\theta \in [\rho_{T_r}, \rho^{T_r}])$$

is an isomorphism of $[\rho_{T_r}, \rho^{T_r}]$ onto $[\varepsilon_{S/\rho_{T_r}}, \mu_{rS/\rho_{T_r}}]$.

PROOF. The mapping φ is the restriction to $[\rho_{T_r}, \rho^{T_r}]$ of the isomorphism $\theta \rightarrow \theta/\rho_{T_r}$ of $[\rho_{T_r}, \omega]$ onto $\text{Con } S/\rho_{T_r}$. Obviously $(\rho_{T_r})^{T_r} = \rho^{T_r}$ so that by Lemma 5.6, $\rho^{T_r}/\rho_{T_r} = \mu_{rS/\rho_{T_r}}$. One now shows easily that φ is a bijection of $[\rho_{T_r}, \rho^{T_r}]$ onto $[\varepsilon_{S/\rho_{T_r}}, \mu_{rS/\rho_{T_r}}]$ and thus also an isomorphism of complete lattices.

The proof of the next result follows along the same lines.

5.9. PROPOSITION. *For any congruence ρ on S the mapping*

$$\varphi: \theta \rightarrow \theta/\rho_T \quad (\theta \in [\rho_T, \rho^T])$$

is an isomorphism of $[\rho_T, \rho^T]$ onto $[\varepsilon_{S/\rho_T}, \mu_{S/\rho_T}]$.

5.10. PROPOSITION. *For each $e \in E(S)$, let \mathfrak{N}_e be the lattice of normal subgroups of H_e . Then the mapping*

$$\varphi: \rho \rightarrow (e\rho, e \in E(S)) \quad (\rho \in [\varepsilon, \mu])$$

is a complete monomorphism of $[\varepsilon, \mu]$ into $\prod_{e \in E(S)} \mathfrak{N}_e$.

PROOF. If $\rho \in [\varepsilon, \mu]$, then $\rho \subseteq \mathcal{H}$, so $e\rho \subseteq H_e$ for each $e \in E(S)$. Since $\rho|_{H_e}$ is a congruence on the group H_e , $e\rho$ is a normal subgroup of H_e . Hence

$$(e\rho, e \in E(S)) \in \prod_{e \in E(S)} \mathfrak{N}_e.$$

Let \mathcal{F} be a family of congruences in $[\varepsilon, \mu]$. For $e \in E(S)$, $e(\bigcap_{\rho \in \mathcal{F}} \rho) = \bigcap_{\rho \in \mathcal{F}} e\rho$, and it follows that φ is a complete \bigcap -homomorphism. Let $x \in e(\bigvee_{\rho \in \mathcal{F}} \rho)$, $e \in E(S)$. Then there exist $\rho_1, \dots, \rho_n \in \mathcal{F}$ and $x_0, \dots, x_n \in S$ such that $e = x_0 \rho_1 x_1 \rho_2 x_2 \cdots \rho_n x_n = x$. Since $\rho_i \subseteq \mathcal{H}$ for all i , we have that $x_0, \dots, x_n \in H_e$ and $e = x_0(\rho_1|_{H_e})x_1(\rho_2|_{H_e})x_2 \cdots (\rho_n|_{H_e})x_n = x$ in the group H_e . Consequently $x \in \bigvee_{\rho \in \mathcal{F}} e\rho$. Conversely if $x \in \bigvee_{\rho \in \mathcal{F}} e\rho$, then x is $(\bigvee_{\rho \in \mathcal{F}} \rho)$ -related to e in the group H_e and thus also in S . We conclude that for $e \in E(S)$, $e(\bigvee_{\rho \in \mathcal{F}} \rho) = \bigvee_{\rho \in \mathcal{F}} e\rho$. Hence φ is also a complete \bigvee -homomorphism.

If ρ and θ are different congruences in $[\varepsilon, \mu]$, then from $\text{tr } \rho = \varepsilon|_{E(S)} = \text{tr } \theta$ and Lemma 1.2 it follows that $\ker \rho \neq \ker \theta$. Consequently there exists $e \in E(S)$ such that $e\rho \neq e\theta$. Therefore φ is injective.

5.11. COROLLARY. *The lattice $[\varepsilon, \mu]$ of idempotent separating congruences is a complete modular lattice.*

PROOF. Since the lattice of normal subgroups of a group is modular, and since by Proposition 5.10 $[\varepsilon, \mu]$ is embeddable into the lattice of normal subgroups of a group, we conclude that $[\varepsilon, \mu]$ is modular.

5.12. COROLLARY [11, THEOREM 3.4]. *If ρ is a congruence on S , then $[\rho_T, \rho^T]$ is a complete modular lattice.*

PROOF. This is immediate from Theorem 5.9 and Corollary 5.11.

We now turn to the properties of the congruence K .

5.13. EXAMPLE [9, §3]. Let S be the direct product of a nontrivial group and a two-element semilattice. Then the relation K on $\text{Con } S$ is not a congruence.

We nevertheless have the following.

5.14. PROPOSITION. *Let ρ be a congruence on S . Then $\rho K = [\rho_K, \rho^K]$ is a complete sublattice of $\text{Con } S$.*

PROOF. That ρK is an interval follows from Lemmas 2.4 and 2.9. The remaining part in the statement is clear from the fact that an interval of a complete lattice is a complete sublattice.

5.15. DEFINITION. A congruence ρ on S is *idempotent pure* if $\ker \rho = E(S)$. Furthermore, S is *E -disjunctive* if ε is the only idempotent pure congruence on S .

5.16. LEMMA. *A congruence ρ on S is idempotent pure if and only if $\rho \in [\varepsilon, \varepsilon^K]$.*

PROOF. If $\rho \in [\varepsilon, \varepsilon^K]$, then $\ker \rho \subseteq \ker \varepsilon^K = E(S)$ implies $\ker \rho = E(S)$. Conversely, if ρ is idempotent pure, then $\rho \subseteq \varepsilon^K$ since ε^K is the greatest congruence with kernel $\ker \varepsilon = E(S) = \ker \rho$.

5.17. LEMMA. *For a congruence ρ on S we have*

$$\rho^K/\rho = (\varepsilon_{S/\rho})^K.$$

PROOF. If $a\rho \in \ker(\rho^K/\rho)$, then $(a\rho)(\rho^K/\rho)(a\rho)^2 = a^2\rho$ and thus $a\rho^K a^2$. From this we infer that $a \in \ker \rho^K = \ker \rho$ and consequently that $a\rho \in E(S/\rho)$. We conclude that $\ker(\rho^K/\rho) = E(S/\rho) = \ker(\varepsilon_{S/\rho})^K$ and thus $\rho^K/\rho \subseteq (\varepsilon_{S/\rho})^K$.

Let $a\rho, b\rho \in S/\rho$ be such that $(a\rho)(\varepsilon_{S/\rho})^K(b\rho)$. In view of Lemma 2.4 we have for all $x, y \in S^1$,

$$(x\rho)(a\rho)(y\rho) \in E(S/\rho) \Leftrightarrow (x\rho)(b\rho)(y\rho) \in E(S/\rho),$$

or in other words, because of Result 1.2, we obtain that

$$xay \in \ker \rho \Leftrightarrow xby \in \ker \rho.$$

Therefore we have

$$(a\rho)(\varepsilon_{S/\rho})^K(b\rho) \Rightarrow a\pi_{\ker \rho} b \Leftrightarrow a\rho^K b.$$

Consequently, $(\varepsilon_{S/\rho})^K \subseteq \rho^K/\rho$.

5.18. COROLLARY. *Let ρ be a congruence on S . Then $\rho = \rho^K$ if and only if S/ρ is E -disjunctive.*

PROOF. This follows easily from Lemmas 5.16 and 5.17.

The proof of the next result follows along the same lines as the proof of Proposition 5.8.

5.19. PROPOSITION. *For any congruence ρ on S , the mapping*

$$\varphi: \theta \rightarrow \theta/\rho_K \quad (\theta \in [\rho_K, \rho^K])$$

is an isomorphism of $[\rho_K, \rho^K]$ onto $[\varepsilon_{S/\rho}, (\varepsilon_{S/\rho})^K]$.

6. Congruence triples. In the light of the results obtained in §4 we are now able to present a refinement of the concept of congruence pairs of §2. We arrive here at congruence triples for a regular semigroup which are analogous to admissible triples used for describing congruences on Rees matrix semigroups (see [5, III.4]).

6.1. NOTATION. Let S be a regular semigroup, $\gamma \in \text{Eq}(S/\mathcal{L})$ and $\delta \in \text{Eq}(S/\mathcal{R})$. For any $a, b \in S$, we put

$$a\bar{\gamma}b \Leftrightarrow L_a\gamma L_b, \quad a\bar{\delta}b \Leftrightarrow R_a\delta R_b.$$

It is evident that $\bar{\gamma}, \bar{\delta} \in \text{Eq } S$, where $\mathcal{L} \subseteq \bar{\gamma}$ and $\mathcal{R} \subseteq \bar{\delta}$.

We introduce next the following concepts.

6.2. DEFINITION. An equivalence $\gamma \in \text{Eq}(S/\mathcal{L})$ is *normal* if

$$\gamma = (\bar{\gamma}^0 \vee \mathcal{L})/\mathcal{L}$$

and an equivalence $\delta \in \text{Eq}(S/\mathcal{R})$ is *normal* if

$$\delta = (\bar{\delta}^0 \vee \mathcal{R})/\mathcal{R}.$$

If ρ is a congruence on S , then $(\rho \vee \mathcal{L})/\mathcal{L} [(\rho \vee \mathcal{R})/\mathcal{R}]$ is the \mathcal{L} -part [\mathcal{R} -part] of ρ .

6.3. LEMMA. *An equivalence $\delta \in \text{Eq}(S/\mathcal{R})$ is normal if and only if δ is the \mathcal{R} -part of a congruence ρ on S .*

PROOF. If δ is normal, then δ is the \mathcal{R} -part of the congruence $\bar{\delta}^0$. Suppose conversely that δ is the \mathcal{R} -part of a congruence ρ , that is $\delta = (\rho \vee \mathcal{R})/\mathcal{R}$, or in other words, $\bar{\delta} = \rho \vee \mathcal{R}$. Clearly $\rho \subseteq \bar{\delta}^0$ so that $\rho \vee \mathcal{R} \subseteq \bar{\delta}^0 \vee \mathcal{R}$. On the other hand $\bar{\delta}^0 \subseteq \rho \vee \mathcal{R}$ and thus $\bar{\delta}^0 \vee \mathcal{R} \subseteq \rho \vee \mathcal{R}$. We therefore have $\delta = (\rho \vee \mathcal{R})/\mathcal{R} = (\bar{\delta}^0 \vee \mathcal{R})/\mathcal{R}$ and δ is normal.

Our final basic concept is the following.

6.4. DEFINITION. A triple (γ, K, δ) consisting of normal equivalences $\gamma \in \text{Eq}(S/\mathcal{L})$ and $\delta \in \text{Eq}(S/\mathcal{R})$ and a normal subset $K \subseteq S$, is a *congruence triple* if

- (i) $\bar{\gamma} = (\bar{\gamma} \cap \bar{\delta})^0 \vee \mathcal{L}$, $\bar{\delta} = (\bar{\gamma} \cap \bar{\delta})^0 \vee \mathcal{R}$,
- (ii) $K \subseteq \ker \bar{\gamma}^0$, $\bar{\gamma} \subseteq \theta_K^0 \vee \mathcal{L}$,
- (iii) $K \subseteq \ker \bar{\delta}^0$, $\bar{\delta} \subseteq \theta_K^0 \vee \mathcal{R}$.

If this is the case, we define

$$\rho_{(\gamma, K, \delta)} = (\bar{\gamma} \cap \theta_K \cap \bar{\delta})^0.$$

For the principal result of this section, we prove first an auxiliary result which is also of independent interest (cf. Theorem 4.7).

6.5. LEMMA. *The mapping*

$$\rho \rightarrow \rho \vee \mathcal{R} \quad (\rho \in \text{Con } S)$$

is a complete homomorphism of $\text{Con } S$ into $\text{Eq } S$ which induces T_r .

PROOF. Let \mathcal{F} be a family of congruences on S . Let $a, b \in S$ be such that $a(\bigcap_{\rho \in \mathcal{F}}(\rho \vee \mathcal{R}))b$. It follows that $a(\rho \vee \mathcal{R})b$ for every $\rho \in \mathcal{F}$. If $e \in E(R_a)$ and $f \in E(R_b)$, then the foregoing implies that $e(\rho \vee \mathcal{R})f$ for every $\rho \in \mathcal{F}$. Hence $e(\bigcap_{\rho \in \mathcal{F}}(\rho \vee \mathcal{R}))f$. Applying Theorem 4.7 we have $e((\bigcap_{\rho \in \mathcal{F}} \rho) \vee \mathcal{R})f$ and thus also $a((\bigcap_{\rho \in \mathcal{F}} \rho) \vee \mathcal{R})b$. We have proved that

$$\bigcap_{\rho \in \mathcal{F}}(\rho \vee \mathcal{R}) \subseteq \left(\bigcap_{\rho \in \mathcal{F}} \rho \right) \vee \mathcal{R}.$$

The reverse inclusion is obvious and so the equality prevails. It follows that the mapping $\rho \rightarrow \rho \vee \mathcal{R}$ ($\rho \in \text{Con } S$) is a complete lattice homomorphism of $\text{Con } S$ into $\text{Eq } S$.

If $\rho, \theta \in \text{Con } S$ are such that $\rho T_r \theta$, then $\text{rtr } \rho = \text{rtr } \theta$, so that, by Lemma 4.3(i),

$$\rho \vee \mathcal{R} = \mathcal{R}(\text{rtr } \rho)\mathcal{R}(\text{rtr } \rho)\mathcal{R} = \mathcal{R}(\text{rtr } \theta)\mathcal{R}(\text{rtr } \theta)\mathcal{R} = \theta \vee \mathcal{R}.$$

Conversely if $\rho \vee \mathcal{R} = \theta \vee \mathcal{R}$, then

$$\text{rtr } \rho = \text{tr}(\rho \vee \mathcal{R})^0 = \text{tr}(\theta \vee \mathcal{R})^0 = \text{rtr } \theta.$$

The next result describes congruences on a regular semigroup in terms of congruence triples. It represents an analogue of Theorem 2.13.

6.6. THEOREM. *If (γ, K, δ) is a congruence triple for S , then $\rho_{(\gamma, K, \delta)}$ is the unique congruence ρ on S such that γ is the \mathcal{L} -part of ρ , $K = \ker \rho$ and δ is the \mathcal{R} -part of ρ . Conversely, if ρ is a congruence on S , then $(\gamma, K, \delta) = ((\rho \vee \mathcal{L})/\mathcal{L}, \ker \rho, (\rho \vee \mathcal{R})/\mathcal{R})$ is a congruence triple for S and $\rho = \rho_{(\gamma, K, \delta)}$.*

PROOF. Let (γ, K, δ) be a congruence triple. Then

$$\begin{aligned} \ker \rho_{(\gamma, K, \delta)} &= \ker(\bar{\gamma} \cap \theta_K \cap \bar{\delta})^0 \\ &= \ker(\bar{\gamma}^0 \cap \theta_K^0 \cap \bar{\delta}^0) \quad (\text{by Result 1.4}) \\ &= \ker \bar{\gamma}^0 \cap \ker \theta_K^0 \cap \ker \bar{\delta}^0 \quad (\text{by Lemma 2.5}) \\ &= \ker \bar{\gamma}^0 \cap K \cap \ker \bar{\delta}^0 \quad (\text{since } K \text{ is a normal subset of } S) \\ &= K \quad (\text{since } K \subseteq \ker \bar{\gamma}^0 \text{ and } K \subseteq \ker \bar{\delta}^0). \end{aligned}$$

Further,

$$\begin{aligned} \rho_{(\gamma, K, \delta)} \vee \mathcal{R} &= (\bar{\gamma} \cap \theta_K \cap \bar{\delta})^0 \vee \mathcal{R} \\ &= ((\bar{\gamma} \cap \bar{\delta})^0 \cap \theta_K^0) \vee \mathcal{R} \quad (\text{by Result 1.4}) \\ &= ((\bar{\gamma} \cap \bar{\delta})^0 \vee \mathcal{R}) \cap (\theta_K^0 \vee \mathcal{R}) \quad (\text{by Lemma 6.5}) \\ &= \bar{\delta} \cap (\theta_K^0 \vee \mathcal{R}) \quad (\text{since } \bar{\delta} = (\bar{\gamma} \cap \bar{\delta})^0 \vee \mathcal{R}) \\ &= \bar{\delta} \quad (\text{since } \bar{\delta} \subseteq \theta_K^0 \vee \mathcal{R}), \end{aligned}$$

and thus

$$(\rho_{(\gamma, K, \delta)} \vee \mathcal{R})/\mathcal{R} = \delta$$

is the \mathcal{R} -part of $\rho_{(\gamma, K, \delta)}$. Dually, γ is the \mathcal{L} -part of $\rho_{(\gamma, K, \delta)}$.

Let (γ, K, δ) be a congruence triple and $\theta \in \text{Con } S$ be such that γ is the \mathcal{L} -part of θ , $K = \ker \theta$ and δ is the \mathcal{R} -part of θ . Then $\ker \theta = K = \ker \rho_{(\gamma, K, \delta)}$. Further,

$$\begin{aligned} \text{tr } \theta &= \text{ltr } \theta \cap \text{rtr } \theta \quad (\text{by Lemma 4.3(iv)}) \\ &= \text{tr}(\delta \vee \mathcal{L})^0 \cap \text{tr}(\theta \vee \mathcal{R})^0 = \text{tr } \bar{\gamma}^0 \cap \text{tr } \bar{\delta}^0, \end{aligned}$$

and similarly

$$\text{tr } \rho_{(\gamma, K, \delta)} = \text{tr } \bar{\gamma}^0 \cap \text{tr } \bar{\delta}^0.$$

From Lemma 2.10 we may now conclude that $\theta = \rho_{(\gamma, K, \delta)}$.

Let $\rho \in \text{Con } S$ and let $\gamma = (\rho \vee \mathcal{L})/\mathcal{L}$, $K = \ker \rho$ and $\delta = (\rho \vee \mathcal{R})/\mathcal{R}$. By Lemma 6.3 and its dual γ and δ are normal equivalences on $\text{Eq}(S/\mathcal{L})$ and $\text{Eq}(S/\mathcal{R})$, respectively. By Lemma 2.4, K is a normal subset of S . We note that $\bar{\gamma} = \rho \vee \mathcal{L}$ and $\bar{\delta} = \rho \vee \mathcal{R}$.

Since $\mathcal{R} \subseteq \bar{\delta}$, it follows that $(\bar{\gamma} \cap \bar{\delta})^0 \vee \mathcal{R} \subseteq \bar{\delta}$. Further, since $\rho \subseteq (\bar{\gamma} \cap \bar{\delta})^0$ we have $\text{tr } \rho \subseteq (\bar{\gamma} \cap \bar{\delta})^0$. This together with Lemma 2.6(iii) gives

$$\bar{\delta} = \rho \vee \mathcal{R} = \mathcal{R}(\text{tr } \rho)\mathcal{R}(\text{tr } \rho)\mathcal{R} \subseteq (\bar{\gamma} \cap \bar{\delta})^0 \vee \mathcal{R}.$$

Therefore $\bar{\delta} = (\bar{\gamma} \cap \bar{\delta})^0 \vee \mathcal{R}$ and dually, $\bar{\gamma} = (\bar{\gamma} \cap \bar{\delta})^0 \vee \mathcal{L}$.

From $\rho \subseteq (\rho \vee \mathcal{R})^0 = \bar{\delta}^0$ it immediately follows that $K = \ker \rho \subseteq \ker \bar{\delta}^0$. Further, $\ker \rho = K$ gives $\rho \subseteq \theta_K^0$ whence $\bar{\delta} = \rho \vee \mathcal{R} \subseteq \theta_K^0 \vee \mathcal{R}$. A dual reasoning yields $K \subseteq \bar{\gamma}^0$ and $\bar{\gamma} \subseteq \theta_K^0 \vee \mathcal{L}$.

We have proved that (γ, K, δ) is a congruence triple. From the first part of the proof we may now conclude that $\rho = \rho_{(\gamma, K, \delta)}$.

6.7. PROPOSITION. Let $\text{Ct } S$ be the poset of all congruence triples for S under the partial order given by

$$(\gamma, K, \delta) \subseteq (\gamma', K', \delta') \Leftrightarrow \gamma \subseteq \gamma', K \subseteq K', \delta \subseteq \delta'.$$

Then the mappings

$$\rho \rightarrow ((\rho \vee \mathcal{L})/\mathcal{L}, \ker, \rho, (\rho \vee \mathcal{R})/\mathcal{R}), \quad (\gamma, K, \delta) \rightarrow \rho_{(\gamma, K, \delta)}$$

are mutually inverse isomorphisms of $\text{Con } S$ and $\text{Ct } S$.

PROOF. This follows easily from Theorem 6.6.

As an immediate consequence of Lemma 6.5 and Theorem 6.6 we have

6.8. COROLLARY. The mapping

$$\rho_{(\gamma, K, \delta)} \rightarrow \delta \quad ((\gamma, K, \delta) \in \text{Ct } S)$$

is a complete homomorphism of $\text{Con } S$ into $\text{Eq}(S/\mathcal{R})$ inducing T_r . The normal equivalences on S/\mathcal{R} form a complete sublattice of $\text{Eq}(S/\mathcal{R})$.

6.9. PROPOSITION. Let $(\gamma_i, K_i, \delta_i)$, $i = 1, 2$, be congruence triples for S . Let $\rho_i = \rho_{(\gamma_i, K_i, \delta_i)}$, $i = 1, 2$. Then

$$\rho_1 T_l \rho_2 \Leftrightarrow \gamma_1 = \gamma_2,$$

$$\rho_1 K \rho_2 \Leftrightarrow K_1 = K_2,$$

$$\rho_1 T_r \rho_2 \Leftrightarrow \delta_1 = \delta_2.$$

PROOF. We have

$$\rho_1 T_r \rho_2 \Leftrightarrow \rho_1 \vee \mathcal{R} = \rho_2 \vee \mathcal{R} \quad (\text{by Theorem 6.5})$$

$$\Leftrightarrow \bar{\delta}_1 = \bar{\delta}_2 \quad (\text{by Theorem 6.6})$$

$$\Leftrightarrow \delta_1 = \delta_2.$$

This together with its dual gives the first and the third part in the statement of the proposition in view of Corollary 4.8. The second part follows immediately from Theorem 6.6.

A simple expression for $\rho_{(\gamma, K, \delta)}$ is provided by the following.

6.10. PROPOSITION. Let (γ, K, δ) be a congruence triple for S . Then for any $a, b \in S$,

$$a \rho_{(\gamma, K, \delta)} b \Leftrightarrow L_a \gamma L_b, \quad R_a \delta R_b \text{ and } ab' \in K \text{ for some [all] } b' \in V(b).$$

PROOF. Let $\tau = \text{tr } \rho_{(\gamma, K, \delta)}$. Then

$$\bar{\delta} = \rho_{(\gamma, K, \delta)} \vee \mathcal{R} \quad (\text{by Theorem 6.6})$$

$$= \mathcal{R} \tau \mathcal{R} \tau \mathcal{R} \quad (\text{by Lemma 2.6(iii)}).$$

Consequently, for $a, b \in S$ we have

$$R_a \delta R_b \Leftrightarrow a (\mathcal{R} \tau \mathcal{R} \tau \mathcal{R}) b$$

and dually

$$L_a \gamma L_b \Leftrightarrow a (\mathcal{L} \tau \mathcal{L} \tau \mathcal{L}) b.$$

The result now follows from Lemma 2.10.

The following two results relate our representation of congruences by congruence triples with our findings of §2.

6.11. PROPOSITION. *Let $\gamma \in \text{Eq}(S/\mathcal{L})$ and $\delta \in \text{Eq}(S/\mathcal{R})$ be normal equivalences such that 6.4(i) is satisfied. Then $\tau = (\bar{\gamma} \cap \bar{\delta})|_{E(S)} = \text{tr}(\bar{\gamma} \cap \bar{\delta})^0$ is a normal equivalence on $E(S)$ and*

$$\gamma = (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L})/\mathcal{L}, \quad \delta = (\mathcal{R}\tau\mathcal{R}\tau\mathcal{R})/\mathcal{R}.$$

Conversely, if τ is a normal equivalence on $E(S)$, then $\gamma = (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L})/\mathcal{L} \in \text{Eq}(S/\mathcal{L})$ and $\delta = (\mathcal{R}\tau\mathcal{R}\tau\mathcal{R})/\mathcal{R} \in \text{Eq}(S/\mathcal{R})$ are normal equivalences such that 6.4(i) is satisfied and $\tau = (\bar{\gamma} \cap \bar{\delta})|_{E(S)}$.

PROOF. We first suppose that γ and δ are normal equivalences satisfying 6.4(i). We put $\rho = (\bar{\gamma} \cap \bar{\delta})^0$ and $\tau = \text{tr } \rho$. Then

$$\begin{aligned} \tau &= \tau\mathcal{L}\tau \cap \tau\mathcal{R}\tau \quad (\text{by Lemma 2.6(v)}) \\ &= (\rho \vee \mathcal{L})|_{E(S)} \cap (\rho \vee \mathcal{R})|_{E(S)} \quad (\text{by Lemma 2.6(iv)}) \\ &= \bar{\gamma}|_{E(S)} \cap \bar{\delta}|_{E(S)} \quad (\text{since 6.4(i) is satisfied}) \\ &= (\bar{\gamma} \cap \bar{\delta})|_{E(S)}. \end{aligned}$$

We conclude that $\tau = \text{tr}(\bar{\gamma} \cap \bar{\delta})^0 = (\bar{\gamma} \cap \bar{\delta})|_{E(S)}$ is a normal equivalence on $E(S)$ by Lemma 2.3.

If, conversely, τ is a normal equivalence on $E(S)$, then there exists $\rho \in \text{Con } S$ such that $\tau = \text{tr } \rho$. By Theorem 6.6 $\gamma = (\rho \vee \mathcal{L})/\mathcal{L}$ and $\delta = (\rho \vee \mathcal{R})/\mathcal{R}$ are normal equivalences on S/\mathcal{L} and S/\mathcal{R} , respectively, such that condition 6.4(i) is satisfied. By Lemma 2.6(iii) we have $\gamma = (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L})/\mathcal{L}$ and $\delta = (\mathcal{R}\tau\mathcal{R}\tau\mathcal{R})/\mathcal{R}$. Moreover,

$$\begin{aligned} (\bar{\gamma} \cap \bar{\delta})|_{E(S)} &= ((\rho \vee \mathcal{L}) \cap (\rho \vee \mathcal{R}))|_{E(S)} \\ &= ((\rho \vee \mathcal{L})|_{E(S)} \cap (\rho \vee \mathcal{R})|_{E(S)}) \\ &= \tau\mathcal{L}\tau \cap \tau\mathcal{R}\tau \quad (\text{by Lemma 2.6(iv)}) \\ &= \tau \quad (\text{by Lemma 2.6(v)}). \end{aligned}$$

6.12. PROPOSITION. *Let (γ, K, δ) be a congruence triple. If $\tau = (\bar{\gamma} \cap \bar{\delta})|_{E(S)}$, then (K, τ) is a congruence pair and $\rho_{(\gamma, K, \delta)} = \rho_{(K, \tau)}$.*

Let (K, τ) be a congruence pair. If $\gamma = (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L})/\mathcal{L}$ and $\delta = (\mathcal{R}\tau\mathcal{R}\tau\mathcal{R})/\mathcal{R}$, then (γ, K, δ) is a congruence triple and $\rho_{(K, \tau)} = \rho_{(\gamma, K, \delta)}$.

PROOF. Let (γ, K, δ) be a congruence triple. Then we can show as in the proof of Theorem 6.6 that

$$\text{tr } \rho_{(\gamma, K, \delta)} = \text{tr } \bar{\gamma}^0 \cap \text{tr } \bar{\delta}^0.$$

By Proposition 6.11 it thus follows that

$$\tau = (\bar{\gamma} \cap \bar{\delta})|_{E(S)} = \text{tr}(\bar{\gamma} \cap \bar{\delta})^0 = \text{tr } \rho_{(\gamma, K, \delta)}.$$

On the other hand $K = \ker \rho_{(\gamma, K, \delta)}$ by Theorem 6.6. By Theorem 2.13 we may conclude that (K, τ) is a congruence pair and that $\rho_{(\gamma, K, \delta)} = \rho_{(K, \tau)}$.

Conversely, let (K, τ) be a congruence pair. Then

$$\begin{aligned} \gamma &= (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L})/\mathcal{L} \\ &= (\mathcal{L}(\text{tr } \rho_{(K, \tau)})\mathcal{L}(\text{tr } \rho_{(K, \tau)})\mathcal{L})/\mathcal{L} \quad (\text{by Theorem 2.13}) \\ &= (\rho_{(K, \tau)} \vee \mathcal{L})/\mathcal{L} \quad (\text{by Lemma 2.6(iii)}) \end{aligned}$$

and similarly

$$\delta = (\rho_{(K,\tau)} \vee \mathcal{R})/\mathcal{R},$$

whereas

$$K = \ker \rho_{(K,\tau)}$$

by Theorem 2.13. By Theorem 6.6 we have that (γ, K, δ) is a congruence triple and that $\rho_{(\gamma, K, \delta)} = \rho_{(K, \tau)}$.

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DIENST HOGERE MEETKUNDE, RIJKSUNIVERSITEIT TE GENT, B-9000 GENT, BELGIUM

Current address (Francis Pastijn): Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, Wisconsin 53233