

A PARAMETRIX FOR STEP-TWO HYPOELLIPTIC DIFFUSION EQUATIONS ¹

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ABSTRACT. In this paper I construct a parametrix for the hypoelliptic diffusion equations $(\partial/\partial t - L)u = 0$, where $L = \sum_{a=1}^n g_a^2$ and where the g_a are vector fields which satisfy the property that they, together with all of the commutators $[g_a, g_b]$ for $a < b$, are at each point linearly independent and span the tangent space.

1. Introduction. Let L be the Laplace-Beltrami operator on a complete Riemannian manifold M . In this situation, L is essentially selfadjoint on $C_0^\infty(M)$. Let $P(t, x)$ be the fundamental solution of the heat equation on M ; i.e.,

$$\left(\frac{\partial}{\partial t} - L\right)P(t, x) = 0 \quad \text{and} \quad \lim_{t \rightarrow 0} P(t, x) = \delta_x,$$

where δ_x is the δ -function at x . Suppose that μ is an invariant measure for the semigroup e^{tL} so that the density $p_\mu(t, x, y)$ of $P(t, x)$ with respect to μ satisfies $p_\mu(t, x, y) = p_\mu(t, y, x)$ and is a smooth function of (t, x, y) for $t > 0$. It also follows that e^{tL} is a selfadjoint semigroup on $L^2(M, d\mu)$ and, if M is compact, that

$$\text{trace}(e^{tL}) = \int p_\mu(t, x, x) d\mu(x).$$

Thus, $p_\mu(t, x, x)$ is a kind of "local trace" for e^{tL} . Let Δ be the elliptic operator defined by "freezing the coefficients of L " at $\zeta \in M$, i.e., the operator which is the constant part of L with respect to some coordinate system centered at ζ . Thus, $(L - \Delta)$ is a second-order partial differential operator which vanishes at ζ . Since the asymptotics of $p_\mu(t, x, x)$ are locally determined quantities at ζ , it can be (and has been—see McKean and Singer [13]) assumed that $M = R^m$, that $(L - \Delta)$ is of compact support and that μ is Lebesgue measure. Let q_μ be the (Gaussian) fundamental solution of $(\partial_t - \Delta)$. It is easy to see that

$$(\partial_t - \Delta)[p(t, x, y) - q(t, x, y)] = (L - \Delta)p(t, x, y).$$

Thus, one sees that there is a formal expansion

$$p(t, x, y) = \sum_{N=0}^{\infty} [(\partial_t - \Delta)^{-1}(L - \Delta)]^N q(t, x, y).$$

It turns out that this formal expansion converges that uniformly, on compacta, and is indeed equal to $p(t, x, y)$ (see §3 of McKean and Singer [13]). Now, the

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Schwartz kernel of $(\partial_t - \Delta)^{-1}$ is just the Gaussian heat kernel so that, expanding $(L - \Delta)$ in a formal power series, one obtains a formal series for $p(t, x, y)$ in terms which are homogeneous under dilations. The terms of this expansion are integrals involving integrands which are products of Gaussians and polynomials and so may be computed explicitly. Evaluating these integrals when $x = y$ leads to an asymptotic expansion

$$p(t, x, x) \sim (4\pi t)^{-(d/2)} [1 + c_1(x)t + c_2(x)t^2 + \cdots],$$

where $d = \dim(M)$.

Weyl first obtained the leading term $(4\pi t)^{-(d/2)}$. Minakshisundaram [14, 15, 16] first proved the existence of this asymptotic expansion.

Later work by various authors [1, 7, 17] demonstrated profound connections of the coefficients of this expansion with Riemannian and topological invariants of M .

Now, let M be a smooth manifold of dimension $m = \frac{1}{2}n(n+1)$. Consider a partial differential operator L such that locally there are vector fields $\{g_a\}_{a=1}^n$ such that the g_a 's together with all the vector fields $[g_a, g_b]$ span the tangent space at each point and such that $L = \sum g_a^2$. By a theorem of Hörmander [10], L is hypoelliptic, as is also the diffusion operator $(\partial/\partial t - L)$. Motivated by the work of Rothschild and Stein [18], I will call such an operator a “step-two free hypoelliptic operator”. (“Step-two” because no iterated commutators are necessary to span the tangent space; “free” because the vector fields g_a and $[g_b, g_c]$ are linearly independent at every point.)

It turns out that a step-two free hypoelliptic operator L is associated with a certain geometrical structure, just as elliptic operators are associated with Riemannian geometry.

In particular, since L is a second-order partial differential operator, it induces a quadratic form G^* on T^*M . For functions $f_1, f_2 \in C^\infty(M)$ such that $f_1(\zeta) = f_2(\zeta) = 0$, $\zeta \in M$,

$$G^*(df_1(\zeta), df_2(\zeta)) = [L(f_1 f_2)](\zeta) = \sum_{a=1}^n df_1(g_a(\zeta)) df_2(g_a(\zeta)).$$

Since L is not elliptic, G^* is a degenerate form. In fact, $\text{Rank } G^* = n$ so that $\dim \ker G^* = \frac{1}{2}n(n-1)$. However, G^* induces a nondegenerate quadratic form G on the subbundle $\mathcal{D} \subset T^*M$, defined by $\mathcal{D}_\zeta = \text{span}\{g_a(\zeta)\}_{a=1}^n$.

This geometric structure is an example of what has been named a “singular Riemannian geometry”, and it has many properties which are very similar to Riemannian geometry [3].

In fact, one may construct a parametrized family G_ϵ^* of nondegenerate quadratic forms on T^*M such that $\lim_{\epsilon \rightarrow \infty} G_\epsilon^* = G^*$. Then the geodesic flows of the G_ϵ^* 's converge to the hamiltonian flow associated to G^* .

For the case that L is a step-two free hypoelliptic operator, the singular Riemannian geometry implies a reduction of the bundle of frames on M , with structure group a certain semidirect product of $O(n)$ with a vector space. Thus, one can, as usual, construct connections and various invariants of the structure (details will appear elsewhere).

On the other hand, assume for the moment that L has an extension which is the generator of a strongly continuous contradiction semigroup on $C_0(M)$ (= the

space of bounded continuous functions vanishing at ∞). Thus, one can obtain a fundamental solution of $(\partial/\partial t - \Delta)$, which is a smooth measure for $t > 0$, just as in the Riemannian case.

In this paper, I construct a parametrix for the diffusion operator $(\partial/\partial t - L)$. The construction involves the use of perturbation methods with respect to exact solutions of certain left-invariant hypoelliptic diffusion equations on step-two free nilpotent groups. It is, in fact, no loss of generality to assume that L has an extension which is a generator, because a parametrix is a purely local object, and I can always replace L by a generator which is equal to L on a compact set. The form of this parametrix implies an asymptotic expansion at $t = 0$, in half-integral powers of t , of the form

$$p(t, \zeta, \zeta) \sim K t^{-(n/2)^2} (1 + C_1(\zeta)t + C_{3/2}(\zeta)t^{3/2} + C_2 t^2 + \cdots),$$

where K is a number which depends on n but not on ζ .

Note that there is no $t^{1/2}$ term in this expansion; it is a consequence of the geometry of the problem that the coefficient of $t^{1/2}$ vanishes.

In attempting to construct a parametrix for step-two hypoelliptic diffusion equations, one's first hope might be that the Minakshisundaram approach could be applied directly to the computation of the fundamental solution and asymptotics of the fundamental solutions for hypoelliptic diffusion equations. However, when one freezes the coefficients of a nonparabolic diffusion equation, the result is a nonhypoelliptic diffusion equation, so the necessary techniques of analysis fail to apply.

Rothschild and Stein [18] have been successful in approximating the local behavior of hypoelliptic operators by invariant hypoelliptic operators on free nilpotent groups. This motivates me to be interested in nilpotent groups in the context of hypoelliptic diffusions. On the other hand, results in control theory concerning Volterra series have provided a context for the following theorem of Brockett [3].

THEOREM. *Let $g_1, \dots, g_n$ be vector fields on $R^{n(n+1)/2}$ which generate a Lie algebra that spans at step two (and hence is free, by dimensionality). Then there is a coordinate system centered at zero, $\{x^a, y^{bc}\}$, for $b < c$ such that*

$$\tilde{g}_a = \left(\frac{\partial}{\partial x^a} + \frac{1}{2} \sum_{b=1}^n x^b \frac{\partial}{\partial y^{ab}} \right) + h_a,$$

where the vector field $\partial/\partial y^{ab}$ is defined to be $-\partial/\partial y^{ba}$ when $b < a$, and where h_a has the properties that its coefficients vanish to second order in x and y .

REMARK. Note that this theorem is entirely local in nature, so the same result holds for vector fields on an $n(n+1)/2$ -dimensional manifold M on a neighborhood of every point $\zeta \in M$.

Thus, if Δ is defined to be the operator

$$\Delta = \sum_{a=1}^n \left(\frac{\partial}{\partial x^a} + \frac{1}{2} \sum_{b=1}^n x^b \frac{\partial}{\partial y^{ab}} \right)^2$$

in the coordinate system defined by Brockett's theorem, $(L - \Delta)$ is a (locally defined) operator which vanishes to (slightly more than) second order at $\zeta \in M$. Thus Δ contains both the zeroth- and first-order behavior of the coefficients of L at ζ .

Moreover, as I will discuss in §2, the exponential map gives a global coordinate system on step-two free nilpotent groups such that the left-invariant “subelliptic Laplacian” of such a group has exactly the same form as Δ . The corresponding diffusion equation has fundamental solutions which are easy to characterize in terms of certain partial Fourier transforms. These diffusion equations have been recently discussed in the papers [6 and 9].

Now, since $(\partial/\partial t - L)$ is hypoelliptic, the operator $(\partial/\partial t - L)^{-1}$ is pseudolocal. Since the Schwartz kernel of $(\partial/\partial t - L)^{-1}$ is the fundamental solution $P(t, \zeta)$ of $(\partial/\partial t - L)$, it follows that $p_\mu(t, \zeta, \xi)$ is determined, modulo smooth functions on $R \times M$, by the Taylor series of the coefficients of L at the point ζ . In particular, if ϕ is a smooth function of compact support such that $\phi \equiv 1$ on a neighborhood of ζ , and if $L' = \Delta + \phi(L - \Delta)$, then on the domain $U \subset M$ where Δ is defined, $(\partial/\partial t - L')P(t, x)$ is a smooth measure (which vanishes to infinite order at $t = 0$).

Thus, it is sufficient to consider operators on step-two free nilpotent groups which differ from the subelliptic Laplacian by an operator which is smooth and of compact support. For these operators it turns out to be easy to apply perturbation methods to obtain a parametrix for L in terms of the coefficients of L (in the Brockett coordinate system) and in terms of the fundamental solution of $(\partial/\partial t - \Delta)$. I develop these results in §3.

Note that the leading term of the asymptotic expansion in the elliptic case is exactly the function of time which normalizes the mass of the heat kernel of the heat equation for the standard Laplacian operator on R^n . Thus, one might expect (and I will show in §3) that the hypoelliptic analog of this term will be given by the asymptotics of the fundamental solution of the diffusion equation for the subelliptic Laplacian on step-two free nilpotent groups. From a dilation symmetry of this diffusion equation I deduce in §2 a particular (power law) functional dependence, and a calculation involving the partial Fourier transform of fundamental solutions yields the specific coefficient. I perform this calculation explicitly in the case $n = 2$; in higher dimensions I leave the answer as an integral of a certain function on $R^{n(n-1)/2}$.

In §4 I apply the parametrix to compute coefficients of the asymptotic expansion of $p(t, \zeta, \zeta)$ for several examples of three-dimensional hypoelliptic diffusion equations. The case of the left-invariant hypoelliptic diffusion equation on the Heisenberg group is particularly easy, because the dilation homogeneity discussed above implies that $C_k(\zeta) = 0$ for $k \geq 1$.

I also discuss the case when $L = g_1^2 + g_2^2$, where g_1 and g_2 are independent left-invariant vector fields on $SL(2, R)$, which generate noncompact one parameter subgroups, which are orthogonal with respect to the Killing pseudometric. In this case I compute that

$$p(t, \zeta, \zeta) \sim 1/16t^2 - 1/64t + O(t^{-1/2}).$$

The Weyl unitary trick may be applied to this computation to yield the asymptotic expansion for the diffusion equation on $SO(3)$ with $L = g_1^2 + g_2^2$, where g_1, g_2 are independent rotation generators which are orthonormal with respect to the Killing metric. In this case

$$p(t, \zeta, \zeta) \sim 1/16t^2 + 1/64t + O(t^{-1/2}).$$

REMARK. It is possible to verify this asymptotic expansion by using the representation theory of $SO(3)$ to compute the spectrum of L and thence to compute an asymptotic expansion for the trace of e^{tL} . Dividing this expansion by the volume of $SO(3)$ yields the result.

A number of recent papers are of a similar spirit and overlap with this paper when $u = 2$, particularly Beals, Greiner and Stanton [2], Stanton and Tartakoff [19] and M. Taylor [20].

2. Invariant hypoelliptic diffusion equations on step-two free nilpotent groups. Let V be an n -dimensional vector space. Let $V \wedge V$ be the antisymmetric tensor product of V with itself. The vector space $N(n) = V \oplus (V \wedge V)$ may be given the structure of a Lie algebra as follows: for $v, w \in V$ set $[v, w] = \frac{1}{2}v \wedge w$; set the commutator of anything with an element of $V \wedge V$ to be zero. This Lie algebra is called the step-two free Lie algebra on n generators. Clearly, $N(n)$ has dimension $\frac{1}{2}n(n+1)$; the ideal $V \wedge V$ has dimension $\frac{1}{2}n(n-1)$.

$V \oplus (V \wedge V)$ may also be given the structure of a Lie group; in fact, it is given by the exponential of the adjoint action of $N(n)$ on itself. One may compute that the group multiplication is $(v, u, w) \cdot (\bar{v}, \bar{u}, \bar{w}) = (v + \bar{v}, u + \bar{u}, w + \bar{w} + \frac{1}{2}v \wedge \bar{v})$, and that the group inverse is the same as the vector-space inverse. Let $N(n)$ denote this Lie group. Consider the vector space $V^* \oplus (V^* \wedge V^*)$. A choice of basis for V determines a basis $\{x^a, y^{bc}\}$, where $a, b, c = 1, \dots, n$ and $b < c$, for $V^* \oplus (V^* \wedge V^*)$. This basis may be regarded as a global coordinate system on $N(n)$. It is easy to show that the vector fields

$$V_a = \frac{\partial}{\partial x^a} + \frac{1}{2} \sum_{b=1}^n x^b \frac{\partial}{\partial y^{ab}}, \quad \text{for } a = 1, \dots, n,$$

where the vector field $\partial/\partial y^{ab}$ is defined to be $-\partial/\partial y^{ba}$ when $a > b$, are left invariant and generate the Lie algebra of $N(n)$. It follows from Hörmander's theorem that the operators $\Delta = \sum_{a=1}^n V_a^2$ and $(\partial/\partial t - \Delta)$ are both hypoelliptic.

The fundamental solutions $Q(t, \zeta)$ of $(\partial/\partial t - \Delta)$ are a set of time-dependent measures on $N(n)$ which are parametrized by $\zeta \in N(n)$ and (weakly) annihilated by $(\partial/\partial t - \Delta)$. Then $Q(t, \zeta)$ also satisfies

$$\int_{N(n)} dQ(t, \zeta) = 1, \quad \lim_{t \rightarrow 0} Q(t, \zeta) = \delta_\zeta,$$

where δ_ζ is the δ -function supported at the point $\zeta \in N(n)$. The hypoellipticity of $(\partial/\partial t - \Delta)$ implies that $Q(t, \zeta)$ is a smooth measure.

In addition, the volume form

$$\mu = dx^1 \wedge dx^2 \wedge \dots \wedge dx^n \wedge dy^{12} \wedge dy^{13} \wedge \dots \wedge dy^{n-1, n}$$

on $N(n)$ is invariant under right or left translations, so Δ is formally selfadjoint. The function μ defines a smooth measure on $N(n)$ such that the density of $Q(t, \zeta)$ with respect to μ is a smooth function $q(t, \zeta, \xi)$ on $(0, \infty) \times N(n) \times N(n)$ which is symmetric with respect to the interchange of ζ and ξ . In addition, for each fixed ζ , $q(t, \zeta, \cdot)$ is in the Schwartz class on $N(n)$, as is $q(t, \cdot, \xi)$ for each $\xi \in N(n)$ [5].

Consider for example $n = 2$. Then $N(2)$ is diffeomorphic to R^3 and is the Heisenberg group. We can use the traditional coordinates x, y, z on R^3 and write

$$V_1 = \frac{\partial}{\partial x} + \frac{1}{2}y \frac{\partial}{\partial z}, \quad V_2 = \frac{\partial}{\partial y} - \frac{1}{2}x \frac{\partial}{\partial z}.$$

It then follows that

$$\Delta = [\partial_x^2 + \partial_y^2 + \frac{1}{4}(x^2 + y^2)\partial_z^2] + [(x\partial_y - y\partial_x)\partial_z].$$

Now, I want to compute the measure $Q(t, 0) = \exp(t\Delta)\delta_{(0,0,0)}$. Since both the delta function and Δ may be seen to be invariant under rotations about the z -axis, it follows that $Q(t, 0)$ is also invariant under rotations about the z -axis. Note that the term in Δ in the second set of brackets contains the generator of rotations about the z -axis as one factor. Thus, the second term in brackets annihilates $Q(t, 0)$. $Q(t, 0)$ is thus also the fundamental solution of the diffusion equation for the operator Δ_0 given by the first term in brackets, which is elliptic everywhere except on the line $x = y = 0$. Let $q(t, x, y, z; 0)$ be the density of $Q(t, 0)$.

Define the Fourier transform in the z variable of q by

$$\hat{q}(t, x, y, k; 0) = \int e^{ikz} q(t, x, y, z; 0) dz.$$

Then $\hat{q}$ satisfies the equation

$$(III.4) \quad \frac{\partial}{\partial t} \hat{q} = \left(\partial_x^2 + \partial_y^2 - \frac{1}{4}k^2(x^2 + y^2) \right) \hat{q}, \quad \hat{q}(0, x, y, k; 0) = \delta_0(x)\delta_0(y).$$

The separation of variables method is applicable to this equation. In fact, $\hat{q} = q^x q^y$, where q^x satisfies the equation

$$\partial a^x / \partial t = (\partial_{x^2} - \frac{1}{4}k^2 x^2) q^x, \quad q^x(0, x, k; 0) = \delta_0(x),$$

and q^y satisfies the analogous equation.

However, Feynman's book [4, pp. 49–51] (also Gaveau [6]) tells us that in this case

$$q^x(t, x, k; 0) = [k/4\pi \sinh(kt)]^{1/2} \exp[-(k/4) \coth(kt)(x^2)].$$

Thus, $\hat{q}(t, x, y, k; 0)$ is equal to

$$[k/4\pi \sinh(kt)] \exp[-(k/4) \coth(kt)(x^2 + y^2)].$$

Recall that Δ is left invariant. This implies that the fundamental solution starting at (s, v, u) , $p(t, x, y, z; s, v, u)$, is the left translation of $p(t, x, y, z; 0)$. We obtain the partial Fourier transform of this left translation to be

$$\hat{q}(t, x, y, k; s, v, u) = e^{ik(u+sy-vx)} q(t, x-s, y-v, k; 0).$$

Now, in general, one may show that for arbitrary n , the generator which we are interested in is

$$\begin{aligned} \sum_{a=1}^n \left(\frac{\partial}{\partial x^a} \right)^2 + \sum_{a=1}^n \sum_{b=1}^n x^b \frac{\partial}{\partial x^a} \frac{\partial}{\partial y^{ab}} \\ + \frac{1}{4} \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n x^a x^b \frac{\partial}{\partial y^{ca}} \frac{\partial}{\partial y^{cb}}. \end{aligned}$$

Denote by Y the antisymmetric matrix with coefficients y^{ab} . Define A to be an element of the dual space of the space of all such Y 's; let α_{ab} be the (a, b) th coefficient of A with respect to the basis determined by the y^{ab} 's. Then

$$A(Y) = \sum \alpha_{ba} y_{ab}.$$

In these terms, the Fourier transform of q in the y variables is given by

$$\hat{q}(t, x, A; 0) = (2\pi)^{-n(n-1)/4} \int e^{iA(Y)/2} q(t, x, Y; 0) \prod_{a < b} dy^{ab}.$$

Then we have the Fourier-transformed diffusion operator

$$\sum_{a=1}^n \left[\left(\frac{\partial}{\partial x^a} \right)^2 + \sum_{b=1}^n ix^b \frac{\partial}{\partial x^a} \alpha_{ab} - \frac{1}{4} \sum_{b=1}^n \sum_{c=1}^n x^a x^b \alpha_{ca} \alpha_{cb} \right].$$

Then the diffusion equation in matrix-vector notation is

$$(\partial_t - [\langle \partial_x, \partial_x \rangle - i \langle Ax, \partial_x \rangle - \frac{1}{4} \langle Ax, Ax \rangle]) \hat{q} = 0.$$

Now, I claim that the operator $\langle Ax, \partial_x \rangle$ commutes with both $\langle \partial_x, \partial_x \rangle$ and $\langle Ax, Ax \rangle$. Indeed, since A is antisymmetric, $\langle Ax, \partial_x \rangle$ is the infinitesimal generator of a rotation in R^n , and the Laplacian $\langle \partial_x, \partial_x \rangle$ is invariant under rotations. Also,

$$[\langle Ax, \partial_x \rangle, \langle Ax, Ax \rangle] = \langle A^2 x, Ax \rangle + \langle Ax, A^2 x \rangle = 0$$

by antisymmetry. Thus, as in the $n = 2$ case, $\hat{q}(t, x, A; 0)$ is annihilated by the term involving a rotation. Therefore, we are left with the partial differential equation

$$[\partial_t - \langle \partial_x, \partial_x \rangle + \frac{1}{4} \langle Ax, Ax \rangle] \hat{q} = 0, \quad \lim_{t \rightarrow 0} \hat{q}(t, x, A; 0) = \delta_0(x).$$

LEMMA. *The following function satisfies the above equations:*

$$\det \left(iA [4\pi \sinh(iAt)]^{-1} \right)^{1/2} \exp [-\langle x, (iA/4) \coth(iAt)x \rangle].$$

REMARK. The functions $z/\sinh(z)$ and $z[\coth(z)]$ are easily seen to be smooth for all real z . Therefore, since A is antisymmetric and hence diagonalizable with imaginary eigenvalues, it follows that the above matrix functions are defined, and C^∞ in A and in $t > 0$.

PROOF. Most of the proof is straightforward matrix calculus. I now describe the only tricky detail. Expand $\det(iA[4\pi \sinh(iAt)]^{-1})$ as a product in terms of the eigenvalues of A . Then one easily computes that

$$\begin{aligned} (d/dt) \det(iA[4\pi \sinh(iAt)]^{-1}) \\ = -\text{trace}[iA \coth(iAt)] \det(iA[4\pi \sinh(iAt)]^{-1}). \quad \text{Q.E.D.} \end{aligned}$$

Let $D(s)$ be the group of dilations on $N(n)$ defined by the pullbacks

$$D(s)^* x^a = e^s x^a, \quad D(s)^* y^{ab} = e^{2s} y^{ab}.$$

Recall that vector fields push forward. Note that

$$D(s)_* V_a = e^{-s} V_a, \quad D(s)_* \frac{\partial}{\partial y^{ab}} = e^{-2s} \frac{\partial}{\partial y^{ab}}.$$

Since $\Delta = \sum V_a^2$, it follows that $D(s)_* \Delta = e^{-2s} \Delta$. Note that the (Lebesgue) measure μ satisfies $D(s)_* \mu = e^{-n^2 s} \mu$.

Let $\tilde{D}(s) = D(s) \otimes \exp[2st(d/dt)]$. This is a dilation on $N(n) \times R$, since $\exp[2st(d/dt)]$ is a dilation on R . Then $\tilde{D}(s)_*(\partial_t - \Delta) = e^{-2s}(\partial_t - \Delta)$. Now, $[Q(t, 0) dt]$ is Green's measure for our diffusion operator:

$$(\partial_t - \Delta_X^*)[Q(t, 0) dt] = \delta_0(X) \delta_0(t),$$

but $\tilde{D}(s)_* \delta_0(X) \delta_0(t) = \delta_0(X) \delta_0(t)$, since the δ -function is invariant under dilations. Thus, $[Q dt]$ is determined by an equation homogeneous under $\tilde{D}(s)$. It follows that Q is also homogeneous and that its homogeneity is determined by the homogeneity of the other terms. Indeed, because $\tilde{D}_*(s) \delta_0(x) \delta_0(t) = \delta_0(x) \delta_0(t)$, we have

$$\begin{aligned} \tilde{D}(s)_* \{(\partial_t - \Delta^*)[Q(t, 0) dt]\} &= (\partial_t - \Delta^*)[Q(t, 0) dt] \\ &= [\tilde{D}(s)_*(\partial_t - \Delta^*)\tilde{D}(-s)_*] \tilde{D}(s)_*[Q(t, 0) dt] \\ &= e^{2s}(\partial_t - \Delta^*)\tilde{D}(s)_*[Q(t, 0) dt]. \end{aligned}$$

Thus, we get that

$$\tilde{D}(s)_*[Q(t, 0) dt] = e^{-2s}[Q(t, 0) dt].$$

But $Q(t, 0) dt = q(t, 0, X) dX dt$, where q is a smooth function of (t, X) for $t > 0$. Then

$$\begin{aligned} e^{-2s}[Q(t, 0) dt] &= \tilde{D}(s)_*[Q(t, 0) dt] \\ &= [\tilde{D}(-s)^* q(t, 0, X)] [e^{-(n^2+2)s} dX dt]. \end{aligned}$$

Thus, $\tilde{D}(-s)^* q(t, 0, X) = e^{n^2 s} q(t, 0, X)$. But the submanifold $X = 0$ is invariant under the action $\tilde{D}(s)$, so this equation implies that

$$q(t, 0, 0) \sim (\text{constant}) t^{-n^2/2}.$$

Thus, to compare this result to the elliptic case, we see that $q(t, 0, 0)$ gets one factor of $t^{-(1/2)}$ for every x -direction ("ordinary direction") and a factor of t^{-1} for every y -direction ("exceptional direction"). For example, when $n = 2$, $q(t, 0, 0)$ is proportional to t^{-2} , so, as far as Weyl's theorem goes, a three-dimensional singular Riemannian manifold looks like a four-dimensional Riemannian manifold.

Recall that the partial Fourier transform $\hat{q}$ of q in the y variables is given by the function

$$\begin{aligned} \hat{q}(t, x, A; 0) &= \det\{iA[4\pi \sinh(iAt)]^{-1}\}^{1/2} \\ &\quad \cdot \exp[-\langle x, (iA/4) \coth(iAt)x \rangle], \end{aligned}$$

where $i = \sqrt{-1}$ and A is the antisymmetric matrix representation of the Fourier-transformed variable of the y . Now,

$$q(t, x, y; 0) = \left(\frac{1}{2\pi}\right)^{n(n-1)/2} \int e^{-(i/2) \text{trace}(AY)} \hat{q}(t, 0, (x, A)) dA.$$

Thus,

$$\begin{aligned} q(t, 0, 0) &= \left(\frac{1}{2\pi}\right)^{n(n-1)/2} \int q(t, 0, (0, A)) dA \\ &= \left(\frac{1}{2\pi}\right)^{n(n-1)/2} \int \det\{iA[4\pi \sinh(iAt)]^{-1}\} dA. \end{aligned}$$

EXAMPLE. When $n = 2$, the space of A 's is one dimensional. Thus this space is parametrized by the eigenvalues of A , $\pm i\alpha$. Thus, $\det\{iA[\sinh(iAt)]\} = [\alpha^2/16\pi^2 \sinh^2(\alpha t)]$, so

$$\begin{aligned} q(t, 0, 0) &= \frac{1}{2\pi} \int \left[\frac{\alpha}{4\pi \sinh(\alpha t)} \right] d\alpha \\ &= \left(\frac{1}{8\pi^2}\right) t^{-2} \int \left[\frac{\alpha}{\sinh \alpha} \right] d\alpha. \end{aligned}$$

But Gradshteyn and Ryzhik [8, §3.521] give us that $q(t, 0, 0) = (16t^2)^{-1}$.

3. The construction of the parametrix.

A. *Background mathematics.* Recall from §1 the remarks following Brockett's theorem: for an arbitrary step-two free system of vector fields on $R^{n(n+1)/2}$, Brockett's theorem defines coordinates x, y in a neighborhood U_{xy} of every point ζ . These coordinates define a local diffeomorphism from U_{xy} onto a neighborhood of the identity in $N(n)$. This diffeomorphism can be used to pull back objects on $N(n)$ to U_{xy} . In particular, the dilation $D(s)$ can be pulled back to a local group of local diffeomorphisms on U_{xy} (call it $D(s)$ again). These local diffeomorphisms fix the point ζ . The vector fields h_a of Brockett's theorem are

$$h_a = \left(A_{abc}^{de} x^b x^c \frac{\partial}{\partial y^{de}} \right) + j_a,$$

where one sums on all repeated indices and where j_a has a formal power series at ζ in terms homogeneous under $D(s)$ of order greater than or equal to one. We can also pull back the vector fields V_a to vector fields (called V_a) on U_{xy} . Let Δ be the partial differential operator $\sum V_a^2$. Note that Δ is homogeneous under $D(s)$ of order -2 . Let $L = \sum g_a^2$. A consequence of Brockett's theorem is that on U_{xy}

$$L = \Delta + L^{(-1)} + L^{(0)},$$

$$L = \Delta + \left(V_a A_{abc}^{de} x^b x^c \frac{\partial}{\partial y^{de}} + \text{symmetric} \right) + L^{(0)},$$

where $L^{(-1)}$ is homogeneous of order -1 under $D(s)$ and $L^{(0)}$ has a formal power series at ζ with terms homogeneous of order greater than or equal to zero.

Let $P(t, W)$ be the fundamental solution for $(\partial_t - L)$, with density $p(t, W, X)$,

$$(\partial_t - L(X)^*)p(t, W, X) dX = 0, \quad \lim_{t \rightarrow 0} p(t, W, X) dX = \delta_W(X),$$

where X and W each denote the coordinates x and y on $N(n)$ and where, recalling that $\delta_W(X)$ is a measure, one includes the factor dX in order to keep track of variances. Recall that p depends only on the formal power series of L^* at $X = W$, mod smooth functions which vanish to infinite order at $t = 0$ (because $(\partial_t - L)$ is hypoelliptic). Therefore, since we are interested only in the small time asymptotics of p , it suffices to restrict our study to operators L on $N(n)$. Particularly, it is sufficient to consider only operators of the form

$$L = \Delta + \xi = \sum (V_a + \phi h_a)^2,$$

where ϕ is smooth of compact support and equal to one on a neighborhood of ζ .

Recall that $\tilde{D}(s)$ is the dilation on $N(n) \times R$ given by $D(s) \otimes \exp(2st(d/dt))$. If π is the canonical projection on the left factor, then the following diagram commutes:

$$\begin{array}{ccc} N(n) \times R & \xrightarrow{\tilde{D}(s)} & N(n) \times R \\ \pi \downarrow & & \pi \downarrow \\ N(n) & \xrightarrow{D(s)} & N(n) \end{array}$$

Define L_s to be $e^{2s} D(s)^* L D(-s)^*$ and ξ_s to be $(L_s - \Delta)$. Define p_s to be

$$p_s(t, W, X) = e^{n^2 s} \tilde{D}(s)^* p(t, W, X),$$

where $\tilde{D}$ acts on t and the X variable. It is easy to see that the fundamental solution for $(\partial_t - L_s)$ is $P_s(t, W) = p_s(t, W, X) dX$.

B. *A perturbation argument.* Now, the R -linear span of the $V_a + \phi h_a$ are all complete vector fields. Therefore, it follows [21, Chapter 4] that Δ and, for each s , L_s (specifically, a closure) generate a strongly continuous contraction semigroup on the space of bounded continuous functions vanishing at infinity, $C_0(N(n))$, considered as a Banach space with sup norm. I now wish to apply the following theorem, which is in Kato's book [12, p. 502].

THEOREM. *Let T and $\{T_n\}_{n=1}^\infty$ generate contraction semigroups. If $(T_n - \lambda)^{-1}$ converges strongly to $(T - \lambda)^{-1}$ for some λ with $\operatorname{re}(\lambda) > 0$, then $\exp(tT_n)$ converges strongly to $\exp(tT)$ uniformly for t in any finite interval.*

To apply this theorem to L_s and Δ , I will need to prove some lemmas.

LEMMA 1. *The coefficients of ξ_s approach zero as $s \rightarrow -\infty$ uniformly on compact subsets of $N(n)$.*

PROOF. We can represent ξ_0 as a sum

$$\sum_{i,j} f^{ij}(X) u^{ij}(X) \frac{\partial}{\partial X^i} \frac{\partial}{\partial X^j} + \sum_k f^k(X) u^k(X) \frac{\partial}{\partial X^k},$$

where $i, j = 1, \dots, n(n+1)/2$, u^{ij} and u^k are smooth functions of compact support and the terms

$$f^{ij} \frac{\partial}{\partial X^i} \frac{\partial}{\partial X^j} \quad \text{and} \quad f^k \frac{\partial}{\partial X^k}$$

(no sum) are homogeneous with polynomial coefficients of orders d_{ij} and d_k greater than or equal to -1 . Thus, the ij th and k th coefficients of ξ_s are

$$\exp[s(d_{ij} + 2)] u^{ij}(D(s)^* X) f^{ij}(X)$$

and

$$\exp[s(d_k + 2)] u^k(D(s)^* X) f^k(X).$$

Now, $u(D(s)^* X)$ is $u(e^s x, e^{2s} y)$ and so approaches the constant function $u(0)$ uniformly on compact domains as $s \rightarrow -\infty$, since u is a continuous function. But the f 's are polynomials and therefore bounded on compact domains while $\exp[s(d+2)]$ goes to zero as s goes to $-\infty$. Q.E.D.

LEMMA 2. *For all functions $g \in C_c^\infty$, $\lim_{s \rightarrow \infty} \|\xi_s g\| = 0$.*

PROOF. All derivatives of g are again in C_c^∞ . But the coefficients of ξ_s approach zero uniformly on compact domains, particularly on $\operatorname{supp}(g)$. Q.E.D.

LEMMA 3. *C_c^∞ is dense in the domain $\mathcal{D}(\Delta)$ of Δ when $\mathcal{D}(\Delta)$ is considered as a Banach space with norm $\|\cdot\| + \|\Delta \cdot\|$.*

PROOF. For $\lambda > 0$, $(\Delta - \lambda)$ is an isomorphism of $\mathcal{D}(\Delta)$ with C_0 . It suffices to show that $(\Delta - \lambda)C_c^\infty$ is dense in C_0 . Suppose the contrapositive. Then there exists a measure μ of finite nonzero variation such that integration against μ annihilates the image of C_c^∞ , i.e., for all $g \in C_c^\infty$, $\langle \mu, (\Delta - \lambda)g \rangle = 0$. Now, let $R(X)$ denote right translation by $X \in N(n)$. Since right translation maps C_c^∞ to C_c^∞ and leaves the sup norm invariant, it follows that for every $g \in C_c^\infty$, $u(X) = \langle \mu, R(X)g \rangle$ is

a bounded continuous function on $N(n)$. On the other hand, u is also a weak solution of $(\Delta - \lambda)u = 0$; hence, by hypoellipticity it is a pointwise solution, so by the maximum principle $u \equiv 0$ [11]. But this implies that $\mu = 0$. Q.E.D.

LEMMA 4. *Strong- $\lim_{s \rightarrow -\infty} (L_s - \lambda)^{-1} = (\Delta - \lambda)^{-1}$.*

PROOF. We have formally

$$\frac{1}{L_s - \lambda} = \frac{1}{\Delta - \lambda} - \frac{1}{L_s - \lambda} (L_s - \Delta) \frac{1}{\Delta - \lambda}.$$

It is obvious that this equation is true on the domain $(\Delta - \lambda)C_c^\infty$, and on the domain

$$\begin{aligned} \frac{1}{L_s - \lambda} (L_s - \Delta) \frac{1}{\Delta - \lambda} &= L_s \frac{1}{L_s - \lambda} \frac{1}{\Delta - \lambda} - \frac{1}{L_s - \lambda} \Delta \frac{1}{\Delta - \lambda} \\ &= \left\{ 1 + \frac{\lambda}{L_s - \lambda} \right\} \frac{1}{\Delta - \lambda} - \frac{1}{L_s - \lambda} \left\{ 1 + \frac{\lambda}{\Delta - \lambda} \right\}. \end{aligned}$$

However, this last operator is bounded and has norm less than $4/\text{re}(\lambda)$ because L_s and Δ generate contraction semigroups. Since, by Lemma 3, $(\Delta - \lambda)C_c^\infty$ is dense in C_0 , it follows that $(L_s - \lambda)^{-1}(L_s - \Delta)(\Delta - \lambda)^{-1}$ extends uniquely to a bounded operator on C_0 with the same norm.

Now, let $g \in C_0$, and let $\{f_n\}_{n=1}^\infty$ in $(\Delta - \lambda)C_c^\infty$ be such that $f_n \rightarrow g$ in C_0 . For all $\varepsilon > 0$, let $n^* \in \mathbb{Z}^+$ be such that for all $n > n^*$, $\|f_n - g\| < (\varepsilon \text{re}(\lambda)/8)$. Then for all s ,

$$\begin{aligned} \left\| \frac{1}{L_s - \lambda} (L_s - \Delta) \frac{1}{(\Delta - \lambda)} g \right\| &\leq \left\| \frac{1}{L_s - \lambda} (L_s - \Delta) \frac{1}{\Delta - \lambda} f_n \right\| \\ &\quad + \left\| \frac{1}{L_s - \lambda} (L_s - \Delta) \frac{1}{\Delta - \lambda} (f_n - g) \right\| \\ &\leq \left\| \frac{1}{L_s - \lambda} (L_s - \Delta) \frac{1}{\Delta - \lambda} f_n \right\| + \frac{\varepsilon}{2} \\ &\leq \frac{1}{\text{re}(\lambda)} \|(L_s - \Delta)\tilde{f}_n\| + \frac{\varepsilon}{2}, \end{aligned}$$

where $\tilde{f}_n = (\Delta - \lambda)^{-1}f_n \in C_c^\infty$. By Lemma 2, $\|(L_s - \Delta)\tilde{f}_n\|$ goes to zero as s goes to $-\infty$; hence we can choose s^* sufficiently small such that, for all $s < s^*$, $\|(L_s - \Delta)\tilde{f}_n\| < (\varepsilon \text{re}(\lambda)/2)$. Q.E.D.

If we combine Lemma 4 with Kato's theorem we have

THEOREM 1. *Strong- $\lim_{s \rightarrow -\infty} \exp(tL_s) = \exp(t\Delta)$ uniformly for t in finite intervals.*

Let $Q(t, W)$ denote the fundamental solution of our model diffusion equation $(\partial_t - \Delta)f = 0$.

COROLLARY 1. *As $s \rightarrow -\infty$, $P_s(t, W) \rightarrow Q(t, W)$ in the weak* topology in C_0^* uniformly for t in nonnegative finite intervals.*

PROOF. Because P_s and Q are the Schwartz kernels for $\exp(tL_s)$ and $\exp(t\Delta)$, the strong convergence of these implies pointwise convergence of $\exp(tL_s)f$ to

$\exp(t\Delta)f$ for every $f \in C_0$, and by definition of the Schwartz kernel, of P_s to Q in the weak* topology in C_0^* uniformly for t in nonnegative finite intervals. Q.E.D.

C. *A parametriz.* Now, we have that

$$(\partial_t - L_s^*)P_s(t, W) = 0, \quad (\partial_t - \Delta^*)Q(t, W) = 0,$$

$$\lim_{t \rightarrow 0} P_s(t, W)(\lambda) = \lim_{t \rightarrow 0} Q(t, W)(X) = \delta_W(X).$$

Thus, for $t > 0$, we can subtract the P.D.E. for Q from the P.D.E. for P_s , and rearrange to get

$$(\partial_t - \Delta)(P_s - Q)(t, W) = (L_s - \Delta)P_s(t, W).$$

Therefore, since $P_s = Q + (P_s - Q)$, we can formally write

$$(P_s - Q) = \sum_{m=1}^{\infty} [(\partial_t - \Delta)^{-1}(L_s - \Delta)]^m Q.$$

If in this series we substitute the expansion of $(L_s - \Delta)$ in terms homogeneous under $D(s)$, we then have, again formally, a parametriz for P_s in terms which are homogeneous under $\tilde{D}(s)$. Thus, we are motivated to study the functional analysis involved with hypoelliptic diffusion equations in order to clarify in what sense these formal expansions might have validity.

From now on I use the same symbol for the density of Q and P_s as for Q and P_s themselves. First of all, $Q(t, W, X)$ is defined on $N(n) \times [0, \infty)$ for each W . Extend Q by the zero function to the domain $N(n) \times R$, and denote this extended function again by Q . This function is smooth everywhere except at $t = 0$, $W = X$. In fact, recall that in the distributional sense

$$(\partial_t - \Delta(X)^*)Q(t, W, X) dX dt = \delta_W(X) \delta_0(t).$$

Therefore, $(\partial_t - \Delta)^{-1}$ is an integral operator with kernel Q . Also, $(\partial_t - \Delta)^{-1}$ is homogeneous under $\tilde{D}(s)^*$ of order two. Thus, in order to understand our parametriz, we want to understand Q and, hence, $\exp(t\Delta)$ in the context of distributions.

Let $\mathcal{S}$ be the Schwartz class of functions on $N(n)$. Recall that $\mathcal{S}$ is a Fréchet space with respect to the following set of seminorms: for all $k \in Z^+$, and for $f \in \mathcal{S}$,

$$\|f\|_k = \sum_{I, J \leq k} \sup |X^I V^J f(X)|,$$

where I and J are multi-indices and V^J is a polynomial in left-invariant vector fields on $N(n)$ ordered according to some fixed choice of ordered basis. Note that $\|f\|_k \leq \|f\|_{k+1}$ for all f and k . Recall that a sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{S}$ converges in $\mathcal{S}$ if it is Cauchy with respect to $\|\cdot\|_k$ for all k .

Let T be an operator mapping $\mathcal{S}$ to $\mathcal{S}$. Then T is continuous or bounded if for every k there is some $j \in Z^+$ and some $c \in R^+$ such that, for all $f \in \mathcal{S}$, $\|Tf\|_k \leq c\|f\|_j$. Define $\|T\|_{k,j}$ to be the infimum of such c . Note that $\|T\|_{k-a, j+b} \leq \|T\|_{k,j}$ for all a, b in Z^+ .

LEMMA 5. *For each $t \geq 0$, $\exp(t\Delta)$ is a continuous map $\mathcal{S} \rightarrow \mathcal{S}$, and in fact there are constants $c_1, c_2 > 0$ which depend only on k such that $\|\exp(t\Delta)f\|_k \leq c_1 e^{c_2 t} \|f\|_k$.*

PROOF. The proof is based on Proposition 3.2 of [11], which in this situation states that $\exp(t\Delta)$ defines a strongly continuous semigroup on B_k (= the closure

of $\mathcal{S}$ with respect to the norm $\|\cdot\|_k$ if left translation on $N(n)$ acts strongly continuously on B_k . Thus, it is sufficient to prove that left translation is strongly continuous on B_k .

For $\zeta \in N(n)$, let

$$\zeta = \exp \left(\sum_{a=1}^n t^a V_a + \sum_{a < b} t^{ab} \frac{\partial}{\partial y^{ab}} \right).$$

Since $\exp$ is a global diffeomorphism for $N(n)$, the choice of a basis $(V_a, \partial/\partial y^{ab})$ determines uniquely the numbers $\{t^a, t^{ab}\}$. Thus,

$$|\zeta| = \sum_a |t^a| + \sum_{a < b} |t^{ab}|$$

defines a function on $N(n)$ such that $(\zeta, \xi) \rightarrow |\zeta \xi^{-1}|$ is a distance function.

Let $U(\zeta)f$ denote left translation of $f \in B_k$ by $\zeta \in N(n)$; i.e.,

$$[U(\zeta)f](\xi) = f(\zeta^{-1}\xi).$$

The group multiplication law for $N(n)$ gives us the estimates

$$\begin{aligned} |U(\zeta)x^a| &\leq |x^a| + |t^a|, \\ |U(\zeta)y^{ab}| &\leq |y^{ab}| + |t^{ab}| + |t^a x^b - t^b x^a|. \end{aligned}$$

Hence,

$$(*)1 \quad |U(\zeta)x^a| \leq e^{|t^a|}(1 + |x^a|) \leq e^{|\zeta|}(1 + |x^a|),$$

$$\begin{aligned} (*)2 \quad |U(\zeta)y^{ab}| &\leq e^{|t^{ab}|}(1 + |y^{ab}|) + |t^a| |x^b| + |t^b| |x^a| \\ &\leq e^{|\zeta|}(1 + |y^{ab}| + |x^b| + |x^a|). \end{aligned}$$

Therefore, we have for $f \in B_k$

$$\sup |X^I V^J U(\zeta)f| = \sup |X^I U(\zeta) V^J f|$$

(since the V^J are left invariant)

$$= \sup |U(\zeta)([U(\zeta^{-1})X^I]V^J f)| = \sup |[U(\zeta^{-1})X^I]V^J f|$$

(since $\sup$ is invariant under translation)

$$= \sup (|U(\zeta^{-1})X^I| |V^J f|).$$

Now, since translation is an automorphism of the ring of continuous functions, we have, for $N = n(n+1)/2$,

$$|U(\zeta^{-1})(X^1)^{i_1}(X^2)^{i_2} \dots (X^N)^{i_N}| \leq e^{k|\zeta|} \sum_{|I| < k} |X^I|,$$

so

$$\sup |X^I V^J U(\zeta)f| \leq e^{k|\zeta|} \|f\|_k.$$

Thus, there is a constant $c > 0$ such that

$$\|U(\zeta)f\|_k \leq c e^{k|\zeta|} \|f\|_k.$$

Now, for all $f \in \mathcal{S}$ and all multi-indices I and J , $X^I V^J f \in \mathcal{S}$ and so vanishes at infinity. Thus, if the sequence $\{f_n\}_{n=0}^\infty \subset \mathcal{S}$ is Cauchy in B_k , $\{X^I V^J f_n\}_{n=0}^\infty$ is

Cauchy in C_0 if $|I|, |J| \leq k$, so for each $f \in B_k$ and each $\varepsilon > 0$ there is a compact set $K(\varepsilon) \subset N(n)$ such that

$$\sum_{|I|, |J| \leq k} \sup_{x \notin K(\varepsilon)} |X^I V^J f| < \varepsilon.$$

Let $u^\varepsilon \in C_c^\infty$ be such that $0 \leq u^\varepsilon \leq 1$ and $u^\varepsilon|_{K(\varepsilon)} \equiv 1$. Then

$$\begin{aligned} (*3) \quad \| [U(\zeta) - 1]f \|_k &\leq \sum_{|I|, |J| \leq k} \sup |X^I [U(\zeta) - 1] u^\varepsilon V^J f| \\ &\quad + \sum_{|I|, |J| \leq k} \sup |X^I [U(\zeta) - 1] (1 - u^\varepsilon) V^J f|. \end{aligned}$$

But by definition of $K(\varepsilon)$, the second sum is less than $\varepsilon [C e^{k|\zeta|}]$, where C is independent of ε .

On the other hand, $u^\varepsilon V^J f$ is continuous and of compact support and, hence, also in C_0 , and left translations are strongly continuous on C_0 . Also, estimates (*1) and (*2) imply that there is a compact set $K \subset N(n)$ such that the support of $U(\zeta) u^\varepsilon V^J f$ is contained in K for all $|\zeta|$ sufficiently small. The strong continuity of left translations on C_0 then implies that for all $\varepsilon' > 0$ and for $|\zeta|$ sufficiently small

$$\sum_{|I|, |J| \leq k} \sup \| [U(\zeta) - 1] u^\varepsilon V^J f \| < \varepsilon',$$

so

$$\sum_{|I|, |J| \leq k} \sup |X^I [U(\zeta) - 1] u^\varepsilon V^J f| \leq \varepsilon' \sup_{x \in K, |I| \leq k} |X^I|.$$

This means that we can make the second term on the right of (*3) arbitrarily small for all $|\zeta|$ smaller than some fixed value by choosing ε small enough. Even though this choice of ε may make the first term on the right of (*3) large for a fixed value of $|\zeta|$, the first term may also be made arbitrarily small by choosing $|\zeta|$ small enough. Thus,

$$\lim_{|\zeta| \rightarrow 0} \| [U(\zeta) - 1]f \|_k = 0.$$

Thus, indeed left translation is strongly continuous on B_k , so according to Proposition 3.2 of [11] the measure $Q(t, 0)$ has the property that $\int dQ(t, 0, \zeta) U(\zeta)$ is a strongly continuous semigroup on B_k . But this integral is just the semigroup $e^{t\Delta}$ on C_0 restricted to B_k .

Since $e^{t\Delta}$ is strongly continuous, it follows that there are constants $c_1, c_2 > 0$ such that $\|e^{t\Delta} f\|_k < c_1 e^{c_2 t} \|f\|_k$ for all $f \in B_k$ and, hence, for all $f \in \mathcal{S}$. Since this is true for all k , it follows that $e^{t\Delta}$ is a continuous operator on $\mathcal{S}$. Q.E.D.

LEMMA 6. *For all $f \in \mathcal{S}$, the map $t \rightarrow \exp(t\Delta)f$ of $[0, \infty) \rightarrow \mathcal{S}$ is continuous.*

PROOF. The proof of Lemma 5 shows that the semigroup $e^{t\Delta}$ on C_0 restricts to strongly continuous semigroups on B_k for each k . Thus, for $f \in \mathcal{S} = \bigcap_k B_k$, $e^{t\Delta} f$ is continuous with respect to the seminorm $\|\cdot\|_k$. Q.E.D.

LEMMA 7. *Let f be a smooth function on $N(n) \times R$ of the following type: $t \rightarrow f(t, \cdot)$ is a continuous map into $\mathcal{S}$ and $f(t, \cdot)$ is zero in $\mathcal{S}$ for t sufficiently*

small (sufficiently large). Then $(\partial_t - \Delta)^{-1}f$ (respectively, $(\partial_t + \Delta)^{-1}f$) exists and is a function of the same type.

PROOF. It is sufficient to prove the $(\partial_t - \Delta)$ case since the other is related by $t \rightarrow -t$. Now $\|\exp(t\Delta)f\|_k$ is continuous in t , so $\|\exp[(t-v)\Delta]f(v, \cdot)\|_k$ is continuous and, hence, measurable in v . Thus, according to Yoshida [22, p. 134], $\exp[(t-v)\Delta]f(v, \cdot)$ is Bochner integrable in the Banach space B_k (the completion of $\mathcal{S}$ with respect to $\|\cdot\|_k$). But this is true for all $k \in \mathbb{Z}^+$, so the integral is in $\mathcal{S}$. Also, as an $\mathcal{S}$ -valued function the integral

$$X^I V_J \int_{-\infty}^t dv \exp[(t-v)\Delta]f(v, \cdot) = \int_{-\infty}^t dv X^I V_J \exp[(t-v)\Delta]f(v, \cdot)$$

is in $\mathcal{S}$ for each t in R , since $X^I V_J$ maps B_k into $B_{k+\max(|I|, |J|)}$.

In the above integrals we consider $\exp[(t-v)\Delta]$ to be zero for $v > t$, so the above integrals are zero in $\mathcal{S}$ for t sufficiently small. The fact that $(\partial_t - \Delta)$ can be moved across the integral sign implies that $(\partial_t - \Delta)^{-1}$ is defined on this class of functions by the above Bochner integral and that Q is its Schwartz kernel. The hypoellipticity of $(\partial_t - \Delta)$ implies smoothness. Q.E.D.

REMARK. $(\partial_t - \Delta)^{-1}$ is "bounded on B_k " in the sense that for each continuous function $f: R \rightarrow B_k$, such that $f(s) = 0$ for all $s < t_0$ for some $t_0 \in R$, the function $t \rightarrow \|[(\partial_t - \Delta)^{-1}f](t)\|_k$ is a locally bounded function of R . Indeed,

$$\begin{aligned} \|[(\partial_t - \Delta)^{-1}f](t)\|_k &= \left\| \int_{-\infty}^t dv e^{(t-v)\Delta} f(v) \right\|_k \\ &\leq \int_{-\infty}^t dv \|e^{(t-v)\Delta} f(v)\|_k \\ &\leq \int_{s_0}^t dv c_1 e^{c_2(t-v)} \|f(v)\|_k \end{aligned}$$

(by Lemma 5, for $t > s_0$)

$$\leq c_1 e^{c_2(t-s_0)} \int_{s_0}^t dv \|f(v)\|_k.$$

We are now in a position to prove

LEMMA 8. For every $N' > 0$, in a (tempered) distributional sense P_s is equal to

$$\sum_{N=0}^{N'} [(\partial_t - \Delta^*)^{-1}(L_s - \Delta)^*]^N Q + [(\partial_t - \Delta^*)^{-1}(L_s - \Delta)^*]^{N'} (P_s - Q).$$

PROOF. For $f \in C_c^\infty(N(n) \times R)$,

$$\langle f, P_s - Q \rangle = \langle f, (\partial_t - \Delta^*)^{-1} \xi_s^* P_s \rangle = -\langle \xi_s (\partial_t + \Delta)^{-1} f, P_s \rangle,$$

where the extra signs are because ∂_t is skew symmetric with respect to $dX dt$. Note that $\xi_s (\partial_t + \Delta)^{-1} f$ vanishes for t sufficiently large, and P_s vanishes in $\mathcal{S}'$ for $t < 0$, so the last integral makes sense. But $P_s = Q + (P_s - Q)$, so by Lemma 7 it makes sense to iterate. Q.E.D.

LEMMA 9. For all $f \in \mathcal{S}$, $\lim_{s \rightarrow -\infty} e^{-s} \xi_s f = L^{(-1)} f$ strongly in $\mathcal{S}$.

PROOF. Recall the notation of Lemma 1. I claim that the operator of multiplication by $D(s)^* u$ ($u = u^{ij}$ or u^k) converges strongly to multiplication by the constant $u(0)$. Indeed,

$$(*) \quad \|X^I V_J u(D(s)^* X) f\| \leq \sum_{K \leq J} C_K \|X^I [V_{J-K} u(D(s)^* X)] V_K f\|,$$

where by $K \leq J$, I mean that every component of the multi-index K is less than or equal to the corresponding component of the multi-index J . But, $V_{J-K} D(s)^* u = e^{-ds} D(s)^* [V_{J-K} u]$, where d is the homogeneity of V_{J-K} and is a nonpositive number, zero only in the case that $J = K$. Thus, since $[D(s)^* V_{J-K} u](X)$ is uniformly bounded in s and X , the same reasoning as in Lemma 1 gives us that $V_{J-K} u(D(s)^* X)$ converges uniformly to zero when J is not equal to K . In this case, therefore, $X^I [V_{J-K} u(D(s)^* X)] V_K f \rightarrow 0$ as $s \rightarrow -\infty$. Thus in the limit of $s \rightarrow -\infty$, $X^I V_J [u(D(s)^* X) f]$ is the same as the limit of $u(D(s)^* X) [X^I V_J f]$, which, as in the proof of Lemma 1, converges uniformly on compacta. But uniform convergence on compacta in C_0 implies uniform convergence: Since $u(D(s)^* X)$ is uniformly bounded in s and X , for each f and for every $\varepsilon > 0$ there is a compact subset E such that on E^c , $|u^{ij}(D(s)^* X) f| < \varepsilon$ uniformly in s and in s^* such that, for $s < s^*$, $|u(D(s)^* X) - u(0)| < \varepsilon$ uniformly on E . This proves my claim.

Now, as in the proof of Lemma 1, the coefficients of ξ_0 are $f^{ij} u^{ij}$ and $f^k u^k$, where $f^{ij} \partial_i \partial_j$ and $f^k \partial_k$ are homogeneous of order d_{ij} , d_k greater than or equal to -1 . When $d = -1$, $e^{-s} \exp[s(d+2)]$ is independent of s . When $d > -1$, this exponential goes to zero as $s \rightarrow -\infty$. The assertion follows as in Lemma 1. Q.E.D.

LEMMA 10. $s \rightarrow D(s)^*$ is a strongly continuous group in $\mathcal{S}$.

PROOF.

$$\begin{aligned} \|X^I V_J D(s)^* f\| &= \exp[(d_I + d_J)s] \|D(s)^* X^I V_J f\| \\ &= \exp[(d_I + d_J)s] \|X^I V_J f\| \end{aligned}$$

since sup norm is invariant under dilations, where the d_I and d_J are the homogeneities of X^I and V_J , respectively. Note that this implies $\|D(s)^* f\|_k < C e^{k|s|} \|f\|_k$ for some $C > 0$.

In addition,

$$\begin{aligned} \|X^I V_J [1 - D(s)^*] f\| &= \|[1 - \exp[s(d_I + d_J)]] D(s)^* X^I V_J f\| \\ &\leq \|(1 - \exp[(d_I + d_J)s]) X^I V_J f\| \\ &\quad + \exp(d_I + d_J)s \|[1 - D(s)^*] X^I V_J f\|. \end{aligned}$$

But $X^I V_J f \in C_0$, and dilations are strongly continuous on C_0 . Q.E.D.

LEMMA 11. Let $f \in C_c^\infty(N(n) \times R)$. Then $[(L_s - \Delta)(\partial_t + \Delta)^{-1}]^n f = O(e^{ns})$ at $s = -\infty$, and, in particular,

$$\lim_{s \rightarrow -\infty} [e^{-s} (L_s - \Delta)(\partial_t + \Delta)^{-1}]^n f = [L^{(-1)}(\partial_t + \Delta)^{-1}]^n f$$

for all $n \in \mathbb{Z}^+$.

PROOF. For each t , we have the following k -norm estimate on $\mathcal{S}(N(n))$:

$$\begin{aligned} & \| [L^{(-1)}(\partial_t + \Delta)^{-1}]^n f - [e^{-s}(L_s - \Delta)(\partial_t + \Delta)^{-1}]^n f \|_k \\ & \leq \sum_{m=0}^{n-1} \| [L^{(-1)}(\partial_t + \Delta)^{-1}]^m [L^{(-1)} - e^{-s}\xi_s] \\ & \quad \times (\partial_t + \Delta)^{-1} [e^{-s}\xi_s(\partial_t + \Delta)^{-1}]^{n-m-1} f \|_k \\ & \leq C \sum_{m=0}^{n-1} \| [L^{(-1)} - e^{-s}\xi_s](\partial_t + \Delta)^{-1} \\ & \quad \times [e^{-s}\xi_s(\partial_t + \Delta)^{-1}]^{n-m-1} f \|_{k+2m} \end{aligned}$$

for some $C > 0$ since, in the terminology of Lemma 7, $L^{(-1)}$ maps B_{k+2} into B_k and $(\partial_t + \Delta)^{-1}$ maps B_k to B_k for each k and each t .

Now, Lemma 10 implies (because ξ_0 is second order of compact support) that, for each s and for some $p > 2$, $\|e^{-s}\xi_s\|_{k,k+p} < \infty$. I claim that p can be chosen so that this operator norm is uniformly bounded for $s < 0$. Indeed, in the notation of Lemmas 1 and 9, for $g \in \mathcal{S}$,

$$\begin{aligned} \|e^{-s}\xi_s g\|_k &= e^{-s} \left\| \sum_{ij} [D(s)^* u^{ij} f^{ij}] \partial_i \partial_j g \right\|_k \\ &\leq \sum_{ij} \exp[(d_{ij} + 1)s] \| [D(s)^* u^{ij}] f^{ij} \partial_i \partial_j g \|_k, \end{aligned}$$

where d_{ij} is the homogeneity of $f^{ij} \partial_i \partial_j$ (for the sake of brevity, we omit the terms of the form $[D(s)^* u^i f^i] \partial_i$; the estimates are of exactly the same form). But $f^{ij} \partial_i \partial_j g$ is in $\mathcal{S}$, so, as in the proof of Lemma 9, inequality (*) holds. Thus, we have that each term in the above sum is less than a sum over I and J of terms of the form

$$\begin{aligned} & \| X^I V_J [D(s)^* u^{ij}] f^{ij} \partial_i \partial_j g \| \\ & \leq C \sum_{K \leq J} e^{-sd_{J-K}} \| [D(s)^* (V_{J-K} u^{ij})] X^I V_K f^{ij} \partial_i \partial_j g \|. \end{aligned}$$

But each term in this sum is

$$\begin{aligned} & \leq e^{-sd_{J-K}} \| D(s)^* (V_{J-K} u^{ij}) \| \| X^I V_K f^{ij} \partial_i \partial_j g \| \\ & \leq e^{-sd_{J-K}} \| V_{J-K} u^{ij} \| \| X^I V_K f^{ij} \partial_i \partial_j g \|. \end{aligned}$$

But since the u^{ij} are in C_c^∞ , there is a constant $C_J > 0$ such that

$$\| X^I V_J D(s)^* [u^{ij} f^{ij} \partial_i \partial_j g] \| < C_J \sum_{K \leq J} \| X^I V_K f^{ij} \partial_i \partial_j g \|,$$

since $\exp(-sd_{J-K})$ and $\exp[s(d_{ij} + 1)]$ are ≤ 1 for $s < 0$. But each component of K is less than the corresponding component of J , so the sum over I and J gives, for $s < 0$,

$$\| e^{-s} [D(s)^* (u^{ij} f^{ij})] \partial_i \partial_j g \|_k \leq C \| f^{ij} \partial_i \partial_j g \|_k,$$

for some $C > 0$. Since f^{ij} are polynomials, there is an integer $p > 1$ such that, for all i, j , $\| f^{ij} \partial_i \partial_j g \|_k < C_{ij} \| g \|_{k+p}$ for some constants C_{ij} sufficiently large. Thus,

for each k there is a constant C_k sufficiently large such that $\|e^{-s}\xi_s g\|_k < C_k \|g\|_{k+p}$. This proves my claim.

Therefore, by substituting $e^{-s}\xi_s = L^{(-1)} + (e^{-s}\xi_s - L^{(-1)})$, we can bound

$$\|[L^{(-1)} - e^{-s}\xi_s](\partial_t + \Delta)^{-1}[e^{-s}\xi_s(\partial_t - \Delta)^{-1}]^{n-m-1}f\|_{k+2m}$$

by a sum of terms, each of the form

$$\begin{aligned} & \|[q \text{ factors in } L^{(-1)} \text{ or } e^{-s}\xi_s \text{ times } (\partial_t + \Delta)^{-1}], \\ & \quad [L^{(-1)} - e^{-s}\xi_s](\partial_t + \Delta)^{-1}[L^{(-1)}(\partial_t + \Delta)^{-1}]^{n-m-1-q}f\|_{k+2m} \\ & < C_q \|[L^{(-1)} - e^{-s}\xi_s](\partial_t + \Delta)^{-1}[L^{(-1)}(\partial_t + \Delta)^{-1}]^{n-m-1-q}f\|_{k+pq} \end{aligned}$$

for some $C_q > 0$. But for each t , $[L^{(-1)}(\partial_t + \Delta)^{-1}]^{n-m-1-q}f \in \mathcal{S}$, so by Lemma 9 this quantity goes to zero as $s \rightarrow -\infty$. Q.E.D.

COROLLARY 2. (a) $[(\partial_t - \Delta^*)^{-1}\xi_s^*]^n Q$ is $O(e^{ns})$ in $\mathcal{S}'$ at $s = -\infty$.

(b) $[(\partial_t - \Delta^*)^{-1}\xi_s^*]^n (P_s - Q)$ is $O(e^{(n+1)s})$ in $\mathcal{S}'$ at $s = -\infty$.

PROOF. (a) Duality as in the proof of Lemma 8 applied to Lemma 11.

(b) Duality again, with Lemma 11 and Corollary 1 give us that $[(\partial_t - \Delta^*)^{-1}\xi_s^*]^n \times (P_s - Q)$ is $o(e^{ns})$. This plus the expansion of Lemma 8 implies that $(P_s - Q)$ is $O(e^s)$, and, hence, that $[(\partial_t - \Delta^*)^{-1}\xi_s^*]^n (P_s - Q)$ is $O(e^{(n+1)s})$. Q.E.D.

REMARK. The positive t -axis is an invariant set for $D(s)$. Also, t is a homogeneous function of order 2. This implies, in particular, that

$$P_s(t, 0, 0) = e^{n^2 s} P(e^{2s} t, 0, 0).$$

COROLLARY 3. $P(t, 0, 0) \sim Q(t, 0, 0) + O[t^{(1-n^2)/2}]$.

PROOF. The expansion in Lemma 8 equals $Q(t, 0, 0) + O[t^{1/2}Q(t, 0, 0)]$. Q.E.D.

LEMMA 12. Suppose that T is a second-order partial differential operator of compact support. Suppose that T has a formal power series at zero in terms homogeneous of order $\geq q$, under $D(s)$, where q is the biggest such integer. Then $[e^{qs}D(s)^*TD(-s)^*]$ converges strongly on $\mathcal{S}$ to $T^{(q)}$ (where $T^{(q)}$ is the polynomial coefficient partial differential operator equal to the part of the formal series for T which is homogeneous of order q).

PROOF. The same as Lemma 9. Q.E.D.

LEMMA 13. Let $T_1, \dots, T_N$ be operators of orders $q_1, \dots, q_N$, as in Lemma 12. Then as $s \rightarrow -\infty$,

$$D(s)^* \left\{ \prod_{i=1}^n [e^{(q_i+2)s} T_i (\partial_t + \Delta)^{-1}] \right\} D(-s)^*$$

converges strongly on $\mathcal{S}$ to $\prod_{i=1}^n T^{(q_i)} (\partial_t + \Delta)^{-1}$.

PROOF. As in Lemma 11. Q.E.D.

Now recall that $L = \sum (V_a + \phi h_a)^2$. Since $\phi = 1$ on a neighborhood of zero, it follows that for all $q \in \mathbb{Z}^+$, $L = \Delta + L_{-1} + L_0 + L_1 + \dots + L_q + L^{q+1}$. Here, for $k = 1, \dots, q$, L_k is of the form $u_k L_k^{(k)}$, with u_k a smooth function of compact support which is identically one on a neighborhood of zero, $L_k^{(k)}$ is homogeneous of degree k and L^{q+1} has a formal power series consisting of terms homogeneous of order $\geq (q+1)$. We have the following lemma.

LEMMA 14. For all $n \in \mathbb{Z}^+$, $D(s)^* L_k D(s)^* = e^{ks} L_k^{(k)} + O(e^{ns})$ strongly in $\mathcal{S}$ at $s = -\infty$.

PROOF. $L_k = L_k^{(k)} - (1 - u_k) L_k^{(k)}$, so

$$D(s)^* L_k D(-s)^* = e^{ks} [L_k^{(k)} - (1 - D(s)^* u_k) L_k^{(k)}].$$

But $L_k^{(k)}$ is a bounded operator on $\mathcal{S}$, so it suffices to show that $(1 - D(s)^* u_k)$ vanishes strongly to infinite order as an operator on $\mathcal{S}$.

Now,

$$\begin{aligned} X^I V_J [D(s)^* (1 - u_k)] f \\ = \exp[-s(d_I + d_J)] D(s)^* X^I V_J (1 - u_k) D(-s)^* f. \end{aligned}$$

But $X^I V_J (1 - u_k)$ vanishes on a neighborhood, call it U , of zero. The statement follows since U is open and f restricted to $D(s)U^c$ vanishes to infinite order at $s = -\infty$, since $f(e^s x, e^{2s} y)$ vanishes to infinite order in $\{e^s\}$ as $s \rightarrow -\infty$. Q.E.D.

LEMMA 15. For each $N \in \mathbb{Z}^+$, $[\xi_s(\partial_t + \Delta)^{-1}]^N$ is strongly asymptotic to a sum of terms homogeneous of integer order greater than N . In addition, there are only a finite number which are homogeneous of a given order.

PROOF. In light of Lemma 14, the expansion of ξ_s implied by Lemma 9 and the discussion before Lemma 14 extends strongly to a formal power series about $s = -\infty$ (we consider ξ_s as an operator-valued function of s). The coefficients of this expansion are the $L_k^{(k)}$, which are homogeneous. $(\partial_t + \Delta)^{-1}$ is also homogeneous, so $[L_k^{(k)}(\partial_t + \Delta)^{-1}]$ is homogeneous, as is any product of such factors. The homogeneity of such a product is the sum of the homogeneities of the factors. There are only a finite number of ways to add together positive integers to obtain a given integer or to order a given finite set of positive integers. Q.E.D.

THEOREM 2. P_s has an asymptotic expansion about $s = -\infty$, in terms of the $L_K^{(k)}$, $(\partial_t - \Delta)^{-1}$ and Q (i.e., a parametrix for P_s).

PROOF. Consider the expansion of Lemma 8. According to Corollary 2 the term $[(\partial_t - \Delta^*)^{-1}(L_s - \Delta)^*]^{N'}(P_s - 0)$ is $O(e^{(N'+1)s})$. Lemma 15 gives us that every other term in the expansion of Corollary 2 is asymptotic to a finite sum of terms which are homogeneous of order no more than order N' , plus a term which is $O(e^{(N'+1)s})$. But this is true for every $N' \in \mathbb{Z}^+$. Q.E.D.

COROLLARY 4. $P(t, 0, 0)$ has an asymptotic expansion in powers of $t^{1/2}$ with lowest-order term $t^{-n^2/2}$.

PROOF. The function t is homogeneous of order 2. P has an asymptotic expansion in terms of distributions homogeneous of order q for all q bigger than or equal to $-n^2$. Q.E.D.

THEOREM 3. The coefficient of $t^{(1-n^2)/2}$ in the asymptotic expansion for $P(t, 0, 0)$ vanishes.

PROOF. The coefficient is $\int_0^t dv dX Q(t-v, X, 0) L^{(-1)} Q(v, 0, X)$. But the operator $L^{(-1)}$ is explicitly represented by

$$\begin{aligned} & V_a A_{ade}^{bc} x^d x^e \frac{\partial}{\partial y^{bc}} + A_{ade}^{bc} x^d x^e \frac{\partial}{\partial y^{bc}} V_a \\ &= 2A_{ade}^{bc} x^d x^e \frac{\partial}{\partial y^{bc}} \frac{\partial}{\partial x^a} + A_{aae}^{bc} x^e \frac{\partial}{\partial y^{bc}} \\ &+ A_{ade}^{bc} x^d x^e x^f \frac{\partial}{\partial y^{bc}} \frac{\partial}{\partial y^{af}} \end{aligned}$$

(sum on repeated indices). Note that every term is antisymmetric with respect to the reflection $x \rightarrow -x$. But Q is invariant under this reflection. Therefore the integrand is antisymmetric under this reflection, and so the integral with respect to X is zero. Q.E.D.

4. The local trace on $\text{SL}(2, R)$. Let us identify the space of all 2×2 real matrices with R^4 . Define $\psi: R^4 \rightarrow \text{gl}(2, R)$ as follows. Let w, x, y, z denote the standard Cartesian coordinates on R^4 . Let g_{ij} , for $ij = 1, 2$, be the matrices with a in the ij position and zero everywhere else. Let f_{ij} be the linear functional on $\text{gl}(2, R)$ given by $f_{ij}(M) = \text{trace}(g_{ij}M)$. Then $\psi^* f_{11} = w + x$, $\psi^* f_{22} = w - x$, $\psi^* f_{12} = y + z$, $\psi^* f_{21} = y - z$.

Consider the differential equation on $\text{gl}(2, R)$

$$\dot{M}(t) = GM = \begin{vmatrix} u & v \\ v & -u \end{vmatrix} M(t),$$

where u and v are real-valued parameters. This amounts to the following differential equation on R^4 :

$$\begin{vmatrix} \dot{w} \\ x \\ y \\ z \end{vmatrix} = \begin{vmatrix} 0 & u & v & 0 \\ u & 0 & 0 & -v \\ v & 0 & 0 & u \\ 0 & -v & u & 0 \end{vmatrix} \begin{vmatrix} w \\ x \\ y \\ z \end{vmatrix},$$

with associated vector fields

$$g_1 = (x\partial_w + w\partial_x + z\partial_y + y\partial_z), \quad g_2 = (y\partial_w - z\partial_x + w\partial_y - x\partial_z).$$

Note that these vector fields each generate a 1-parameter Lie group of diffeomorphisms which is isomorphic to R . Define g_3 to be the commutator $g_3 = [g_2, g_1] = 2(w\partial_z - z\partial_w + y\partial_x - x\partial_y)$. Because G is in $\text{sl}(2, R)$, it follows that the determinant function $\det: \text{gl}(2, R) \rightarrow R$ is independent of time when evaluated on $M(t)$; i.e., $\det$ is annihilated by $\psi_* g_a$, $a = 1, 2, 3$. Thus, we can restrict our attention to a three-dimensional submanifold, S , given by $\det = 1$. But this amounts to the submanifold of R^4 given by $w^2 - x^2 - y^2 + z^2 = 1$. At the point $(1, 0, 0, 0)$ in S the functions x, y, z define a local coordinate system for some neighborhood. Since the ∂_w component of the g 's is zero at this point, we can factor out $w = \sqrt{1 + x^2 + y^2 - z^2}$. In these coordinates the g 's can be written as

$$\begin{aligned} g_1 &= \sqrt{1 + x^2 + y^2 - z^2} \partial_x + z\partial_y + y\partial_z, \\ g_2 &= -z\partial_x + \sqrt{1 + x^2 + y^2 - z^2} \partial_y - x\partial_z, \\ g_3 &= y\partial_z - z\partial_y + \sqrt{1 + x^2 + y^2 - z^2} \partial_z. \end{aligned}$$

We can expand the square root in a power series around the point $x = y = z = 0$ (it is analytic in an open neighborhood) to obtain the vector fields

$$\begin{aligned} g_1 &= \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} + z \frac{\partial}{\partial y} + \frac{1}{2}(x^2 + y^2 - z^2) \frac{\partial}{\partial x} + \text{higher order}, \\ g_2 &= \frac{\partial}{\partial y} - x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x} + \frac{1}{2}(x^2 + y^2 - z^2) \frac{\partial}{\partial y} + \text{higher order}. \end{aligned}$$

Consider a position-dependent rotation of the vector fields g_1 and g_2 :

$$\begin{aligned} \begin{vmatrix} \bar{g}_1 \\ \bar{g}_2 \end{vmatrix} &= \exp \left(z \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \right) \begin{vmatrix} g_1 \\ g_2 \end{vmatrix} \\ &= \begin{vmatrix} g_1 - zg_2 \\ g_2 + zg_1 \end{vmatrix} + \text{second order in } z. \end{aligned}$$

Thus, up to terms homogeneous of order bigger than one, we can write the vector fields as

$$\begin{aligned} \bar{g}_1 &= \left[1 + \frac{1}{2}(x^2 + y^2) \right] \partial_x + (y - xz) \partial_z, \\ \bar{g}_2 &= \left[1 + \frac{1}{2}(x^2 + y^2) \right] \partial_y - (x - yz) \partial_z, \end{aligned}$$

which is of the form given in Brockett's theorem.

One can calculate that

$$\begin{aligned} L &= g_1^2 + g_2^2 = \bar{g}_1^2 + \bar{g}_2^2 + (x\partial_x - y\partial_y) + \text{h.o.} = \Delta + L_0 + \text{h.o.} \\ &= \Delta + (x^2 + y^2)[(\partial_x^2 + \partial_y^2) + (y\partial_x - x\partial_y)\partial_z] \\ &\quad + (x\partial_x - y\partial_y) + (x\partial_x + y\partial_y) + z\partial_z(y\partial_y - x\partial_x) + \text{h.o.} \end{aligned}$$

Now, the $(y\partial_x - x\partial_y)$ term does not contribute to the integral since Q is cylindrically symmetric. Likewise, the $(y\partial_y - x\partial_x)$ term does not contribute, since it is antisymmetric under the transformation $x \leftrightarrow y$, whereas Q is symmetric.

Recall that the Fourier transform of Q is

$$F(Q)(t, k, x, y; 0) = \frac{ke^{-(1/2)k \coth(2kt)[x^2 + y^2]}}{2\pi \sinh(2kt)},$$

and

$$F(L_0) = (x^2 + y^2)(\partial_x^2 + \partial_y^2) + ik(x^2 + y^2)(y\partial_x - x\partial_y) + (x\partial_x + y\partial_y).$$

Recall also that

$$\partial_x^2 \exp[-(c/2)x^2] = [c^2 x^2 - c] \exp[-(c/2)x^2].$$

Therefore

$$\begin{aligned} F(L_0 Q) &= F(L_0)F(Q) \\ &= (x^2 + y^2)[k^2(x^2 + y^2) \coth^2(2kt) - 3k \coth(2kt)]F(Q). \end{aligned}$$

Thus, we need to evaluate the integral

$$\int_0^t dv \int dx dy dz \left\{ \int \left[dk \frac{k e^{-ikz} e^{-(k/2)[\coth[2k(t-v)](x^2+y^2)]}}{4\pi^2 \sinh[2k(t-v)]} \right] \right. \\ \times \left[\int dk' \frac{k' e^{-ik'z} e^{-(k'/2) \coth[2k'v](x^2+y^2)}}{4\pi^2 \sinh[2k(t-v)]} \right. \\ \left. \left. \times (x^2 + y^2)(k'^2 \coth^2(2k'v)(x^2 + y^2) - 3k' \coth(2k'v)) \right] \right\}.$$

Since the integrand vanishes rapidly at infinity in the variables x, y, z, k, k' for t and v bigger than zero, we can first integrate out the z dependence. The $e^{-i(k+k')z}$ integrates to give $2\pi\delta(k+k')$. Thus, we are left with

$$\int_0^t dv \int dk \left\{ \int dx dy \left[\frac{k^4 \coth^2(2kv)(x^2 + y^2) - 3k^3 \coth(2kv)}{8\pi^3 \sinh[2k(t-v)] \sinh(2kv)} \right] \right. \\ \times [(x^2 + y^2) \exp[-(k/2) \coth(2kv)(x^2 + y^2)] \\ \left. \times \exp[-(k/2) \coth[2k(t-v)](x^2 + y^2)] \right] \left. \right\}.$$

Now we integrate with respect to x and y . Recall that $dx dy$ is equal to $r dr d\theta$. The integral with respect to θ gives us a simple factor of 2π . Recall that the integrals of $r^5 e^{-cr^2}$ and $r^3 e^{-cr^2}$ from 0 to ∞ are

$$\int_0^\infty dr r^5 e^{-cr^2} = \frac{1}{c^3}, \quad \int_0^\infty dr r^3 e^{-cr^2} = \frac{1}{2c^2}.$$

Thus we get

$$\int_0^t dv \int dk \left\{ \left[\frac{1}{4\pi^2 \sinh[2k(t-v)] \sinh(2kv)} \right] \right. \\ \times \left[\frac{(6k) \coth[2k(t-v)] \coth(2kv) - (2k) \coth^2(2kv)}{[\coth[2k(t-v)] + \coth(2kv)]^2} \right] \left. \right\}.$$

Now use the fact that $\coth = \cosh/\sinh$, clear the denominator, and use the identity for $\sinh$ of the sum of two angles. The integrand becomes

$$\frac{(3k) \cosh[2k(t-v)] \sinh[2k(t-v)] \sinh(4kv)}{-4\pi^2 \sinh^3(2kt)} \\ - \frac{(2k) \cosh^2(2kv) \sinh^2(2kv)}{-4\pi^2 \sinh^3(2kt)}.$$

If we now apply the identities for the $\sinh$ and $\cosh$ of the sum of two angles to the numerator, we see that it can be written as

$$(k/2)[\cosh(4kt) - 2\cosh[4k(t-2v)] + 1],$$

plus a term which integrates to zero. Thus our integral can be written as

$$\int_0^t dv \int dk \frac{(k/2)[2\cosh[4k(t-2v)] - \cosh(4kt) - 1]}{4\pi^2 \sinh^3(2kt)}.$$

I claim that the integrand is absolutely integrable. Indeed, the numerator vanishes to third order in k with coefficient of magnitude less than $4t^2$, so the integrand is bounded. The derivative of the integrand with respect to v is

$$-(8k) \sinh[4k(t-2v)]/[4\pi^2 \sinh^3(2kt)].$$

This is equal to zero only on the line $v = t/2$. Any extremum of the integrand will therefore occur on the lines $v = 0, t/2, t$. Thus the magnitude of the integrand is bounded by the maximum of $(k/2)[\cosh(4kt) - 1]/4\pi^2 \sinh^3(2kt)$, which is obviously a bounded function. Denote this bound by C . Obviously, we can bound the magnitude of the numerator by $(2k) \cosh(4kt)$. Therefore we can bound the magnitude of the integrand by a function equal to C for $k < K_0$, for some $K_0 > 0$, and equal to $(2k) \cosh(4kt)/4\pi^2 \sinh^3(2kt)$ on the rest of $[0, t] \times R$. But this function is integrable, so that my claim follows. Thus, by the Lebesgue-Fubini theorem we can switch the order of integration of v and k . Now

$$2 \int_0^t dv \cosh[4k(t-2v)] = \frac{1}{2k} \sinh(4kt).$$

Therefore, our integral is equal to

$$\begin{aligned} & - \int dk \left\{ \frac{kt + (kt) \cosh(4kt) - \frac{1}{2} \sinh(4kt)}{8\pi^2 \sinh^3(2kt)} \right\} \\ & = -t^{-1} \int du \left\{ \frac{2u + (2u) \cosh(4u) - \sinh(4u)}{16\pi^2 \sinh^3(2u)} \right\}. \end{aligned}$$

Now,

$$\begin{aligned} & \int_{-\infty}^{\infty} du \frac{2u + (2u) \cosh(4u) - \sinh(4u)}{16\pi^2 \sinh^3(2u)} \\ & = \int_{-\infty}^{\infty} du \frac{4u + (4u) \sinh^2(2u) - \sinh(4u)}{16\pi^2 \sinh^3(2u)} \\ (*) \quad & = \int_{-\infty}^{\infty} du \frac{u + (u) \sinh^2(u) - \frac{1}{2} \sinh(2u)}{16\pi^2 \sinh^3(u)} \\ & = \int_{-\infty}^{\infty} du \frac{u}{16\pi^2 \sinh(u)} + \int_{-\infty}^{\infty} du \frac{u - \frac{1}{2} \sinh(2u)}{16\pi^2 \sinh^3(u)}. \end{aligned}$$

But

$$\int du \frac{\sinh(2u)}{\sinh^3(u)} = -\frac{2}{\sinh(u)}.$$

Also, in Gradshteyn and Ryzhik [8, p. 126, number 2.477.19], we find

$$\int du \frac{u}{\sinh^3(u)} = -\frac{1}{2} \left\{ \frac{(u) \cosh(u)}{\sinh^2(u)} + \frac{1}{\sinh(u)} + \int du \frac{u}{\sinh(u)} \right\}.$$

Thus (*) is equal to

$$\begin{aligned} & \frac{1}{2} \int_{-\infty}^{\infty} du \frac{u}{16\pi^2 \sinh(u)} - \lim_{u \rightarrow \infty} \left\{ \frac{(u) \cosh(u) - \sinh(u)}{\sinh^2(u)} \right\} \\ & = \frac{1}{2} \int_{-\infty}^{\infty} du \frac{u}{16\pi^2 \sinh(u)}. \end{aligned}$$

But, Gradshteyn and Ryzhik #3.521.1 is

$$\int_0^\infty du \frac{u}{\sinh(u)} = \frac{\pi^2}{4},$$

so $(*)$ is $1/64$. Thus,

$$P(t, 0, 0) \sim 1/16t^2 - 1/64t.$$

REMARK. The Weyl unitary trick converts representations of $\mathfrak{sl}(2, R)$ into representations of $\mathfrak{su}(2)$. We may apply this to our hypoelliptic operator on $\mathrm{SL}(2, R)$ to obtain asymptotics for a hypoelliptic diffusion equation on $\mathrm{SU}(2)$. Begin by considering the differential equation

$$\dot{M} = i \begin{vmatrix} u & v \\ v & u \end{vmatrix} M$$

on $\mathfrak{gl}(2, C)$. This amounts to the differential equation on C^4

$$\begin{vmatrix} \dot{w} \\ x \\ y \\ z \end{vmatrix} = \begin{vmatrix} 0 & iu & iv & 0 \\ iu & 0 & 0 & -iv \\ iv & 0 & 0 & iu \\ 0 & -iv & iu & 0 \end{vmatrix} \begin{vmatrix} w \\ x \\ y \\ z \end{vmatrix}.$$

Apply to this differential equation the unitary change of basis

$$U = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{vmatrix}$$

to obtain the differential equation

$$\begin{vmatrix} \dot{w} \\ x \\ y \\ z \end{vmatrix} = \begin{vmatrix} 0 & -u & -v & 0 \\ u & 0 & 0 & -v \\ v & 0 & 0 & u \\ 0 & v & -u & 0 \end{vmatrix} \begin{vmatrix} w \\ x \\ y \\ z \end{vmatrix}.$$

To translate these matters into geometry, one merely multiplies the vector fields g_1 and g_2 by i and transforms coordinates by $w \rightarrow w$, $x \rightarrow ix$, $y \rightarrow iy$, $z \rightarrow z$. One easily computes in this situation that

$$\begin{aligned} L = g_1^2 + g_2^2 &= \Delta - (x^2 + y^2) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \\ &\quad + \text{terms of higher homogeneity} \\ &\quad + \text{terms which do not contribute.} \end{aligned}$$

An inspection of the calculation in the last section reveals that on $\mathrm{SU}(2)$, therefore,

$$p(t, \varsigma, \varsigma) \sim 1/16t^2 + 1/64t + O(t^{-1/2}).$$

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