

TORSION FREE GROUPS

PAUL HILL¹ AND CHARLES MEGIBBEN

ABSTRACT. In this paper we introduce the class of torsion free k -groups and the notion of a knice subgroup. Torsion free k -groups form a class of groups more extensive than the separable groups of Baer, but they enjoy many of the same closure properties. We establish a role for knice subgroups of torsion free groups analogous to that played by nice subgroups in the study of torsion groups. For example, among the torsion free groups, the balanced projectives are characterized by the fact that they satisfy the third axiom of countability with respect to knice subgroups. Separable groups are characterized as those torsion free k -groups with the property that all finite rank, pure knice subgroups are direct summands. The introduction of these new classes of groups and subgroups is based on a preliminary study of the interplay between primitive elements and $*$ -valuated coproducts. As a by-product of our investigation, new proofs are obtained for many classical results on separable groups. Our techniques lead naturally to the discovery that a balanced subgroup of a completely decomposable group is itself completely decomposable provided the corresponding quotient is a separable group of cardinality not exceeding $\aleph_1$; that is, separable groups of cardinality $\aleph_1$ have balanced projective dimension ≤ 1 .

1. Introduction. In this paper, we examine the fundamental concepts underlying the theory of torsion free abelian groups. This is done in the spirit of the seminal work of Baer [1] and involves a reappraisal of some of the most basic notions in the light of new ideas introduced in our recent third axiom of countability characterization of p -local Warfield groups in [8]. Indeed the present paper should be viewed in the context of our ongoing study of isotype subgroups of simply presented groups initiated in [7]. In particular, we generalize Baer's notion of a primitive element which, when coupled with our concept of a $*$ -valuated coproduct, leads not only to new results but also to new proofs of a number of classical theorems.

Throughout this paper, G denotes an additively written torsion free abelian group. By a *height sequence* we understand a sequence $s = (s_p)_{p \in \mathbf{P}}$, indexed by the set $\mathbf{P}$ of primes, where each s_p is either a nonnegative integer or the symbol ∞ . Height sequences are, of course, ordered pointwise and in fact form a complete distributive lattice with the meet operation defined by $s \wedge t = (s_p \wedge t_p)_{p \in \mathbf{P}}$ where $s_p \wedge t_p = \min\{s_p, t_p\}$. With each $x \in G$, we associate its height sequence $|x|$, where $|x|_p$ is the height in G of x at the prime p , that is, $|x|_p = n$ if $x \in p^n G \setminus p^{n+1} G$ and $|x|_p = \infty$ if $x \in p^n G$ for all $n < \omega$. Each height sequence s determines a fully

Received by the editors June 18, 1985. Presented to the Society at the 92nd annual meeting, New Orleans, Louisiana, January 7–11, 1986.

1980 *Mathematics Subject Classification.* Primary 20K15, 20K20.

Key words and phrases. Torsion free group, primitive element, $*$ -valuated coproduct, free $*$ -valuated subgroup, k -group, separable group, knice subgroup, completely decomposable, third axiom of countability, balanced projective dimension.

¹Research supported, in part, by N.S.F. Grant #MCS-8102470.

invariant subgroup $G(s) = \{x \in G : |x| \geq s\}$. Notice that $s \leq t$ implies $G(s) \supseteq G(t)$. Recall that height sequences s and t are said to be *equivalent* provided (i) $s_p = t_p$ for all but finitely many p and (ii) $s_p = \infty$ if and only if $t_p = \infty$. Agreeing that $\infty - \infty = 0$, we observe that s and t are equivalent if and only if $\sum_{p \in \mathbf{P}} |s_p - t_p|$ is finite. It is, of course, clear that $|x|$ and $|y|$ are equivalent if there exist nonzero integers m and n such that $mx = ny$. An equivalence class of height sequences is called a *type*, and the lattice relations among height sequences induce in the obvious manner corresponding relations on the set of types. In a *rank one group* (i.e., a subgroup of the additive group of rationals), all nonzero elements are of the same type. We also find it convenient to define a multiplication of height sequences by positive integers as follows: ns is the height sequence $(t_p)_{p \in \mathbf{P}}$ given by $t_p = s_p + n_p$, where p^{n_p} is the highest power of p dividing n . Thus s and t are equivalent if and only if there exist positive integers m and n such that $ms = nt$. Notice that $G(ns) = nG(s)$. In addition to the subgroup $G(s)$, we shall also require the fully invariant subgroup $G(s^*)$ generated by those $x \in G(s)$ such that $|x|$ is not equivalent to s ; i.e., each element of $G(s^*)$ is a sum of elements $x \in G(s)$ with $\sum_{p \in \mathbf{P}} (|x|_p - s_p) = \infty$. If $t = ns$, then $G(t^*) = nG(s^*)$. Occasionally we shall need to consider the fully invariant subgroup $G(\sigma) = \sum_{s \in \sigma} G(s) = \bigcup_{s \in \sigma} G(s)$ determined by the type σ , as well as the similarly defined $G(\sigma^*) = \sum_{s \in \sigma} G(s^*) = \bigcup_{s \in \sigma} G(s^*)$.

Finally, recall that a torsion free group G is said to be *completely decomposable* if it decomposes into a direct sum of rank one subgroups, and that G is *separable* (in the sense of Baer) if each finite subset of G can be imbedded in a finite rank completely decomposable direct summand.

2. Primitive elements and *-valuated coproducts. Our notion of a primitive element x in G is motivated by the requirement that when G is completely decomposable, the pure subgroup $\langle x \rangle_*$ generated by x be a summand. Baer's requirement that a primitive element x of type σ satisfy $|x| \geq |x+g|$ for all $g \in G(\sigma^*)$ is, however, too stringent and is indeed not appropriate for groups which fail to be separable. To provide insight into our more general definition of primitivity, we consider a simple example. Suppose $G = A \oplus B$ is a rank two group containing elements $a \in A$ and $b \in B$ such that $|a| = (1, 1, 1, \dots)$ and $|b| = (0, \infty, 1, \dots)$. Let $x = a + b$ and $y = a' + b$, where $2a' = a$. Then both x and y have $s = (0, 1, 1, \dots)$ as their height sequence. But $\langle y \rangle_*$ is a direct summand of G with B serving as a complement (see 2.6 below), while the pure subgroup $\langle x \rangle_*$ fails to be a summand of G . The source of this difference between x and y resides in the fact that b is an element of $G(s^*)$ such that $x - b$ has greater height than x at the prime $p = 2$. Agreeing to write $G(s^*, p)$ for $G(s^*) + G(ps)$ and being guided by the foregoing example, we now formulate our version of primitivity.

DEFINITION 2.1. Let x be an element of the torsion free group G . If $x \notin G(s^*, p)$ for each prime p and each height sequence s equivalent to $|x|$ for which $s_p = |x|_p$ and $|x|_p \neq \infty$, then we say that x is *primitive* in G .

If $\langle x \rangle_*$ is a direct summand of G , then it is trivial to verify that x is primitive. Moreover, it is not difficult, using well-known facts about finite rank completely decomposable groups, to give an ad hoc proof that $\langle x \rangle_*$ is a summand provided x is primitive in the completely decomposable group G . But this latter observation arises quite naturally in the general developments to follow (see 2.9 below). Notice that 0 is primitive, as is any x with $|x|_p = \infty$ for all primes p . A simple and

frequently useful observation about primitive elements is the following: If x is primitive in G and if $x \in G(s^*, p)$, then either $\sum_{p \in \mathbf{P}} (|x|_p - s_p) = \infty$ or $x \in G(ps)$.

LEMMA 2.2. *Let $x \in G$ and suppose n is a nonzero integer. Then x is primitive in G if and only if nx is primitive in G .*

PROOF. That nx being primitive implies the same for x is a consequence of the fact that $G(t^*, p) = nG(s^*, p)$ if $t = ns$. Conversely, assume that x is primitive and observe that it suffices to consider the case where n is a prime. Assume by way of contradiction that $nx \in G(s^*, p)$, where s is equivalent to $|nx|$ and $s_p = |nx|_p \neq \infty$. If $n = p$, then $|nx|_p = |x|_p + 1$ and there is a height sequence t such that $pt = s$ and $t_p = |x|_p$. In this case, $nx = px \in G(s^*, p) = pG(t^*, p)$ and $x \in G(t^*, p)$, contradicting the primitivity of x . Suppose, however, that $n \neq p$ and choose integers l and m such that $1 = lp + mn$. Clearly there is a height sequence t equivalent to s such that $t \leq |x|$, $t \leq s$ and $t_p = s_p = |nx|_p = |x|_p$. Then $x = lp_x + mn_x$ is in $G(pt) + G(s^*, p) \subseteq G(t^*, p)$, once again contradicting the fact that x is primitive.

LEMMA 2.3. *If x is primitive in G with $s = |x|$, then each element of the coset $x + G(s^*)$ is primitive with s as its height sequence.*

PROOF. Let $y = x + z$, where $z \in G(s^*)$. Assume by way of contradiction that $y \in G(t^*, p)$, where t is equivalent to $|y|$ and $t_p = |y|_p \neq \infty$. Notice that $|y| \geq |x| \wedge |z| \geq s$ and hence $t_p \geq s_p$. Then $x = y - z \in G(t^*, p) + G(s^*) \subseteq G((s \wedge t)^*, p)$. But this contradicts the fact that x is primitive since $s \wedge t$ is equivalent to $|x|$ and $s_p \wedge t_p = s_p = |x|_p$. Finally, observe that $|y|_p > |x|_p \neq \infty$ for some prime p would also contradict the primitivity of x .

Having settled on our definition of primitive element, the next problem we wish to consider is the formulation of a suitably general condition relating two independent primitive elements so that the pure subgroup generated by them will be a summand when the containing group is completely decomposable. As Baer [1] observed, using his notion of primitivity, there is no real difficulty if the elements are of different types. The case when the primitive elements are of the same type, however, seems not to have been dealt with successfully in the literature.

Consider the direct sum $A = \langle x_1 \rangle \oplus \langle x_2 \rangle$, where x_1 and x_2 are independent primitive elements in G . If there is any hope for $B = \langle x_1 \rangle_* \oplus \langle x_2 \rangle_*$ to be a direct summand of G , then we must have $|n_1 x_1 + n_2 x_2| = |n_1 x_1| \wedge |n_2 x_2|$ for all integers n_1 and n_2 . This observation leads naturally to the following definition. A direct sum $A_1 \oplus A_2$ of independent subgroups of G is said to be a *valuated coproduct* in G if $|a_1 + a_2| = |a_1| \wedge |a_2|$ for all $a_1 \in A_1$ and $a_2 \in A_2$. An equivalent formulation is that $a_1 + a_2 \in G(s)$ implies $a_1, a_2 \in G(s)$ for all height sequences s . This definition generalizes in the obvious manner to arbitrary direct sums $\bigoplus_{i \in I} A_i$ of independent subgroups of G . If B_i/A_i is torsion for all i , then $\bigoplus_{i \in I} B_i$ is a valuated coproduct in G if and only if $\bigoplus_{i \in I} A_i$ is. Indeed this follows from the fact that G is torsion free and $G(ns) = nG(s)$ for all n and s . Another easy but important observation is the fact that the valuated coproduct $\bigoplus_{i \in I} A_i$ is pure in G if each A_i is a pure subgroup of G . The requirement that $A = \langle x_1 \rangle \oplus \langle x_2 \rangle$ be a valuated coproduct in G does not suffice, however, to make $B = \langle x_1 \rangle_* \oplus \langle x_2 \rangle_*$ a direct summand of the completely decomposable group G . To understand why this is so, we consider another example.

Let $G = A_1 \oplus A_2 \oplus A_3$ be a rank three completely decomposable group containing elements $x_1 \in A_1$, $x_2 \in A_2$ and $x_3 \in A_3$ such that $|x_1| = (0, 0, 0, \dots)$, $|x_2| = (\infty, 0, 0, \dots)$ and $|x_3| = (0, \infty, 0, \dots)$. Then $y = x_1 + x_2 + x_3$ is a primitive element by Lemma 2.3 with $|y| = |x_1|$, and $A = \langle x_1 \rangle \oplus \langle y \rangle$ is readily seen to be a valuated coproduct in G . Nonetheless, the pure subgroup $B = \langle x_1 \rangle_* \oplus \langle y \rangle_*$ fails to be a direct summand of G . Indeed, well-known and easily proved facts about summands of completely decomposable groups forbid G having a direct summand isomorphic to B . A more intrinsic insight into the failure of B to be a summand of G is gained from the observation that $y - x_1 = x_2 + x_3 \in G(s^*)$, where $s = |y| = |x_1|$ and neither y nor x_1 is in $G(s^*)$. Employing the subgroups $G(s^*, p)$ again, we show that the defect observed in the preceding example can be overcome by the introduction of a notion more stringent than that of a valuated coproduct.

DEFINITION 2.4. Let $A = \bigoplus_{i \in I} A_i$ be a valuated coproduct in G and represent each $a \in A$ as a sum $a = \sum_{i \in I} a_i$, where $a_i \in A_i$ for all i . If for each prime p and each height sequence s it is the case that $a \in G(s^*)$ implies $a_i \in G(s^*)$ for all i and also $a \in G(s^*, p)$ implies $a_i \in G(s^*, p)$ for all i , then we say that $A = \bigoplus_{i \in I} A_i$ is a **-valuated coproduct*.

Observe that in the special case of a valuated coproduct $A = A_1 \oplus A_2$, where $A_2 = \langle x \rangle$ with x primitive in G , one needs only verify the latter of the two conditions to establish that $A = A_1 \oplus A_2$ is a *-valuated coproduct. Once again, if B_i/A_i is torsion for all i , the direct sum $\bigoplus_{i \in I} B_i$ is a *-valuated coproduct in G if and only if $\bigoplus_{i \in I} A_i$ is a *-valuated coproduct in G . The notion of a *-valuated coproduct provides us with precisely the right tool to resolve the question of when the pure subgroup generated by two independent primitive elements is a summand of the completely decomposable group containing them. To see that this is so, however, requires a series of lemmas.

LEMMA 2.5. *If $N \oplus \langle x \rangle$ is a *-valuated coproduct in G with x primitive and if $y = x + z$, where $z \in N$ and $|y| = |x|$, then y is primitive and $N \oplus \langle y \rangle$ is a *-valuated coproduct in G .*

PROOF. First observe that $N \oplus \langle x \rangle$ being a *-valuated coproduct forces $y = x + z$ to be primitive since $y \in G(s^*, p)$ implies $x \in G(s^*, p)$, where $|y| = |x|$. To show that $N \oplus \langle y \rangle$ is at least a valuated coproduct, it is enough to verify that $|w + ny| \leq |ny|$ whenever $w \in N$. But $|w + ny| = |w + nz| \wedge |nx| \leq |nx| = |ny|$. It remains to argue that $w + ny \in G(s^*, p)$ implies $ny \in G(s^*, p)$ whenever $w \in N$. Since $w + ny = (w + nz) + nx$, we have $nx \in G(s^*, p)$. But nx is primitive and hence either $\sum_{p \in \mathbf{P}} (|nx|_p - s_p) = \infty$ or $|nx|_p > s_p$. Because $|y| = |x|$, the first possibility implies $ny \in G(s^*)$ and the second implies $ny \in G(ps)$. In either case, $ny \in G(s^*, p)$.

COROLLARY 2.6. *If $G = \langle x \rangle_* \oplus K$, where $|x| = s$, and if $y = x + z$ with $z \in G(s^*)$, then $G = \langle y \rangle_* \oplus K$.*

PROOF. Notice that $N = \langle x \rangle \oplus K$ is a *-valuated coproduct with G/N torsion. Since $x = y - z$ is primitive with $z \in G(s^*) \subseteq K$, $|x| = |y|$ and therefore $N = \langle y \rangle \oplus K$ is a valuated coproduct by 2.5. Since $\langle y \rangle_* \oplus K$ is a pure subgroup of G , the desired conclusion follows.

The next technical lemma is crucial to all that follows.

LEMMA 2.7. Suppose $N = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle$ is a $*$ -valuated coproduct in G , where $x_1, x_2, \dots, x_n$ are all primitive elements of the same type. Then every element of N is primitive in G . Moreover, if $y_1 = x_1 + x_2 + \cdots + x_n$, then there exist elements $y_2, \dots, y_n$ in N such that $N = \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \cdots \oplus \langle y_n \rangle$ is a $*$ -valuated coproduct in G .

PROOF. A straightforward induction reduces the proof to the case $n = 2$. By Lemma 2.2, the first assertion will follow once we show that $y_1 = x_1 + x_2$ is primitive. Assume by way of contradiction that $y_1 \in G(s^*, p)$, where s is equivalent to $|y_1| = |x_1| \wedge |x_2|$ and $s_p = |y_1|_p \neq \infty$. We may suppose without loss of generality that $|x_1|_p \leq |x_2|_p$. But then the fact that $N = \langle x_1 \rangle \oplus \langle x_2 \rangle$ is a $*$ -valuated coproduct with $|x_1|$ and $|x_2|$ equivalent implies that $x_1 \in G(s^*, p)$, where $s_p = |x_1|_p$ and s is equivalent to $|x_1|$. This, however, contradicts the hypothesis that x_1 is primitive.

Since $|x_1|$ and $|x_2|$ are equivalent, the sets $A_1 = \{p \in \mathbf{P} : |x_1|_p < |x_2|_p \neq \infty\}$ and $A_2 = \{p \in \mathbf{P} : |x_2|_p < |x_1|_p \neq \infty\}$ are finite. Therefore there are relatively prime integers m and n such that $|mx_1|_p = |x_2|_p$ for all $p \in A_1$ and $|nx_2|_p = |x_1|_p$ for all $p \in A_2$. In fact, we may make such a choice with m divisible only by primes in A_1 and n divisible only by primes in A_2 . Now select integers k and l such that $1 = km + ln$ and let $y_2 = -kx_1 + lx_2$. If $p \in A_1$, then $|y_1|_p = |x_1|_p$ and

$$|y_2|_p = |kx_1|_p \wedge |lx_2|_p = |kx_2|_p \wedge |lx_2|_p = |x_2|_p$$

since k and l are relatively prime. Similarly, if $p \in A_2$, then $|y_1|_p = |x_2|_p$ and $|y_2|_p = |x_1|_p$; while if $p \notin A_1 \cup A_2$, the four elements x_1, x_2, y_1 and y_2 all have the same height at p . Since the matrix of the transformation between the y_i 's and x_i 's is unimodular, we have $N = \langle y_1 \rangle \oplus \langle y_2 \rangle$. We shall first show that this direct sum is at least a valuated coproduct. We need to argue that $|n_1y_1 + n_2y_2|_p$ equals $|n_1y_1|_p \wedge |n_2y_2|_p$ for all integers n_1, n_2 and all primes p . Of course, we need only verify this equality under the assumption that $|n_1y_1|_p = |n_2y_2|_p$. Moreover, since G is torsion free, common p -power factors can be canceled and hence we may assume that at least one of the integers n_1 or n_2 is prime to p . But since $|y_1|_p < |y_2|_p$ for $p \in A_1 \cup A_2$ and $|y_1|_p = |y_2|_p$ for $p \notin A_1 \cup A_2$, we see that $p \nmid n_1$ if $p \in A_1 \cup A_2$ and that both n_1 and n_2 may be taken to be prime to p when $p \notin A_1 \cup A_2$. Now observe that for any p we have $|n_1y_1 + n_2y_2|_p = |m_1x_1|_p \wedge |m_2x_2|_p$, where $m_1 = n_1 - kmn_2$, $m_2 = n_1 + lnn_2$ and $m_2 - m_1 = n_2$. If $p \in A_1$, $|n_1x_1|_p = |n_1y_1|_p = |n_2y_2|_p = |y_2|_p = |x_2|_p$ and thus the same power of p divides n_1 as divides m ; that is, at least that power of p divides m_1 and consequently p does not divide m_2 . Therefore, for $p \in A_1$,

$$|m_2x_2|_p = |n_2y_2|_p = |n_1y_1|_p \leq |m_1x_1|_p.$$

For $p \in A_2$, a similar analysis shows that

$$|m_1x_1|_p = |n_2y_2|_p = |n_1y_1|_p \leq |m_2x_2|_p.$$

On the other hand, if $p \notin A_1 \cup A_2$, the equation $m_2 - m_1 = n_2$ insures that p divides at most one of the integers m_1 and m_2 , and consequently in this case we also have $|m_1x_1|_p \wedge |m_2x_2|_p = |n_1y_1|_p \wedge |n_2y_2|_p$.

Finally, it remains to explain why $N = \langle y_1 \rangle \oplus \langle y_2 \rangle$ is actually a $*$ -valuated coproduct. Suppose that $0 \neq y = n_1y_1 + n_2y_2 \in G(s^*, p)$ and recall that y is necessarily primitive. Then either y has type strictly greater than that determined by s or else $y \in G(ps)$. Since $N = \langle y_1 \rangle \oplus \langle y_2 \rangle$ is a valuated coproduct, the first

possibility forces both n_1y_1 and n_2y_2 to be in $G(s^*)$, while the second implies that both are in $G(ps)$.

We now have all the ingredients required to establish the following important exchange property.

THEOREM 2.8. *Suppose $N = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_m \rangle$ is a $*$ -valuated coproduct in G where each of the x_i 's is primitive. If $y_1 \neq 0$ is a primitive element contained in N , then there exist primitive elements $y_2, \dots, y_m$ such that $N' = \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \cdots \oplus \langle y_m \rangle$ is a $*$ -valuated coproduct with N/N' finite.*

PROOF. Write $y_1 = n_1x_1 + n_2x_2 + \cdots + n_mx_m$ and observe that there is no loss of generality in assuming that each n_i is nonzero. But then passing immediately to the $*$ -valuated coproduct $N' = \langle n_1x_1 \rangle \oplus \langle n_2x_2 \rangle \oplus \cdots \oplus \langle n_mx_m \rangle$, we may further assume that $y_1 = x_1 + x_2 + \cdots + x_m$. Notice then that we have $|y_1| \leq |x_i|$ for each i , and therefore we can rearrange that x_i 's so that $x_1, \dots, x_k$ have the same type as y_1 and the remaining x_i 's are of strictly larger types. Now write $y_1 = y + g$ where $y = x_1 + \cdots + x_k$ and g is the sum of the remaining x_i 's. Next we make the crucial observation that the primitivity of y_1 implies that $|y_1| = |y|$. By Lemma 2.7, there are primitive elements $y_2, \dots, y_k$ such that $N = \langle y \rangle \oplus \langle y_2 \rangle \oplus \cdots \oplus \langle y_k \rangle \oplus \langle x_{k+1} \rangle \oplus \cdots \oplus \langle x_m \rangle$. Finally, an application of Lemma 2.5 allows us to replace y by y_1 .

COROLLARY 2.9. *If x is a primitive element in the separable group G , then $\langle x \rangle_*$ is a direct summand of G .*

PROOF. By separability, x is contained in a direct summand $A = A_1 \oplus A_2 \oplus \cdots \oplus A_m$, where each A_i is a rank one subgroup of G . Thus we will have x contained in a $*$ -valuated coproduct $N = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_m \rangle$, where $x_i \in A_i$ for each i and A/N is torsion. But then Theorem 2.8 yields a $*$ -valuated coproduct $N' = \langle x \rangle \oplus \langle y_2 \rangle \oplus \cdots \oplus \langle y_m \rangle$, where N/N' is finite. Since $B = \langle x \rangle_* \oplus \langle y_2 \rangle_* \oplus \cdots \oplus \langle y_m \rangle_*$ is pure in G and A/N' is torsion, $B = A$ and $\langle x \rangle_*$ is a summand of G .

COROLLARY 2.10 (BAER [1]). *A finite rank summand of a separable group is completely decomposable.*

PROOF. Suppose $G = A \oplus K$ is separable and A has finite rank. The proof is by induction on the rank of A . Choose $0 \neq x \in A$ to be of maximal type σ in A . By 2.9 and the implicit induction hypothesis, it suffices to show that x is primitive. Now if $s \in \sigma$ and $g \in G(s^*)$, the choice of x implies that $g \in K$. But then $|x - g|_p = |x|_p \wedge |g|_p \leq |x|_p$ for all primes p and hence x is primitive.

Because of their frequent occurrence in the remainder of this paper, we introduce the term *free $*$ -valuated subgroup* to refer to any subgroup F of G that can be represented as a $*$ -valuated coproduct $G = \bigoplus_{i \in I} \langle x_i \rangle$, where the x_i 's are nonzero primitive elements of G . Under these circumstances, we shall say that the x_i 's form a *set of free generators* of F .

THEOREM 2.11. *If F and N are free $*$ -valuated subgroups of G , where N has finite rank and $N \subseteq F$, then there is a $*$ -valuated coproduct $F' = N \oplus M$, where F/F' is finite and M is also a free $*$ -valuated subgroup of G .*

PROOF. Suppose we have $N = \langle y_1 \rangle \oplus \cdots \oplus \langle y_m \rangle$, where the y_i 's form a set of free generators of N . Proceeding by induction, we may assume that we have a $*$ -valuated coproduct $F_1 = \langle y_1 \rangle \oplus \cdots \oplus \langle y_{n-1} \rangle \oplus M_1$, where F/F_1 is finite and M_1

is a free $*$ -valuated subgroup of G . Then some nonzero multiple y'_n of y_n is in F_1 . Let $s = |y'_n|$. Now we can write $y'_1 + \cdots + y'_{n-1} + y'_n = y + g$ where the y'_i 's are multiples of the y_i 's, $g \in M_1 \cap G(s^*)$ and y is a primitive element in M_1 having the same type as y'_n . Since y'_n is primitive and $\langle y_1 \rangle \oplus \cdots \oplus \langle y_n \rangle$ is a $*$ -valuated coproduct, it follows that $|y| = |y'_n|$. Then, just as in the proof of 2.8, we have a $*$ -valuated coproduct $M'_1 = \langle y \rangle \oplus M$, where M_1/M'_1 is finite and M is a free $*$ -valuated subgroup containing g . Using Lemma 2.5, we conclude that we have a $*$ -valuated coproduct $F'_1 = \langle y_1 \rangle \oplus \cdots \oplus \langle y_{n-1} \rangle \oplus \langle y'_n \rangle \oplus M$ with F/F'_1 finite. Since y_n has finite order modulo $\langle y'_n \rangle$, we still have a $*$ -valuated coproduct when $\langle y'_n \rangle$ is replaced by $\langle y_n \rangle$, yielding thereby the desired F' .

COROLLARY 2.12. *If N is a finite rank, free $*$ -valuated subgroup of the separable group G , then the pure closure of N is a direct summand of G .*

PROOF. Let $N = \langle y_1 \rangle \oplus \cdots \oplus \langle y_n \rangle$, where the y_i 's are free generators of N . Since G is separable, N is contained in a direct summand $A = A_1 \oplus \cdots \oplus A_m$ of G , where each A_i is a rank one subgroup. Each A_i is locally cyclic and therefore we have nonzero x_i 's in the corresponding A_i 's such that $F = \langle x_1 \rangle \oplus \cdots \oplus \langle x_n \rangle$ contains N . By 2.11, we have a $*$ -valuated coproduct $F' = N \oplus M$, where F/F' is finite. Thus A is the pure closure of F' and we clearly have a direct decomposition $A = B \oplus C$, where B and C are, respectively, the pure closures of N and M in G .

COROLLARY 2.13 (BAER [1]). *Any finite rank, pure subgroup of a homogeneous separable group is a direct summand.*

PROOF. Let A be a nonzero finite rank pure subgroup of the homogeneous separable group G and let N be a free subgroup of A with A/N torsion. As in the proof of 2.12, N is contained in a finite rank, free $*$ -valuated subgroup F of G . Now notice that Lemma 2.7 implies that all the elements of N are primitive in G . By Theorem 2.8, we have $*$ -valuated coproduct $F' = \langle y_1 \rangle \oplus \cdots \oplus \langle y_n \rangle$, where F/F' is finite, the y_i 's are primitive in G and y_1 is a nonzero element of N . A straightforward induction then leads to a free $*$ -valuated subgroup N' of G with N/N' finite. Since A is the pure closure of N' , the desired conclusion follows from 2.12.

3. k -groups. In this section, we study a class of torsion free groups more general than the separable groups. We call G a k -group if each finite subset can be imbedded in a finite rank, free $*$ -valuated subgroup.

EXAMPLE 3.1. A k -group need not be separable. Let $K = \mathbb{Z}^{\aleph_0}$ and take H to be the corresponding direct sum of $\aleph_0$ copies of $\mathbb{Z}$. Then for any prime p , $G = H + pK$ is an $\aleph_1$ -free group that is not separable (see p. 114 of [4]). On the other hand, G is a k -group since it is $\aleph_1$ -free. Indeed, any finite subset is imbeddable in a finite rank, pure free subgroup $F = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle$. As F is pure, this direct decomposition is a valuated coproduct. Furthermore $G(s^*) = 0$ for all height sequences s since G is homogeneous of type $(0, 0, \dots, 0, \dots)$. Consequently, each x_i is primitive and $F = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle$ is a $*$ -valuated coproduct in G .

Example 3.1 notwithstanding, we can prove the following result.

PROPOSITION 3.2. *A finite rank summand of a k -group is completely decomposable.*

PROOF. Let A be a finite rank summand of the k -group G . Then there is a finitely generated subgroup F of A with A/F torsion. Since G is a k -group, F is contained in a free $*$ -valuated subgroup N of G . The pure closure B of the subgroup N is a completely decomposable group containing A . But then A is a direct summand of B and therefore A is completely decomposable by 2.10.

Separable groups have received much attention in the literature. The following result provides a useful characterization of these groups.

THEOREM 3.3. *G is separable if and only if G is a k -group with the property that the pure closure of each finite rank, free $*$ -valuated subgroup is a direct summand.*

PROOF. Since separable groups are obviously k -groups, the proposition is an immediate consequence of 2.12 and 3.2.

Like separable groups, countable k -groups are completely decomposable. But the proof of this fact requires the following lemma.

LEMMA 3.4. *Let N be a finite rank, free $*$ -valuated subgroup of the k -group G and suppose S is a finite subset of G . Then there exists a finite collection of primitive elements $y_1, y_2, \dots, y_m$ such that $N' = N \oplus \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \dots \oplus \langle y_m \rangle$ is a $*$ -valuated coproduct in G with $\langle S, N' \rangle / N'$ finite.*

PROOF. Let $x_1, x_2, \dots, x_n$ be free generators of N and take $S' = S \cup \{x_1, x_2, \dots, x_n\}$. Since G is a k -group, we can select a finite rank, free $*$ -valuated subgroup F containing S' . Then $N \subseteq F$ and, by Theorem 2.11, we have a $*$ -valuated coproduct $N' = N \oplus \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \dots \oplus \langle y_m \rangle$, where the y_i 's are primitive and F/N' is finite. Since $S \subseteq F$, the proof is complete.

THEOREM 3.5. *A countable k -group is completely decomposable.*

PROOF. Suppose $x_1, x_2, \dots, x_n, \dots$ is an enumeration of the elements of G and let $X_n = \{x_i : i < n\}$ for each $n < \omega$. Using 3.4, we define inductively an ascending sequence $\{S_n\}$, where each S_n is a finite set of primitive elements serving as a free basis of a free $*$ -valuated subgroup F_n with $\langle X_n, F_n \rangle / F_n$ finite. Then $F = \bigcup_{n < \omega} F_n$ is a free $*$ -valuated subgroup of G with G/F torsion. Thus G is the pure closure of F and therefore G is completely decomposable.

We next want to show that the class of k -groups is closed under the operation of taking direct summands. Our proof of this fact requires a technical lemma, which also turns out to be useful later. The statement of this lemma, however, requires a couple of preliminary definitions. If $H \oplus K$ is a $*$ -valuated coproduct in G and if F is a subgroup of G , then we say that F *splits along H and K* provided $F = (F \cap H) \oplus (F \cap K)$ and we say that F is *quasi-splitting along H and K* if $F/(F \cap H) \oplus (F \cap K)$ is torsion.

LEMMA 3.6. *Suppose $H \oplus K$ is a $*$ -valuated coproduct in G and let F be a free $*$ -valuated subgroup that is quasi-splitting along H and K . If the finite rank, free $*$ -valuated subgroup A is contained in F , then there is a $*$ -valuated coproduct $F' = A \oplus B \oplus C$, where F/F' is torsion, B has finite rank and C splits along H and K .*

PROOF. The proof is by induction on the rank of A and we impose the further hypothesis that B be a free $*$ -valuated subgroup having a set of free generators each

of which has the same type as one of the members of a fixed set of free generators of A . The proof also requires the following sublemma: If N is quasi-splitting along H and K and if $N' = M \oplus L$ is a $*$ -valuated coproduct, where N/N' is torsion and M splits along H and K , then there exists a $*$ -valuated coproduct $N'' = M \oplus L'$, where N/N'' is torsion and L' also splits along H and K . Indeed we need only take $N'' = N' \cap H + N' \cap K$ and $L' = H_1 \oplus K_1$, where $H_1 = H \cap [(M \cap K) \oplus L]$ and $K_1 = K \cap [(M \cap H) \oplus L]$.

Suppose $A = A_1 \oplus \langle x \rangle$ is a $*$ -valuated coproduct where x is a primitive element of maximal type in A . By our induction hypothesis, we have a $*$ -valuated coproduct $F_1 = A_1 \oplus B_1 \oplus C_1$, where F/F_1 is torsion, B_1 satisfies the appropriate conditions and C_1 splits along H and K . Notice that F being quasi-splitting along H and K is essential here even in the case $A_1 = 0$. Then some nonzero multiple x' of x lies in F_1 and we write $x' = a + b + c_1 + c_2$, where $a \in A_1$, $b \in B_1$, $c_1 \in C_1 \cap H$ and $c_2 \in C_1 \cap K$. Let $s = |x'|$. Since $|x'| \leq |a|$, either $a = 0$ or else a has maximal type in A_1 . Thus Lemma 2.7 implies that a and $x' - a = b + c_1 + c_2$ are primitive elements. Moreover, since $A_1 \oplus \langle x' \rangle$ is a valuated coproduct and $|x'| \leq |b + c_1 + c_2|$, it follows that $|b + c_1 + c_2| = s$. Even though c_1 and c_2 need not themselves be primitive elements, there are, we claim, elements $z_1 \in H \cap G(s^*)$ and $z_2 \in K \cap G(s^*)$ such that $h = c_1 - z_1$ and $k = c_2 - z_2$ are primitive. As the arguments are parallel, we provide the details only for the choice of z_1 . If $c_1 \in G(s^*)$, then we need only take $z_1 = c_1$. Assume then that $c_1 \notin G(s^*)$. As c_1 is contained in the free $*$ -valuated subgroup F and $s \leq |c_1|$, we can write $c_1 = v + w$, where $w \in G(s^*) \cap F$ and v is a primitive element having the same type as c_1 . Moreover, since F is quasi-splitting along H and K , some nonzero multiple of w is expressible as a sum $z_1 + w'$, where $z_1 \in F \cap H$ and $w' \in F \cap K$. Replacing x' by some nonzero multiple of itself if necessary, we may assume that $c_1 = v + z_1 + w'$ and that also $z_1 \in F_1$. Notice then that $z_1 \in G(s^*)$ since $H \oplus K$ is a $*$ -valuated coproduct. Then $h = c_1 - z_1 = v + w'$ is primitive by Lemma 2.3 because $|v| = |h| \wedge |w'|$ since $h \in H$ and $w' \in K$.

Observe that the choice of x and the condition on B_1 force $F_1 \cap G(s^*) \subseteq C_1$. Thus from our choice of z_1 above, $h \in C_1$ and, similarly, $k = c_2 - z_2 \in C_1$. By Theorem 2.11 and the fact that C_1 is contained in the free $*$ -valuated subgroup F , we have a $*$ -valuated coproduct $C'_1 = \langle h \rangle \oplus \langle k \rangle \oplus C$, where C_1/C'_1 is torsion and $C_1 \cap G(s^*) \subseteq C$. By our sublemma above, we may further assume that C splits along H and K . Now clearly $F_2 = A_1 \oplus B_1 \oplus \langle h \rangle \oplus \langle k \rangle \oplus C$ is a $*$ -valuated coproduct with F/F_2 torsion. Let $z = x' - z_1 - z_2 = a + b + h + k$ and note that z is a primitive element with $|z| = s$ by Lemma 2.3. Also $y_1 = b + h + k$ is primitive with $|y_1| = |b + c_1 + c_2| = s$ for the same reason. Since $|x'| \leq |b|$, either $b = 0$ or b is a primitive element in B_1 of maximal type. In either case, we have a $*$ -valuated coproduct $B'_1 = B_0 \oplus \langle b \rangle$, where B_1/B'_1 is finite and B_0 is a free $*$ -valuated subgroup of G . Consider then the $*$ -valuated coproduct $F' = A_1 \oplus B_0 \oplus \langle b \rangle \oplus \langle h \rangle \oplus \langle k \rangle \oplus C$ and apply Lemma 2.7 to express F' as the $*$ -valuated coproduct $F' = A_1 \oplus B_0 \oplus \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \langle y_3 \rangle \oplus C$, where y_2 and y_3 are also primitive. Lemma 2.5 allows us to replace y_1 by z , and since $z_1, z_2 \in C$ another application of Lemma 2.5 enables us to replace z by x' . Thus we have a $*$ -valuated coproduct $F' = A_1 \oplus \langle x' \rangle \oplus B \oplus C$, where $B = B_0 \oplus \langle y_2 \rangle \oplus \langle y_3 \rangle$, F/F' is torsion and C splits along H and K . Finally, as x has finite order modulo $\langle x' \rangle$, we may replace $A_1 \oplus \langle x' \rangle$ by $A = A_1 \oplus \langle x \rangle$.

THEOREM 3.7. *A direct summand of a k -group is a k -group.*

PROOF. Suppose $G = H \oplus K$ is a k -group and let S be a finite subset of H . Then $S \subseteq A$ for some finite rank, free $*$ -valuated subgroup A of G . Using 3.4, we define inductively a pair of ascending sequences $\{T_n\}_{n<\omega}$ and $\{S_n\}_{n<\omega}$ of finite subsets of G satisfying the following conditions:

(i) T_n is a set of free generators of a free $*$ -valuated subgroup F_n of G with $F_0 = A$.

(ii) $F_n \subseteq \langle S_n \rangle$ and $S_n = (S_n \cap H) \cup (S_n \cap K)$.

(iii) $\langle S_n, F_{n+1} \rangle / F_{n+1}$ is finite for each n .

Then $F = \bigcup_{n<\omega} F_n$ is a free $*$ -valuated subgroup of G , and conditions (ii) and (iii) imply that F is quasi-splitting along H and K . Then we have a $*$ -valuated coproduct $F' = A \oplus B \oplus C$, where F' , B and C satisfy the conditions stated in Lemma 3.6. Recalling the sublemma used in the proof of 3.6, we see that there is a $*$ -valuated coproduct $F'' = A_1 \oplus B_1 \oplus C$, where F/F'' is torsion and $A_1 = H \cap [A \oplus B \oplus (C \cap K)]$. Notice that A_1 contains S and that A_1 is necessarily a finite rank subgroup of G . Now let L and M be the pure closures of A_1 and F , respectively. Then M is completely decomposable and, since M is also the pure closure of F'' , L is a direct summand of M . By 2.10, L is completely decomposable and clearly then S is contained in a direct sum $\langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \cdots \oplus \langle y_m \rangle$, where the y_i 's are primitive in M and this direct sum is a $*$ -valuated coproduct in M . That the y_i 's are actually primitive in G and that $\langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \cdots \oplus \langle y_m \rangle$ is a $*$ -valuated coproduct in G follow from the crucial fact that $M \cap G(s^*, p) = M(s^*, p)$ for all height sequences s and all primes p . Indeed from the construction of M , $M = \bigoplus_{i<\omega} M_i$ is a $*$ -valuated coproduct in G where each M_i is the pure subgroup of G generated by some primitive element x_i of G . Thus $M \cap G(s^*, p) = M(s^*, p)$ reduces to $M_i \cap G(s^*, p) = M_i(s^*, p)$ for each i , and this latter fact is an easy consequence of the fact that M_i is pure in G and each element of M_i is primitive in G .

COROLLARY 3.8 (FUCHS [2]). *A direct summand of a separable group is separable.*

PROOF. Suppose $G = H \oplus K$ is separable. Then H is a k -group by Theorem 3.7. Clearly $H \cap G(s^*) = H(s^*)$ and $H \cap G(s^*, p) = H(s^*, p)$ for all height sequences s and all primes p . The latter observation implies that each free $*$ -valuated subgroup of H is a free $*$ -valuated subgroup of G . That H is separable follows now from Proposition 3.3.

4. Knice subgroups. In this section, we introduce a class of subgroups that plays a role for torsion free groups analogous to that played by nice subgroups in the study of torsion groups. Indeed, following the theme of our earlier paper [8], we elevate to the status of a definition the property exhibited in 3.4 of finite rank, free $*$ -valuated subgroups of k -groups.

DEFINITION 4.1. A subgroup N of the torsion free group G is said to be a *knice subgroup* if for each finite subset S of G there exist primitive elements $y_1, y_2, \dots, y_m$ such that $N' = N \oplus \langle y_1 \rangle \oplus \langle y_2 \rangle \cdots \langle y_m \rangle$ is a $*$ -valuated coproduct with $\langle S, N' \rangle / N'$ finite.

It is easy to see that N is knice in G if and only if its pure closure is knice in G . If N is both pure and knice in G , then a routine argument shows that the y_i 's can be chosen so that $S \subseteq N'$. Note then that G is a k -group if and only if 0 is

a knice subgroup of G . That summands of k -groups are knice is a nontrivial fact that follows from Theorem 3.7.

PROPOSITION 4.2. *If $N' = N \oplus \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle$ is a $*$ -valuated coproduct in G with N a knice subgroup of G and each x_i primitive in G , then N' is a knice subgroup of G .*

PROOF. By induction, it suffices to consider the case $n = 1$. Assume then that $N' = N \oplus \langle x \rangle$ is a $*$ -valuated coproduct in G with N knice and x primitive. Let S be a finite subset of G and take $S' = S \cup \{x\}$. Since N is knice in G , there is a $*$ -valuated coproduct $F = N \oplus \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \cdots \oplus \langle y_m \rangle$, where the y_i 's are primitive and $\langle S', F \rangle / F$ is finite. In particular, some nonzero multiple x' of x is contained in F . Thus we can write $x' = z + t_1 y_1 + t_2 y_2 + \cdots + t_m y_m$, where $z \in N$. Since $N \oplus \langle x' \rangle$ is a $*$ -valuated coproduct, if all the $t_i y_i$'s had type greater than the type of x' , the primitivity of x' would be contradicted. Then $x' = z + y + g$ where the primitive element y is the sum of the $t_i y_i$'s having the same type as x' . Observe that $x' - z = y + g$ is also primitive since if $x' - z \in G(t^*, p)$ where t is equivalent to $|x' - z| = |x'| \wedge |z| = |x'|$ and $t_p = |x' - z|_p = |x'|_p \neq \infty$, then $N \oplus \langle x' \rangle$ being a $*$ -valuated coproduct implies $x' \in G(t^*, p)$ contrary to the fact that x' is primitive. But then $g \in G(s^*)$, where $s = |x'| = |x' - z|$ and hence the equation $x' - z = y + g$ forces us to conclude that $|y| = |x' - z| = |x'|$. By Lemma 2.7, x' is contained in a $*$ -valuated coproduct $F' = N \oplus \langle y \rangle \oplus \langle z_2 \rangle \oplus \cdots \oplus \langle z_m \rangle$ where F/F' is finite and, of course, the z_i 's are primitive. Lemma 2.5 then allows us to replace y by x' in the representation of F' above. Finally, we note that $H = N \oplus \langle x \rangle \oplus \langle z_2 \rangle \oplus \cdots \oplus \langle z_m \rangle$ is a $*$ -valuated coproduct with $\langle S, H \rangle / H$ finite, that is, $N \oplus \langle x \rangle$ is a knice subgroup of G .

Recall that a pure subgroup H of the torsion free group G is said to be *balanced* if each coset $x + H$ contains an element y such that $|y| = |x + H|$, where the height sequences are computed in G and G/H , respectively.

THEOREM 4.3. *A pure subgroup H of G is a knice subgroup if and only if H is balanced in G and G/H is a k -group.*

PROOF. First assume that H is a knice and pure subgroup of G . Then for any x in G we can write $x = z + y$, where $z \in H$ and $H \oplus \langle y \rangle$ is a valuated coproduct in G . It follows that $|x + h| \leq |y|$ for all $h \in H$, that is, $|x + H| = |y|$ and H is balanced. Since H is balanced in G , $(G/H)(s) = G(s) + H/H$ and $(G/H)(s^*, p) = G(s^*, p) + H/H$ for all height sequences s and all primes p . To prove that G/H is a k -group it is clearly enough to show that if $H \oplus \langle y_1 \rangle \oplus \cdots \oplus \langle y_m \rangle$ is a $*$ -valuated coproduct in G with primitive y_i 's, then the $y_i + H$'s are primitive and $\langle y_1 + H \rangle \oplus \cdots \oplus \langle y_m + H \rangle$ is a $*$ -valuated coproduct in G/H . But these facts are easily established using the observations above about $(G/H)(s)$ and $(G/H)(s^*, p)$.

Conversely, assume that H is balanced in G and that G/H is a k -group. Now consider any $*$ -valuated coproduct $\langle y_1 + H \rangle \oplus \cdots \oplus \langle y_m + H \rangle$ in G/H , where the $y_i + H$'s are primitive and $|y_i| = |y_i + H|$ for each i . It is trivial then that the y_i 's are primitive in G , and obviously it is enough to argue that $H \oplus \langle y_1 \rangle \oplus \cdots \oplus \langle y_m \rangle$ is a $*$ -valuated coproduct in G . Let $x = h + t_1 y_1 + \cdots + t_m y_m$, where $h \in H$. First suppose $x \in G(s)$. Then, for each i , $t_i y_i + H \in (G/H)(s)$ and hence $t_i y_i \in G(s)$ since $|t_i y_i| = |t_i y_i + H|$. Thus $H \oplus \langle y_1 \rangle \oplus \cdots \oplus \langle y_m \rangle$ is at least a valuated coproduct. Next suppose $x \in G(s^*, p)$. Then once again for each i , $t_i y_i + H \in (G/H)(s^*, p)$.

But since $t_i y_i + H$ is primitive, either $t_i y_i + H$ is in $(G/H)(ps)$ or $|t_i y_i + H|$ is not equivalent to s . In either case, $|t_i y_i| = |t_i y_i + H|$ implies $t_i y_i \in G(s^*, p)$.

Since the ideas involved in the proof of Theorem 3.4 can be used to show that every countable subset of a k -group is contained in a countable completely decomposable pure subgroup, the following corollary implies that pure knice subgroups are $\aleph_1$ -pure.

COROLLARY 4.4. *If H is a pure knice subgroup of G and if G/H is countable, then H is a direct summand of G .*

PROOF. As is well known, H is a summand of G if H is balanced in G and G/H is completely decomposable. Thus the result follows from 4.3 and 3.5.

COROLLARY 4.5. *If H is balanced in G and K/H is a pure knice subgroup of G/H , then K is a pure knice subgroup of G .*

PROOF. If H is balanced in G and K/H is balanced in G/H , then K is balanced in G . Thus Theorem 4.3 and the canonical isomorphism $G/K \simeq (G/H)/(K/H)$ yield the desired conclusion.

COROLLARY 4.6. *Let H be a pure subgroup of G . If H is knice in G and N/H is knice in G/H , then N is knice in G .*

PROOF. It suffices to observe that the hypotheses imply that the pure closure K of N is a knice subgroup of G . But since K/H is the pure closure in G/H of the knice subgroup N/H , we conclude that K is knice in G by 4.5.

Formalizing a notion that has already played a role in 3.7 and 3.8, we say that a pure subgroup H of G is **-pure* if $H \cap G(s^*) = H(s^*)$ and $H \cap G(s^*, p) = H(s^*, p)$ for all height sequences s and all primes p . We have already seen that summands are *-pure subgroups, and so are rank one pure subgroups generated by primitive elements. Clearly the ascending union of *-pure subgroups is *-pure. Also a *-valuated coproduct $H = \bigoplus_{i \in I} H_i$ in G will be a *-pure subgroup if each H_i is *-pure in G .

PROPOSITION 4.7. *A pure knice subgroup is *-pure.*

PROOF. Suppose H is a pure knice subgroup of G and let $x \in H \cap G(s^*, p)$. Then we can write $x = z_1 + z_2 + \cdots + z_n + g$, where $g \in G(ps)$ and each z_i is an element of $G(s)$ with $|z_i|$ not equivalent to s . Since H is a pure knice subgroup of G , $\{z_1, z_2, \dots, z_n, g\}$ is contained in a *-valuated coproduct $H \oplus \langle y_1 \rangle \oplus \cdots \oplus \langle y_m \rangle$, where the y_i 's are primitive in G . Then we can write $g = h + t_1 y_1 + \cdots + t_m y_m$ and, for each i , $z_i = h_i + t_1^{(i)} y_1 + \cdots + t_m^{(i)} y_m$ where h and each h_i is in H . Since $|g| \leq |h|$ and $|z_i| \leq |h_i|$ for $i = 1, 2, \dots, n$, $x = z_1 + \cdots + z_n + g = h_1 + h_2 + \cdots + h_n + h$ is in $H(s^*, p)$. Likewise, if $x \in H \cap G(s^*)$, then clearly $x \in H(s^*)$.

Exploiting Lemma 3.6 again, we shall prove that a pure knice subgroup of a k -group is itself a k -group.

THEOREM 4.8. *The pure knice subgroup H of G is a k -group if and only if G is a k -group.*

PROOF. First consider the case where H is a k -group and let S be a finite subset of G . Since H is a pure knice subgroup of G , S is contained in a *-valuated

coproduct $H \oplus \langle y_1 \rangle \oplus \cdots \oplus \langle y_m \rangle$, where the y_i 's are primitive in G . But then $S \subseteq \langle T \rangle \oplus \langle y_1 \rangle \oplus \cdots \oplus \langle y_m \rangle$ for some finite subset T of H . Since H is a k -group, there are primitive elements $x_1, \dots, x_n$ in H such that $\langle x_1 \rangle \oplus \cdots \oplus \langle x_n \rangle$ is a $*$ -valuated coproduct in H containing T . Then 4.7 implies that $F = \langle x_1 \rangle \oplus \cdots \oplus \langle x_n \rangle \oplus \langle y_1 \rangle \oplus \cdots \oplus \langle y_m \rangle$ is a free $*$ -valuated subgroup of G that contains S .

Conversely, assume that G is a k -group. The proof that H is a k -group will involve an elaboration on the technique used to prove Theorem 3.7, and we shall need to use Proposition 4.2 in exploiting the fact that H is knice in order to generate an appropriate K with $H \oplus K$ a $*$ -valuated coproduct in G . Once again we begin with a finite rank, free $*$ -valuated subgroup A of G which contains some fixed finite subset S of H . But this time we define inductively, in addition to a pair of ascending sequences $\{T_n\}_{n < \omega}$ and $\{S_n\}_{n < \omega}$ of finite subsets of G , a third sequence $\{B_n\}_{n < \omega}$ of finite rank, pure knice subgroups such that the following conditions are satisfied:

(i) T_n is a set of free generators of a free $*$ -valuated subgroup F_n of G with $F_0 = A$.

(ii) $H \oplus B_0 \oplus \cdots \oplus B_n$ is a $*$ -valuated coproduct in G that contains F_n .

(iii) $F_n \subseteq \langle S_n \rangle$ and $S_n = (S_n \cap H) \cup (S_n \cap (B_0 \oplus \cdots \oplus B_n))$.

(iv) $\langle S_n, F_{n+1} \rangle / F_{n+1}$ is finite for each n .

Then $F = \bigcup_{n < \omega} F_n$ is a free $*$ -valuated subgroup of G that is quasi-splitting along H and $K = \bigoplus_{n < \omega} B_n$. The proof of the theorem is now completed exactly as in the case of Theorem 3.7.

The preceding theorem leads quickly to a generalization of the important observation of Fuchs that summands of separable groups are separable.

COROLLARY 4.9. *A pure knice subgroup of a separable group is itself separable.*

PROOF. Suppose H is a pure knice subgroup of the separable group G . Since H is $*$ -pure in G by 4.7, each free $*$ -valuated subgroup of H is a free $*$ -valuated subgroup of G . That H is separable then follows from 4.8 and 3.3.

In [9], Rangaswamy introduces the notion of a strongly balanced subgroup and his Theorem 7 asserts that both H and G/H are separable if H is strongly balanced in the separable group G . Thus strongly balanced subgroups of separable groups are actually pure knice subgroups and hence 4.9 can also be viewed as a partial generalization of Rangaswamy's theorem. Notice, however, that in the context of separable groups, the pure knice subgroups form a more comprehensive class than the strongly balanced subgroups. Indeed if H is a pure knice subgroup of the separable groups G , then G/H need not necessarily be separable. This latter observation follows from 3.1, 4.3 and the well-known fact that every torsion free group is a homomorphic image of a completely decomposable group with balanced kernel.

We close this section by recording some further facts about pure knice subgroups.

PROPOSITION 4.10. *Let H and K be pure subgroups of the torsion free group G with $H \subseteq K$.*

(i) *If H is knice in K and K is knice in G , then H is knice in G .*

(ii) *If H is knice in G and K/H is knice in G/H , then K is knice in G .*

(iii) *If K is knice in G , then K/H is knice in G/H .*

(iv) *If H and K are both knice in G , then H is knice in K .*

PROOF. Of course, (ii) has already been proved in 4.6. To prove the transitivity result (i), we take S to be a finite subset of G . Since we are assuming that K is pure and knice in G , we have S contained in a $*$ -valuated coproduct $K \oplus \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \cdots \oplus \langle y_m \rangle$, where the y_i 's are primitive in G . But then S has a corresponding finite projection T in K , that is, $S \subseteq \langle T, y_1, y_2, \dots, y_m \rangle$, where T is a finite subset of K . But because H is assumed to be knice in K , T is contained in a $*$ -valuated coproduct (in K) $H \oplus \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_n \rangle$, where x_j 's are primitive in K . By 4.7, the x_j 's are primitive in G and $H \oplus \langle x_1 \rangle \oplus \cdots \oplus \langle x_n \rangle \oplus \langle y_1 \rangle \oplus \cdots \oplus \langle y_m \rangle$ is a $*$ -valuated coproduct in G containing S ; that is, H is knice in G .

That K being knice in G implies K/H is knice in G/H is an easy consequence of Theorem 4.3. Indeed under these circumstances, K/H is clearly balanced in G/H and $(G/H)/(K/H) \simeq G/K$ is a k -group. Finally, to prove (iv), we assume that both H and K are knice in G . Then G/H is a k -group by 4.3, and therefore (iii) and Theorem 4.8 imply that K/H is a k -group. Noting that H is balanced in K since it is balanced in G by 4.3, we see that yet another application of 4.3 yields that desired conclusion that H is knice in K .

COROLLARY 4.11. *If $H = A \oplus B$ is a pure knice subgroup of the k -group G , then A and B are pure knice subgroups of G .*

PROOF. By Theorem 4.8, H itself is a k -group. But as noted earlier, it is a consequence of Theorem 3.7 that summands of k -groups are knice subgroups. The conclusion that A and B are knice in G follows from 4.10(i).

5. The third axiom of countability. A torsion free group G is said to satisfy the *third axiom of countability with respect to knice subgroups* provided there is a family $\mathcal{C}$ of knice subgroups of G such that the following three conditions hold:

- (0) $0 \in \mathcal{C}$.
- (1) $\mathcal{C}$ is closed with respect to the group union of an arbitrary number of groups.
- (2) If $A \in \mathcal{C}$ and S is a countable subset of G , then there exists a $B \in \mathcal{C}$ such that B/A is countable and $B \supseteq \langle A, S \rangle$.

PROPOSITION 5.1. *If $\mathcal{C}$ is a family of knice subgroups of G satisfying (0), (1) and (2), then $\mathcal{C}$ contains a subfamily $\mathcal{C}'$ of pure knice subgroups satisfying (0), (1) and (2).*

PROOF. By Theorem 2 of [5], there is a family $\mathcal{P}$ of pure subgroups of G satisfying (0), (1) and (2). Let $\mathcal{C}' = \mathcal{C} \cap \mathcal{P}$. Clearly $\mathcal{C}'$ satisfies (0) and (1). Now suppose $A \in \mathcal{C}'$ and let S be a countable subset of G . Exploiting condition (2) for $\mathcal{C}$ and $\mathcal{P}$ separately, we generate inductively a nested ascending sequence

$$B_0 \subseteq C_0 \subseteq B_1 \subseteq C_1 \subseteq \cdots \subseteq B_n \subseteq C_n \subseteq \cdots,$$

where $\langle A, S \rangle \subseteq B_0$, B_0/A is countable, each C_n is in $\mathcal{P}$ and both C_n/B_n and B_{n+1}/C_n are countable for all n . Then $B = \bigcup_{n < \omega} B_n = \bigcup_{n < \omega} C_n$ is in $\mathcal{C}'$, $\langle A, S \rangle \subseteq B$ and B/A is countable. Thus $\mathcal{C}'$ satisfies (2), as desired.

THEOREM 5.2. *If G is a torsion free group satisfying the third axiom of countability with respect to knice subgroups, then any direct summand of G also satisfies the third axiom of countability with respect to knice subgroups.*

PROOF. Suppose $G = H \oplus K$ and let $\mathcal{C}$ be a family of knice subgroups of G satisfying (0), (1) and (2). By 5.1, we may assume that the members of $\mathcal{C}$ are

pure and knice in G . Let $\mathcal{C}'$ consist of all those subgroups A of H for which there corresponds an $N \in \mathcal{C}$ such that $N = A \oplus (N \cap K)$. It is a familiar fact (see, e.g., the proof of 81.5 in [3]) that $\mathcal{C}'$ also satisfies (0), (1) and (2). It remains to show that the members of $\mathcal{C}'$ are knice in H . But this is an easy consequence of 4.3 and 3.7. Indeed suppose $A \in \mathcal{C}'$ and $N = A \oplus (N \cap K) \in \mathcal{C}$. It is then a routine argument to show that N being balanced in G implies that A is balanced in H ; while H/A is a k -group since it is canonically isomorphic to a direct summand of the k -group G/N .

Given the well-known third axiom of countability characterization of simply presented torsion groups, it should not be surprising that the class of torsion free groups we have introduced in this section consists precisely of the simply presented ones. This is indeed the content of our next theorem.

THEOREM 5.3. *A torsion free group satisfies the third axiom of countability with respect to knice subgroups if and only if it is completely decomposable.*

PROOF. Assume first that $G = \bigoplus_{i \in I} G_i$, where each G_i is a torsion free rank one group. For each subset J of I , let $G(J) = \sum_{i \in J} G_i$. As a summand of the k -group G , each $G(J)$ is a knice subgroup of G . If $\mathcal{C}'$ consists of all $G(J)$ as J ranges over all subsets of I , then it is clear that $\mathcal{C}$ satisfies (0), (1) and (2).

Conversely, assume that $\mathcal{C}$ is a family of pure knice subgroups of G that satisfies (0), (1) and (2). It is routine to show that G is the union of a well-ordered family $\{H_\alpha\}_{\alpha < \mu}$ satisfying the following conditions:

- (i) $H_0 = 0$, $H_\alpha \subseteq H_\beta$ if $\alpha < \beta$ and $H_\alpha = \bigcup_{\beta < \alpha} H_\beta$ whenever α is a limit ordinal;
- (ii) For all $\alpha < \mu$, $H_\alpha \in \mathcal{C}$ and $H_{\alpha+1}/H_\alpha$ is countable.

The proof is completed by showing that, for each α , $H_{\alpha+1} = H_\alpha \oplus L_\alpha$, where L_α is completely decomposable; for if this can be established, it will follow that $G = \bigoplus_{\alpha < \mu} L_\alpha$. Thus, by 3.5, 4.3 and 4.4, it is enough to prove that H_α is knice in $H_{\alpha+1}$ for each α . But since H_α is balanced in G and $H_{\alpha+1}/H_\alpha$ is pure in G/H_α , it is routine to verify that H_α is balanced in $H_{\alpha+1}$. By 4.3 it remains only to prove that $H_{\alpha+1}/H_\alpha$ is a k -group. It is, however, a triviality to show that $H_{\alpha+1}$ being a pure knice subgroup G implies that $H_{\alpha+1}/H_\alpha$ is a pure knice subgroup of the k -group G/H_α . Finally, an application of 4.8 completes the proof.

Notice that 5.2 and 5.3 combine to yield the Baer-Kulikov-Kaplansky theorem which asserts that summands of completely decomposable groups are themselves completely decomposable. Of course, theorems insuring that certain subgroups of completely decomposable groups are completely decomposable are difficult to come by, and all too often such theorems involve artificial assumptions concerning the set of types assumed by elements of the containing group. As a final application of the ideas developed in this paper, however, we shall present a fairly general criterion concerning subgroups of completely decomposable groups which can be paraphrased as follows: Separable groups of cardinality at most $\aleph_1$ have balanced projective dimension ≤ 1 .

THEOREM 5.4. *If H is a balanced subgroup of the completely decomposable group G and if G/H is a separable group of cardinality not exceeding $\aleph_1$, then H is completely decomposable.*

PROOF. We begin our proof with a few preliminary observations. First we note that a separable group K of cardinality $\aleph_1$ is the union of a smooth well-ordered chain $\{K_\alpha\}_{\alpha < \omega_1}$, where each K_α is a countable, separable pure subgroup of K . Indeed beginning with a well-ordering $\{x_\alpha\}_{\alpha < \omega_1}$ of the elements of K , we construct the K_α 's inductively in such a fashion that (a) $x_\alpha \in K_{\alpha+1}$ and (b) each finite subset S of K_α can be imbedded in a finite rank summand A of K with $A \subseteq K_\alpha$. Assume that the requisite K_α 's have been constructed for all $\alpha < \beta$. If β is a limit ordinal, we let $K_\beta = \bigcup_{\alpha < \beta} K_\alpha$ and observe that condition (b) is inherited by K_β . On the other hand, if $\beta = \alpha + 1$ for some α , then we choose an enumeration $\{y_n\}_{n < \omega}$ of the elements of K_α and take $K_\beta = \bigcup_{n < \omega} A_n$, where the A_n 's form an ascending sequence of finite rank summands of K with A_n containing $\{y_0, \dots, y_n, x_\alpha\}$.

Our remaining preliminary observations involve the notion of *global compatibility* in the sense of [6]. If A and B are subgroups of G , then we write $A \parallel B$ to indicate that the following condition is satisfied: If $(a, b) \in A \times B$ and if $s \leq |a + b|$ for some height sequence s , then there exists a $b' \in A \cap B$ such that $s \leq |a + b'|$. If H is a balanced subgroup of G and if S is a countable subset of G , then there is a countable subgroup B of G such that $S \subseteq B$ and $B \parallel H$ (see the proof of Lemma 1 in [6]). Finally, we wish to note that $A \cap H$ will be a balanced subgroup of A provided H is balanced in G and A is a pure subgroup of G with $A \parallel H$. Indeed suppose $s = |a + A \cap H|$, where heights are computed in $A/A \cap H$, or equivalently in $G/A \cap H$ since A is pure. But then $|a + H| \geq s$ and since H is balanced in G , there is an $h \in H$ such that $|a + h| = |a + H|$. Recalling that $A \parallel H$, we see that $|a + h'| \geq s$ for some $h' \in A \cap H$ and it follows that $A \cap H$ is balanced in A .

We shall first prove the theorem under the further restriction that $|G| \leq \aleph_1$. Then as the proof of 5.3 clearly indicates, it suffices to show that H is the union of a family $\mathcal{C}$ of countable pure knice subgroups where $\mathcal{C}$ satisfies (0), (2) and the following weakened version of (1):

(1') $\mathcal{C}$ is closed under the union of countable chains.

We fix a direct decomposition $G = \bigoplus_{i \in I} G_i$, where each G_i is a rank one group, and we let $G(J) = \sum_{i \in J} G_i$ whenever J is a subset of I . Since G/H is separable, one of our preliminary observations tells us that G/H is the union of a family $\mathcal{D}$ of countable, separable pure subgroups where $\mathcal{D}$ also satisfies the closure property (1'). We shall call a subset J of K "special" provided J satisfies the following conditions:

- (i) J is countable;
- (ii) $G(J) \parallel H$;
- (iii) $G(J) + H/H$ is in $\mathcal{D}$.

Given the inductive nature of (ii) and our earlier observations, a standard back-and-forth argument shows that $\{G(J) : J \text{ is "special"}\}$ satisfies (0), (1') and (2). Thus if we set $H(J) = H \cap G(J)$, the family $\mathcal{C} = \{H(J) : J \text{ is "special"}\}$ inherits the properties (0), (1'), (2). It remains to show that the members of $\mathcal{C}$ are knice in H . Initially we recognize each $H(J)$ with J "special" as a knice subgroup of $G(J)$ by 4.3 since, as observed in the preceding paragraph, (ii) implies that $H(J)$ is balanced in $G(J)$ and (iii) implies that $G(J)/H(J) \simeq G(J) + H/H$ is a k -group. Since the summands $G(J)$ of the k -group G are knice, 4.10(i) tells us that the members of

C are knice in G . Finally, note that 4.10(iv) then implies that the $H(J)$'s with J "special" are actually knice in H .

We still have the problem of removing the cardinality restriction on G . Fortunately, this difficulty can be overcome by a routine variant of Schanuel's trick. First note that, since $|G/H| \leq \aleph_1$ and G is completely decomposable, G/H is canonically the image of a summand G_0 of G with $|G_0| \leq \aleph_1$. Furthermore, we can choose G_0 such that each element of G/H has a preimage in G_0 with the same height sequence and therefore $H_0 = H \cap G_0$ will be balanced in G_0 . By the case covered by our restricted version of the theorem, H_0 is completely decomposable. Since it is routine that pullbacks of balanced exact sequences are balanced exact, we obtain two balanced exact sequences $0 \rightarrow H \rightarrow K \rightarrow G_0 \rightarrow 0$ and $0 \rightarrow H_0 \rightarrow K \rightarrow G \rightarrow 0$, where K is the obvious subgroup of $G \oplus G_0$. Since G and G_0 are completely decomposable, these sequences split and we have an isomorphism $H \oplus G_0 \simeq H_0 \oplus G$. The Baer-Kulikov-Kaplansky theorem yields the desired conclusion that H is completely decomposable.

COROLLARY 5.5. *If G is a completely decomposable group of cardinality not exceeding $\aleph_1$, then every strongly balanced subgroup of G is completely decomposable.*

REFERENCES

1. R. Baer, *Abelian groups without elements of finite order*, Duke Math. J. **3** (1937), 68–122.
2. L. Fuchs, *Summands of separable abelian groups*, Bull. London Math. Soc. **2** (1970), 205–208.
3. —, *Infinite abelian groups*, Vol. II, Academic Press, New York, 1973.
4. P. Griffith, *Infinite abelian groups*, Univ. of Chicago Press, Chicago, 1970.
5. P. Hill, *The third axiom of countability for abelian groups*, Proc. Amer. Math. Soc. **82** (1981), 347–350.
6. —, *Isotype subgroups of totally projective groups*, Lecture Notes in Math., vol. 874, Springer-Verlag, New York, 1981, pp. 305–321.
7. P. Hill and C. Megibben, *On the theory and classification of abelian p -groups*, Math. Z. **190** (1985), 17–38.
8. —, *Axiom 3 modules*, Trans. Amer. Math. Soc. **295** (1986), 715–734.
9. K. Rangaswamy, *On strongly balanced subgroups of separable torsion-free abelian groups*, Lecture Notes in Math., vol. 1006, Springer-Verlag, New York, 1983, pp. 268–274.

DEPARTMENT OF MATHEMATICS, AUBURN UNIVERSITY, AUBURN, ALABAMA 36849

DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY, NASHVILLE, TENNESSEE 37235 (Current address of Charles Megibben)

Current address (Paul Hill): Department of Mathematics, Baylor University, Waco, Texas 76798