QUANTIZATION AND HAMILTONIAN G-FOLIATIONS

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O et praesidium et dulce decus meum

To J. Dixmier on his sixtieth birthday

ABSTRACT. As it was recognized twenty five years ago by A. A. Kirillov, in the unitary representation theory of nilpotent Lie groups a crucial role is played by orbits of the coadjoint representation. B. Kostant noted that, for any connected Lie group, these orbits admit a symplectic structure and lend themselves to an intrinsic characterization. The present author later observed, that already for the purposes of unitary representation theory of solvable Lie groups, this concept has to be enlarged and replaced by that of a generalized orbit. One objective of this paper is their intrinsic characterization. Other results prepare the way for the geometric construction of the corresponding unitary representations, to be developed later.

Introduction. Let G be a connected and simply connected solvable Lie group with the Lie algebra $\mathfrak{g}$. In a previous paper (cf. [7]) we established a canonical bijection between the set of all quasi-equivalence classes of normal representations of G and a family (F, say) of geometric objects (cf. loc. cit. Theorem 3, p.134) which we had proposed to call generalized orbits (cf. [5, Chapter II, in particular p. 539]). If G is of type I our theory reproduces the description, due to L. Auslander and B. Kostant, of the unitary dual of such a group (cf. [1]). In the latter case one has a canonical map from F onto the set of all coadjoint orbits on the dual $\mathfrak{g}^*$ of the underlying space of $\mathfrak{g}$. The preimage of each orbit is a principal homogeneous space of a certain multitorus. A similar proposition holds true in the general case, too, with the essential difference that the notion of a coadjoint orbit has to be replaced by that of an R-orbit which, for an arbitrary (that is, not necessarily solvable) G, is defined as follows (cf. Lemma 1 below). We consider the set S of all equivalence relations on $\mathfrak{g}^*$, which are G-invariant with locally closed orbits. One can show (cf. loc. cit.) that S contains a well-determined finest element; we call it R. A coadjoint orbit is an R-orbit if and only if it is locally closed. This is always true, for instance, if the radical of $\mathfrak{g}$ is nilpotent. In general, however, R-orbits are not G-transitive.

The objective of the present paper is to initiate a geometric study, with a view on applications in representation theory, of generalized orbits. It has been inspired by B. Kostant's work on quantization and unitary representations (cf. [3]) which, incidentally, well demonstrates that the scope of the subject much transcends unitary representation theory.

The immediate goals which we pursue are (1) Intrinsic characterization of generalized orbits, (2) Geometric construction of normal representations, in the same spirit as this has been done e.g. in the type I solvable case in [1]. The paper consists of five sections. §I is preparatory. §§II and III deal with (1), whereas IV and V prepare (2), but are of an independent interest. We shall return to (2) in a subsequent publication. In many instances the assumption of solvability is superfluous. Therefore, to place things in the proper perspective, we often formulated our results for general Lie groups even when, for the time being, no interpretation in representation theory seems to be available. Solvability is assumed only in §§ III and V.

We now proceed to outline the background and results of this paper. Prior to doing this, however, we are obliged to say a few words about the notion in the title. In this manner, by the way, we shall fix some definitions and notations which, in general, will not be repeated later. The reader finds a list of notational conventions, observed throughout this paper, in the Appendix (quoted as A in the sequel).

Let (M, ω) be a symplectic manifold. We denote by $C(M)$ the collection of all real-valued C^∞ functions on M. The following construction is classical (cf. e.g. [3, p. 166]). Given $\phi \in C(M)$, there is a smooth vector field X_ϕ on M well-determined by $\iota(X_\phi)\omega = d\phi$. Such vector fields are called Hamiltonian; we denote by $\text{Ham}(M, \omega)$ their totality. Given $\phi, \psi \in C(M)$, their Poisson bracket $\{\phi, \psi\}$ is defined as $\omega(X_\phi, X_\psi)$ ($\equiv X_\psi\phi$, $\equiv -X_\phi\psi$). One has $[X_\phi, X_\psi] = -X_{\{\phi,\psi\}}$, and thus $\{\ ,\ \}$ gives rise on $C(M)$ to the structure of a Lie algebra such that, writing $j(\phi) = -X_\phi$, we have

$$0 \to \mathbb{R} \overset{\text{id}}{\to} C(M) \overset{j}{\to} \text{Ham}(M, \omega) \to 0.$$

Below, unless stated otherwise, G is a connected and simply connected Lie group with the Lie algebra $\mathfrak{g}$. If M is a C^∞-manifold acted upon smoothly by G, given $m \in M$ and $l \in \mathfrak{g}$ we shall write

$$\sigma_m(l) = (d/dt)(\exp(tl)m)\big|_{t=0} \in T_m(M).$$

Fundamental for the whole theory is the observation (cf. [3, pp. 182–184]), that *any coadjoint orbit O carries canonically the structure of a symplectic manifold*, *defined by a closed 2-form ω_O satisfying*

$$\omega_O\big(\sigma_g(x), \sigma_g(y)\big) = ([x, y], g) \qquad (x, y \in \mathfrak{g};\ g \in O).$$

Let us stop here for a moment to consider an R-orbit $\mathcal{O}$, which is not G-transitive. It can be shown (cf. Lemma 1 below), that it carries a unique C^∞-structure, which turns it into an imbedded submanifold. Comparison with the transitive case, just quoted, reveals that it is a foliation by symplectic manifolds. We formalize this observation through the following generalization, to be called *symplectic foliation*, of

the notion of a symplectic manifold (for slightly different versions cf. [**8**, p. 456] and [**9**, 5.1, p. 61]). *A symplectic foliation* (s.f.) *is a C^∞-manifold M with an involutive distribution $\mathscr{D}$, the leaves of which are dense in M. There is given, moreover, a smooth assignment, to each $m \in M$, of a nondegenerate skew-symmetric bilinear form ω_m on $\mathscr{D}_m \times \mathscr{D}_m$, where $\mathscr{D}_m \subset T_m(M)$ stands for the value of $\mathscr{D}$ at m, which, on each leaf, gives rise to a closed 2-form.* There is a natural extension of the notion of a Hamiltonian vector field (cf. [**9**, Definition 2, p. 62]) based on the following remark. Given a leaf $O \in M/\mathscr{D}$, we write ι_O for the inclusion map $O \to M$. Then for any $\phi \in C(M)$ there is a smooth vector field Y_ϕ such that, for each O, Y_ϕ and $X_{\phi|O}$ are ι_O-related. One defines the Poisson bracket, to be denoted again by $\{\ ,\ \}$, through the conditon that, if ϕ, $\psi \in C(M)$, we have for any O: $\iota_O^*(\{\phi, \psi\}) = \{\iota_O^*(\phi), \iota_O^*(\psi)\}$ (cf. loc. cit. Definition 3). Finally with notations, analogous to those of the transitive case, we have once more the exact sequence of Lie algebras

$$0 \to \mathbf{R} \overset{\mathrm{id}}{\to} C(M) \overset{j}{\to} \mathrm{Ham}(M, \omega) \to 0.$$

Below ω will be quoted as the canonical 2-form of our symplectic foliation.

Returning to the case of a coadjoint orbit, the key notion, due to Kostant, in our present context is that of a Hamiltonian G-space (H. G-sp.) (cf. [**3**, pp. 176–177]). It arises by noting two additional features in this example. Given $l \in \mathfrak{g}$, let us write $\sigma(l)$ for the vector field, satisfying $(\sigma(l))_m = \sigma_m(l)$ ($m \in M$) on M. Then we have (1) $\sigma(l) \in \mathrm{Ham}(O, \omega_O)$, implying the existence of a morphism of linear spaces λ: $\mathfrak{g} \to C(M)$ such that $\sigma(l) = X_{\lambda_l}$, (2) λ can even be chosen to be a morphism of Lie algebras. The notion of a H.G-sp. arises by postulating all these properties. In other words, *this is a symplectic space (M, ω), where M is G-homogeneous, and for which there is given a morphism of Lie algebras λ: $\mathfrak{g} \to C(M)$ satisfying:*

$$
\begin{array}{ccc}
C(M) & \overset{j}{\to} & \mathrm{Ham}(M, \omega) \\
{\scriptstyle \lambda} \nwarrow & & \nearrow {\scriptstyle -\sigma} \\
& \mathfrak{g} &
\end{array}
$$

This notion permits a simple intrinsic characterization of a coadjoint orbit (cf. [**3**, Corollary of Theorem 5.4.1, p. 190] and also Lemma 9 below).

By virtue of our previous remarks concerning nontransitive R-orbits, the following generalization imposes itself (this definition, to be repeated in Definition 1 below, is a slightly modified form of [**8**, p. 457] and [**9**, Definition 3, p. 65]). *A Hamiltonian G-foliation* (H.G-f.) $\mathscr{H}$ *is a symplectic foliation $(M, \mathscr{D}, \omega)$, where $\mathscr{D}$ results from a G-action (or $D_m = \sigma_m(\mathfrak{g})$ for each $m \in M$) such that there is a Lie algebra morphism λ: $\mathfrak{g} \to C(M)$ satisfying:*

$$
\begin{array}{ccc}
C(M) & \overset{j}{\to} & \mathrm{Ham}(M, \omega) \\
{\scriptstyle \lambda} \nwarrow & & \nearrow {\scriptstyle -\sigma} \\
& \mathfrak{g} &
\end{array}
$$

To make explicit the ingredients, we shall write $\mathscr{H} = (M, G, \omega, \lambda)$. Any R-orbit $\mathcal{O}$ is canonically a H.G-f. by choosing $\lambda(p) \equiv (l, p)$ ($p \in \mathcal{O}$; $l \in \mathfrak{g}$, fix). In fact, by what we said earlier, it is enough to verify the analogous statement for coadjoint orbits,

for which case cf. [3, Theorem 5.3.1 (p. 184)]. *Below, when required by the context, any $\mathcal{O} \in \mathfrak{g}^*/R$ will stand also for the H.G-f. structure just described.* The set of all H.G-f.'s can be turned into a category by a persuasive notion of morphism (cf. Definition 2 below, inspired by [3, p. 178]). To a morphism $\phi: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ there corresponds a smooth map from the underlying manifold of $\mathcal{H}_1$ into that of $\mathcal{H}_2$ (usually denoted by the same letter). ϕ is called an isomorphism if the latter is a diffeomorphism.

The principal results of §II are Theorems 1 and 2. Given a H.G-f., $\mathcal{H}$, say, the first claims the existence of an R-orbit $\mathcal{O}$ and of a morphism $\tau: \mathcal{H} \rightarrow \mathcal{O}$; both $\mathcal{O}$ and τ are well determined. The smooth map into $\mathcal{O}$, given rise to by τ, is called the momentum map. The corresponding transitive result, due to Kostant and serving as our model, asserts that any H.G-sp. admits a unique morphism into a well-determined coadjoint orbit (cf. [3, Theorem 5.4.1, p. 187]). Theorem 2 deals with a situation which is new to the nontransitive case. It claims, for a given H.G-f. $\mathcal{H}$, the possibility to extend its canonical 2-form to a closed 2-form (in the usual sense) of the underlying manifold. We shall call such a 2-form admissible and shall denote by $\mathcal{F}(\mathcal{H})$ their totality. Theorem 2 generalizes a previous result of ours asserting the same for the special case of an R-orbit of a solvable group (cf. [9, Lemma 8, p. 48]). We note that the analogous statement for symplectic foliations is wrong.

In §III we assume that G is solvable. It is shown that any generalized orbit (g.o.) underlies a canonically defined H.G-f. (cf. Remark 5 in §III) and the main result (Theorem 3) provides a characterization of this structure within the category of H.G-f.'s. In more detail we recall first that any R-orbit $\mathcal{O}$ carries a canonically defined principal torus bundle $\mathcal{B}(\mathcal{O})$ (cf. [5, Chapter II, in particular pp. 529–539] or [9, 3, p.48]). This is also a G-space, and a g.o. is just a G-orbit closure (cf. [5, loc. cit.] or Lemma 1 and Definition 3 below). We note, incidentally, that if the G-orbits in $\mathcal{O}$ are integral (that is $\omega_O \in (Z^2(\mathcal{O}))^{\text{int}}$ for all $O \in \mathcal{O}/G$), then the underlying set of $\mathcal{B}(\mathcal{O})$ is just the collection of all pairs composed of a point g in $\mathcal{O}$ and of an equivalence class of line bundles with connection, the curvature form being equal to ω_O, where $O = Gg$ (cf. [3, p. 131]); G acts trivially on the second component. The bundle $\mathcal{B}(\mathcal{O})$ is trivial. The obstruction to a G-invariant cross-section is represented by an equivalence class of multipliers with values in the structure group; we denote it by $[O]$ (for an algorithm to determine an element of $[O]$ in terms of $\mathcal{F}(\mathcal{O})$ cf. [9, Theorem 2, p. 58]). Given a H.G-f. we say that it is of vanishing obstruction (v.o.) if $\tau^*([\mathcal{O}])$ is trivial, where $\mathcal{O}$ and τ are as in Theorem 1. This being so Theorem 3 claims that a H.G-f. is isomorphic to a g.o. if and only if (1) It is of v.o., (2) Any of its morphisms into a H.G-f. of v.o. is an isomorphism. Upon deleting (1), we obtain a characterization of R-orbits (cf. Lemma 9).

In §IV G is again permitted to be an arbitrary connected and simply connected Lie group. In applications to representation theory particular interest belongs to the following situation. There is given a H.G-f. $\mathcal{H} = (M, G, \omega, \lambda)$, a complex line bundle $L \overset{p}{\rightarrow} M$ and a smooth lifting of the G-action on M to a morphism of G into the group of bundle automorphisms such that if, for $m \in M$ fix, G_m acts on $L_m = p^{-1}(\{m\})$ by multiplication through $\psi \in X(G_m)$, then $\psi_*(l) = -2\pi i \lambda_l(m)$ ($l \in \mathfrak{g}_m$). A lifting of the described kind will be said to be *admissible*. If for $\mathcal{H}$ this can

be arranged, we shall write $\mathcal{H} = (C)$ (cf. Definition 13). Let us put $(\mathcal{F}(\mathcal{H}))^{\text{int}} = \mathcal{F}(\mathcal{H}) \cap (Z^2(M))^{\text{int}}$ ($\mathcal{F}(\mathcal{H})$ as in Theorem 2). The principal result of §IV is Theorem 4 which, in particular, claims that we have $\mathcal{H} = (C)$ if and only if $(\mathcal{F}(\mathcal{H}))^{\text{int}}$ is nonempty. Suppose now that M is G-homogeneous (or that $\mathcal{H}$ is a H.G-sp.) in which case $\mathcal{F}(\mathcal{H}) = \{\omega\}$. If $[\omega]$ is integral, then $(\mathcal{F}(\mathcal{H}))^{\text{int}}$ is nonempty and hence, by Theorem 4, there is a line bundle with admissible lifting; this is Theorem 5.1.1, p. 178 in [3]. Let us set, for a given $m \in M$, $G_m^{\#} = \{\chi; \chi \in X(G_m), \chi_*(l) = -2\pi i \lambda_l(m) \ (l \in \mathfrak{g}_m)\}$. Then, if $[\omega]$ is integral, clearly $G_m^{\#}$ is nonempty. Theorem 5.7.1 (p. 203) in [3] asserts the opposite implication. It is a measure of the distance between the transitive and nontransitive case that, even under additional assumptions, in the general case the quoted statement admits no analogue. In fact, the assumption $\mathcal{H} = (C)$ gives rise to what we call an admissible field of characters (cf. Definition 10, (2) and Example 1 in §IV). This is an assignment to each $m \in M$ of a $\chi(m) \in G_m^{\#}$ with certain invariance and regularity properties. If M is G-homogeneous and $m \in M$ fixed, there is an obvious bijection between $G_m^{\#}$ and the set of all admissible fields of characters. On the other hand, in the general case (1) The existence of the latter is not assured, even if we assume that $G_m^{\#}$ is nonvoid for all $m \in M$ (cf. Example 3 in §IV), (2) Even if it exists (cf. e.g. Examples 2 in §IV), by Theorem 4, we do not necessarily have $\mathcal{H} = (C)$ (cf. Lemma 18).

In §V we assume again that G is solvable. To motivate the principal result of this section (cf. Theorem 5) we recall that, given $O \in \mathfrak{g}^*/G$, the representation of G obtained through the prequantization-quantization procedure of [3] is useful only if the latter is applied to an appropriate covering of O, which we shall call *standard* (cf. resp. [5, Theorem 1, p. 512] and Definition 14 below). If $g \in O$, the covering in question is $G/\overline{G}_g$, where $\overline{G}_g$ is the reduced stabilizer of g (cf. [5, Definition 4.1, p. 492] or III.1(1) below). There is also a simple intrinsic characterization (cf. Lemma 19). Let $\mathcal{H} = (M, G, \omega, \lambda)$ be a H.G-f. and τ the momentum map (cf. Theorem 1). If $O \in M/G$, $(O, \tau \mid O)$ is a covering of the coadjoint orbit $\tau(O)$ (cf. e.g. [3, Theorem 5.6.1, p. 187] or Lemma 7 below). We shall say that $\mathcal{H}$ is standard if, for any $O \in M/G$, so is $(O, \tau \mid O)$. Let us write $(\mathcal{F}(\mathcal{H}))^{\text{rat}} = \mathcal{F}(\mathcal{H}) \cap (Z^2(M))^{\text{rat}}$. We shall say that $\mathcal{H}$ is rational if $(F(\mathcal{H}))^{\text{rat}}$ is nonvoid. This being so, Theorem 5 claims, for any given R-orbit $\mathcal{O}$, the existence of a H.G-f. belonging to $\mathcal{O}$ in the sense of Theorem 1, and which is also rational, standard and of vanishing obstruction. The solution of the analogous problem in the transitive case is given in Lemma 19. It is also shown (cf. Lemma 21) that, even when $\mathcal{O}$ itself is a g.o., the H.G-f. in the statement cannot be expected to be a covering.

Once more, the reader is advised to consult the Appendix, which specifies some general assumptions and notational conventions observed throughout this paper.

Some of the results of this paper were announced in the author's talk at the Summer Institute for C^*-algebras in Kingston (cf. [8]).

I. Preliminaries. The objective of this section is partly to supplement the Introduction by some remarks (cf. I.1), and partly to present two lemmas basic for this paper

(cf. I.2). *If M is a C^∞-manifold, it will be assumed to satisfy the second axiom of countability.*

I.1. (1) Let $\mathscr{D}$ be an involutive distribution on M. As in the Introduction, we write D_m for the value of $\mathscr{D}$ at $m \in M$. We denote by $\mathscr{V}(M)$ the collection of all smooth vector fields on M. $\mathscr{V}_{\mathscr{D}}(M)$ will stand for the collection of all those elements X of $\mathscr{V}(M)$ for which $X_m \in D_m$ for all $m \in M$. Since $\mathscr{D}$ is involutive, $\mathscr{V}_{\mathscr{D}}(M)$ is a Lie subalgebra of $\mathscr{V}(M)$. We write $M/\mathscr{D}$ for the set of all connected maximal integral manifolds of $\mathscr{D}$. Given $O \in M/\mathscr{D}$, ι_O will stand for the inclusion map $O \to M$. This being so, a closed 2-form on M relative to $\mathscr{D}$, η, say (cf. [9, 2.2, p. 45]), or $\eta \in Z^2(M, \mathscr{D})$ means an assignment to each $m \in M$ of a skew-symmetric bilinear form η_m on $D_m \times D_m$, such that, given $X, Y \in \mathscr{V}_{\mathscr{D}}(M)$, the function $m \to \eta(X_m, Y_m)$ on M is smooth and for each $O \in M/\mathscr{D}$, $\iota_O^*(\eta)$ is a closed 2-form on O. *We shall often write η_O for the latter.* A simple example arises by considering a closed 2-form ω on M, and by defining, given $m \in M$, η_m as the restriction of ω_m to $D_m \times D_m$. *In this case we shall write $\eta = \omega | \mathscr{D}$.* Let $\mathscr{D}$ and $\mathscr{E}$ be distributions on M and N resp. If $\phi \colon M \to N$ is smooth, such that $\phi_{*m}(D_m) \subseteq E_{\phi(m)}$ for each $m \in M$, we shall write $\phi_*(\mathscr{D}) \subseteq \mathscr{E}$. More specifically, $\phi_*(\mathscr{D}) = \mathscr{E}$ will mean $\phi_{*m}(D_m) \equiv E_{\phi(m)}$. Suppose now that $\mathscr{D}$ and $\mathscr{E}$ are involutive. Given $\mu \in Z^2(N, \mathscr{E})$, if $\phi_*(\mathscr{D}) \subseteq \mathscr{E}$, we can form $\phi^*(\mu) \in Z^2(M, \mathscr{D})$ (cf. [9, c), p. 56 top]). In §IV we shall use also 1-forms relative to $\mathscr{D}$ as above.

(2) If $\mathscr{D}$ as above results, as in most cases in this paper, from the smooth action of a connected Lie group G on M, we shall often take the liberty of replacing $\mathscr{D}$ by G in our notations. Thus, for instance, given $\omega \in Z^2(M)$, we have $\omega | G \in Z^2(M, G)$.

(3) Below G will stand for a connected and simply connected Lie group with the Lie algebra $\mathfrak{g}$.

We recall from the Introduction that a *symplectic foliation* $(M, \mathscr{D}, \mu)$ *means a C^∞-manifold M, an involutive distribution $\mathscr{D}$ and a $\mu \in Z^2(M, \mathscr{D})$ such that, for each $O \in M/\mathscr{D}$, μ_O is nondegenerate on O. In addition it is being assumed that any such O is dense in M.* We recall also (cf. loc. cit.), that in this case we can form the exact sequence of Lie algebras

$$0 \to \mathbb{R} \overset{\mathrm{id}}{\to} C(M) \overset{j}{\to} \mathrm{Ham}(M, \mu) \to 0,$$

where, for $\phi \in C(M)$, $j(\phi)$ is defined as $-Y_\phi$.

The following is the key notion of this paper. It is a slightly modified version of Definition 3, p. 65 in [9].

DEFINITION 1. *A Hamiltonian G-foliation (H.G-f.) $\mathscr{H} = (M, G, \mu, \lambda)$ is a C^∞-manifold M, acted upon smoothly by G, the action giving rise to an involutive distribution $\mathscr{D}$ on M. μ is an element of $Z^2(M, \mathscr{D})$, such that $(M, \mathscr{D}, \mu)$ is a symplectic foliation. Finally λ is a morphism of Lie algebras $\lambda \colon \mathfrak{g} \to C(M)$ satisfying:*

$$C(M) \quad \overset{j}{\to} \quad \mathrm{Ham}(M, \mu)$$

$$\lambda \nwarrow \qquad\qquad \nearrow -\sigma$$

$$\mathfrak{g}$$

Here, given $l \in \mathfrak{g}$, $\sigma(l)$ is the element of $\mathscr{V}_{\mathscr{D}}(M)$ such that $(\sigma(l))_m = (d/dt)(\exp(tl)m)|_{t=0}$ $(m \in M)$.

In the special case when M is G-homogeneous, our definition essentially reduces to Kostant's definition of a Hamiltonian G-space (H.G-sp.) (cf. [3, p. 176]). Below we shall retain this terminology. Observe that given $O \in M/G$, $(O, G, \mu_O, \iota_O^*(\mu))$ is a H.G-sp., which we shall denote by $\mathscr{H}\,|\,O$.

I.2. For the solvable case, the following statement was proved in [5, Proposition 2.1, p. 521].

LEMMA 1. *Let G be a connected and simply connected Lie group with the Lie algebra $\mathfrak{g}$. (1) There is an equivalence relation R on $\mathfrak{g}^*$, well-determined by the following conditions. Let $\mathcal{O}$ be an R-orbit. Then (i) $\mathcal{O}$ is locally closed in $\mathfrak{g}^*$, (ii) $\mathcal{O}$ is G-invariant containing any of its G-orbits as a dense subset. (2) $\mathcal{O}$ carries a unique C^∞-structure, such that the inclusion map $\mathcal{O} \to \mathfrak{g}^*$ is an imbedding.*

PROOF. (1) *Uniqueness.* To establish this, it obviously suffices to verify the following: *For $j = 1, 2$, let O_j be G-invariant and locally closed in $\mathfrak{g}^*$, each having the property that, if $x \in O_j$, then $O_j \subset \overline{Gx}$. Assuming $O_1 \cap O_2 \neq \varnothing$, we have $O_1 = O_2$.* To see this we note first that if $E \subset \mathfrak{g}^*$ is G-invariant and $E \subset \overline{Gx}$ for some $x \in E$, then $\overline{E} = \overline{Gx}$. Hence $\overline{O}_1 = \overline{O}_2$, $= F$, say. If $O_1 - O_2$ (say) is nonempty and $x \in O_1 - O_2$, we have the inclusions $Gx \subset O_1 - O_2 \subset F - O_2$. Since O_2 is locally closed in $\mathfrak{g}^*$, $F - O_2$ is closed and thus $\overline{Gx} \subset F - O_2$, contradicting $O_2 \subset \overline{Gx}$. Hence $O_1 \cap O_2 \neq \varnothing$ implies, under our assumptions, that $O_1 = O_2$.

Existence. We start by noting that there is an equivalence relation on $\mathfrak{g}^*$, such that $x, y \in \mathfrak{g}^*$ are equivalent if and only if $\overline{Gx} = \overline{Gy}$. Let H be one of its orbits; it is clearly G-invariant. We remark that if $x \in H$, then we have $H \subset \overline{Gx}$. In fact, if $y \in H$, one has $y \in \overline{Gy} = \overline{Gx}$, and thus $y \in \overline{Gx}$ and $H \subset \overline{Gx}$. In this manner, to complete our proof of (1), it is enough to establish that H is locally closed in $\mathfrak{g}^*$. For the following construction, cf. e.g. [7, 1.1, p. 84]. Let $\tilde{\mathfrak{g}}$ be an algebraic Lie algebra, such that $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ and $\mathfrak{g}^2 = \tilde{\mathfrak{g}}^2$. To obtain $\tilde{\mathfrak{g}}$, it suffices to consider a faithful linear representation of $\mathfrak{g}$, and to take the smallest algebraic Lie algebra containing the image. Let $\tilde{G}$ be the connected and simply connected Lie group with the Lie algebra $\tilde{\mathfrak{g}}$. Since G is invariant in $\tilde{G}$, the latter operates on $\mathfrak{g}^*$, and the $\tilde{G}$-orbits are locally closed (cf. e.g. [5, Theorem, p. 379]). This being so, we note that H is contained in a $\tilde{G}$-orbit O. In fact, if not, we have two such orbits O_j ($j = 1, 2$) such that $O_1 \neq O_2$ and $O_j \cap H \neq \varnothing$ ($j = 1, 2$). But then also $\overline{H} \subset \overline{O}_j$ ($j = 1, 2$). In fact, if $x \in H \cap O_1$ (say), then $H \subset \overline{Gx} \subset \overline{O}_1$; similarly for $j = 2$. But then the sets $\overline{H} \cap O_j$ ($j = 1, 2$) are both dense and open in $\overline{H}$ with an empty intersection. If O is a $\tilde{G}$-orbit and $x \in O$, since G is invariant in $\tilde{G}$, $G\tilde{G}_x$ is an invariant subgroup of $\tilde{G}$; we denote its closure by D. Note that D depends on O only. Since O is locally closed in $\mathfrak{g}^*$, the map $a\tilde{G}_x \mapsto ax$ from $\tilde{G}/\tilde{G}_x$ onto O is a homeomorphism. From this we conclude at once that the collection of D-orbits on O coincide with that of G-orbit closures. Hence, if $H \subset O$, H is contained in a D-orbit. To complete our proof it is enough to show that H coincides with a D-orbit. But, by what we have just seen, this follows from the trivial observation that, if $x, y \in O$ are such that $\overline{Gx} \cap O = \overline{Gy} \cap O$, then also $\overline{Gx} = \overline{Gy}$.

For later use we note the following implication of the above proof. *Given* $\mathcal{O} \in \mathfrak{g}^*/R$, *there is a connected and simply connected Lie group* G_1 *such that* $G_1 \supset G$, $G_1^2 = G^2$ *and* $\mathcal{O}$ *is a* G_1-*orbit*. In fact, with notations as above, the connected component of the identity in D will have all the requisite properties. Note that G_1, in general, depends on $\mathcal{O}$.

(2) We recall that if M is a C^∞-manifold and A a subset of M with the induced topology, then A carries at most one C^∞-structure, whereby the inclusion map is an imbedding. In this fashion, to complete the proof of Lemma 1, it is enough to show that, given $\mathcal{O} \in \mathfrak{g}^*/R$, it carries at all a C^∞-structure, making it a submanifold of $\mathfrak{g}^*$. Suppose that $x \in O$ is fix. Let G_1 belong to $\mathcal{O}$ as at the end of (1). Since $\mathcal{O}$ is locally closed in $\mathfrak{g}^*$, the map $aG_1 \mapsto ax$ is a homeomorphism from $G_1/(G_1)_x$ onto $\mathcal{O}$. Hence it suffices to define a C^∞-structure on the latter by transfer.   Q.E.D.

REMARK 1. As stated earlier, according to context, $O \in \mathfrak{g}^*/R$ may signify below also the corresponding structure of a H.G-f. (for this cf. Introduction or [9, Example, p. 65]). In this case we shall write for λ and μ, as in Definition 1, λ^f and μ^f resp.; in addition σ^f and $\mathcal{D}^f$ in place of σ and $\mathcal{D}$ resp.

For the solvable case, the following lemma reproduces Lemma 5 in [9, p. 44].

LEMMA 2. *Given* $\mathcal{O} \in \mathfrak{g}^*/R$, *there is a connected and simply connected Lie group* G' *with the Lie algebra* $\mathfrak{g}'$ (*in general depending on* $\mathcal{O}$), *such that* $\mathfrak{g}' \supset \mathfrak{g}$, $(\mathfrak{g}')^2 = \mathfrak{g}^2$ *and, for any* $x \in \mathcal{O}$: $\mathcal{O} = G'x$ *and* $(G'_x)_0 = (G_x)_0$.

PROOF. Let $\mathfrak{g}_1$ be the Lie algebra of G_1 (as at the end of (1), proof of Lemma 1). For some $x \in \mathcal{O}$ fix, let $\mathfrak{a}$ be a supplementary subspace to $\mathfrak{g} + (\mathfrak{g}_1)_x$ in $\mathfrak{g}_1$. We set $\mathfrak{g}' = \mathfrak{g} + \mathfrak{a}$. Since $\mathfrak{g}' \supset \mathfrak{g}$, $\mathfrak{g}^2 = \mathfrak{g}_1^2$, $\mathfrak{g}'$ is an ideal in $\mathfrak{g}_1$. Let G' be the corresponding connected subgroup of G_1. We claim that G' has all the properties of the statement above. We have clearly $(\mathfrak{g}')^2 = \mathfrak{g}^2$. Also $\mathfrak{g}'_x = \mathfrak{g}' \cap (\mathfrak{g}_1)_x = \mathfrak{g} \cap (\mathfrak{g}_1)_x = \mathfrak{g}_x$. In this manner it suffices to show that $G'x$ is open in $\mathcal{O}$. We have

$$\mathfrak{g}_1/(\mathfrak{g}_1)_x = \big(\mathfrak{g}' + (\mathfrak{g}_1)_x\big)/(\mathfrak{g}_1)_x = \mathfrak{g}'/\big(\mathfrak{g}' \cap (\mathfrak{g}_1)_x\big) = \mathfrak{g}'/\mathfrak{g}_x$$

and thus, in particular, $\dim(\mathfrak{g}_1/(\mathfrak{g}_1)_x) = \dim(\mathfrak{g}'/\mathfrak{g}'_x)$ proving our statement. Q.E.D.

II. The momentum map for a Hamiltonian G-foliation.

As explained in the Introduction, our principal objectives in this section are Theorems 1 and 2.

Given a symplectic foliation $\mathscr{S} = (M, \mathscr{D}, \omega)$ (cf. I.1(3)) we recall (cf. Introduction), that $\eta \in Z^2(M)$ is *admissible* if $\eta \,|\, \mathscr{D} = \omega$ (cf. I.1(1)). We denote by $\mathscr{F}(\mathscr{S})$ the totality of all such forms. Below we show that $\mathscr{F}(\mathscr{S})$ is nonempty if $\mathscr{S}$ underlies the H.G-f. given rise to by an R-orbit $\mathcal{O}$ (cf. Remark 1), in which case we write $\mathscr{F}(\mathcal{O})$. In general, however, $\mathscr{F}(\mathscr{S})$ can very well be empty. In the solvable case, the following lemma reproduces Lemma 8 in [9, p. 48].

LEMMA 3. *Let* G *be a connected and simply connected Lie group with the Lie algebra* $\mathfrak{g}$. *Given* $\mathcal{O} \in \mathfrak{g}^*/R$, *there is* $\omega \in Z^2(\mathcal{O})$ *such that* $\omega \,|\, G = \omega^f$. *Hence* $\mathscr{F}(\mathcal{O})$ *is nonempty.*

PROOF. (i) We assume that G' and $\mathfrak{g}'$ correspond to $\mathcal{O}$ as in Lemma 2. Let π be the restriction map $(\mathfrak{g}')^* \mapsto \mathfrak{g}^*$. We claim that $\overset{-1}{\pi}(0)$ is a G' orbit in $(\mathfrak{g}')^*$. This is implied at once by the following proposition. *Let x be arbitrary in $\mathcal{O}$ and $y \in (\mathfrak{g}')^*$ such that $y \mid \mathfrak{g} = x$. Then we have $(G_x)_0 y = y + \mathfrak{g}^\perp$.* This is implied by Lemma 6 (p. 44) in [9] or can be proved directly as follows. (a) Let u, v be in $\mathfrak{g}_x$; we claim that $uv \cdot y = 0$. In fact, we have for any $l \in \mathfrak{g}'$: $(l, uv \cdot y) = ([v[u, l]], x)$. But, by $(\mathfrak{g}')^2 \subset \mathfrak{g}$ and $v \in \mathfrak{g}_x$, the right-hand side is equal to zero. We conclude from this that $(G_x)_0 y = y + \mathfrak{g}_x \cdot y$, and thus it is enough to show that $\mathfrak{g}_x \cdot y = \mathfrak{g}^\perp$. (b) Since, clearly, $\mathfrak{g}_x \cdot y \subset (\mathfrak{g}')^*$ is orthogonal to $\mathfrak{g}$, the desired conclusion will follow from (c) $\dim(\mathfrak{g}_x \cdot y) = \operatorname{codim}(\mathfrak{g})$. In fact, let B be the skew-symmetric bilinear form defined on $\mathfrak{g}' \times \mathfrak{g}'$ by $B(u, v) = ([u, v], y)$ $(u, v \in \mathfrak{g}')$. Then $\mathfrak{g}_x = \mathfrak{g}'_x$ is the orthogonal subspace with respect to B of $\mathfrak{g}$; therefore we have $\dim(\mathfrak{g}_x) = \dim(\mathfrak{g}') - \dim(\mathfrak{g}) + \dim(\mathfrak{g}_y)$. Hence finally: $\dim(\mathfrak{g}_x y) = \dim(\mathfrak{g}_x) - \dim(\mathfrak{g}_y) = \dim(\mathfrak{g}') - \dim(\mathfrak{g}) = \operatorname{codim}(\mathfrak{g})$, completing our demonstration of $(G_x)_0 y = y + \mathfrak{g}^\perp$ in $(\mathfrak{g}')^*$.

(ii) Let $\mathfrak{b}$ be a subspace of $\mathfrak{g}'$ such that $\mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{b}$. We define the map ι: $\mathfrak{g}^* \to (\mathfrak{g}')^*$ such that, if $g \in \mathfrak{g}^*$, we have $\iota(g) \mid \mathfrak{g} = g$, $\iota(g) \mid \mathfrak{b} = 0$; hence $\pi \circ \iota$ is the identity on $\mathfrak{g}$. This, along with what we have seen in (i), implies that there is a G'-orbit O' in $(\mathfrak{g}')^*$ such that $\iota(O) \subseteq O'$. Let ω' be the canonical 2-form on O'. We set $\omega = \iota^*(\omega') \in Z^2(\mathcal{O})$.

(iii) To conclude our demonstration of Lemma 3 we show that $\omega \mid G = \omega^f$. To this end it is enough to prove that, if $x \in \mathcal{O}$ and k, $l \in \mathfrak{g}$, then $\omega(\sigma_x(l), \sigma_x(k)) = ([l, k], x)$. We select again $y \in (\mathfrak{g}')^*$ such that $y \mid \mathfrak{g} = x$. We have for any real t: $\exp(tl)y - \iota(\exp(tl)x) \in \ker(\pi) = \mathfrak{g}^\perp$. By what we saw in (i), $\mathfrak{g}^\perp$ is equal to $\mathfrak{g}_x \cdot y$. Hence there is $\bar{l}$ in $\mathfrak{g}_x$ such that $\iota_{*x}(\sigma_x(l)) = (l - \bar{l})y$. Similarly, we can find $\bar{k} \in \mathfrak{g}_x$ satisfying $\iota_{*x}(\sigma_x(k)) = (k - \bar{k})y$. Hence finally, by $(\mathfrak{g}')^2 \subset \mathfrak{g}$ and $y \mid \mathfrak{g} = x$, we get

$$\omega(\sigma_x(l), \sigma_x(k)) = \omega'(\iota_{*x}(\sigma_x(l)), \iota_{*x}(\sigma_x(k))) = ([l - \bar{l}, k - \bar{k}], x) = ([l, k], x),$$

completing the proof of Lemma 3.   Q.E.D.

REMARK 2. One can show by examples (cf. Remark 3 below), that the image of $\mathscr{F}(\mathcal{O})$ in $H^2(\mathcal{O})$, in general, is "large." This suggests the replacement of $\mathscr{F}(\mathcal{O})$ by the set of all G-invariant elements $(\mathscr{F}(\mathcal{O}))^G$. The latter, however, can be empty; below we sketch an example to show this. Let θ be a fixed irrational number. Let $\mathfrak{g}$ be the Lie algebra spanned over $\mathbb{R}$ by the elements $\{e_0, \ldots, e_4\}$ satisfying the commutation relations

$$[e_0, e_1] = 2\pi e_2, \quad [e_0, e_2] = -2\pi e_1,$$
$$[e_0, e_3] = 2\pi \theta e_4, \quad [e_0, e_4] = -2\pi \theta e_3,$$

the remaining brackets being equal to zero. We write G for the connected and simply connected Lie group corresponding to $\mathfrak{g}$. Let g be a fixed element of $\mathfrak{g}^*$. If $g = x_0 e_0^* + \cdots + x_4 e_4^*$, setting $x_1 + i x_2 = r_1 e^{2\pi i \tau_1}$, $x_3 + i x_4 = r_2 e^{2\pi i \tau_2}$ we shall assume $r_1 r_2 > 0$.

(i) Let O be the G-orbit of g. Putting for τ, $u \in \mathbb{R}$:

$$(\tau, u) = u e_0^* + r_1 \cdot \operatorname{Re}\left(e^{2\pi i(\tau_1 + \tau)}(e_1^* - i e_2^*)\right) + r_2 \cdot \operatorname{Re}\left(e^{2\pi i(\tau_2 + \tau)}(e_3^* - i e_4^*)\right)$$

we have $O = \{(\tau, u); \tau, u \in \mathbb{R}\}$. One can show that $\omega_O = d\tau \wedge du$.

(ii) The R-orbit, containing O as above, is just its closure; we denote it by O. Given the real numbers u_1, u_2, u, we define the element (u_1, u_2, u) of $\mathfrak{g}^*$ as

$$ue_0^* + r_1 \operatorname{Re}\!\big(e^{2\pi i u_1}(e_1^* - ie_2^*)\big) + r_2 \operatorname{Re}\!\big(e^{2\pi i u_2}(e_3^* - ie_4^*)\big).$$

Then we have $\mathcal{O} = \{(u_1, u_2, u);\ u_1,\ u_2,\ u \in \mathbb{R};\ u_1,\ u_2 \bmod 1\}$. Let ω be a smooth 2-form on $\mathcal{O}$, $\omega = a(du_1 \wedge du_2) + b(du_1 \wedge du) + c(du_2 \wedge du)$ say. One can show that $\omega \,|\, G = \omega^f$ if and only if $b + \theta c \equiv 1$.

(iii) We write (t, z_1, z_2) for the element $g = \exp(y_1 e_1 + \cdots)\exp(te_0)$ of G, where $z_1 = y_1 + iy_2$, $z_2 = y_3 + iy_4$. We have

$$g(u_1, u_2, u) = \big(u_1 + t, u_2 + t, u + \nu(g;\, u_1, u_2)\big),$$

where

$$\nu(g;\, u_1, u_2) = 2\pi\big(r_1 \operatorname{Im}\!\big(e^{2\pi i(u_1 + t)}\bar{z}_1\big) + r_2 \cdot \operatorname{Im}\!\big(e^{2\pi i(u_2 + \theta t)}\bar{z}_2\big)\big).$$

We set for $g \in G$ fixed and $x = (u_1, u_2, u) \in \mathcal{O}$:

$$d\nu = \lambda_1(g;\, x)\, du_1 + \lambda_2(g;\, x)\, du_2.$$

(a) With the previous notations, given a fixed g in G, we have $g^*(\omega) = \omega$ if and only if, identically in $x \in \mathcal{O}$:

$$a(gx) + b(gx)\lambda_1(g;\, x) - c(gx)\lambda_2(g;\, x) \equiv a(x),$$
$$b(gx) \equiv b(x), \qquad c(gx) \equiv c(x).$$

We conclude from this that if ω is G-invariant, there are real numbers β, $\gamma \in \mathbb{R}$ such that $b(x) \equiv \beta$, $c(x) \equiv \gamma$. (b) We set $T = (t, 0, 0)$ and note that by (a), we have identically in $t \in \mathbb{R}$ and $x \in \mathcal{O}$: $a(Tx) \equiv a(x)$. Since $x = (u_1, u_2, u)$, this implies that $a(x)$ depends on u only; accordingly we shall write $a(x) \equiv a(u)$. (c) We infer from (b) that ω is, in addition, closed, if and only if $a_u \equiv 0$. Hence, by (1) and (2), ω is closed and G-invariant if and only if there are numbers α, β, γ such that $\omega = \alpha(du_1 \wedge du_2) + \beta(du_1 \wedge du) + \gamma(du_2 \wedge du)$, and β, γ satisfy $\beta\lambda_1(g;\, x) \equiv \gamma\lambda_2(g;\, x)$ for all $g \in G$ and $x \in \mathcal{O}$. Hence clearly $\beta = 0 = \gamma$. (d) By what we saw in (ii) above, for $\omega \,|\, G = \omega^f$ we ought to have $\beta + \theta\gamma = 1$. Therefore $(\mathscr{F}(\mathcal{O}))^G$ is empty, as claimed in Remark 2.

REMARK 3. For later reference we note here that, in our previous example, the image of $\mathscr{F}(\mathcal{O}) \subseteq Z^2(\mathcal{O})$ in $H^2(\mathcal{O})$ is equal to the whole space. In fact, this is clear by noting that (1) We have $(du_1 \wedge du_2)\,|\, G = 0$, (2) By (ii) in Remark 2, the 2-form $\omega = a(du_1 \wedge du_2) + b(du_1 \wedge du) + c(du_2 \wedge du)$ satisfies $\omega \,|\, G = \omega^f$ if and only if $b + \theta c \equiv 1$.

LEMMA 4. *Suppose that $(M, \mathscr{D}, \omega)$ and $(M', \mathscr{D}', \omega')$ are symplectic foliations (cf. I.1(3)). Let $f\colon M \to M'$ be a smooth map, such that $f_*(\mathscr{D}) = \mathscr{D}'$ (cf. I.1(3)) and $f^*(\omega') = \omega$. Assume that $\phi \in C(M')$ and write $\psi = f^*(\phi)$. Then Y_ϕ and Y_ψ are f-related.*

PROOF. Suppose that $m \in M$; we have to show that $f_{*m}((Y_\psi)_m) = (Y_\phi)_{f(m)}$. We have, by definition, $\iota((Y_\phi)_{f(m)})\omega'_{f(m)} = d\phi \,|\, D'_{f(m)}$. Let us put $F = f_{*m}$; by assumption, it is an isomorphism $D_m \to D'_{f(m)}$. If $v \in D_m$ is such that $F(v) = (Y_\phi)_{f(m)}$, we have

$$\iota(v)\omega_m = F^*\big(d\phi \,|\, D'_{f(m)}\big) = d\psi \,|\, D_m = \iota\big((Y_\psi)_m\big)\omega_m$$

and hence, since ω_m is nondegenerate on $D_m \times D_m$, $v = (Y_\psi)_m$ and

$$f_{*m}\left((Y_\psi)_m\right) = (Y_\phi)_{f(m)}. \qquad \text{Q.E.D.}$$

LEMMA 5. *Given the Hamiltonian G-foliation $\mathcal{H} = (M, G, \omega, \lambda)$ define $\tau \colon M \to \mathfrak{g}^*$ by $(l, \tau(m)) \equiv \lambda_l(m)$ ($m \in M$, fix; $l \in \mathfrak{g}$). Then τ is G-equivariant.*

PROOF. This follows [3, p. 187].

(i) Suppose that M and N are G-spaces and $f \colon M \to N$ is a smooth map. Below we distinguish notions, analogous to those on M, on N by a prime. We observe (cf. [3, Remark 4.4.1, p. 172]) that *if, for any $l \in \mathfrak{g}$, the operators $\sigma(l)$ and $\sigma'(l)$ are f-related, then f is a G-map.* In fact, given $m \in M$ and $l \in \mathfrak{g}$, fix, let us define $\gamma \colon \mathbb{R} \to N$ by $\gamma(t) \equiv f(\exp(tl)m)$. Then we have

$$\dot{\gamma}(t) = f_{*\exp(tl)m}\left(\sigma_{\exp(tl)m}(l)\right) = \sigma'_{\gamma(t)}(l),$$

where we have just used our assumption. On the other hand, putting $\delta(t) \equiv \exp(tl)f(m)$, we have $\dot{\delta}(t) \equiv \sigma'_{\delta(t)}(l)$. Since $\gamma(0) = f(m) = \delta(0)$, we can conclude that $\gamma(t) \equiv \delta(t)$. Therefore, for any $l \in \mathfrak{g}$ and $m \in M$, $f(\exp(l)m) = \exp(l)f(m)$. Hence, since G is connected, f is a G-map.

(ii) We assume next that $N = \mathfrak{g}^*$ and write σ^f in place of σ'. Given $l \in \mathfrak{g}$, fix, let us define $\lambda_l^f \in C(\mathfrak{g}^*)$ by $\lambda_l^f(g) \equiv (l, g)$ ($g \in \mathfrak{g}^*$). This being so we note first that the map $\tau \colon M \to \mathfrak{g}^*$ is smooth since, for any fix $l \in \mathfrak{g}$, $\lambda_l^f(\tau(m)) = (l, \tau(m)) \equiv \lambda_l(m)$ depends smoothly on $m \in M$. By the same token, $\tau^*(\lambda_l^f) = \lambda_l$. *We complete our proof by showing that $\sigma(l)$ and $\sigma^f(l)$ are τ-related.* To this end it is enough to prove that, for any $k \in \mathfrak{g}$ and $m \in M$, we have

$$\tau_{*m}\left(\sigma_m(l)\right)\lambda_k^f = \sigma_{\tau(m)}^f(l) \cdot \lambda_k^f.$$

But, by what we have just seen, the left-hand side is equal to

$$\sigma_m(l)\lambda_k = \{\lambda_k, \lambda_l\}(m) = \lambda_{[k,l]}(m) = ([k, l], \tau(m))$$

which clearly coincides with the right-hand side.   Q.E.D.

Below we shall call τ (as above) the momentum map of H.

LEMMA 6. *With the previous notations, there is an R-orbit $\mathcal{O}$ (cf. Lemma 1) such that $\tau(M) \subset \mathcal{O}$.*

PROOF. We write $E = \tau(m) \subset \mathfrak{g}^*$. By what we saw in the proof of Lemma 1, to prove our claim it is enough to show that $\overline{Gx} = \overline{Gy}$ for x, $y \in \mathfrak{g}$. Suppose $x \in E$, $x = \tau(m)$, say. By Lemma 5 we have $Gx = \tau(Gx)$, and hence, since Gm is dense in M (cf. Definition 1), also $E \subseteq \overline{Gx}$. This implies $\overline{Gx} = \overline{E}$ for any $x \in E$, proving our lemma.   Q.E.D.

The following definition is inspired by that of Kostant (cf. [3, p. 178]).

DEFINITION 2. *Suppose $\mathcal{H}$ and $\mathcal{H}'$ are Hamiltonian G-foliations, $\mathcal{H} = (M, \mathcal{D}, \omega, \lambda)$ etc., and $f \colon M \to M'$ is a smooth map. We shall say that f is a morphism of $\mathcal{H}$ into $\mathcal{H}'$ if $f_*(\mathcal{D}) = \mathcal{D}'$, $\omega = f^*(\omega')$ (cf. I.1) and $\lambda = f^*(\lambda')$. If, in addition, f is a diffeomorphism, it will be called an isomorphism of $\mathcal{H}$ onto $\mathcal{H}'$.*

In the transitive case the following result is due to Kostant (cf. [3, Theorem 5.4.1, p. 187]). We recall (cf. Remark 1 above) that $\mathcal{O}$ may stand also for the H.G-f. $(\mathcal{O}, \mathcal{D}^f, \omega^f, \lambda^f)$.

THEOREM 1. *Suppose $\mathcal{H}$ is a Hamiltonian G-foliation. Then there is an R-orbit $\mathcal{O}$ and a morphism τ of $\mathcal{H}$ into $\mathcal{O}$. Both τ and $\mathcal{O}$ are well determined, and τ coincides with the momentum map (cf. Lemma 5).*

PROOF. We assume $\mathcal{H} = (M, \mathcal{D}, \omega, \lambda)$.

Existence. We have seen in the proof of Lemma 5 that the momentum map τ: $M \to \mathfrak{g}^*$ is smooth. Let $\mathcal{O}$ be such that $\tau(M) \subset \mathcal{O}$ (cf. Lemma 6). Since $\mathcal{O}$ is an imbedded submanifold of $\mathfrak{g}^*$, τ: $M \to \mathcal{O}$ is smooth. We know also that $\lambda^f = \tau^*(\lambda)$.

(i) Let us prove that $\tau_*(\mathcal{D}) = \mathcal{D}^f$. If $m \in M$ and $u \in D_m$, there is $l \in \mathfrak{g}$ such that $u = \sigma_m(l)$. Hence, since τ, by Lemma 5, is a G-map, we conclude that $\tau_{*m}(u) = \sigma^f_{\tau(m)}(l)$. Since $D^f_{\tau(m)} = \sigma^f_{\tau(m)}(\mathfrak{g})$, this yields the desired conclusion.

(ii) To complete this part of the proof, it is enough to show that $\tau^*(\omega^f) = \omega$. Let u, v be elements of D_m for some $m \in M$ fix; then there is l, $k \in \mathfrak{g}$ such that $u = \sigma_m(l)$, $v = \sigma_m(k)$. Our claim is implied by the following relations:

$$\tau^*(\omega^f)(u,v) = \omega^f\big(\tau_{*m}(u), \tau_{*m}(v)\big) = \omega^f\big(\sigma^f_{\tau(m)}(l), \sigma^f_{\tau(m)}(k)\big)$$

$$= ([l,k], \tau(m)) = \lambda_{[l,k]}(m) = \{\lambda_l, \lambda_k\}(m)$$

$$= \omega_m\big(Y_{\lambda_l}, Y_{\lambda_k}\big) = \omega\big(\sigma_m(l), \sigma_m(k)\big) = \omega_m(u,v)$$

and thus $\tau^*(\omega^f) = \omega$. Summing up, if τ is the momentum map and $\mathcal{O}$ an R-orbit such that $\tau(M) \subseteq \mathcal{O}$, then τ is a morphism of $\mathcal{H}$ into $\mathcal{O}$.

Uniqueness. Suppose that $\mathcal{O}'$ is an R-orbit, and ρ a morphism of $\mathcal{H}$ into $\mathcal{O}'$. We are going to show that $\mathcal{O} = \mathcal{O}'$ and $\tau = \rho$. Given $m \in M$, we have

$$(l, \tau(m)) = \lambda_l(m) = \rho^*(\lambda^f_l)(m) = (l, \rho(m))$$

for all $l \in \mathfrak{g}$. Hence $\tau(m) \equiv \rho(m)$ $(m \in M)$, and thus $\mathcal{O} = \mathcal{O}'$ and $\tau = \rho$.   Q.E.D.

Given a H.G-f. $\mathcal{H}$, below we shall write $o(\mathcal{H})$ for the R-orbit which corresponds to it by virtue of Theorem 1.

In analogy with the terminology introduced in Lemma 3, given $\mathcal{H}$ as above, we shall say that $\eta \in Z^2(M)$ is admissible if $\eta | G = \omega$ (cf. I.1(2)). We denote by $\mathcal{F}(\mathcal{H})$ the set of all admissible 2-forms.

THEOREM 2. *Suppose that $\mathcal{H}$ is a Hamiltonian G-foliation. Then $\mathcal{F}(\mathcal{H})$ is nonvoid.*

PROOF. With the notations as in Theorem 1 we recall first that, by Lemma 3, $\mathcal{F}(\mathcal{O})$ is nonempty. Let ζ be an element of $\mathcal{F}(\mathcal{O})$. Writing $\eta = \tau^*(\zeta)$, we claim that η is admissible. To see this it is enough to note that $\eta | G = \tau^*(\zeta | G) = \tau^*(\omega^f)$, and the right-hand side, by Theorem 1, is equal to ω.   Q.E.D.

LEMMA 7. *Suppose $\mathcal{H}$ and $\mathcal{H}'$ are Hamiltonian G-foliations such that $\mathcal{H} = (M, G, \omega, \lambda)$, etc. A smooth map f: $M \to M'$ is a morphism $\mathcal{H} \to \mathcal{H}'$ if and only if (1) f is a G-map, (2) We have $o(\mathcal{H}) = o(\mathcal{H}') (= \mathcal{O}$, say) and:*

$$\begin{array}{ccc} M & \overset{f}{\to} & M' \\ & & \\ \tau \searrow & & \swarrow \tau' \\ & O & \end{array}$$

In this case $f | O$ is a covering map for any $O \in M/G$.

PROOF. *Necessity.* Assume that f is a morphism $\mathscr{H} \to \mathscr{H}'$.

(i) We prove first that $f: M \to M'$ is a G-map. For given $l \in \mathfrak{g}$, let $\sigma(l)$ ($\sigma'(l)$) correspond to $\mathscr{H}$ ($\mathscr{H}'$ resp.). Reasoning as in the proof of Lemma 5, it is enough to show that $\sigma(l)$ and $\sigma'(l)$ are f-related. Since, by Definition 1, $\sigma(l) = Y_{\lambda_l}$, $\sigma'(l) = Y_{\lambda'_l}$, and, by assumption, $\lambda_l = f^*(\lambda'_l)$, $f_*(\mathscr{D}) = \mathscr{D}'$ and $\omega = f^*(\omega')$, the desired conclusion follows from Lemma 4.

(ii) We write again τ (τ') for the momentum map of $\mathscr{H}$ ($\mathscr{H}'$ resp.) and set $\mathscr{O} = o(\mathscr{H})$, $\mathscr{O}' = o(\mathscr{H}')$. By Theorem 1, τ is a morphism of $\mathscr{H}$ into $\mathscr{O}$ and $\tau' \circ f$ a morphism of $\mathscr{H}$ into $\mathscr{O}'$, and thus, by the same statement, $\mathscr{O} = \mathscr{O}'$ and $\tau = \tau' \circ f$.

Sufficiency. Here we assume that $f: M \to M'$ is a G-map, $o(\mathscr{H}) = o(\mathscr{H}')$, $= \mathscr{O}$ (say), and

$$
M \xrightarrow{f} M'
$$

$$
\tau \searrow \qquad \swarrow \tau
$$

$$
\mathscr{O}
$$

We shall show that f is a morphism $\mathscr{H} \to \mathscr{H}'$.

(i) Since f is a G-map, we have $f_*(\mathscr{D}) = \mathscr{D}'$.

(ii) We infer from Theorem 1 that $\omega = \tau^*(\omega^f)$ and $f^*(\omega') = f^*((\tau')^*(\omega^f))$; but, by assumption, the right-hand sides coincide and thus $\omega = f^*(\omega')$.

(iii) By the same token, we have also $\lambda = \tau^*(\lambda^f) = f^*((\tau')^*(\lambda^f)) = f^*(\lambda')$ or $\lambda = f^*(\lambda')$, and thus f is a morphism $\mathscr{H} \to \mathscr{H}'$.

To complete the proof of Lemma 7, we proceed to show that, in either case, for all $O \in M/G$, $f|O$ is a covering map. Let $m \in M$ be given; we have to prove that $\mathfrak{g}_m = \mathfrak{g}_{f(m)}$. Let us put $F = f_{*m}$. Then we have $F(\sigma_m(l)) = \sigma'_{f(m)}(l)$ for all $l \in \mathfrak{g}$ and thus $F(D_m) = D'_{f(m)}$. Hence it is enough to show that F is nonsingular. Since $F^*(\omega'_{f(m)}) = \omega_m$, this is so, since ω_m is nonsingular on $D_m \times D_m$.   Q.E.D.

The following statement is an extension of Remark 5.1.2 (p. 180) in [3].

LEMMA 8. *Assume that $\mathscr{H} = (M, G, \omega, \lambda)$ is a Hamiltonian G-foliation. Let N be a G-space, such that the G-action gives rise to an involutive distribution $\mathscr{D}$, and in which any G-orbit is dense. Suppose $f: N \to M$ is a smooth G-map, such that $f|O$ is a covering map for any $O \in N/G$. There is a Hamiltonian G-foliation $\mathscr{H}_0 = (N, \mathscr{D}, \eta, \kappa)$, well-determined by the condition that f be a morphism $\mathscr{H}_0 \to \mathscr{H}$.*

PROOF. Now we distinguish the notions, corresponding to σ and $\mathscr{D}$ on M, by a prime. The uniqueness part of our lemma being clear, let us prove the existence.

(i) Since f is a G-map, we have $f_*(\mathscr{D}) = \mathscr{D}'$ and thus can form $\eta = f^*(\omega) \in Z^2(N, G)$. Since $f|O$ is a covering map for all $O \in N/G$, given $n \in N$, f_{*n}: $D_n \to D'_{f(n)}$ is an isomorphism. In this manner η_n is nondegenerate on $D_n \times D_n$. All this implies that $(N, \mathscr{D}, \eta)$ is a symplectic foliation (cf. I.1(3)) and hence we can form the exact sequence of Lie algebras

$$
0 \to \mathbb{R} \xrightarrow{\mathrm{id}} C(N) \xrightarrow{j} \mathrm{Ham}(N, \eta) \to 0.
$$

(ii) We define next the morphism of linear spaces κ: $\mathfrak{g} \to C(N)$ as $f^*(\lambda)$. We claim that κ is, in fact, a morphism of Lie algebras. To see this we note that by Lemma 4, Y_{κ_l} and Y_{λ_l} are f-related ($l \in \mathfrak{g}$). Hence we have for any l, $k \in \mathfrak{g}$ and $n \in N$ fix:

$$\{\kappa_l, \kappa_k\}(n) = \eta_n(Y_{\kappa_l}, Y_{\kappa_k}) = \omega_{f(n)}(f_{*n}(Y_{\kappa_l}), f_{*n}(Y_{\kappa_k}))$$

$$= \omega_{f(n)}((Y_{\lambda_l})_{f(n)}, (Y_{\lambda_k})_{f(n)}) = \{\lambda_l, \lambda_k\}(f(n))$$

$$= \lambda_{[l,k]}(f(n)) = \kappa_{[l,k]}(n)$$

or $\{\kappa_l, \kappa_k\}(n) = \kappa_{[l,k]}(n)$, proving our statement.   Q.E.D.

LEMMA 9. *A Hamiltonian G-foliation is isomorphic to an R-orbit if and only if any of its morphisms into another Hamiltonian G-foliation is an isomorphism.*

PROOF. To establish sufficiency, it is enough to recall that given a H.G-f., by Theorem 1 it admits a morphism into some R-orbit.

Assume next that $\mathcal{O}$ is some R-orbit, and f a morphism of $\mathcal{O}$ into a H.G-f. $\mathcal{H}$, with the underlying space M. We claim that f is an isomorphism. To prove this, by Definition 2 we have a show that f: $\mathcal{O} \to M$ is a diffeomorphism.

(i) We note first that, by Lemma 7, $\tau \circ f$ is the identity map of $\mathcal{O}$ onto itself. We conclude from this that f is an immersive injection with a closed image in M.

(ii) By the same lemma, $f(\mathcal{O})$ is G-invariant and hence dense in M. Thus f is bijective.

(iii) Our manifolds satisfy the second axiom of countability. But then an immersive bijection is a diffeomorphism.   Q.E.D.

III. Characterization of generalized orbits. In this section $\mathfrak{g}$ will be assumed solvable. The principal result of this section is Theorem 3. We shall show that the generalized orbits (cf. Introduction) are canonically H.G-f.'s, and shall provide a characterization for them within this category. For the representation-theoretic motivation we refer to the Introduction.

III.1. In this introductory subsection we present a brief summary of the results, to be used in the sequel, on the geometry of the coadjoint representation of a connected and simply connected solvable Lie group G with the Lie algebra $\mathfrak{g}$. The principal definitions are given in (3) below. For more details as well as proofs we refer to [9, 3, p. 48].

(1) We start by recalling that any connected subgroup of G is closed and simply connected. In particular, so is the connected component of the identity $(G_y)_0$ in the stabilizer of any element y of $\mathfrak{g}^*$. Hence it admits a character χ_y well-determined by the condition that we have $(\chi_y)_* = -2\pi i(y | g_y)$. The reduced stabilizer $\overline{G}_y$ of y is defined as

$$\overline{G}_y = \{a; a \in G_y \text{ such that } \chi_y([a, b]) = 1 \text{ for all } b \in G_y\}.$$

$\overline{G}_y$ is an open subgroup of G_y. We set

$$\hat{\overline{G}}_y = \{\chi; \chi \in X(\overline{G}_y) \text{ such that } \chi | (G_y)_0 = \chi_y\}.$$

Although not used in this section, it is useful to bear in mind that χ_y extends to G_y, or $\overline{G}_y = G_y$, if and only if the de Rham class of the canonical 2-form ω_{G_y} is integral (cf. [3, Theorem 5.7.1, p. 203] or §IV below). In this case there is a canonical bijection between $\hat{G}_y$ and the set of all equivalence classes of complex line bundles with connection and invariant Hermitian structure on Gy, the curvature form being equal to ω_{G_y}. If $\overline{G}_y \neq G_y$, a similar statement holds true by replacing Gy through $G/\overline{G}_y$.

(2) Let $\mathcal{O}$ be an R-orbit in $\mathfrak{g}^*$ (cf. Lemma 1). Putting $L = [G, G]$ we observe that the subgroup $\overline{G}_y L$ of G is independent of $y \in \mathcal{O}$. In fact, assume that G' corresponds to $\mathcal{O}$ as in Lemma 2. Since $[G', G'] = L$ and $\mathcal{O}$ is G'-homogeneous it is enough to note that for $y \in \mathcal{O}$ and $a \in G'$ we have $\overline{G}_{ay} = a\overline{G}_y a^{-1}$. Let K be the closed subgroup of G satisfying $K \equiv \overline{G}_y L$ ($y \in \mathcal{O}$). The group $\Pi = K/K_0$ is free abelian and hence, writing $\phi: K \to \Pi$ for the canonical morphism, the group $\hat{\Pi} = \phi^*(X(\Pi))$ ($\subset X(K)$) is isomorphic to a multitorus. Let us note that, trivially, given $y \in O$, fix, $\hat{\Pi}$ operates simply transitively on $\hat{G}_y$ by $\chi \mapsto (\psi | \overline{G}_y)\chi$ ($\psi \in \hat{\Pi}$, fix; $\chi \in \hat{G}_y$).

(3) O being as in (2) we can associate to it a principal bundle $\mathcal{B}(\mathcal{O}) \to \mathcal{O}$ with the structure group $\hat{\Pi}$ as follows. (a) The underlying set of $\mathcal{B}(\mathcal{O})$ is $\{(y, \chi); y \in \mathcal{O}, \chi \in \hat{G}_y\}$, (b) We define a projection $t: \mathcal{B}(\mathcal{O}) \to \mathcal{O}$ by $t((y, \chi)) = y$, (c) We define an action of $\hat{\Pi}$ on $\mathcal{B}(\mathcal{O})$ by $\psi(y, \chi) = (y, (\psi | \overline{G}_y)\chi)$. (We note that (a)–(c) determines on $\mathcal{B}(\mathcal{O})$ the structure of an (abstract) principal $\hat{\Pi}$-bundle over $\mathcal{O}$.) (d) Next we recall the following result (cf. [9, Lemma 9, p. 49]). *Given $a \in \Pi$, fix, there is a smooth map $b: \mathcal{O} \to G$ such that $b(y) \in \overline{G}_y$ for all $y \in \mathcal{O}$, and $\phi(b(y)) \equiv a$.* For convenience of language, we shall call such a $b: \mathcal{O} \to G$ an *admissible* map. This being so one verifies easily that there is a C^∞ structure on $\mathcal{B}(\mathcal{O})$, well-determined by the condition that for any admissible b as above, the function f_b defined on $\mathcal{B}(\mathcal{O})$ by $f_b((y, \chi)) \equiv \chi(b(y))$ be smooth. Summing up, (a)–(d) defines $\mathcal{B}(\mathcal{O})$ as a principal bundle over $\mathcal{O}$ with the structure group $\hat{\Pi}$. It is trivial as a $\hat{\Pi}$-bundle.

(4) Given $y \in \mathfrak{g}^*$, $\chi \in \hat{G}_y$ and $a \in G$, we define $a\chi \in X(\overline{G}_{ay})$ by $(a\chi)(b) = \chi(a^{-1}ba)$ ($b \in \overline{G}_{ay}$); clearly $a\chi \in \hat{G}_{ay}$. The rule $a(y, \chi) = (ay, a\chi)$ defines a smooth G-action on $\mathcal{B}(\mathcal{O})$, which commutes with the action of $\hat{\Pi}$.

Since $\mathcal{B}(\mathcal{O})$ is trivial, there is a smooth section s from $\mathcal{O}$ into $\mathcal{B}(\mathcal{O})$. We define μ: $G \times \mathcal{O} \to \hat{\Pi}$ by $\mu(a, y) \equiv a(s(y))/s(ay)$. This is a $\hat{\Pi}$-valued cocycle, the image of which in $H^1(\mathcal{O})$ (cf. [9, p. 57]) is independent of the choice of s. We shall write $[\mathcal{O}]$ for the latter. Let us note, incidentally, that Theorem 2 in [9] (p. 58) permits us to write down explicitly a cocycle of class $[\mathcal{O}]$ by aid of the canonical 2-form of $\mathcal{O}$. $[\mathcal{O}]$ is equal to zero if and only if there is a cross-section s satisfying $a(s(y)) \equiv s(ay)$ ($a \in G$, $y \in \mathcal{O}$). This is trivially the case, whenever $\mathcal{O}$ is G-transitive.

III.2. The following statement is a simplified version of Proposition 7.1 (p. 539) in [5].

LEMMA 10. *Suppose that $\mathcal{O}$ is an R-orbit. There is an equivalence relation S on $\mathcal{B}(\mathcal{O})$ such that S-orbits and G-orbit closures coincide.*

PROOF. (i) Assume that G' is as in Lemma 2. We can extend the action of G, on $\mathcal{B}(\mathcal{O})$, to G' by $a(y, \chi) = (ay, a\chi)$ ($a \in G'$). Since $[G', G'] = L = [G, G]$, the action

of G' commutes with that of $\hat{\Pi}$. Let us define $\mathscr{G} = G' \times \hat{\Pi}$. Then $\mathscr{B}(\mathcal{O})$ is a homogeneous $\mathscr{G}$-space, and the action of $\mathscr{G}$ is smooth.

(ii) Suppose that q is some point of $\mathscr{B}(\mathcal{O})$. Since $[\mathscr{G}, \mathscr{G}] = [G, G]$, the subgroup $\mathscr{G}_q \cdot G$ of $\mathscr{G}$ is independent of the choice of q. Writing Δ for its closure, we clearly have $\Delta p = \overline{Gp}$ for all $p \in \mathscr{B}(\mathcal{O})$, which proves our lemma.   Q.E.D.

REMARK 4. For later use we note that writing $D = \Delta_0$, D is a connected Lie group containing G as a closed subgroup such that $[D, D] = [G, G]$ and $\mathscr{B}(\mathcal{O})/S = \mathscr{B}(\mathcal{O})/D$.

The following notion was introduced in [5, p. 539].

DEFINITION 3. *A generalized orbit (g.o.) is an S-orbit in $\mathscr{B}(\mathcal{O})$ for some choice of the R-orbit $\mathcal{O}$.*

LEMMA 11. *Let O be a generalized orbit in $\mathscr{B}(\mathcal{O})$. Then (1) O carries a C^∞-structure, well-determined by the condition that the inclusion map $O \to \mathscr{B}(\mathcal{O})$ be an imbedding, (2) O underlies a Hamiltonian G-foliation, for which $t \mid O$ is the momentum map.*

PROOF. (1) This will be similar to the reasoning of (2) in the proof of Lemma 1. It is enough to establish the existence; the question of uniqueness can be settled as loc. cit. Suppose that D is as in Remark 4. If q is fix in O, the map $aD_q \mapsto aq$ ($a \in D$) is a homeomorphism from D/D_q onto O. We define a C^∞-structure on O by transfer. Since D acts smoothly on $\mathscr{B}(\mathcal{O})$, the inclusion map $O \to \mathscr{B}(\mathcal{O})$ is smooth.

(2) We shall apply Lemma 8 by substituting loc. cit. N by O and $\mathscr{H}$ by $\mathcal{O}$. Clearly, in O every G-orbit is dense. Since O is D-homogeneous and G is invariant in D (cf. Remark 4), there is an involutive distribution $\mathscr{D}$ on O such that $O/G = O/D$. Hence it is enough to note that $t \mid O$ is G-equivariant, and its restriction to any G-orbit is bijective.   Q.E.D.

REMARK 5. Given a g.o., just as in the case of an R-orbit (cf. Remark 1), whenever convenient it will be considered as representing the corresponding H.G-f. (as in the last lemma).

DEFINITION 4. *Given a Hamiltonian G-foliation $\mathscr{H}$, we shall say that it is of vanishing obstruction (or $\mathscr{H} =$ v.o.) if $\tau^*([O]) = 0$ (cf.III.1(4)) for $O = o(\mathscr{H})$ (cf. Theorem 1).*

The language of the following definition has been chosen in view of IV.3 below.

DEFINITION 5. *Given a Hamiltonian G-foliation $\mathscr{H}$ with the underlying manifold M, we shall say that it has property (B′) (or $\mathscr{H} =$ (B′)) if there is a map χ assigning to each $m \in M$ an element $\chi(m)$ of $\hat{G}_{\tau(m)}$ (cf. III.1(1)) such that (1) $\chi(am) \equiv a(\chi(m))$ ($a \in G, m \in M$), (2) For any b: $\mathcal{O} \to G$ admissible (cf. III.1(3), (d)), the map $m \to \chi(m)((\tau^*b)(m))$ is smooth.* We shall refer to χ as a weakly admissible field of characters (w.a.f.ch.).

REMARK 6. Let us observe that the two definitions just given are equivalent. In fact, assume first that $\mathscr{H} =$ v.o. (cf. Definition 4). Given a section s: $\mathcal{O} \to \mathscr{B}(\mathcal{O})$, there is then a smooth map ψ: $M \to \hat{\Pi}$ satisfying $a((\tau^*s)(m))/(\tau^*s)(am) \equiv \psi(am)/\psi(m)$ ($a \in G, m \in M$). Supposing $s(y) \equiv (y, \zeta(y))$ ($y \in \mathcal{O}$), let us define

$\chi(m)$ as $\psi(m)\zeta(\tau(m))$ $(m \in M)$. Clearly $\chi(m) \in \hat{G}_{\tau(m)}$. We have, on the other hand, by the definition of ψ:

$$a(\chi(m)) = \psi(m)a(\zeta(\tau(m))) = \psi(am)\zeta(\tau(am)) = \chi(am)$$

or $a(\chi(m)) = \chi(am)$ $(a \in G, m \in M)$ which is condition (1) in Definition 5. The other condition is evidently fulfilled. Thus $\mathscr{H} =$ v.o. implies $\mathscr{H} =$ (B'). Conversely, if $\mathscr{H} =$ (B'), $\chi =$ w.a.f.ch. and s is as above, then there is a map $\psi: M \to \hat{\Pi}$ such that $(\tau^*\zeta)(m) \equiv \psi(m)\chi(m)$. We have therefore

$$a((\tau^*s)(m))/(\tau^*s)(am) \equiv a((\tau^*\zeta)(m))/(\tau^*\zeta)(am)$$
$$\equiv (\psi(m)/\psi(am))(a(\chi(m))/\chi(am)) \equiv \psi(m)/\psi(am)$$

for all $a \in G$ and $m \in M$. In this manner it suffices to show that $\psi: M \to \hat{\Pi}$ is smooth. Let $b: \mathcal{O} \to G$ be admissible (cf. III.1(3)) such that $\phi(b(y)) \equiv a \in \Pi$, say. Then we have

$$\psi(m)(a) \equiv (\tau^*\zeta)(m)((\tau^*b)(m))/\chi(m)((\tau^*b)(m)),$$

whence the conclusion.

LEMMA 12. *Any generalized orbit is of vanishing obstruction.*

PROOF. By what we have just seen, it suffices to display a w.a.f.ch. (cf. Definition 5). As trivially verified, to this end it is enough to define $\chi(m) \equiv \chi$ for $m = (y, \chi) \in O$. Q.E.D.

LEMMA 13. *A Hamiltonian G-foliation $\mathscr{H}$ is of vanishing obstruction if and only if it admits a morphism into a generalized orbit.*

PROOF. Assume $\mathscr{H} = (M, G, \mu, \lambda)$.

(A) Let us suppose first that $\mathscr{H} =$ v.o. We put $\mathcal{O} = o(\mathscr{H})$.

(i) By Remark 6, our assumption implies the existence of a w.a.f.ch. χ on M. We define a map $\rho: M \to \mathscr{B}(\mathcal{O})$ by $\rho(m) = (\tau(m), \rho(m))$, where τ is the momentum map. Clearly, ρ is smooth.

(ii) We show next that ρ is a G-map. In fact, we have for $m \in M$, $a \in G$:

$$\rho(am) = (\tau(am), \chi(am)) = (a\tau(m), a(\chi(m))) = a(\tau(m), \chi(m)) = a\rho(m),$$

whence $\rho(am) = a(\rho(m))$.

(iii) We observe that there is a g.o. $O \subset \mathscr{B}(\mathcal{O})$ such that $\rho(M) \subset O$. In fact, let us select $m_0 \in M$ and set $O = \overline{G\rho(m_0)}$. By Lemma 10 and Definition 3, O is a g.o. We have by (ii): $\rho(Gm_0) \subset O$ and hence, since Gm_0 is dense in M, also $\rho(M) \subset O$. Let us note that, by Lemma 11, the map $\rho: M \to O$ is smooth.

(iv) We infer from what we have just seen that ρ is a morphism of $\mathscr{H}$ into O. In fact, by Lemma 7 it is enough to note that (1) By (ii) above, ρ is a G-map, (2) By the definition of ρ clearly $\tau = t \circ \rho$.

(B) Assume now that O is a g.o. and ρ a morphism of $\mathscr{H}$ into O; we shall show that $\mathscr{H} =$ v.o. In fact, let us write: $\rho(m) = (\tau(m), \chi(m))$ $(m \in M)$. To attain our goal, by Remark 6 it is enough to show that $m \mapsto \chi(m)$ is a w.a.f.ch. We have certainly $\chi(m) \in \hat{G}_{\tau(m)}$. By Lemma 7, ρ is a G-map and hence $\rho(am) = a(\rho(m))$

$(a \in G, m \in M)$. Finally, if $b: \mathcal{O} \to G$ is admissible (cf. III.1(3), (d)), since ρ is smooth, $m \mapsto \chi(m)((\tau^*b)(m))$ is certainly smooth.   Q.E.D.

REMARK 7. With the previous notations, given a g.o. $O \subset \mathcal{B}(\mathcal{O})$ and $\alpha \in \hat{\Pi}$, αO ($= O'$, say) is again a g.o. and the map $u \to \alpha u$ ($u \in O$) is an isomorphism of H.G-f.'s $O \to O'$. Hence O (as in the proof of the previous lemma) could be replaced by any other g.o. in $\mathcal{B}(\mathcal{O})$.

THEOREM 3. *Let $\mathcal{H}$ be a Hamiltonian G-foliation. It is isomorphic to a generalized orbit if and only if* (1) *It is of vanishing obstruction (cf. Definition 4),* (2) *Any morphism of $\mathcal{H}$ into another Hamiltonian G-foliation of vanishing obstruction is an isomorphism.*

PROOF. Let us prove first the sufficiency. If $\mathcal{H} =$ v.o., by Lemma 13 there is a g.o. $O \subset \mathcal{B}(\mathcal{O})$ ($\mathcal{O} = o(\mathcal{H})$) and a morphism of $\mathcal{H}$ into O. By Lemma 12, $O =$ v.o. and therefore, by assumption, $\mathcal{H}$ is isomorphic to O.

Conversely, let O be a g.o.; then $O =$ v.o. Let $\mathcal{H}$ be a H.G-f. such that $\mathcal{H} =$ v.o., and f a morphism of O into $\mathcal{H}$. We have to show that f is an isomorphism. Since $\mathcal{H} =$ v.o., by Lemma 13 and Remark 7 there is a morphism ρ of $\mathcal{H}$ into O and we can assume that $\rho \circ f$ is the identity map $O \to O$. We conclude from this that f is an immersive injection of O into the underlying manifold M of $\mathcal{H}$, such that $f(O)$ is closed. Since, by Lemma 7, $f(O)$ is G-invariant, f, in fact, is bijective. From here we conclude that $f: O \to M$ is a diffeomorphism, as in the proof of Lemma 9. Hence any morphism of a g.o. into $\mathcal{H}$, such that $\mathcal{H} =$ v.o., is an isomorphism.   Q.E.D.

IV. Conditions of integrality. As explained in the Introduction, the principal result of this section is Theorem 4. It spells out relations among four conditions, bearing on a Hamiltonian G-foliation, to be introduced below in the first four subsections. In the transitive case, by virtue of results due to Kostant (cf. in particular [**3**, Theorem 5.7.1, p. 203]), they are all equivalent. In the general case, however, this is not so. Their consideration is motivated by the unitary representation theory of Lie groups. For the same reason, a further restriction, to be discussed in §V, will be necessary.

IV.1. Let $\mathcal{S} = (M, \mathcal{D}, \mu)$ be a symplectic foliation (cf. I.1(3)). Similarly, as in Theorem 2 above, we write $\mathcal{F}(\mathcal{S})$ for the collection of all those closed 2-forms on M, which satisfy $\eta | \mathcal{D} = \mu$ (cf. I.1(1)). An element of $\mathcal{F}(\mathcal{S})$ will be called *admissible*. We set $(\mathcal{F}(\mathcal{S}))^{\text{int}} = \mathcal{F}(\mathcal{S}) \cap (Z^2(M))^{\text{int}}$ and $(\mathcal{F}(\mathcal{S}))^{\text{rat}} = \mathcal{F}(\mathcal{S}) \cap (Z^2(M))^{\text{rat}}$.

DEFINITION 6. *We call a symplectic foliation $\mathcal{S}$ integral* (rational) *if* $(\mathcal{F}(\mathcal{S}))^{\text{int}}$ $((\mathcal{F}(\mathcal{S}))^{\text{rat}}$ resp.)) *is nonempty*. In this case we shall write $\mathcal{S} = (I)$ ($\mathcal{S} = (Q)$ resp.).

REMARK 8. Evidently, if $\mathcal{S}$ is integral, $O \in M/\mathcal{D}$ and we set $\mu_O = \iota_O^*(\mu) \in Z^2(O)$ (cf. I.1(1)), then $[\mu_O]$ is integral; analogously, if $\mathcal{S}$ is rational. *The converse statement is false.* This will follow directly from Lemma 18 below. Still, the following indirect reasoning is of some interest. Let $\mathfrak{g}$ be a solvable Lie algebra and G and corresponding connected and simply connected Lie group. We showed previously (cf. [**9**, p. 61, top]) that if $\mathcal{O}$ is a rational R-orbit, then the canonical projection from any generalized orbit over $\mathcal{O}$ (cf. Definition 3 in III.2 above) onto $\mathcal{O}$ is a covering

map. On the other hand, one can arrange the choice of $\mathfrak{g}$ as above, and that of an R-orbit $\mathcal{O}$ in such a fashion that, for any $O \in \mathcal{O}/G$, $[\mu_O]$ is integral, $\mathcal{B}(\mathcal{O})$ (cf. III.1(3)) is of a positive fiber dimension and itself a generalized orbit (cf. [5, p. 511]). Hence here $\mathcal{O}$ cannot be rational.

IV.2. Here again, the principal definition (cf. Definition 8 below) involves only symplectic foliations. The main conclusion (cf. Proposition 1) claims the equivalence of the property so introduced with integrality (cf. Definition 6 above).

DEFINITION 7. *Suppose that M is a C^∞-manifold, $\mathcal{D}$ a distribution on M and $L \xrightarrow{p} M$ a complex line bundle. Given $m \in M$, we set $L_m = \overset{-1}{p}(\{m\})$. (1) For $m \in M$ fix, a $\mathcal{D}$-connection ∇ at m is a bilinear map $D_m \times \Gamma(M) \to L_m$ such that, if $v \in D_m$, $s \in \Gamma(M)$ and $f \in C(M)$, then we have $\nabla_v(fs) = vf \cdot s(m) + f(m) \cdot \nabla_v s$, (2) A $\mathcal{D}$-connection ∇ on L is an assignment to each point m of M of a $\mathcal{D}$-connection at m (cf. (1)) such that if $X \in \mathcal{V}_{\mathcal{D}}(M)$ and $s \in \Gamma(M)$, then the map $m \to \nabla_{X_m} s$ defines a smooth section $M \to L$.*

EXAMPLE. Suppose that ∇' is a connection on L. We can define a $\mathcal{D}$-connection ∇ on L by the stipulation that for any $m \in M$, $v \in D_m$ and $s \in \Gamma(M)$ we have $\nabla_v s = \nabla'_v s$. In this case we shall write $\nabla = \nabla' | \mathcal{D}$. The following lemma provides a partial converse of this construction.

LEMMA 14. *Suppose that M is a C^∞-manifold, $\mathcal{D}$ an involutive distribution on M and $L \xrightarrow{p} M$ a complex line bundle. Given a $\mathcal{D}$-connection ∇ on L, there is a connection ∇' such that $\nabla = \nabla' | \mathcal{D}$.*

PROOF. (i) As in [9, 4.1, (ii) (p. 55)] we shall say that an open subset U of M is $\mathcal{D}$-open if there are coordinates $\{x_j;\ 1 \leqslant j \leqslant \dim(M)\}$ on U such that, denoting by $\partial_j \in \mathcal{V}(U)$ the operator of derivation according to x_j, $\{\partial_j;\ 1 \leqslant j \leqslant \dim(\mathcal{D})\}$ spans D_m for each $m \in M$. (a) Since $\mathcal{D}$ is involutive, by the theorem of Frobenius, each point of M admits a $\mathcal{D}$-open neighborhood. Hence there is a locally finite, contractible covering $\mathcal{U} = \{U_j;\ j = 1, 2, \dots\}$ of M, and a partition of the unity $\{\phi_j;\ j = 1, 2, \dots\}$ subordinated to $\mathcal{U}$. (b) We note that if U is $\mathcal{D}$-open and α is a complex-valued 1-form on U relative to $\mathcal{D}|U$, there is a 1-form β on U such that $\alpha = \beta | (\mathcal{D}|U)$.

(ii) (a) Since U_j is contractible, it carries a never vanishing smooth section s_j ($j = 1, 2, \dots$). We define the relative 1-form $\alpha_j \in \Lambda_1(U_j, \mathcal{D}|U_j)$ by the stipulation that if $m \in U_j$ and $v \in D_m$, we have $\alpha_j(v) = \nabla_v s_j / 2\pi i s_j(m)$ ($j = 1, 2, \dots$). By (i)(b) above, we can find, for each j, $\beta_j \in \Lambda_1(U_j)$ such that $\beta_j | (\mathcal{D}|U_j) = \alpha_j$. Below, if $v \in T_m(M)$ and m is outside U_j, we set $\alpha_j(v) = 0$. Also, given some $f: U_j \to \mathbf{C}$, we define $f\phi_j: M \to \mathbf{C}$ to be zero outside U_j; similar convention will apply to $\phi_j s_j$ ($j = 1, 2, \dots$). (b) Given a section $s \in \Gamma(M)$, we define $g_j \in C^\infty(U_j)$ by $g_j(m) = s(m)/s_j(m)$ ($m \in M$), and shall write $s = \{g_j\}$. We note that for any $m \in M$: $s(m) = \sum_{j=1}^\infty \phi_j(m) g_j(m) s_j(m)$.

(iii) Suppose that $s \in \Gamma(M)$ and $s = \{g_j\}$. Given $m \in M$ and $v \in T_m(M)$ we define

$$F_v s = \sum_{j=1}^\infty \phi_j(m) \big(v g_j + 2\pi i \beta_j(v) g_j(m) \big) s_j(m) \in L_m.$$

It is known that F is a connection on L. In fact, since $f \in C(M)$ implies $fs = \{fs_j\}$, we get at once $F_v(fs) = vf \cdot s(m) + f(m)F_v s$. Suppose now that $X \in \mathscr{V}(\mathscr{M})$ is given. If $m_0 \in M$, since $\mathscr{U}$ is locally finite, the set $J = \{j;\ m_0 \in \overline{U}_j\}$ is finite. Let us write $J' = \{j;\ j \in J,\ m_0 \in \partial U_j\} \subset J$ and $I = J - J'$. There is a neighborhood $V(m_0)$ of m_0 such that, for $j \in J'$ supp$(\phi_j) \cap V(m_0)$ is empty, $V(m_0) \subset \bigcap_{j \in I} U_j$ and $V(m_0) \cap U_j$ is empty if $j \notin J$. We have on $V(m_0)$:

$$F_{X_m} s = \sum_{j \in I} \phi_j(m) \big(X_m g_j + 2\pi i g_j(m) \beta_j(X_m) \big) s_j(m)$$

and hence, to see that $m \to F_{X_m} s$ is a section, it is enough to note that the jth summand is smooth on $V(m_0)$.

(iv) We complete our proof of Lemma 14 by showing that $F \mid \mathscr{D} = \nabla$. In fact, it will suffice then to take F in place of ∇' in the statement. If $m \in M$ and $v \in D_m$, writing $J_0 = \{j;\ m \in U_j\}$, we have by our construction:

$$F_v s = \sum_{j \in J_0} \phi_j(m) \big(v g_j + 2\pi i \beta_j(v) g_j(m) \big) s_j(m)$$

$$= \sum_{j \in J_0} \phi_j(m) \big(v g_j + 2 i \alpha_j(v) g_j(m) \big) s_j(m).$$

In this fashion it is enough to observe that if $m \in U_j$, we have

$$\nabla_v s = \nabla_v (g_j s_j) = v g_j \cdot s_j(m) + g_j(m) \nabla_v s_j = \big(v g_j + 2\pi i \alpha_j(v) g_j(m) \big) s_j(m).$$

$$\text{Q.E.D.}$$

Suppose that M and N are differentiable manifolds, and $\mathscr{D}$ ($\mathscr{E}$) is a distribution on M (N resp.). Let $\phi: M \to N$ be a smooth map such that $\phi_*(\mathscr{D}) \subseteq \mathscr{E}$ (cf. I.1(1)). If $L \to N$ is a complex line bundle and ∇ an $\mathscr{E}$-connection on L (cf. Definition 7 above), then we can form, in an obvious fashion, the $\mathscr{D}$-connection $\phi^*(\nabla)$ on $\phi^*(L) \to M$. In particular, if (M, ϕ) is an integral manifold of $\mathscr{E}$, in this manner we obtain a connection (in the usual sense) on $\phi^*(L)$ and can take its curvature form curv$(\phi^*(L), \phi^*(\nabla)) \in (Z^2(M))^{\text{int}}$ (cf. [3, pp. 103–104]).

LEMMA 15. *Let M be a differentiable manifold, $\mathscr{D}$ an involutive distribution on M and μ a closed 2-form on M relative to $\mathscr{D}$ (cf. I.1(1)). The following conditions are equivalent*: (1) *There is $\zeta \in Z^2(M)$ of an integral de Rham class such that $\zeta \mid \mathscr{D} = \mu$,* (2) *There is a complex line bundle $L \to M$ and a $\mathscr{D}$-connection ∇ on L such that, for each $O \in M/\mathscr{D}$, we have* curv$(\iota_O^*(L), \iota_O^*(\nabla)) = \mu_O$.

PROOF. (1) $\Rightarrow$ (2). Since, by assumption, $\zeta \in (Z^2(M))^{\text{int}}$, there is a complex line bundle $L \to M$ and a connection ∇' on L, such that $\zeta = \text{curv}(L, \nabla')$ (cf. [3, Proposition 2.1.1, p. 133]). We form the $\mathscr{D}$-connection $\nabla = \nabla' \mid \mathscr{D}$ (cf. Example above) and proceed to show that (L, ∇) is as in (2) of the statement. We have, in fact, by $\zeta \mid \mathscr{D} = \mu$, for $O \in M/\mathscr{D}$;

$$\mu_O = \iota_O^*(\zeta) = \iota_O^*(\text{curv}(L, \nabla')) = \text{curv}(\iota_O^*(L), \iota_O^*(\nabla'))$$

$$= \text{curv}(\iota_O^*(L), \iota_O^*(\nabla))$$

or $\mu_O = \text{curv}(\iota_O^*(L), \iota_O^*(\nabla))$ proving our statement.

(2) $\Rightarrow$ (1). Let (L, ∇) be as stipulated in (2) of our statement. By virtue of Lemma 14, there is a connection ∇' on L, such that $\nabla = \nabla' | \mathscr{D}$. We define $\zeta \in Z^2(M)$ as curv(L, ∇'); it is of integral de Rham class (cf. [3, p. 133]). To complete the proof of our lemma, it is enough to show that $\zeta | \mathscr{D} = \mu$. This, however, by (2) follows from

$$\iota_O^*(\zeta) = \iota_O^*(\text{curv}(L, \nabla')) = \text{curv}(\iota_O^*(L), \iota_O^*(\nabla'))$$
$$= \text{curv}(\iota_O^*(L), \iota_O^*(\nabla)) = \mu_O$$

and thus $\iota_O^*(\zeta) = \mu_O$ and $\zeta | \mathscr{D} = \mu$.   Q.E.D.

DEFINITION 8. *Suppose $\mathscr{S} = (M, \mathscr{D}, \mu)$ is a symplectic foliation. We shall say that $\mathscr{S}$ has the property* (A) *(or $\mathscr{S} = $ (A)) if there is a complex line bundle $L \to M$ and a $\mathscr{D}$-connection ∇ on L such that, for any $O \in M/\mathscr{D}$, we have* curv$(\iota_O^*(L), \iota_O^*(\nabla)) = \mu_O$.

By virtue of Lemma 15 we have the

PROPOSITION 1. *The symplectic foliation $\mathscr{S}$ satisfies $\mathscr{S} = $ (A) if and only if it is integral (cf. Definition 6).*

IV.3. The aim of this subsection is to introduce a new property for Hamiltonian G-foliations (cf. Definition 11 below). In the transitive case it is equivalent to integrality, in general, however, it is not (cf. Theorem 4). Below G will stand, unless stated otherwise, for an *arbitrary* connected and simply connected Lie group with the Lie algebra $\mathfrak{g}$.

We recall that given a group Γ, a subgroup K of Γ, $\chi \in X(K)$ and an element g of Γ, $g\chi \in X(gKg^{-1})$ is defined by $(g\chi)(\kappa) = \chi(k)$ if $\kappa = gkg^{-1}$ $(k \in K)$. Note that if M is a G-space, $m \in M$ and $a \in G$, we have $aG_m a^{-1} = G_{am}$.

DEFINITION 9. *Suppose that M is a C^∞-manifold, which is a G-space. A field of characters χ (f.ch.) is a rule which assigns to each point m of M a character $\chi(m)$ of the stabilizer G_m, such that* (1) *For all $m \in M$ and $a \in G$: $\chi(am) = a(\chi(m))$,* (2) *If U is open in M and $b: U \to G$ is smooth such that, for all $m \in U$; $b(m) \in G_m$, then the map $m \mapsto \chi(m)(b(m))$ $(m \in U)$ is smooth.*

EXAMPLE 1. Let $L \overset{p}{\to} M$ be a complex line bundle. Suppose there is a smooth lifting of the G-action on M as a morphism of G into the group of bundle automorphisms. Then there is a field of characters χ, well-determined by the condition that for any $m \in M$, G_m acts on L_m through multiplication by $\chi(m)$ $(\in X(G_m))$.

REMARK 9. If M is a transitive G-space, it is known that any f.ch. can be obtained as in the above example. In general, however, this is false (cf. Theorem 4 below).

DEFINITION 10. *Let $\mathscr{H} = (M, G, \mu, \lambda)$ be a Hamiltonian G-foliation.* (1) *Suppose $m \in M$, given. We call the character χ of G_m admissible if $\chi_*(l) = -2\pi i \lambda_l(m)$ $(l \in \mathfrak{g}_m)$. We write $G_m^\#$ for the (possibly void) set of admissible characters of G_m.* (2) *A field of characters (cf. Definition 9 above) on M will be said to be admissible if $\chi(m) \in G_m^\#$ for all $m \in M$.*

DEFINITION 11. *We shall say that the Hamiltonian G-foliation $\mathscr{H}$ has the property* (B), *or $\mathscr{H} = $ (B), if it admits an admissible field of characters.*

EXAMPLE 2. Assume now that $\mathfrak{g}$ is solvable, and that $\mathcal{O}$ is an R-orbit in $\mathfrak{g}^*$. Suppose, furthermore, that any G-orbit in $\mathcal{O}$ is simply connected. *Then $\mathcal{O}$ carries a*

well-determined field of characters. In fact, in this case, for any $y \in \mathcal{O}$, $G_y^{\#}$ contains precisely one element (cf. III.1(1)). The smoothness condition can be trivially verified by aid of Lemma 2.

EXAMPLE 3. There are H.G-f.'s for which $G_m^{\#}$ is nonempty for all $m \in M$, and still no admissible field of characters exists. To see this, it is enough to consider the example of Remark 8 in IV.1. Observe that if $[\mu_O]$ is integral for all $O \in M/G$, then $G_m^{\#}$ is nonvoid ($m \in M$; cf. IV.5 below).

IV.4. DEFINITION 12. *Suppose that $\mathcal{H} = (M, G, \mu, \lambda)$ is a Hamiltonian G-foliation. Let $L \to M$ be a complex line bundle to which we are given a smooth lifting of the G-action on M as a morphism of G into the group of bundle automorphisms. We shall call the lifting admissible if it gives rise to an admissible field of characters* (cf. Example 1 and Definition 10(2) in IV.3 above).

DEFINITION 13. *Let $\mathcal{H}$ be as above. We shall say that $\mathcal{H}$ has the property* (C) (*or $\mathcal{H} = $ (C)) if there is a complex line bundle $L \to M$ and an admissible lifting of the G-action on M to L.*

Our objective in the rest of §IV is the proof of Theorem 4, which will claim that (1) Given a H.G-f., the properties (I), (A) and (C) introduced above are equivalent, (2) Any of them implies property (C) but, in general, not conversely.

LEMMA 16. *Let $\mathcal{H} = (M, G, \mu, \lambda)$ be a Hamiltonian G-foliation such that $\mathcal{H} = $ (C). Let $\mathcal{S} = (M, G, \mu)$ be the underlying symplectic foliation. Then we have $\mathcal{S} = $ (A) (cf. Definition 8).*

PROOF. Let $L \overset{p}{\to} M$ be the line bundle with the admissible lifting, provided by the assumption $\mathcal{H} = $ (C).

(i) Let E be the complex linear space of all smooth maps from L^* into the complex numbers, which are homogeneous of degree -1 (that is, if $f \in E$, $l \in L^*$ and $c \in C^*$, we have $f(cl) = c^{-1}f(l)$). Given $s \in \Gamma(M)$, we define the complex-valued function Ts on L^* by $(Ts)(l) \equiv s(p(l))/l$ ($l \in L^*$). We recall that $T: \Gamma(M) \to E$ is a linear space isomorphism. In fact, given $s \in \Gamma(M)$, Ts is clearly homogeneous of degree -1. If U is an open subset of M with a never vanishing smooth section, then we have $(Ts)(\sigma(m)) \equiv s(m)/\sigma(m)$ and the right-hand side depends smoothly on $m \in U$. Hence Ts is in E. Since T is evidently injective, it remains to show that it is also surjective. Given $f \in E$, $f(l) \cdot l$ depends only on $p(l)$. Hence there is a map $F: M \to L^*$ such that, if $l \in L_m$, we have $l \cdot f(l) = F(m)$. We are through by showing that $F \in \Gamma(M)$. But, with U and σ as above, we have $F(m) \equiv \sigma(m)f(\sigma(m))$ and, again, the right-hand side depends smoothly on $m \in U$.

(ii) We define the morphism L of G into $\text{End}(\Gamma(M))$ by $(L(a)s)(m) \equiv a(s(a^{-1}m))$ ($a \in G$, fix; $s \in \Gamma(M)$, $m \in M$). Similarly, we define the morphism $\mathcal{L}$ from G into $\text{End}(E)$ by $(\mathcal{L}(a)f)(m) \equiv f(a^{-1}m)$ ($a \in G$, fix; $f \in E, m \in M$). This being so, we have $TL(a) \equiv \mathcal{L}(a)T$ ($a \in G$).

(iii) Let $\mathfrak{g}$ be the Lie algebra of G. Given $l \in \mathfrak{g}$, we define the endomorphism $\pi(l)$ of $\Gamma(M)$ by $T^{-1}\mathcal{L}_*(l)T$. We claim that given $s \in \Gamma(M)$, $m \in M$ and $l \in \mathfrak{g}_m$, we have $(\pi(l)s)(m) + 2\pi i\lambda_l(m)s(m) = 0$. In fact, we note first that (a) Given $\sigma \in \Gamma(M)$ and $n \in M$, $\sigma(n) = 0$ is equivalent to $(T\sigma)(u) = 0$ for any $u \in L_n$. In this

manner, to complete our proof, it is enough to show that (b) Given $f \in E$, $l \in \mathfrak{g}_m$ and $u \in L_m$, we have $(\mathscr{L}_*(l)f)(u) = -2\pi i\lambda_l(m)f(u)$. In fact, let us note first that $(\mathscr{L}_*(l)f)(u) = (d/dt)f(\exp(-tl)u)|_{t=0}$. But, *since our lifting is admissible* (cf. Definition 12), we have $\exp(-tl)u = \exp(2\pi it\lambda_l(m))u$ and hence, since f is homogeneous of degree -1, $f(\exp(-tl)u) = \exp(-2\pi it\lambda_l(m))f(u)$, which gives finally: $(\mathscr{L}_*(l)f)(u) = -2\pi i\lambda_l(m)f(u)$. We can thus conclude that, given $s \in \Gamma(M)$ and $m \in M$, for any $l \in \mathfrak{g}$, the expression $(\pi(l)s)(m) + 2\pi i\lambda_l(m)s(m)$ depends only on $\sigma_m(l)$.

(iv) Our next objective is to show that there is a $\mathscr{D}$-connection ∇ on L (cf. Definition 7(2)), such that if $m \in M$, $v \in D_m$ $(= \sigma_m(\mathfrak{g}))$, $v = -\sigma_m(l)$, say, then we have for any $s \in \Gamma(M)$: $\nabla_v s = (\pi(l)s)(m) + 2\pi i\lambda_l(m)s(m)$. In fact, defining $\nabla_v s$ by the right-hand side, it is enough to prove that, if $f \in C(M)$, we have $\pi(l)(fs)(m) = vf \cdot s(m) + f(m)(\pi(l)s)(m)$. But this is so, since $(T(fs))(u) \equiv (p^*(f))(u)(Ts)(u)$ and, if $\phi \in E$, we have

$$(\mathscr{L}_*(l)(p^*(f)\phi))(u) = -\sigma_{p(u)}(l)f \cdot \phi(u) + f(p(u))(\mathscr{L}_*(l)\phi)(u)$$

for any $u \in L^*$.

(v) By Definition 8, we shall have completed our proof of Lemma 16 by showing that with the $\mathscr{D}$-connection ∇ as in (iv) above, we have for all $O \in M/\mathscr{D}$: $\operatorname{curv}(\iota_O^*(L), \iota_O^*(\nabla)) = \mu_O$. For the following cf. also [3, Proposition 3.4.4, p. 163]. We have to prove that if $X, Y \in \mathscr{V}_{\mathscr{D}}(M)$ and $s \in \Gamma(M)$, then

$$\left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right)s = 2\pi i\mu(X,Y)s.$$

To this end, we can assume $X = \sigma(l)$, $Y = \sigma(k)$ with $l, k \in \mathfrak{g}$ fix. Reasoning as in (iii)(a) above, it is enough to establish the following. Assume that $f \in E$. Then, putting $\eta(l) = \mathscr{L}_*(l) \in \operatorname{End}(E)$, and writing Λ_l in place of $p^*(\lambda_l)$ etc., we have

$$(+) \qquad \left[\eta(l) + 2\pi i\Lambda_l, \eta(k) + 2\pi i\Lambda_k\right]f - \left(\eta([l,k]) + 2\pi i\Lambda_{[l,k]}\right)f$$
$$\equiv 2\pi ip^*(\mu(X,Y))f.$$

We note first that the left-hand side is equal to $[\eta(l), \eta(k)]f + 2\pi i[\eta(l), \Lambda_k]f - 2\pi i[\eta(k), \Lambda_l]f$. But $[\eta(l), \Lambda_k]f = \eta(l)(\Lambda_k f) - \Lambda_k(\eta(l)f)$ and $\eta(l)(\Lambda_k f) = -(\sigma(l)\Lambda_k)f + \Lambda_k(\eta(l)f)$. Given $m \in M$, we have also

$$(\sigma(l)\lambda_k)(m) = (Y_{\lambda_l}\lambda_k)(m) = \mu_m(Y_{\lambda_k}, Y_{\lambda_l}) = \{\lambda_k, \lambda_l\}(m) = -\lambda_{[l,k]}(m).$$

In this manner we obtain $[\eta(l), \Lambda_k]f = \Lambda_{[l,k]}f$. Hence we conclude that the left-hand side of $(+)$ above is equal to $2\pi i\Lambda_{[l,k]}f$. Since $\lambda_{[l,k]}(m) = \mu(\sigma_m(l), \sigma_m(\kappa))$, our proof is complete.   Q.E.D.

The following lemma is a summary of Lemmas 26 and 28 in [9] (p. 65). We refer to loc.cit. for further details and references. We follow closely [3]; cf. in particular, Proposition 4.1.1, p. 166.

Given a H.G-f. $\mathscr{H}$, we shall write $\mathscr{H} = (\mathrm{I})$, if the underlying symplectic foliation has the analogous property (cf. Definition 6 in IV.1).

LEMMA 17. *Suppose that $\mathcal{H} = (M, G, \mu, \lambda)$ is a Hamiltonian G-foliation, such that $\mathcal{H} = $ (I). Then we also have $\mathcal{H} = $ (C) (cf. Definition 13 above).*

PROOF. We have to construct a complex line bundle $L \to M$ and a smooth lifting of the G-action on M as a morphism of G into the group of bundle automorphisms such that if, for $m \in M$, G_m acts on L_m through multiplication by $\chi \in X(G_m)$, then we have $\chi_*(l) = -2\pi i \lambda_l(m)$ $(l \in \mathfrak{g}_m)$.

(i) Since, by assumption, $\mathcal{H} = $ (I), by Definition 6 in IV.1 there is a real $\zeta \in (Z^2(M))^{\mathrm{int}}$ such that $\zeta \,|\, G = \mu$. Hence (cf. [3, Proposition 2.1.1, p. 133]) there is a complex line bundle $L \xrightarrow{p} M$, a connection ∇ on L and a ∇-invariant Hermitian structure H satisfying $\mathrm{curv}(L, \nabla) = \zeta$. Below we shall write $(\mathscr{V}(L^*))^{\mathrm{inv}}$ for the collection of all smooth $\mathbf{C}^*$-invariant vector fields on L^*. (a) Let $s \in L^*$ be fix. We recall that there is an injection $t(s)\colon T_{p(s)}(M) \to T_s(L^*)$, such that if $v \in T_{p(s)}(m)$ and, for some $\delta > 0$, $\gamma\colon (-\delta, \delta) \to M$ is smooth with $\gamma(0) = p(s)$, $\dot\gamma(0) = v$ and $\{s(u);\ -\delta < u < \delta\}$ is an autoparallel section over γ satisfying $s(0) = s$, then $t(s)(v) = \dot{s}(0)$. Given $X \in \mathscr{V}(M)$, there is $\tilde{X} \in (\mathscr{V}(L^*))^{\mathrm{inv}}$ well-determined by $(\tilde{X})_l = t(l)(X_{p(l)})$ for any $l \in L^*$. (b) Given $\phi \in C(M)$, we define $\eta(\phi) \in (\mathscr{V}(L^*))^{\mathrm{inv}}$ as the generator of the flow $(l, t) \mapsto \exp(-2\pi i(p^*\phi)(l)t)$ $(l \in L^*, t \in \mathbb{R})$. This being so, given $\phi \in C(M)$ we set $\delta(\phi) = -(\tilde{Y}_\phi + \eta(\phi)) \in (\mathscr{V}(L^*))^{\mathrm{inv}}$ and claim that $\delta\colon C(M) \to (\mathscr{V}(L^*))^{\mathrm{int}}$ is a morphism of Lie algebras. In fact, (1) Given $X_1, X_2 \in \mathscr{V}(M)$ and $\phi_1, \phi_2 \in C(M)$, we have the following rule of computation: $[\tilde{X}_1 + \eta(\phi_1), \tilde{X}_2 + \eta(\phi_2)] = \tilde{X} + \eta(\phi)$, where $X = [X_1, X_2]$ and $\phi = X_1\phi_2 - X_2\phi_1 + \zeta(X_1, X_2)$. (2) We can thus conclude that $[\delta(\phi), \delta(\psi)] = \tilde{Y} + \eta(\Phi)$, where $Y = [Y_\phi, Y_\psi]$, $\Phi = Y_\phi\psi - Y_\psi\phi + \zeta(Y_\phi, Y_\psi)$. Next, since $\zeta \,|\, G = \mu$, we have $\zeta(Y_\phi, Y_\psi) = \mu(Y_\phi, Y_\psi) = \{\phi, \psi\}$. By $-Y_\phi\psi = \{\phi, \psi\} = Y_\psi\phi$, all this implies that $\Phi = \{\phi, \psi\}$. Since $\{Y_\phi, Y_\psi\} = -Y_{\{\phi,\psi\}}$, in this fashion we can claim that $[\delta(\phi), \delta(\psi)] = -(\tilde{Y}_{\{\phi,\psi\}} + \eta(\{\phi, \psi\})) = \delta(\{\phi, \psi\})$. $\delta(\phi)$ being linear in $\phi \in C(M)$, we have proved that $\delta\colon C(M) \to (\mathscr{V}(L^*))^{\mathrm{int}}$ is a morphism of Lie algebras.

(ii) Next we recall the following result of R. Palais. Suppose that M is a C^∞-manifold, G a connected and simply connected Lie group with the Lie algebra $\mathfrak{g}$, and ρ a morphism of $\mathfrak{g}$ into $\mathscr{V}(M)$ such that, for each $l \in \mathfrak{g}$, the flow of $\rho(l)$ is complete. Then there is a smooth action of G on M such that $\rho = -\sigma$.

(iii) We now apply (ii) by substituting L^* in place of M loc. cit., and by defining $\rho(l) = \delta(\lambda_l)$ $(l \in \mathfrak{g})$. One verifies easily that the flow of $\rho(l)$ is complete. We can thus conclude to the existence of a G-action on L^* such that, given $s \in L^*$ and $l \in \mathfrak{g}$, fix, we have $(\rho(l))_s = -(d/dt)(\exp(tl)s)|_{t=0}$.

(iv) To complete the proof of Lemma 17 we show finally that the above action of G on L^* is an admissible lifting of the G-action on M. To this end, by Definition 12, we have to establish that, for any fix $a \in G$, the map $l \mapsto al$ $(l \in L^*)$ is a bundle automorphism, providing a lifting of $m \mapsto am$ $(m \in M)$ and that, for any $m \in M$ fix, G_m acts on L_m through multiplication by $\chi \in X(G_m)$ such that $\chi_*(l) = -2\pi i \lambda_l(m)$ $(l \in \mathfrak{g}_m)$. We recall that $p\colon L \to M$ is the canonical projection. (a) Since, for any $\phi \in C(M)$, $\delta(l)$ is $\mathbf{C}^*$-invariant, we have for $a \in G$, $c \in \mathbf{C}$ and $s \in L^*$: $a(cs) = c(as)$. In particular, the map $s \mapsto as$ preserves fibers. (b) We show next that $p(as) = a(p(s))$ $(a \in G, s \in L^*)$. To this end it is enough to prove that,

for any $l \in \mathfrak{g}$, $\rho(l) \in (\mathscr{V}(L^*))^{\mathrm{inv}}$ and $-\sigma(l) \in \mathscr{V}(M)$ are p-related (cf. (i) in the proof of Lemma 5). But, given $\phi \in C(M)$ and $s \in L^*$, we have from the definition: $p_{*_s}((\delta(\phi))_s) = -Y_\phi|_{p(s)}$, whence it is enough to take $\phi = \lambda_l$. We infer by aid of (a) and (b), that given $a \in G$ and $m \in M$, $a|L_m$ is a linear isomorphism from L_m onto L_{am}. In particular, G_m acts on L_m through multiplication by some $\chi \in \mathrm{Hom}(G_m, \mathbf{C}^*)$. (c) We claim that $|\chi| \equiv 1$. To this end it suffices to show that if we define $H|^2(s)$ for $s \in L^*$ as the scalar product of s with itself, $|H|^2$ is G-invariant. We recall (cf. [3, Theorem 2.10.1, p. 152; cf. also p. 197]) that given $l \in \mathfrak{g}$ and $s \in L^*$, if σ is an autoparallel section over $t \mapsto \exp(tl)p(s)$ such that $\sigma(0) = s$, then $\exp(tl)s \equiv \exp(-2\pi i \lambda_l(m)t)\sigma(t)$ $(m = p(s))$. Hence it is enough to observe that, since our Hermitian structure is ∇-invariant, the function $t \mapsto |H|^2(\sigma(t))$ is constant. We thus conclude that $\chi \in X(G_m)$. (d) We prove finally that $\chi_*(l) \equiv -2\pi i \lambda_l(m)$ $(l \in \mathfrak{g}_m)$. Given $s \in L^*$ fix, we define $\varepsilon(s): C \to L$ by $\varepsilon(s)z \equiv zs$ $(z \in \mathbf{C})$. Given $s \in L_m^*$ and $l \in \mathfrak{g}_m$, we have

$$\left(\varepsilon(s)\right)_{*1}(\chi_*(l)) = -(\rho(l))_s = \left(\tilde{Y}_{\lambda_l} + \eta(\lambda_l)\right)_s.$$

But $(\tilde{Y}_{\lambda_l})_s = t(s)(\sigma_m(l)) = 0$ by $l \in \mathfrak{g}_m$. On the other hand, from the definition: $(\eta(\lambda_l))_s = (\varepsilon(s))_{*1}(-2\pi i \lambda_l(m))$ which is thus equal to $(\varepsilon(s))_{*1}(\chi_*(l))$. Hence, finally, $\chi_*(l) = -2\pi i \lambda_l(m)$ $(l \in \mathfrak{g}_m)$.   Q.E.D.

IV.5. In order to complete the demonstration of Theorem 4 below, we proceed to prove that the property (B) (cf. Definition 11 in IV.3), in general, does not imply integrality (cf. IV.1), although it does so in the special case of transitive G-actions (cf. Remark 11 below). In view of applications in §V (cf. Lemma 21 below), the following lemma contains more than presently needed.

LEMMA 18. *There is a solvable Lie algebra $\mathfrak{g}$ such that, if G is the corresponding connected and simply connected Lie group, for an R-orbit $\mathcal{O}$ in general position the following is realized*: (1) *If $O \in \mathcal{O}/G$, O is simply connected,* (2) $\mathcal{O} \neq (Q)$ (*cf. Definition 6*), (3) *Any G-equivariant covering of $\mathcal{O}$ by a G-space, in which any G-orbit is dense, is of a finite degree.*

PROOF. Let α be a fixed real number of degree two over the rational field. (To establish (1) and (2), any irrational number would do.)

(i) Let $\mathfrak{v}$ be the Lie algebra spanned over the reals by $\{e_1, e_2, e_3\}$ with the only nonvanishing bracket $[e_1, e_2] = e_3$. We denote by $\mathfrak{w}$ the real vector space, spanned by $\{e_4, \ldots, e_{11}\}$, considered as an abelian Lie algebra. We define a representation of $\mathfrak{v}$ on $\mathfrak{w}$ by

$$[e_1, e_4] = 2\pi e_5, \qquad [e_1, e_5] = -2\pi e_4,$$
$$[e_1, e_6] = 2\pi\alpha e_7, \qquad [e_1, e_7] = -2\pi\alpha e_6,$$
$$[e_2, e_8] = 2\pi e_9, \qquad [e_2, e_9] = -2\pi e_8,$$
$$[e_2, e_{10}] = 2\pi\alpha e_{11}, \qquad [e_2, e_{11}] = -2\pi\alpha e_{10},$$

all other brackets of the form $[e_i, e_j]$ $(1 \leqslant i \leqslant 3, 4 \leqslant j \leqslant 11)$ being equal to zero. Let $\mathfrak{g}$ be the semidirect product so arising. We have $\dim(\mathfrak{g}) = 11$, the greatest nilpotent ideal $\mathfrak{n}$ is abelian, it is spanned by $\{e_3, \ldots, e_{11}\}$, and the center $\mathfrak{c}$ is spanned by e_3.

(ii) Let $g = \sum_{j=1}^{11} x_j e_j^*$ be a fixed element of $\mathfrak{g}^*$, such that $x_3 \cdot \prod_{j=2}^{5}(x_{2j}^2 + x_{2j+1}^2)$ $\neq 0$. We set $O = Gg \in \mathfrak{g}^*/G$. (a) Let us put $N = \exp(\mathfrak{n})$ and $f = g|\mathfrak{n}$. We note that since α is irrational, we have $G_f = N$. (b) We observe next that O is simply connected. In fact, by (a), $G_g \subseteq G_f = N$; hence $G_g = N_g$ and thus G_g is connected. (c) We claim that $G_f g = g + \mathfrak{n}^\perp$. We can show this by imitating the reasoning of Lemma 3. (1) We note first that if $l \in \mathfrak{n}$ and $j \geqslant 2$, then $l^j g = 0$. In fact, if k is arbitrary from $\mathfrak{g}$, we have $(k, l^j g) = ([l, \bar{k}], g)$, where $\bar{k} = (-1)^j (\text{ad } l)^{j-1} \cdot k$. By $j \geqslant 2$, $\bar{k}$ is in $\mathfrak{n}$ and thus, since $\mathfrak{n}^2 = 0$, $l^j g = 0$. We conclude from this that $Ng = g + \mathfrak{n}g$. (2) Again, by $\mathfrak{n}^2 = 0$, $\mathfrak{n}g \subseteq \mathfrak{g}^*$ is orthogonal to $\mathfrak{n}$. (3) In this manner it is enough to show that $\dim(\mathfrak{n}g) = \text{codim}(\mathfrak{n})$. We put $B(l, k) = ([l, k], g)$ for k, $l \in \mathfrak{g}$. Since $\mathfrak{n}_B = \mathfrak{g}_f = \mathfrak{n}$, we have $\dim(\mathfrak{n}) = \dim(\mathfrak{n}_B) = \text{codim}(\mathfrak{n}) + \dim(\mathfrak{n}_g)$ and hence $\dim(\mathfrak{n}g) = \text{codim}(\mathfrak{n})$. (d) The R-orbit, containing O, coincides with the closure of the latter in $\mathfrak{g}^*$; we denote it by $\mathcal{O}$. We infer from (c), that O and $\mathcal{O}$ coincide with the complete inverse image of their projection into $\mathfrak{g}^*$.

(iii) Given $\{y_1, y_2, t_1, t_2\} \subset \mathbb{R}$, we define $\{y_j'; 1 \leqslant j \leqslant 11\}$ by $y_j' = y_j$ $(j = 1, 2)$, $y_3' = x_3$ and

$$y_4' + iy_5' = \exp(2\pi i t_1)(x_4 + ix_5), \quad y_6' + iy_7' = \exp(2\pi i \alpha t_1)(x_6 + ix_7),$$
$$y_8' + iy_9' = \exp(2\pi i t_2)(x_8 + ix_9), \quad y_{10}' + iy_{11}' = \exp(2\pi i \alpha t_2)(x_{10} + ix_{11}).$$

Then the map $(y_1, y_2, t_1, t_2) \mapsto \sum_{j=1}^{11} y' e_j^*$ is a diffeomorphism between $\mathbb{R}^4$ and O. Let ω_O be the canonical 2-form of O. A straightforward computation shows that

$$\omega_O = -\left(x_3(dt_1 \wedge dt_2) + (dy_1 \wedge dt_1) + (dy_2 \wedge dt_2)\right).$$

(iv) We claim that if x_3 is not contained in the field generated by α and the rationals, we have $O \neq (Q)$. In fact, we observe first that

$$\mathcal{O} = \left\{ y; \ y = \sum_{j=1}^{11} y_j e_j^*, \ y_1, \ y_2 \text{ arbitrary}, \ y_3 = x_3, \right.$$

$$y_{2(j+1)} + iy_{2j+3} = \exp(2\pi i u_j)(x_{2(j+1)} + ix_{2j+3})$$

$$\left. (1 \leqslant j \leqslant 4), u_1, \ldots, u_4 \in \mathbb{R} \text{ is arbitrary} \right\}.$$

If $\mathcal{O} = (Q)$, there is a system of rational numbers $\{b_{ij}; 1 \leqslant i < j \leqslant 4\}$ such that if we put $\zeta = \sum_{1 \leqslant i < j \leqslant 4} b_{ij}(du_i \wedge du_j)$ and define $j: \mathbb{R}^2 \to \mathbb{R}^4$ by $j((t_1, t_2)) = (u_1, \ldots, u_4)$, where $u_1 = t_1$, $u_2 = \alpha t_1$, $u_3 = t_2$, $u_4 = \alpha t_2$, then $j^*(\zeta) = x_3(dt_1 \wedge dt_2)$. Since also $j^*(\zeta) = (b_{13} + \alpha(b_{14} + b_{23}) + \alpha^2 b_{24})(dt_1 \wedge dt_2)$, $\mathcal{O} = (Q)$ implies $x_3 \in \mathbb{Q}(\alpha)$. Summing up, by what we have seen so far we can conclude that *if $g = \sum_{j=1}^{11} x_j e_j^*$ is such that $\prod_{j=2}^{5}(x_{2j}^2 + x_{2j+1}^2) \neq 0$, and x_3 is not contained in $\mathbb{Q}(\alpha)$, then the $\mathbb{R}$-orbit containing g is not rational.*

(v) To complete the proof of Lemma 18 it is enough to show that for a $O \in \mathfrak{g}^*/R$ in general position the following holds true. *Let $\tilde{O}$ be a G-space in which any G-orbit is dense, and $\tau: \tilde{O} \to \mathcal{O}$ a G-equivariant covering map. Then the degree of this covering*

is finite. Let W be the subspace spanned by $\{e_1^*, e_2^*\}$ of $\mathfrak{g}^*$ and V a vector space over $\mathbb{R}$ with the basis $\{v_1, \dots, v_4\}$. We set $U = W \oplus V$, and define $\rho: U \to \mathcal{O}$ by

$$\rho\left(\sum_{j=1}^{2} y_j e_j^* + \sum_{k=1}^{4} t_k v_k\right) = \sum_{j=1}^{11} y_j e_j^*,$$

where $y_3 = x_3$, and $y_{2(j+1)} + iy_{2j+3} = \exp(2\pi i t_j)(x_{2(j+1)} + ix_{2j+3})$ $(1 \leqslant j \leqslant 4)$. An easy verification shows that we can endow U with the structure of a G-space, with respect to which ρ is equivariant, as follows. We set $u = \sum_{j=1}^{2} y_j e_j^* + \sum_{k=1}^{4} t_k v_k \in U$. (1) We suppose $l = \sum_{j=4}^{11} a_k e_k$ and write $a = \exp(l)$. Let us set for $j = 1, 2$:

$$\mu_j(a, u) = 2\pi \operatorname{Im}\left((a_{4j} - ia_{4j+1})\exp(2\pi i t_{2j-1})(x_{4j} + ix_{4j+1})\right.$$
$$\left. + (a_{4j+2} - ia_{4j+3})\exp(2\pi i t_{2j})(x_{4j+2} + ix_{4j+3})\right).$$

Then $au = \sum_{j=1}^{2} y_j' e_j^* + \sum_{k=1}^{4} t_k v_k$, where $y_j' = y_j + \mu_j(a, u)$ $(j = 1, 2)$. (2) Define, for $\tau_1, \tau_2 \in \mathbb{R}$:

$$\exp(\tau_1 e_1 + \tau_2 e_2)u = (y_1 + \tau_2 x_3)e_1^* + (y_2 - \tau_1 x_3)e_2^* + (t_1 + \tau_1)v_1$$
$$+ (t_2 + \alpha\tau_1)v_2 + (t_3 + \tau_2)v_3 + (t_4 + \alpha\tau_2)v_4.$$

Let Γ be the lattice generated by $\{v_1, \dots, v_4\}$ in V. For a given subgroup Δ of Γ we define $\tilde{O}$ as U/Δ. Let μ be the corresponding canonical map $U \to \tilde{O}$ and define τ: $\tilde{O} \to O$ such that we have:

$$U$$
$$\mu \downarrow \quad\searrow \rho$$
$$\tilde{\mathcal{O}} \quad\underset{\tau}{\to} \quad \mathcal{O}$$

$\tilde{\mathcal{O}}$ carries the structure of a G-space, well-determined by the condition that μ be G-equivariant; (O, τ) is a G-equivariant covering. Let us write $f_j = v_{2j-1} + \alpha v_{2j}$ $(j = 1, 2)$ and denote by F the subspace spanned by $\{f_1, f_2\}$ of V. We observe that in $\tilde{\mathcal{O}}$ any G-orbit will be dense if and only if $F + \Delta$ is dense in V. This being so, we shall have completed the demonstration of our lemma by proving the following. With notations as above, *assume that $F + \Delta$ is dense in V. Then Δ is cofinite in Γ.* In fact, assume that Δ is such that $F + \Delta$ is dense in V and Γ/Δ is *infinite*. In this case, by choosing in Γ an integral basis, to be denoted again by $\{v_1, \dots, v_4\}$, in an appropriate fashion we can suppose that Δ is generated by $\{v_1, v_2, v_3\}$. We shall also have $f_j = \sum_{k=1}^{4} a_{kj} v_k$ $(j = 1, 2)$, where a_{kj} is of the form $a + \alpha b$ $(a, b \in \mathbb{Z})$. We show next that there is a nonzero element h in V^*, such that $h(\Delta) \subseteq \mathbb{Z}$ and h vanishes on F. This will imply a contradiction with our assumption that $F + \Delta$ is dense in V. If $f \in V^*$, we shall have $f(\Delta) \subset \mathbb{Z}$ if and only if $f = \sum_{j=1}^{4} A_j v_j^*$ and $\{A_1, A_2, A_3\} \subset \mathbb{Z}$. Since $F + \Delta$ is dense in V, we cannot have $a_{41} = 0 = a_{42}$; hence we can and shall suppose that $a_{41} \neq 0$. Replacing f_2 by $f_2 - (a_{42}/a_{41})f_1$, we can assume $a_{42} = 0$. Since $\{a_{k2}; 1 \leqslant k \leqslant 4\} \subset \mathbb{Q}(\alpha)$, *and α is of degree two over the rationals*, we can find $\{A_1, A_2, A_3\} \subset \mathbb{Z}$, not all zero, such that $a_{12}A_1 + a_{22}A_2 + a_{32}A_3 = 0$. Define A_4 by $-(1/a_{41})(a_{11}A_1 + a_{21}A_2 + a_{31}A_3)$ and set $h = \sum_{j=1}^{4} A_j v_j^*$. By virtue of our construction, h is nonzero, assumes integral values on Δ and vanishes on F. But then $F + \Delta$ cannot be dense in V.   Q.E.D.

THEOREM 4. *Suppose* $\mathscr{H} = (M, G, \mu, \lambda)$ *is a Hamiltonian G-foliation. Then* (1) *The properties* (I), (A) *and* (C) (*cf. resp. Definitions* 6, 8 *and* 13) *are equivalent,* (2) *Any of them implies* (B) (*cf. Definition* 11) *but, in general, not conversely.*

We shall see below (cf. Remark 10), that (2) admits a converse if M is G-homogeneous.

PROOF. (Ad 1) (i) We write $\mathscr{S} = (M, G, \mu)$ for the symplectic foliation, which underlies $\mathscr{H}$. By virtue of Proposition 1 in IV.2, the statements $\mathscr{S} = $ (I) and $\mathscr{S} = $ (A) are equivalent.

(ii) It is now enough to recall that, by Lemma 17, (I) implies (C) and, by Lemma 16, (C) implies (A).

(Ad 2) Since trivially (C) implies (B), by (1) also (I) and (A) imply (B). To show that, in general, (B) does not imply any of the properties in (1), we recall that, by Lemma 18, there is a solvable Lie algebra $\mathfrak{g}$ and an R-orbit $\mathcal{O}$ in $\mathfrak{g}^*$ such that any G-orbit in $\mathcal{O}$ is simply connected, and $\mathcal{O}$ *is not rational*. But, by Example 2 in IV.3, we have at the same time $\mathcal{O} = $ (B), completing the proof of Theorem 4.   Q.E.D.

REMARK 10. If M is G-homogeneous, in which case $\mathscr{H}$ is a Hamiltonian G-space in the sense of Kostant (cf. I.1(3), end), all the four properties appearing in Theorem 4 are equivalent. In fact, to show this, it is enough to prove that, in this case, (B) implies (C). We sketch the proof, following [3, p. 197]. Let us choose an m in M. Since $H = $ (B), $G_m^{\#}$ (cf. Definition 10(1) in IV.3) is nonvoid. Let χ be one of its elements. Putting $K = G_m$, we define a right action of K on $G \times \mathbb{C}$ by $(g, z)k = (gk, \overline{\chi(k)}z)$ ($g \in G$, $z \in \mathbb{C}$, $k \in K$). Let L be $(G \times \mathbb{C})/K$ in the quotient topology. Writing j for the canonical map $G \times \mathbb{C} \to L$, there is $p \colon L \to M$ ($= G/K$) such that $p(l) = gK$ if $l = j((g, z))$. $L \xrightarrow{p} M$ can be shown to underlie a (smooth) complex line bundle. Finally, one obtains the requisite admissible lifting (cf. Definition 12 in IV.4) by defining $aj((g, z)) = j((ag, z))$ ($a, g \in G$; $z \in \mathbb{C}$).

REMARK 11. Let $\mathscr{H} = (M, G, \mu, \lambda)$ be a Hamiltonian G-foliation. We recall (cf. I.1(3)), that given $O \in M/G$, $\mathscr{H} \mid O$ signifies the Hamiltonian G-space $(O, G, \mu_O, \iota_O^*(\mu))$. Let us say that $\mathscr{H}$ has the property (I$'$), or that $\mathscr{H}$ is leafwise integral, if $\mathscr{H} \mid O = $ (I) for all $O \in M/G$ etc. Then (1) by what we saw in Remark 10, the properties (I$'$), (A$'$), (B$'$) and (C$'$) are all equivalent, (2) In general, none of these implies even (B). In fact, Example 3 in IV.3 is an instance in point.

V. Question of existence. In this final section $\mathfrak{g}$ will be assumed solvable.

DEFINITION 14 (cf. [8, p. 458]). *Let* $\mathscr{H} = (M, G, \mu, \lambda)$ *be a Hamiltonian G-foliation. We shall say that* $\mathscr{H}$ *is standard if* $G_m \equiv \overline{G}_{\tau(m)}$ *for all* $m \in M$, *where* τ *is the momentum map* (cf. Theorem 1; for $\overline{G}_y$ cf. III.1(1)).

To motivate the above definition we recall (cf. [5, Theorem 1, p. 512]) that, in the case of a standard Hamiltonian G-space, the prequantization-quantization procedure leads to the factor representations, permitting the construction of the normal representations (cf. [5, Theorem 2, p. 551] and [7, §6, p. 133]. The principal objective of this section is Theorem 5, which asserts that given a generalized orbit O (cf. Definition 3 in III.2), there is a H.G-f. which is standard, rational (cf. Definition 6 in

IV.1) and admits a morphism into O. The proof of the analogous transitive result is almost trivial, and implied by Lemma 19 below.

V.1. This subsection aims at providing a simple characterization of a standard H.G-sp. We start by fixing some terminology and notations. Given a coadjoint orbit O, we denote by E the set of all equivalence classes of G-equivariant coverings of O. If (M, p) is such a covering, $[(M, p)]$ will stand for its equivalence class. Let g be a fixed element in O. If $m \in M$ is such that $p(m) = g$, G_m depends only on $[(M, p)]$, $= \mu$ (say), and we shall denote it by G_μ. We recall, incidentally, that $G_g/(G_g)_0$ is a free abelian group and, clearly, G_μ is an open subgroup of G_g. (1) Suppose that μ, ν are elements of E, $\mu = [(M, p)]$, $\nu = [(N, q)]$, say. We shall write $\mu \leqslant \nu$ if there is a G-equivariant map $s: N \to M$ such that:

$$
\begin{array}{ccc}
N & \overset{s}{\to} & M \\
& & \\
q \searrow & & \swarrow p \\
& O &
\end{array}
$$

We have $\mu \leqslant \nu$ if and only if $G_\mu \supseteq G_\nu$ and hence *the map $\mu \mapsto G_\mu$ is an order reversing bijection between E and the set of all open subgroups of G_g.* (2) We shall say that μ in E is *integral* if $\mu = [(M, p)]$ implies that the de Rham class of $p^*(\omega_O)$ is integral. We write $E_{\mathrm{int}} = \{\mu;\ \mu \in E,\ \mu = \text{integral}\}$. (a) The set E_{int} is not void, since the universal covering of O is contractible, (b) Observe that, evidently, $\nu \in E_{\mathrm{int}}$, $\mu \in E$ and $\nu \leqslant \mu$ implies $\mu \in E_{\mathrm{int}}$, (c) We write $E_{\mathrm{min\,int}}$ for the collection of all minimal elements in E_{int}; we show below that $E_{\mathrm{min\,int}}$ is nonvoid.

LEMMA 19. *With the previous notations* (1) *There is an element $\mu \in E_{\mathrm{int}}$ such that $\nu \leqslant \mu$ for any $\nu \in E_{\mathrm{min\,int}}$, and if $\mu' \in E$ admits this property, then $\mu \leqslant \mu'$,* (2) *μ is standard.*

PROOF. We continue to keep $g \in O$ fixed. Since otherwise the proof is trivial, we shall assume $g \,|\, \mathfrak{g}_g \neq 0$. We recall (cf. III.1(1)) that, since $(G_g)_0$ is simply connected, there is a $\chi_g \in X((G_g)_0)$ such that $(\chi_g)_* = -2\pi i(g \,|\, \mathfrak{g}_g)$.

(i) We start by observing that given $\nu \in E_{\mathrm{int}}$, there is a minimal integral τ such that $\tau \leqslant \nu$. In fact (a) We claim that $\mu \in E$ is integral if and only if χ_g extends to an element of $X(G_\mu)$. (1) Suppose there is such a χ. If $\mu = [(M, p)]$, by Lemma 8 in II, M underlies a H.G-sp. $\mathcal{M}$. Let $m \in M$ be such that $p(m) = g$. Our assumption says that $G_m^{\#}$ is nonvoid (cf. Definition 10(1) in IV.3). Since M is G-homogeneous, we have $\mathcal{M} = (B)$ (cf. Definition 11 in IV.3). Hence, by Remark 10 (cf. IV.5), $\mathcal{M}$ is integral, or $\mu \in E_{\mathrm{int}}$. (2) If, on the other hand, μ is integral, then so is $\mathcal{M}$ and thus, by Theorem 4, $G_m^{\#}$ is nonvoid. We set $\overset{\circ}{G}_g = \ker(\chi_g)$, $\Gamma = G_g/\overset{\circ}{G}_g$ and write α for the canonical morphism $G_g \to \Gamma$. (b) We note next that $\mu \in E$ is integral if and only if $\alpha(G_\mu)$ is abelian. To this end it is enough to remark that the last condition is equivalent to the existence of an extension of χ_g to G_μ. Hence the desired conclusion is implied by (a). To complete the proof of (i) suppose now that A is maximal abelian in Γ such that $\alpha(G_\nu) \subseteq A$. We write $F = \overset{-1}{\alpha}(A)$ and note that F is open in G_g. Let τ be the element in E such that $G_\tau = F$; it has all the properties claimed at the start of (i). In particular, we have shown that $E_{\mathrm{min\,int}}$ is nonempty.

(ii) We can now easily establish that any subset $Z \subset E_{\text{int}}$ admits a well-determined least upper bound. In fact, with the notations of (i) let us set $A = \bigcap_{\tau \in Z} \alpha(G_\tau)$. The requisite element $\mu \in E$ is well determined by $G_\mu = \overset{-1}{\alpha}(A)$.

(iii) Since, by (i), $E_{\text{min int}}$ is nonempty, (ii) permits us to conclude that, in particular, $E_{\text{min int}}$ admits a well-defined least upper bound μ. Therefore, to complete the proof of Lemma 19 it suffices to show that $G_\mu = \overline{G}_g$. (a) By what we saw in (i), if $\tau \in E_{\text{min int}}$, $\alpha(G_\tau)$ is maximal abelian. (b) We claim that the intersection of all maximal abelian subgroups ($= A$, say) is equal to the center $\Gamma^{\#}$ of Γ. In fact, to this end it is enough to establish that $A \subseteq \Gamma^{\#}$. Suppose that $a \in \Gamma - \Gamma^{\#}$, and select b such that $[a, b] \neq e$. Let B be the maximal abelian subgroup of Γ, containing b. Then $a \notin B$ and thus also $a \notin A$, which leads to $A \subseteq \Gamma^{\#}$. In this fashion it is sufficient to note that (c) Evidently, $\Gamma^{\#} = \alpha(\overline{G}_g)$. Q.E.D.

V.2. We recall that in this section $\mathfrak{g}$ is solvable.

LEMMA 20. *Any R-orbit $\mathcal{O}$ admits a standard covering.*

PROOF. (i) Suppose that G' is as in Lemma 2, corresponding to $\mathcal{O}$. We recall that G' is a connected and simply connected Lie group, such that $G' \supseteq G$, $[G', G'] = [G, G]$, $= L$ (say), and for $x \in \mathcal{O}$, $G'x = \mathcal{O}$ and $(G'_x)_0 = (G_x)_0$. Below we keep x fixed. Let A' be G'/L and $\lambda: G' \to A'$ the canonical morphism; A' is isomorphic to a vector group. (a) Putting $B = \lambda(G'_x)$ we claim that B is closed in A'. In fact, since L acts unipotently on $\mathfrak{g}^*$, Lx is closed in $\mathfrak{g}^*$, and hence so is $G'_x L$, implying our assertion. (b) Writing $A = \lambda(G)$ and $D = \lambda(G_x)$, A is a vector subgroup and $D = A \cap B$. Hence B/D is free abelian; let Γ be a direct factor to D in B. We set $\overline{D} = \lambda(\overline{G}_x)$ and note that $\overline{D}$, too, is closed in A'. We define Δ as $\overset{-1}{\lambda}(\Gamma \oplus \overline{D}) \cap G'_x$. We clearly have $G'_x \supseteq \Delta \supseteq (G_x)_0$. Let us set $\Omega = G'/\Delta$ and denote by ρ the canonical projection from Ω onto $\mathcal{O}$. Below we propose to show that (Ω, ρ) is a standard covering of $\mathcal{O}$.

(ii)(a) We claim that in Ω any G-orbit is dense. In fact, by homogeneity, it is enough to show that $\overline{Gq} = \Omega$, where $q = \{\Delta\} \in G'/\Delta = \Omega$. Since any G-orbit is dense in $\mathcal{O}$, $G \cdot G'_x$ is dense in G', and hence it suffices to prove that $G\Delta = G \cdot G'_x$. Since both sides contain L, it is enough to verify that $\lambda(G \cdot \Delta) = \lambda(G \cdot G'_x)$. But we have $\lambda(G \cdot \Delta) = \lambda(G) + \lambda(\Delta) = A + (\Gamma + \overline{D}) = A + \Gamma = A + (\Gamma + D) = A + B = \lambda(G \cdot G'_x)$, completing the proof of our claim. (b) We show finally that Ω is standard. Since $\rho(q) = x$ it is enough to check that $G_q = \overline{G}_x$. This, however, is evident from $\Delta \cap G = \overline{G}_x$. Q.E.D.

REMARK 12. For later use, we observe here the following, the easy verification of which, by aid of III.1(3)(d), we leave to the reader. Suppose $\mathcal{O}$ is a given R-orbit. Let (Ω, ρ) be a standard covering of $\mathcal{O}$, such that G operates transitively on the ρ-fibers (e.g. take (Ω, ρ) as in Lemma 20). We define $\mathcal{B}(\Omega)$ as the pullback $\rho^*(\mathcal{B}(\mathcal{O}))$, and denote by P the canonical projection $\mathcal{B}(\Omega) \to \mathcal{B}(\mathcal{O})$. *Then P is a covering map.* There is a C^∞-structure on $\mathcal{B}(\Omega)$, well-determined by the condition that if $b: \Omega \to G$ is smooth and satisfies $b(y) \in G_y$ ($= \overline{G}_{\rho(y)}$) for all $y \in \Omega$, then the map $f_b: \Omega \to T$ defined by $f_b((y, \chi)) \equiv \chi(b(y))$ is smooth. The image under P of a G-orbit

closure in $\mathscr{B}(\Omega)$ is a G-orbit closure in $\mathscr{B}(\mathcal{O})$, that is, a generalized orbit (cf. Lemma 10 and Definition 3 in II). In fact, P establishes a bijection between G-orbit closures in $\mathscr{B}(\Omega)$ and generalized orbits contained in $\mathscr{B}(\mathcal{O})$.

The message of the next lemma in our present context is to show that the objective we set out to accomplish in Theorem 5, in general, cannot be obtained through coverings.

LEMMA 21. *There are standard generalized orbits admitting no covering by a rational Hamiltonian G-foliation.*

PROOF. Take $\mathfrak{g}$ and $\mathcal{O} \in \mathfrak{g}^*/R$ as in Lemma 18. Since any G-orbit is simply connected, $\mathcal{O}$ itself is a generalized orbit and standard. By what we proved loc.cit., given a G-equivariant covering, containing G-orbits as dense subsets, it must be of a finite degree. If it gives rise to a H.G-f. which is rational, then $\mathcal{O}$ itself ought to be rational, in contradiction to Lemma 18.   Q.E.D.

V.3. LEMMA 22. *Suppose that V is a finite-dimensional real vector space, Γ a lattice in V and U a subspace of V such that $\Gamma + U$ is dense in V. We write $\Gamma_1 = \Gamma \cap U$. Suppose finally that f is a skew-symmetric bilinear form on $U \times U$, which assumes integral values on $\Gamma_1 \times \Gamma_1$. Then there is a finite-dimensional real vector space V', a lattice Γ' in V' and a surjective morphism π from V' onto V such that if we write $\bar{\Gamma} = \pi(\Gamma')$, then $\bar{\Gamma}$ is a subgroup of finite index in Γ. There is, moreover, a morphism ρ of U into V', such that $\pi \circ \rho$ is the identity on U, $\rho(\Gamma_1)$ is contained in Γ' and, writing $U' = \rho(U)$, $\Gamma' + U'$ is dense in V'. There is, finally, a skew-symmetric bilinear form g on $V' \times V'$, assuming integral values on $\Gamma' \times \Gamma'$, and satisfying $\rho^* g = f$.*

PROOF. (i) We need first the following observation. *Let E be a finite-dimensional real vector space, Δ a discrete subgroup of E and f a skew-symmetric bilinear form on $E \times E$, which assumes integral values on $\Delta \times \Delta$. Then there is a system $\{ f_j ;$ $1 \leqslant j \leqslant 2s \} \subset E^*$, the elements of which are integral on Δ and such that*

$$f = \sum_{j=1}^{s} \left(f_{2j-1} \wedge f_{2j} \right).$$

In fact, (a) Let $\{ \delta_j ; 1 \leqslant j \leqslant m \}$ be an integral basis in Δ such that $f(\delta_{2j-1}, \delta_{2j}) = a_j$ is a nonzero integer if $1 \leqslant j \leqslant r$ and $f(\delta_i, \delta_j) = 0$ for any other pair (i, j) $(i < j)$. We denote by W the subspace spanned by $\{ \delta_j ; 1 \leqslant j \leqslant 2r \}$. Then the restriction of f to $W \times W$ is nondegenerate. Hence, if Z is the orthogonal complement of W with respect to f, E is the direct sum of W and Z. Let g be the restriction of f to $Z \times Z$. We write Γ for the discrete subgroup generated by $\{ \delta_j ; 2r + 1 \leqslant j \leqslant m \}$. Then $\Gamma \subset W$ is self-orthogonal with respect to g. (b) Suppose now that A, Λ and g are as E, Δ and f resp. in our starting statement but, in addition, assume that Λ is self-orthogonal with respect to g. We settle this special case in the following manner. Let B be the subspace of A spanned by Λ. By assumption, B is self-orthogonal with respect to g. Let $B' \supset B$ be maximal self-orthogonal. We can find a basis $\{ u_i, v_j ;$ $1 \leqslant i, j \leqslant L \}$ in A modulo the radical of g, such that $\{ v_j ; 1 \leqslant j \leqslant L \} \subset B'$,

$g(v_i, u_i) \equiv 1$ $(1 \leqslant i \leqslant L)$, the value of g on any other pair being equal to zero. Let B'' be the subspace spanned by $\{u_j; 1 \leqslant j \leqslant L\}$ of A; A is the direct sum of B' and B''. Choosing elements $\{v_j^*; 1 \leqslant j \leqslant L\}$ in A^* such that they are orthogonal to B'' and satisfy $v_i^*(v_j) = \delta_{ij}$ $(1 \leqslant i, j \leqslant L)$, and similarly a system $\{u_j^*; 1 \leqslant j \leqslant L\}$, orthogonal to B' such that $u_i^*(u_j) = \delta_{ij}$ $(1 \leqslant i, j \leqslant L)$, we have $g = \sum_{j=1}^{L}(v_j^* \wedge u_j^*)$. Let $\{w_k^*; 1 \leqslant k \leqslant K\}$ be a basis for the orthogonal complement of B'' such that $w_k^*(\Lambda) \subseteq \mathbb{Z}$ $(1 \leqslant k \leqslant K)$. Writing $v_j^* = \sum_{k=1}^{K} c_{kj} w_k^*$ $(1 \leqslant j \leqslant L)$ we conclude that

$$g = \sum_{k=1}^{K} \sum_{j=1}^{L} \left(w_k^* \wedge \left(c_{kj} u_j^*\right)\right).$$

This being so it is enough to note that $w_k^*(\Lambda) \subseteq \mathbb{Z}$ and $c_{kj} u_j^*(\Lambda) = 0$ $(1 \leqslant k \leqslant K;$ $1 \leqslant j \leqslant L)$. (c) To complete the proof of (i), we return to the notations of (a). (1) Let $\{h_j; 1 \leqslant j \leqslant 2r\}$ be orthogonal to Z such that $h_i(\delta_j) = \delta_{ij}$ $(1 \leqslant i, j \leqslant 2r)$. We set $f_1 = a_1 h_1$, $f_2 = h_2, \ldots, f_{2r-1} = a_r h_{2r-1}$, $f_{2r} = h_{2r}$. (2) By what we saw in (b) above, we can find a system $\{g_j; 1 \leqslant j \leqslant 2t\}$, orthogonal to W, assuming integral values on Δ and satisfying

$$f | (Z \times Z) = \sum_{j=1}^{t} \left(\left(g_{2j-1} | Z\right) \wedge \left(g_{2j} | Z\right)\right).$$

We define $f_{2r+j} = g_j$ $(1 \leqslant j \leqslant 2t)$ and set $s = r + t$. Then, finally

$$f = \sum_{j=1}^{s} \left(f_{2j-1} \wedge f_{2j}\right)$$

and $f_j(\Delta) \subseteq \mathbb{Z}$ $(1 \leqslant j \leqslant 2s)$.

(ii) We infer from (i) that, with the notations of the statement of our lemma, there is a system $\{f_j; 1 \leqslant j \leqslant 2p\}$, such that $f_j(\Gamma_1) \subseteq \mathbb{Z}$ and $f = \sum_{j=1}^{p}(f_{2j-1} \wedge f_{2j})$. We write $\Lambda_0 = \Gamma^* | U \subset U^*$ and $\mathcal{F} = (\{f_j; 1 \leqslant j \leqslant 2p\} \cup \Lambda_0) \subset U^*$. Let Λ be the $\mathbb{Z}$-hull of $\mathcal{F}$. For later use we note that, if $u \in \Gamma_1$ and $\lambda \in \Lambda$, we have $(u, \lambda) \in \mathbb{Z}$. Given a $\mathbb{Z}$-module A, below we shall write A^* for $\mathrm{Hom}(A, \mathbb{Z})$ and A^{*r} for $\mathrm{Hom}(A, \mathbb{R})$; A^{*r} is a linear space over $\mathbb{R}$. This being so let us define V' as Λ^{*r} and Γ' as Λ^*. Clearly, Γ' is a lattice in V'.

(iii) Below we shall consider Γ^* as a subset of V^*. We define $j \in \mathrm{Hom}(V, (\Gamma^*)^{*r})$ by $j(v)(\gamma^*) = (v, \gamma^*)$ $(\gamma^* \in \Gamma^*; v \in V, \mathrm{fix})$; j is an isomorphism $V \to (\Gamma^*)^{*r}$. In the following, whenever convenient, we shall identify V to $(\Gamma^*)^{*r}$ via j.

We define now a morphism π from V' into V such that, for $\alpha \in V' = \Lambda^{*r}$ fix, we have $\pi(\alpha)(\gamma^*) = \alpha(\gamma^* | U)$ $(\gamma^* \in \Gamma^*)$. We claim that $\pi(\Gamma') \subset \Gamma$. In fact, $(j(\pi(\alpha)), \gamma^*) = \alpha(\gamma^* | U)$ $(\gamma^* \in \Gamma^*)$ and, if $\alpha \in \Gamma' = \Lambda^*$, the right-hand side is integral. We define $\bar{\Gamma} = \pi(\Gamma')$; hence $\bar{\Gamma} \subseteq \Gamma$. We show next that $\bar{\Gamma}$ is of a finite index in Γ. To this end it is enough to show that, given a nonzero γ^* in Γ^*, there is an $\alpha \in \Lambda^*$ such that $\alpha(\gamma^* | U) \neq 0$. This is certainly true if $\gamma^* | U$ is nonzero which, however, follows at once from the assumption that $\Gamma + U$ is dense in V.

(iv) Recalling that $V' = \Lambda^{*r}$, we define the morphism ρ of U into V' by $\rho(u)(\lambda) = (u, \lambda)$ $(\lambda \in \Lambda)$. We claim that $\pi \circ \rho$ is the identity on U. In fact, to this end it is enough to note that, given $u \in U$ and $\gamma^* \in \Gamma^*$, we have $\pi(\rho(u))(\gamma^*) = \rho(u)(\gamma^* | U) = (u, \gamma^*) = j(u)(\gamma^*)$. We observe next that ρ^* gives rise to a $\mathbb{Z}$-isomorphism from Γ'^* onto Λ. Let, in fact, λ be fix in $\Gamma'^* = \Lambda$. We have for any $u \in U$: $\rho^*(\lambda)(u) = \lambda(\rho(u)) = (\rho(u), \lambda) = (u, \lambda)$, proving our statement. We conclude from

this that $\Gamma' + U'$ is dense in V', since otherwise $\rho^*|\Gamma'^*$ would admit a nontrivial kernel.

(v) We have $\{f_j;\ 1 \leq j \leq 2p\} \subset \Lambda = \Gamma'^* \subset V'^*$. Let us define $g \in \Lambda_2(V'^*)$ as $\sum_{j=1}^{p}(f_{2j-1} \wedge f_{2j})$. Then g is integral on $\Gamma' \times \Gamma'$ and $\rho^*(g) = f$.

(vi) To complete our proof of Lemma 22 it remains to show that ρ maps $\Gamma_1 = \Gamma \cap U$ into Γ'. We have for $\gamma \in \Gamma_1$ and $\lambda \in \Lambda$: $\rho(\gamma)(\lambda) = (\gamma, \lambda)$; it is to be proved that this is always integral. Since, as we saw in (ii), Λ is generated over $\mathbb{Z}$ by $\Gamma^*|U$ and $\{f_j;\ 1 \leq j \leq 2p\}$, it is thus enough to recall that, by construction, f_j assumes integral values on Γ_1.   Q.E.D.

THEOREM 5. *Suppose that* $\mathfrak{g}$ *is a solvable Lie algebra and G the corresponding connected and simply connected Lie group. Let $\mathcal{O}$ be an R-orbit in $\mathfrak{g}^*$. There is a Hamiltonian G-foliation $\mathcal{H}$ such that $\mathcal{O} = o(\mathcal{H})$ (cf. Theorem 1) satisfying the following conditions.* (1) $\mathcal{H}$ *is standard* (cf. Definition 14), (2) $\mathcal{H}$ *is of vanishing obstruction* (cf. III, Definition 4) *and* (3) $\mathcal{H}$ *is rational* (cf. IV, Definition 6).

PROOF. If $\mathcal{O}$ is a coadjoint orbit, using Lemma 19 (say) we conclude that the standard covering is integral. Note that in the transitive case the obstruction is always trivial. In this manner we can assume below that $\mathcal{O}$ is not G-transitive.

(i) With the notations of Remark 12, let $\mathcal{O}'$ be the closure of a G-orbit in $\mathscr{B}(\Omega)$. Since $P|O'$ is a G-equivariant covering map of O' onto a generalized orbit, by Lemma 8 in II and Remark 5 in III, it gives rise on O' to the structure of a H.G-f. We claim that to prove Theorem 5, it suffices to establish the following. *There is a G-space $\mathcal{M}$ in which any G-orbit is dense and on which there is a distribution $\mathcal{D}$ such that $D_m = \sigma_m(\mathfrak{g})$ for all $m \in \mathcal{M}$. There is a G-equivariant smooth map Ψ of $\mathcal{M}$ onto O', which is bijective on each G-orbit. Let, finally, $\omega \in Z^2(O', G)$ be the canonical 2-form on O'. There is $\mu \in (Z^2(\mathcal{M}))^{\mathrm{rat}}$ such that $\mu|G = \Psi^*(\omega)$.* In fact, by Lemma 8, Ψ determines on $\mathcal{M}$ the structure of a H.G-f., which we continue to denote by $\mathcal{M}$. It is (1) standard, since so is O', and Ψ is bijective on G-orbits, (2) of vanishing obstruction by Lemma 13 in III, since, by Lemma 7, $(P|O') \circ \Psi$ is a morphism of $\mathcal{M}$ into a generalized orbit, (3) rational by virtue of the existence of μ as above.

(ii) Let (Ω, ρ) be the standard covering of $\mathcal{O}$ as in (i). We claim that *there is a connected subgroup M of G containing $L = [G, G]$, such that $\Omega \overset{p}{\to} \Omega/M$ is a fibration with contractible fibers, the base of which is a principal homogeneous space of $X(\pi_1(\Omega))$.* The following proof is patterned after that of Lemma 13 in [9] (p. 52). For unexplained notations we refer to the proof of Lemma 20 above. (a) We recall (cf. loc.cit.) that $\overline{A + B} = A'$. Given a subset E of A', we denote by $\tilde{E}$ the subspace generated by E. Hence $A + \tilde{B} = A'$ and thus there is a subspace C of A such that $A' = C + B$. (b) Let us define $M \subset G'$ as $\overset{-1}{\lambda}(C + B_0)$. Clearly, M is a connected subgroup of G' containing L. Since $B_0 = \lambda(\Delta_0)$ and $\Delta_0 = (G'_x)_0 = (G_x)_0$, M is contained in G. (c) We recall next that given a connected Lie group G and closed subgroups $(G \supset) T \supset K$, $G/K \to G/T$ is a fibration with fiber equal to T/K. We propose to use this by replacing G, T and K resp. by G', $M \cdot \Delta$ and Δ. To this end we have to show that $M \cdot \Delta$ is closed. Since $M \supset L$, it is enough to check that

$\lambda(M \cdot \Delta)$ is closed in A'. But this is so, since $\lambda(M \cdot \Delta) = \lambda(M) + \lambda(\Delta) = (C + B_0)$ $+ (\overline{D} + \Gamma) = C \oplus (\overline{D} + \Gamma)$ and $\overline{D} + \Gamma$ is closed in $B \subset \tilde{B}$. We conclude from all this that $\Omega \xrightarrow{p} \Omega/M$ is a fibration. (d) We show next Ω/M is a principal homogeneous space of $X(\pi_1(\Omega))$. For this it suffices to establish that $G'/M \cdot \Delta$ is isomorphic to $X(\pi_1(\Omega))$. There is a discrete subgroup Γ_1 such that $\overline{D} = B_0 \oplus \Gamma_1$. Hence, by what we saw in (c), $\lambda(M \cdot \Delta) = C \oplus B_0 \oplus \Gamma_1 \oplus \Gamma$. But we also have $A' = C + \tilde{B} = C \oplus B_0 \oplus \tilde{\Gamma}_1 \oplus \tilde{\Gamma}$. In this fashion it is enough to note that, since $\Omega = G'/\Delta$, we have $\pi_1(\Omega) = \Delta/\Delta_0 = \Gamma_1 \oplus \Gamma$. (e) To complete our proof, we show that the fiber is contractible. Since $M \cdot \Delta/\Delta = M/(M \cap \Delta)$, this will follow from $M \cap \Delta = (G_x)_0$. We have $\lambda(M \cap \Delta) \subset \lambda(M) \cap \lambda(\Delta) = (C + B_0) \cap (\overline{D} + \Gamma) = B_0 = \lambda((G_x)_0)$; hence $M \cap \Delta \subset (G_x)_0 L \cap G'_x$. Since, however, L_x is connected, this is equal to $(G_x)_0$, or $M \cap \Delta \subset (G_x)_0$. The opposite inclusion is trivial from our construction. Hence $M \cap \Delta = (G_x)_0$.

(iii) We recall that O' is a fixed G-orbit closure in $\mathcal{B}(\Omega)$ (cf. (i)). Let M be as in (ii). We claim that $O' \xrightarrow{p} O'/M$ *is a fibration with contractible fibers, and the base is a principal homogeneous space of* $X(\pi_1(O'))$. In fact, (a) We put $\mathcal{G} = G' \times \hat{\Pi}$. Proceeding similarly as in the proof of Lemma 10, we get $\mathcal{G}$ act on $\mathcal{B}(\Omega)$; this action is transitive. Given a point q in $\mathcal{B}(\Omega)$, the closed, connected subgroup $((\overline{G_q \cdot G}))_0$ ($= D$, say) of G is independent of q, and G-orbit closures and D-orbits coincide. (b) It is clear from (ii) that $O' \xrightarrow{p} O'/M$ ($= \mathcal{T}$, say) is a fibration with contractible fibers. $\mathcal{T}$ is a principal homogeneous space of D/MD_q ($= \mathcal{A}$, say). Since $[D, D] = L \subset M$, $\mathcal{A}$ is compact, connected and abelian.

(iv) With notations as above, let β be the canonical morphism $D \to D/D_q \cdot M = \mathcal{A}$. Let $\mathfrak{a}$ be the Lie algebra of $\mathcal{A}$. Then $\Gamma = \ker(\exp_A)$ is a lattice in $\mathfrak{a}$. Recalling that $D \supseteq G$, let us write $\mathfrak{b} = \beta_*(\mathfrak{g}) \subset \mathfrak{a}$. Our next objective is to define a certain $f \in \Lambda_2(\mathfrak{b}^*)$ and to show that along with $V = \mathfrak{a} \supset \Gamma$, $U = \mathfrak{b}$ the conditions of Lemma 22 are realized.

Since the fibers of $O' \xrightarrow{p} \mathcal{T}$, by (iii) above, are contractible, by virtue of the theorem of Leray-Hirsch (cf. [2, Theorem 5.11, p. 50]) we have a canonical isomorphism $p^*: H^2(\mathcal{T}) \to H^2(O')$. Let $(\Lambda_2(\mathcal{T}))^{\mathrm{inv}}$ be the set of all translation-invariant 2-forms on $\mathcal{T}$. Via the Hodge map, we have an isomorphism $H^2(\mathcal{T}) \to (\Lambda_2(\mathcal{T}))^{\mathrm{inv}}$. Since $\mathcal{T}$ is a principal homogeneous space of $\mathcal{A}$, $(\Lambda_2(\mathcal{T}))^{\mathrm{inv}}$ admits a canonical identification with $\Lambda_2(\mathfrak{a}^*)$. Through composition of all these maps we obtain a linear space isomorphism $H^2(O') \to \Lambda_2(\mathfrak{a}^*)$. We denote by L its lifting to $Z^2(O')$. By Theorem 2, there is an $\eta \in Z^2(O')$ such that $\eta|G = \omega$ (cf. (i)). We write $F = L(\eta)$ and define f as the restriction of F to $\mathfrak{b} \times \mathfrak{b}$. To prove that V, Γ, U, f, as just proposed, satisfy the conditions of Lemma 22, it suffices to show that (a) Writing $\Gamma_1 = \Gamma \cap \mathfrak{b}$, we have $f(\Gamma_1 \times \Gamma_1) \subset \mathbb{Z}$, (b) $\Gamma + U$ is dense in V. (a) (1) Let O be a fixed G-orbit in O'. Writing $r = p|O$, $O \xrightarrow{r} O/M$, $= \mathcal{T}'$ (say), is a fibration with contractible fibers. Setting $\mathcal{B} = \beta(G) \subset \mathcal{A}$, $\mathcal{T}'$ is a principal homogeneous space of $\mathcal{B}$. (2) Let $\mathfrak{b}_0$ be the subspace, spanned by Γ_1, of $\mathfrak{b}$. The corresponding subgroup $B_0 = \exp(\mathfrak{b}_0)$ is the maximal torus in $\mathcal{B}$, and we have a canonical isomorphism $H^2(\mathcal{B}) \simeq H^2(\mathcal{B}_0)$. Proceeding as above, we obtain an isomorphism

$H^2(O) \to \Lambda_2(\mathfrak{b}_0^*)$ through

$$H^2(O) \to H^2(\mathscr{T}') \to H^2(\mathscr{B}) \to H^2(\mathscr{B}_0) \to \left(\Lambda_2(\mathscr{B}_0)\right)^{\text{inv}} \to \Lambda_2(\mathfrak{b}_0^*).$$

We denote by L_O its lifting to $Z^2(O)$. We have

$$
\begin{array}{ccc}
Z^2(O') & \xrightarrow{L} & \Lambda_2(\mathfrak{a}^*) \\
\iota_O^* \downarrow & & \downarrow \text{restriction} \\
Z^2(O) & \xrightarrow[L_O]{} & \Lambda_2(\mathfrak{b}_0^*)
\end{array}
$$

(3) We write $\omega' = \iota_O^*(\eta)$ and observe that $\omega' \in (Z^2(O))^{\text{int}}$. In fact, since O is a standard covering of its projection into $\mathscr{O}$, the desired conclusion is implied by Lemma 19. Hence $L_O(\omega') \in \Lambda_2(\mathfrak{b}_0^*)$ is integral on $\Gamma_1 \times \Gamma_1$. But, by (2), we have $L_O(\omega') = f\,|\,(\mathfrak{b}_0 \times \mathfrak{b}_0)$, completing the proof of our statement. (b) We claim that $\Gamma + \mathfrak{b}$ is dense in $\mathfrak{a}$. With the notations of (a) this is equivalent to saying that B is dense in A. This, however, is clear from $T/B = T/G$, along with the reminder that any G-orbit in O' is dense.

(v) By (iv), putting $V = \mathfrak{a} \supset \Gamma$, $\mathfrak{b} = U$, with $f \in \Lambda_2(U^*)$ as above we can invoke Lemma 22 to obtain the objects $V' \supset \Gamma'$, π, ρ, g described loc.cit. We write $\Gamma_0 = \ker(\pi\,|\,\Gamma)$. Since $\Gamma'/\Gamma_0 \simeq \overline{\Gamma}$ is free abelian, there is a subgroup $\Delta \subset \Gamma'$ such that $\Gamma' = \Gamma_0 \oplus \Delta$. Let V_0 and D be the subspace of V' spanned by Γ_0 and Δ respectively. The restriction of π to D is an isomorphism onto V, and below we shall identify the two via this map. In this fashion we get $V' = V_0 \oplus V$ and π is the projection onto the second summand; also $\Gamma' = \Gamma_0 \oplus \overline{\Gamma}$. This being so we now proceed to construct the objects $\mathscr{M}$, Ψ and μ described in (i). (a) We write $T = V_0/\Gamma_0$; this is a multitorus. We define $\mathscr{M}$ as $T \times O'$. (b) We turn $\mathscr{M}$ into a G-space as follows. We recall that we set $\mathscr{B} = \beta(G)$. Since, by Lemma 22, $\pi \circ \rho$ is the identity map on U, by virtue of the identifications as above there is a linear map $\kappa: U \to V_0$ such that $\rho(u) \equiv \rho(u) + u$ $(u \in U)$. We claim that *there is a morphism ζ of B into T such that $\zeta_* = \kappa$*. In fact, by $\mathscr{B} = \mathfrak{b}/\Gamma_1$ and $T = V_0/\Gamma_0$, to this end it is enough to show that $\kappa(\Gamma_1) \subseteq \Gamma_0$. Let us write $\Lambda = \Gamma_0 + \Gamma$. We have, by Lemma 22, $\rho(\Gamma_1) \subseteq \Gamma'$. Therefore, given $\gamma \in \Gamma_1$, we can conclude that $\kappa(\gamma) = \rho(\gamma) - \gamma \in \Gamma' + \Gamma = \Lambda$ and thus $\kappa(\gamma) \in \Lambda \cap V_0 = \Gamma_0$ and $\kappa(\Gamma_1) \subseteq \Gamma_0$. Let us define $\chi \in \text{Hom}(G, T)$ by $\chi = \zeta \circ (\beta\,|\,G)$, and an action of G on M by $a(\tau, y) = (\chi(a)\tau, ay)$ $(\tau \in T, y \in O'; a \in G)$. Let, finally, $\Psi: \mathscr{M} \to O'$ be the projection onto the second factor; this is a G-map.

(vi) We start now to verify that $\mathscr{M}$ and Ψ, as defined above, have the properties described in (i). (a) We claim that given $m \in \mathscr{M}$, the map $\Psi\,|\,Gm$ is bijective. In fact, let us put $\Psi(m) = q$. To prove our point, it is enough to show that $G_m \supseteq G_q$. This, however, is clear from $G_q \subset \ker(\beta)$. (b) We claim that any G-orbit is dense in $\mathscr{M}$. In fact, (1) We set $\mathscr{A}' = T \times \mathscr{A}$. If M is as in (iii) above, $\mathscr{M} \xrightarrow{q} \mathscr{M}/M, = \mathscr{T}'$, say, is a fibration with contractible fibers, and $\mathscr{T}'$ is a principal homogeneous space, in the evident manner, of $\mathscr{A}'$. (2) Let us define $\beta': G \to \mathscr{A}'$ by $\beta'(a) = (\chi(a), \beta(a))$ $(a \in G)$. We write $\mathscr{B}' = \beta'(G)$. Since $\mathscr{T}'/G = \mathscr{T}'/\mathscr{B}'$, it suffices to show that $\mathscr{B}'$ is

dense in $\mathscr{A}'$. (3) Writing again $\Lambda = \Gamma_0 + \Gamma$, we have $\mathscr{A}' = V'/\Lambda$. We put, as in Lemma 22, $U' = \rho(U)$; we have $\mathscr{B} = \exp(U')$. In this fashion it is enough to establish that $\Lambda + U'$ is dense in V'. But, by loc.cit., this is true for the smaller set $\Gamma' + U'$.

(vii) We shall have completed the proof of Theorem 5 by showing the existence of a $\mu \in (Z^2(\mathscr{M}))^{\text{rat}}$ such that $\mu \,|\, G = \Psi^*(\omega)$ (cf. (i)). Let $\mathfrak{a}'$ be the Lie algebra of $\mathscr{A}'$. Since the fibers of $\mathscr{M} \overset{q}{\to} \mathscr{M}/M = \mathscr{T}'$ are contractible, proceeding as in (iv) above, we define maps L' and L by:

$$L' = Z^2(\mathscr{M}) \to H^2(\mathscr{M}) \overset{(q^*)^{-1}}{\to} H^2(\mathscr{T}') \to \left(\Lambda_2(\mathscr{T}')\right)^{\text{inv}} \to \Lambda_2(\mathfrak{a}'^*)$$

$$\Psi^* \uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \pi^*$$

$$L = Z^2(\Omega) \to H^2(\Omega) \overset{(p^*)^{-1}}{\to} H^2(\mathscr{T}) \to \left(\Lambda_2(\mathscr{T})\right)^{\text{inv}} \to \Lambda_2(\mathfrak{a}^*)$$

Hence, in particular:

$$
\begin{array}{ccc}
Z^2(M) & \overset{L'}{\to} & \Lambda_2(\mathfrak{a}'^*) \\[4pt]
\Psi^* \uparrow & & \uparrow \pi^* \\[4pt]
Z^2(\Omega) & \underset{L}{\to} & \Lambda_2(\mathfrak{a}^*)
\end{array}
$$

(b) $\eta \in Z^2(\Omega)$ being as in (iv) above, we define $\mu \in Z^2(\mathscr{M})$ as $q^*(g - L'(\Psi^*(\eta))) + \Psi^*(\eta)$. We claim that $\mu \in (Z^2(\mathscr{M}))^{\text{rat}}$. In fact, we have $L'(\mu) = (g - L'(\Psi^*(\eta))) + L'(\Psi^*(\eta)) = g$. Hence it is enough to recall that (1) By Lemma 22, g is integral on $\Gamma' \times \Gamma'$, (2) $\mathscr{A}' = V'/\Lambda$, $\Gamma_0 \oplus \Gamma = \Lambda \supseteq \Gamma' = \Gamma_0 \oplus \bar{\Gamma}$ and thus Γ' is cofinite in Λ. Hence g is rational on $\Lambda \times \Lambda$. (c) We show finally that $\mu \,|\, G = \Psi^*(\omega)$. In fact, for this it is enough to prove $q^*(g - L'(\Psi^*(\eta))) \,|\, G = 0$. We observe first that this is equivalent to $g \,|\, (U' \times U') = L'(\Psi^*(\eta)) \,|\, (U' \times U')$. As in (iv), we write $F = L(\eta)$ and recall that $f = F \,|\, (\mathfrak{b} \times \mathfrak{b})$. We have by the diagram of (a) above: $L'(\Psi^*(\eta)) = \pi^*(F)$. Hence it suffices to check that $g \,|\, (U' \times U') = \pi^*(F) \,|\, (U' \times U')$. This, however, is clear from $U' = \rho(U)$, along with $\pi \circ \rho =$ identity on U and $\rho^* g = f$, supplied by Lemma 22.   Q.E.D.

Appendix: Some general assumptions and notational conventions.

(1) Given a group G, $X(G)$ stands for the group of its characters. If G carries some topology, only continuous characters will be considered.

(2) A differentiable manifold means a C^∞-manifold. If M is such, $\mathscr{V}(M)$ signifies the Lie algebra of all smooth vector fields on M.

(3) If $\mathscr{D}$ is a distribution on M (= smooth assignment, for each $m \in M$, of a subspace of the tangent space $T_m(M)$), then D_m signifies its value at $m \in M$. We write $\mathscr{V}_{\mathscr{D}}(M)$ for the collection of all those elements of $\mathscr{V}(M)$ which, for all $m \in M$, take their values in D_m. $\mathscr{D}$ is involutive if and only if $\mathscr{V}_{\mathscr{D}}(M)$ is a Lie subalgebra of $\mathscr{V}(M)$.

(4) Let M be as above and suppose that the connected Lie group G with the Lie algebra $\mathfrak{g}$ acts smoothly on M. Given $m \in M$, $\sigma_m\colon \mathfrak{g} \to T_m(M)$ is defined for $l \in \mathfrak{g}$

as $(d/dt)(\exp(tl)m)|_{t=0}$. Given $l \in g$ fix, $\sigma(l) \in \mathcal{V}(M)$ is defined by the condition $(\sigma(l))_m = \sigma_m(l)$ $(m \in M)$.

(5) Let M be as above. Then $\Lambda_k(M)$ $(k \geqslant 0)$ denotes the collection of all smooth k-forms. If $\omega \in Z^k(M)$, we write $[\omega]$ for its image in $H^k(M)$.

(6) $(Z^k(M))^{\text{int}}$ $((Z^k(M))^{\text{rat}})$ will stand for the collection of all those elements of $Z^k(M)$ $(\omega$, say) for which $[\omega]$ is integral (resp. rational).

(7) If G is a connected Lie group with the Lie algebra g, g $(g^* = $ dual of the underlying space of g) will be considered as a G-module with respect to the adjoint (resp. coadjoint) representation. In the same fashion, g (g^*) will be considered as a g-module.

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